Hindawi
Complexity
Volume 2019, Article ID 4823709, 15 pages
https://doi.org/10.1155/2019/4823709

Research Article

Quasi-Matrix and Quasi-Inverse-Matrix Projective Synchronization for Delayed and Disturbed Fractional Order Neural Network

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Received 4 January 2019; Accepted 24 March 2019; Published 11 April 2019

Academic Editor: Toshikazu Kuniya

This paper is concerned with the quasi-matrix and quasi-inverse-matrix projective synchronization between two nonidentical delayed fractional order neural networks subjected to external disturbances. First, the definitions of quasi-matrix and quasi-inverse-matrix projective synchronization are given, respectively. Then, in order to realize two types of synchronization for delayed and disturbed fractional order neural networks, two sufficient conditions are established and proved by constructing appropriate Lyapunov function in combination with some fractional order differential inequalities. And their estimated synchronization error bound is obtained, which can be reduced to the required standard as small as what we need by selecting appropriate control parameters. Because of the generality of the proposed synchronization, choosing different projective matrix and controllers, the two synchronization types can be reduced to some common synchronization types for delayed fractional order neural networks, like quasi-complete synchronization, quasi-antisynchronization, quasi-projective synchronization, quasi-inverse projective synchronization, quasi-modified projective synchronization, quasi-inverse-modified projective synchronization, and so on. Finally, as applications, two numerical examples with simulations are employed to illustrate the efficiency and feasibility of the new synchronization analysis.

1. Introduction

Fractional calculus, which is applied to deal with differentiation and integration of arbitrary noninteger orders, has become an important and powerful tool to research the practical problems in many subjects [1, 2]. Fractional order phenomenon is ubiquitous in the real world and has strong memory and hereditary characteristic, so fractional model can better describe the dynamical properties and internal structure of many classical problems than integer ones. In recent years, many valuable results of fractional order dynamical systems have been obtained and widely applied in many areas, such as mathematical physics [1–11], optimum theory [12], financial problems [13], anomalous diffusion [14], secure communication [15, 16], biological systems [17, 18], and heat transfer process [19]. These research works illustrate the practicality and importance of fractional calculus and promote its development.

Neural network has attracted more and more attention since the introduction of fractional calculus and its dynamical behaviors, such as chaos, hyperchaos, bifurcations [20–22], existence, stability and consensus [23–30], and control and synchronization [31–40], have been widely studied. Recently, its synchronization problem has become a research focus and attracted many researchers. In [32, 33], the authors considered the adaptive pinning synchronization and finite time synchronization for delayed fractional order neural network. In [34], He and his cooperators explored quasi-synchronization problem of heterogeneous dynamic networks via distributed impulsive control.

Moreover, some scholars have studied the synchronization and quasi-synchronization problems for delayed fractional order memristor-based neural network with uncertain parameters [35–40]. These studies have promoted the development of fractional order neural networks to some extent, but most of them are aimed at the special complete synchronization problem, so it is still an important problem to propose a more general and practical synchronization type.

Projective synchronization, where the drive and response systems could be synchronized up to a scaling factor, is an important concept in practical applications. Nowadays, many researchers have introduced the projective synchronization into fractional order neural network. In [41], the authors applied LMI-based method to realize the global projective synchronization for fractional order neural network. In [42–44], the scholars derived some new sufficient conditions and designed the appropriate controllers to guarantee projective synchronization for delayed fractional order memristor-based neural network. In [45], by using comparison principle, Zhang and her cooperators designed suitable controllers to reach projective synchronization for delayed fractional order neural network. In [46], Wu and his cooperators introduced new sliding mode control laws to realize projective synchronization for nonidentical fractional order neural network in finite time. And, in [47, 48], researchers explored projective synchronization and quasi-projective synchronization for fractional order neural network in complex domain.

However, in the above research works, the proportion factor of projective synchronization is a fixed constant, while the simple scaling factor like this maybe does not guarantee high security of the image encryption and text encryption in communication. It is an important and meaningful work to extend the scaling factor to an arbitrary constant matrix and propose a more general synchronization type. So a new synchronization type, i.e., matrix projective synchronization, whose scaling factor is a constant matrix, appears and it can realize faster and safer communication. Additionally, another interesting problem is the inverse case of matrix projective synchronization, that is, when each drive system state synchronizes with a linear combination of response system states. Obviously, complexity of the scaling factors in matrix and inverse-matrix projective synchronization can have important effect in applications. Besides, it is well known that time delay is unavoidable due to finite switching speeds of the amplifiers, and it may cause oscillations or instability of dynamic systems. And external disturbances for the fractional order neural network can result in complicated topological structures because of the complexity and uncertainty of fractional nonlinear systems. Therefore, researching two more general synchronization types for delayed fractional order neural network with external disturbances is a meaningful problem.

According to the aforementioned discussions, this work aims to address these problems and present two more general synchronization types, i.e., fractional quasi-matrix and quasi-inverse-matrix projective synchronization, and establish the synchronization criteria for delayed and disturbed fractional order neural network. The remainder of this paper is organized as follows.

In Section 2, some lemmas of fractional calculus are introduced and n-dimensional delayed and disturbed fractional order neural network is constructed. In Section 3, fractional quasi-matrix and quasi-inverse-matrix projective synchronization are defined and the sufficient criteria for realizing two synchronization types of the delayed and disturbed fractional order neural networks are derived by means of Lyapunov function and some fractional order properties. In Section 4, as applications, quasi-matrix projective synchronization for two 2-dimensional and quasi-inverse-matrix projective synchronization types for two 3-dimensional delayed and disturbed fractional order neural networks are realized, respectively. And numerical simulations demonstrate the feasibility of synchronization analysis. Conclusions are given in Section 5.

2. Preliminaries and System Description

The Caputo derivative of order $\alpha > 0$ for a function $f(t)$ is defined as [1]

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{n}(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi, \quad (t \geq t_0), \tag{1} $$

where t_0 and t are the limits of Caputo derivative operation ${}_{t_0}^{C}D_t^{\alpha}$, $\Gamma(\cdot)$ is Euler's Gamma function, that is, $\Gamma(q) = \int_0^{+\infty} e^{-t} t^{q-1} dt$, $f^{(n)}$ represents the nth-order derivative of $f(x)$, and n is the positive integer satisfying $n-1 < \alpha \leq n$. When $\alpha = 1$, the operation ${}_{t_0}^{C}D_t^{\alpha} f(t)$ coincides with the integer order derivative $df(t)/dt$.

Lemma 1 (see [25]). *If $f(t) \in C^1([0,+\infty), R)$ denotes a continuously differentiable function, the following inequality holds almost everywhere*

$$ {}_{0}^{C}D_t^{\alpha} \left| f(t) \right| \leq \operatorname{sgn}(f(t)) \, {}_{0}^{C}D_t^{\alpha} f(t), \quad (0 < \alpha \leq 1). \tag{2} $$

Lemma 2 (see [29]). *Consider the following fractional order differential inequality with time delay*

$$ \begin{gathered} {}_{0}^{C}D_t^{\alpha} x(t) \leq -a x(t) + b x(t-\tau), \quad 0 < \alpha \leq 1, \\ x(t) = h(t), \quad t \in [-\tau, 0], \; h(t) \geq 0, \end{gathered} \tag{3} $$

and linear fractional order differential system with time delay

$$ \begin{gathered} {}_{0}^{C}D_t^{\alpha} y(t) = -a y(t) + b y(t-\tau), \quad 0 < \alpha \leq 1, \\ y(t) = h(t), \quad t \in [-\tau, 0], \; h(t) \geq 0, \end{gathered} \tag{4} $$

where $x(t)$ and $y(t)$ are continuous and nonnegative in $(0,+\infty)$; if $a > 0$ and $b > 0$, then

$$ x(t) \leq y(t), \quad \forall t \in [0, +\infty). \tag{5} $$

Lemma 3 (see [32]). *Let $V(t) \in R^1$ be a continuously differentiable and nonnegative function and satisfy*

$$ \begin{gathered} {}_{0}^{C}D_t^{\alpha} V(t) \leq -a V(t) + b V(t-\tau), \quad 0 < \alpha \leq 1, \\ V(t) = \varphi(t) \geq 0, \quad t \in [-\tau, 0], \end{gathered} \tag{6} $$

where $t \in [0,+\infty)$. If $a > b > 0$ for all $\varphi(t) \geq 0, \tau > 0$, then $\lim_{t \longrightarrow +\infty} V(t) = 0$.

Consider two nonidentical n-dimensional delayed fractional order neural networks, which are subjected to external disturbances, as the drive system and response system, respectively:

$$
\begin{aligned}
{}_0^C D_t^\alpha x_i(t) = {} & -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) \\
& + \sum_{j=1}^{n} c_{ij} g_j\left(x_j(t-\tau)\right) + d_i \xi_i(t),
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
{}_0^C D_t^\alpha y_i(t) = {} & -\hat{a}_i y_i(t) + \sum_{j=1}^{n} \hat{b}_{ij} f_j\left(y_j(t)\right) \\
& + \sum_{j=1}^{n} \hat{c}_{ij} g_j\left(y_j(t-\tau)\right) + \hat{d}_i \eta_i(t) + u_i(t), \\
& (0 < \alpha < 1),
\end{aligned} \tag{8}
$$

where $i = 1, 2, \ldots, n$ and n is the number of units in a neural network. $(x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ and $(y_1(t), y_2(t), \ldots, y_n(t))^T \in R^n$ denote the state variables. $a_i, \hat{a}_i > 0$ are self-regulating parameters of neurons. $f_j(x_j(t)), f_j(y_j(t))$, $g_j(x_j(t-\tau))$, and $g_j(y_j(t-\tau))$ express neuron activation functions at time t and $t-\tau$. $b_{ij}, \hat{b}_{ij}$ and $c_{ij}, \hat{c}_{ij}$ denote synaptic connection weight of unit j to unit i. $d_i\xi_i(t)$, $\hat{d}_i\eta_i(t)$ are different bounded external disturbances and $|d_i\xi_i(t)| \le P_i$, $|\hat{d}_i\eta_i(t)| \le Q_i$. $u_i(t)$ is the controller to be designed later.

Assumption 4. Neuron activation functions f_j, g_j are continuous and satisfy Lipschitz condition on R with Lipschitz constants $F_j, G_j > 0$ as

$$
\begin{aligned}
& \left| f_j(u) - f_j(v) \right| < F_j |u - v|, \\
& \left| g_j(u) - g_j(v) \right| < G_j |u - v|, \\
& (u, v \in R).
\end{aligned} \tag{9}
$$

3. Main Results

In this section, by using the active control method, we will focus on designing the suitable controllers to realize the quasi-matrix and quasi-inverse-matrix projective synchronization types between systems (7) and (8).

3.1. Fractional Quasi-Matrix Projective Synchronization. Let's first define the quasi-matrix projective synchronization as follows.

Definition 5. Systems (7) and (8) are said to be quasi-matrix projective synchronization with error bound $\delta \ge 0$, if there exists $T \ge t_0$ such that, for all $t \ge T$ and initial values $e_i(0) = y_i(0) - \sum_{j=1}^{n} \Lambda_{ij} x_j(0)$, the synchronization error satisfies $\|e(t)\|_1 = \|y(t) - \Lambda x(t)\|_1 \le \delta$. Here $\Lambda = (\Lambda_{ij})_{n\times n}$ means an arbitrary constant projective matrix, $\|\cdot\|_1 = \sum_{i=1}^{n} |\cdot|$, $e = (e_1, e_2, \ldots, e_n)^T$, and $x_i(0)$ and $y_i(0)$ are the initial values of systems (7) and (8).

Next, let us research the quasi-matrix projective synchronization between systems (7) and (8). Taking Caputo derivative of both sides of error function $e_i = y_i - \sum_{j=1}^{n} \Lambda_{ij} x_j$ and substituting into (7) and (8), the error system can be obtained as

$$
\begin{aligned}
{}_0^C D_t^\alpha e_i(t) = {} & {}_0^C D_t^\alpha y_i(t) - \sum_{j=1}^{n} \Lambda_{ij}\, {}_0^C D_t^\alpha x_j(t) = -\hat{a}_i y_i(t) \\
& + \sum_{j=1}^{n} \hat{b}_{ij} f_j\left(y_j(t)\right) + \sum_{j=1}^{n} \hat{c}_{ij} g_j\left(y_j(t-\tau)\right) + \hat{d}_i \eta_i(t) \\
& - \sum_{j=1}^{n} \Lambda_{ij} \left(-a_j x_j(t) + \sum_{i=1}^{n} b_{ji} f_i\left(x_i(t)\right) \right. \\
& \left. + \sum_{i=1}^{n} c_{ji} g_i\left(x_i(t-\tau)\right) + d_j \xi_j(t) \right) + u_i(t).
\end{aligned} \tag{10}
$$

Constructing the control function $u_i(t)$ as

$$
\begin{aligned}
u_i(t) = {} & \hat{a}_i \sum_{j=1}^{n} \Lambda_{ij} x_j(t) - \sum_{j=1}^{n} \hat{b}_{ij} f_j\left(\sum_{i=1}^{n} \Lambda_{ji} x_i(t) \right) \\
& - \sum_{j=1}^{n} \hat{c}_{ij} g_j\left(\sum_{i=1}^{n} \Lambda_{ji} x_i(t-\tau) \right) + \sum_{j=1}^{n} \Lambda_{ij} \left(-a_j x_j(t) \right. \\
& \left. + \sum_{i=1}^{n} b_{ji} f_i\left(x_i(t)\right) + \sum_{i=1}^{n} c_{ji} g_i\left(x_i(t-\tau)\right) \right) - k_i e_i(t), \\
& (k_i > 0),
\end{aligned} \tag{11}
$$

and substituting it into (10), the error system is changed to

$$
\begin{aligned}
& {}_0^C D_t^\alpha e_i(t) \\
& = \left(-\hat{a}_i - k_i\right) e_i(t) \\
& \quad + \sum_{j=1}^{n} \hat{b}_{ij} \left(f_j\left(y_j(t)\right) - f_j\left(\sum_{i=1}^{n} \Lambda_{ji} x_i(t) \right) \right) \\
& \quad + \sum_{j=1}^{n} \hat{c}_{ij} \left(g_j\left(y_j(t-\tau)\right) - g_j\left(\sum_{i=1}^{n} \Lambda_{ji} x_i(t-\tau) \right) \right) \\
& \quad + \hat{d}_i \eta_i(t) - \sum_{j=1}^{n} \Lambda_{ij} d_j \xi_j(t).
\end{aligned} \tag{12}
$$

Because of different external disturbances for two systems, $\bar{e} = 0$ is not the equilibrium point of system (12). So the complete synchronization between systems (7) and (8) cannot be realized. However, the quasi-matrix projective synchronization can be investigated.

Theorem 6. *Suppose Assumption 4 holds and the following inequality is satisfied:*

$$\min_{1\le i\le n}\left(\widehat{a}_i + k_i - \sum_{j=1}^{n}\left|\widehat{b}_{ji}\right| F_i\right) > \max_{1\le i\le n}\left(\sum_{j=1}^{n}\left|\widehat{c}_{ji}\right| G_i\right), \quad (13)$$

then the drive system (7) and response system (8) with control law (11) will achieve the quasi-matrix projective synchronization with the error bound $D_1/(\lambda_1 - \beta_1) + \varepsilon$, *where* $\lambda_1 = \min_{1\le i\le n}(\widehat{a}_i + k_i - \sum_{j=1}^{n}|\widehat{b}_{ji}|F_i)$, $\beta_1 = \max_{1\le i\le n}(\sum_{j=1}^{n}|\widehat{c}_{ji}|G_i)$, $D_1 = \sum_{i=1}^{n}(Q_i + |\sum_{j=1}^{n}\Lambda_{ij}P_j|)$, *and* $0 < \varepsilon << 1$ *is an arbitrary small constant.*

Proof. Construct the Lyapunov function as $V(t) = \sum_{i=1}^{n}|e_i(t)|$; then $V(t-\tau) = \sum_{i=1}^{n}|e_i(t-\tau)|$. According to Lemma 1 and Assumption 4, taking Caputo derivative of $V(t)$ along trajectory of error equation (12), one can get

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &= {}_0^C D_t^\alpha\left(\sum_{i=1}^{n}|e_i(t)|\right) = \sum_{i=1}^{n}{}_0^C D_t^\alpha |e_i(t)| \\
&\le \sum_{i=1}^{n}\operatorname{sgn}(e_i(t))\,{}_0^C D_t^\alpha e_i(t) \\
&= \sum_{i=1}^{n}\operatorname{sgn}(e_i(t))\Bigg((-\widehat{a}_i - k_i)e_i(t) \\
&\quad + \sum_{j=1}^{n}\widehat{b}_{ij}\left(f_j(y_j(t)) - f_j\left(\sum_{i=1}^{n}\Lambda_{ji}x_i(t)\right)\right) \\
&\quad + \sum_{j=1}^{n}\widehat{c}_{ij}\left(g_j(y_j(t-\tau)) - g_j\left(\sum_{i=1}^{n}\Lambda_{ji}x_i(t-\tau)\right)\right) \\
&\quad + \widehat{d}_i\eta_i(t) - \sum_{j=1}^{n}\Lambda_{ij}d_j\xi_j(t)\Bigg) \\
&\le \sum_{i=1}^{n}\Bigg((-\widehat{a}_i - k_i)|e_i(t)| + \sum_{j=1}^{n}\left|\widehat{b}_{ij}\right| F_j|e_j(t)| \\
&\quad + \sum_{j=1}^{n}\left|\widehat{c}_{ij}\right| G_j|e_j(t-\tau)|\Bigg) + \sum_{i=1}^{n}\left(Q_i + \left|\sum_{j=1}^{n}\Lambda_{ij}P_j\right|\right) \\
&= \sum_{i=1}^{n}\Bigg((-\widehat{a}_i - k_i)|e_i(t)| + \sum_{j=1}^{n}\left|\widehat{b}_{ji}\right| F_i|e_i(t)|\Bigg) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\widehat{c}_{ji}\right| G_i|e_i(t-\tau)| + \sum_{i=1}^{n}\left(Q_i + \left|\sum_{j=1}^{n}\Lambda_{ij}P_j\right|\right) \\
&\le -\sum_{i=1}^{n}\left(\min_{1\le i\le n}\left(\widehat{a}_i + k_i - \sum_{j=1}^{n}\left|\widehat{b}_{ji}\right| F_i\right)\right)|e_i(t)| \\
&\quad + \sum_{i=1}^{n}\left(\max_{1\le i\le n}\left(\sum_{j=1}^{n}\left|\widehat{c}_{ji}\right| G_i\right)\right)|e_i(t-\tau)| + \sum_{i=1}^{n}\Bigg(Q_i \\
&\quad + \left|\sum_{j=1}^{n}\Lambda_{ij}P_j\right|\Bigg) \le -\lambda_1 V(t) + \beta_1 V(t-\tau) + D_1.
\end{aligned} \quad (14)$$

Next, if system

$$\begin{aligned}{}_0^C D_t^\alpha S(t) &= -\lambda_1 S(t) + \beta_1 S(t-\tau) + D_1, \\ &\qquad (S(t) \ge 0,\ S(t) \in R),\end{aligned} \quad (15)$$

has the same initial values with $V(t)$, then ${}_0^C D_t^\alpha V(t) \le {}_0^C D_t^\alpha S(t)$. Using Lemma 2, we have

$$0 < V(t) \le S(t), \quad (\forall t \in [0, +\infty)). \quad (16)$$

By using properties of Caputo derivative, (15) is equivalent to

$$\begin{aligned}{}_0^C D_t^\alpha\left(S(t) - \widetilde{D}_1\right) &= -\lambda_1\left(S(t) - \widetilde{D}_1\right) \\ &\quad + \beta_1\left(S(t-\tau) - \widetilde{D}_1\right),\end{aligned} \quad (17)$$

where $\widetilde{D}_1 = D_1/(\lambda_1 - \beta_1)$. Letting $\widetilde{S}(t) = S(t) - \widetilde{D}_1$, system (17) becomes

$${}_0^C D_t^\alpha \widetilde{S}(t) = -\lambda_1\widetilde{S}(t) + \beta_1\widetilde{S}(t-\tau). \quad (18)$$

Because $\lambda_1 > \beta_1 > 0$, based on Lemma 3, we know $\lim_{t\longrightarrow+\infty}\widetilde{S}(t) = 0$. So

$$\widetilde{S}(t) = S(t) - \widetilde{D}_1 \longrightarrow 0, \quad (t \longrightarrow +\infty). \quad (19)$$

According to (16) and (19), we get $\forall\varepsilon > 0$, $\exists t_0$, when $t > t_0$,

$$0 < V(t) \le S(t) \le \widetilde{D}_1 + \varepsilon; \quad (20)$$

i.e.,

$$\begin{aligned}
V(t) &= \sum_{i=1}^{n}|e_i(t)| = \sum_{i=1}^{n}\left|y_i(t) - \sum_{j=1}^{n}\Lambda_{ij}x_j(t)\right| \\
&\le \widetilde{D}_1 + \varepsilon = \frac{D_1}{\lambda_1 - \beta_1} + \varepsilon \Longleftrightarrow \\
\|e(t)\|_1 &= |y(t) - \Lambda x(t)|_1 \le \frac{D_1}{\lambda_1 - \beta_1} + \varepsilon.
\end{aligned} \quad (21)$$

So, quasi-matrix projective synchronization with error bound $D_1/(\lambda_1-\beta_1)+\varepsilon$ between drive system (7) and response system (8) can be realized. This completes the proof. □

Remark 7. Substitute (11) into (8) and activate controller (11); response system (8) becomes

$$\begin{aligned}{}_{0}^{C}D_t^{\alpha}y_i(t) &= -\widehat{a}_i\left(y_i(t)-\sum_{j=1}^{n}\Lambda_{ij}x_j(t)\right)\\&\quad+\sum_{j=1}^{n}\widehat{b}_{ij}\left(f_j\left(y_j(t)\right)-f_j\left(\sum_{i=1}^{n}\Lambda_{ji}x_i(t)\right)\right)+\sum_{j=1}^{n}\widehat{c}_{ij}\\&\quad\cdot\left(g_j\left(y_j(t-\tau)\right)-g_j\left(\sum_{i=1}^{n}\Lambda_{ji}x_i(t-\tau)\right)\right)\\&\quad+\sum_{j=1}^{n}\Lambda_{ij}\left(-a_jx_j(t)+\sum_{i=1}^{n}b_{ji}f_i(x_i(t))\right.\\&\quad\left.+\sum_{i=1}^{n}c_{ji}g_i(x_i(t-\tau))\right)+\widehat{d}_i\eta_i(t)-k_ie_i(t).\end{aligned}\tag{22}$$

According to error system (12), drive system (7), and controlled response system (22), we can explore the quasi-matrix projective synchronization behaviors between fractional order neural networks (7) and (8).

3.2. Fractional Quasi-Inverse-Matrix Projective Synchronization. Next, let us define the quasi-inverse-matrix projective synchronization.

Definition 8. Systems (7) and (8) are said to be quasi-inverse-matrix projective synchronization with error bound $\delta \geq 0$, if there exists $T \geq t_0$ such that, for all $t \geq T$ and initial values $e_i(0) = x_i(0) - \sum_{j=1}^{n} M_{ij}y_j(0)$, the synchronization error satisfies $\|e(t)\|_1 = \|x(t) - My(t)\|_1 \leq \delta$, where $M = (M_{ij})_{n\times n}$ means an arbitrary invertible projective matrix.

Taking Caputo derivative of both sides of error function $e_i = x_i - \sum_{j=1}^{n} M_{ij}y_j$ and substituting into (7) and (8), the error system is

$$\begin{aligned}{}_{0}^{C}D_t^{\alpha}e_i(t) &= {}_{0}^{C}D_t^{\alpha}x_i(t)-\sum_{j=1}^{n}M_{ij}\,{}_{0}^{C}D_t^{\alpha}y_j(t) = -a_ix_i(t)\\&\quad+\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t)\right)+\sum_{j=1}^{n}c_{ij}g_j\left(x_j(t-\tau)\right)+d_i\xi_i(t)\\&\quad-\sum_{j=1}^{n}M_{ij}\left(-\widehat{a}_jy_j(t)+\sum_{i=1}^{n}\widehat{b}_{ji}f_i(y_i(t))\right.\\&\quad\left.+\sum_{i=1}^{n}\widehat{c}_{ji}g_i(y_i(t-\tau))+\widehat{d}_j\eta_j(t)+u_j(t)\right).\end{aligned}\tag{23}$$

Constructing the control function $u_i(t)$,

$$\begin{aligned}u_j(t) &= \sum_{i=1}^{n}\widehat{M}_{ji}\left(a_i\sum_{j=1}^{n}M_{ij}y_j(t)\right.\\&\quad-\sum_{j=1}^{n}b_{ij}f_j\left(\sum_{i=1}^{n}M_{ji}y_i(t)\right)\\&\quad\left.-\sum_{j=1}^{n}c_{ij}g_j\left(\sum_{i=1}^{n}M_{ji}y_i(t-\tau)\right)+k_ie_i(t)\right)+\widehat{a}_jy_j(t)\\&\quad-\sum_{i=1}^{n}\widehat{b}_{ji}f_i(y_i(t))-\sum_{i=1}^{n}\widehat{c}_{ji}g_i(y_i(t-\tau)),\\&\qquad\left(\widehat{M}=M^{-1},\ k_i>0\right),\end{aligned}\tag{24}$$

and substituting it into (23), error system changes to

$$\begin{aligned}&{}_{0}^{C}D_t^{\alpha}e_i(t)\\&= (-a_i-k_i)e_i(t)\\&\quad+\sum_{j=1}^{n}b_{ij}\left(f_j\left(x_j(t)\right)-f_j\left(\sum_{i=1}^{n}M_{ji}y_i(t)\right)\right)\\&\quad+\sum_{j=1}^{n}c_{ij}\left(g_j\left(x_j(t-\tau)\right)-g_j\left(\sum_{i=1}^{n}M_{ji}y_i(t-\tau)\right)\right)\\&\quad+d_i\xi_i(t)-\sum_{j=1}^{n}M_{ij}\widehat{d}_j\eta_j(t).\end{aligned}\tag{25}$$

Theorem 9. *Suppose Assumption 4 holds and the following inequality is satisfied:*

$$\min_{1\leq i\leq n}\left(a_i+k_i-\sum_{j=1}^{n}\left|b_{ji}\right|F_i\right)>\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|c_{ji}\right|G_i\right),\tag{26}$$

and then drive system (7) and response system (8) with control law (24) will achieve the quasi-inverse-matrix projective synchronization with the error bound $D_2/(\lambda_2-\beta_2)+\varepsilon$, *where* $\lambda_2 = \min_{1\leq i\leq n}(a_i+k_i-\sum_{j=1}^{n}|b_{ji}|F_i)$, $\beta_2 = \max_{1\leq i\leq n}(\sum_{j=1}^{n}|c_{ji}|G_i)$ *and* $D_2 = \sum_{i=1}^{n}(P_i + |\sum_{j=1}^{n} M_{ij}Q_j|)$.

Proof. Choose the Lyapunov function $V(t) = \sum_{i=1}^{n}|e_i(t)|$; then $V(t-\tau) = \sum_{i=1}^{n}|e_i(t-\tau)|$. By using Lemma 1 and Assumption 4, taking Caputo derivative of $V(t)$ along trajectory of error system (25), one can get

$$\begin{aligned}{}_{0}^{C}D_t^{\alpha}V(t) &= {}_{0}^{C}D_t^{\alpha}\left(\sum_{i=1}^{n}\left|e_i(t)\right|\right)=\sum_{i=1}^{n}{}_{0}^{C}D_t^{\alpha}\left|e_i(t)\right|\\&\leq\sum_{i=1}^{n}\operatorname{sgn}(e_i(t))\,{}_{0}^{C}D_t^{\alpha}e_i(t)\\&=\sum_{i=1}^{n}\operatorname{sgn}(e_i(t))\left((-a_i-k_i)e_i(t)\right.\\&\quad+\sum_{j=1}^{n}b_{ij}\left(f_j\left(x_j(t)\right)-f_j\left(\sum_{i=1}^{n}M_{ji}y_i(t)\right)\right)\\&\quad+\sum_{j=1}^{n}c_{ij}\left(g_j\left(x_j(t-\tau)\right)-g_j\left(\sum_{i=1}^{n}M_{ji}y_i(t-\tau)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&+ d_i\xi_i(t) - \sum_{j=1}^{n} M_{ij}\widehat{d}_j\eta_j(t)\Bigg) \\
&\leq \sum_{i=1}^{n}\Bigg((-a_i - k_i)\left|e_i(t)\right| + \sum_{j=1}^{n}\left|b_{ij}\right| F_j \left|e_j(t)\right| \\
&+ \sum_{j=1}^{n}\left|c_{ij}\right| G_j \left|e_j(t-\tau)\right|\Bigg) + \sum_{i=1}^{n}\left(P_i + \left|\sum_{j=1}^{n} M_{ij}Q_j\right|\right) \\
&= \sum_{i=1}^{n}\Bigg((-a_i - k_i)\left|e_i(t)\right| + \sum_{j=1}^{n}\left|b_{ji}\right| F_i \left|e_i(t)\right|\Bigg) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\left|c_{ji}\right| G_i \left|e_i(t-\tau)\right| + \sum_{i=1}^{n}\left(P_i + \left|\sum_{j=1}^{n} M_{ij}Q_j\right|\right) \\
&\leq -\sum_{i=1}^{n}\left(\min_{1\leq i\leq n}\left(a_i + k_i - \sum_{j=1}^{n}\left|b_{ji}\right| F_i\right)\right)\left|e_i(t)\right| \\
&+ \sum_{i=1}^{n}\left(\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left|c_{ji}\right| G_i\right)\right)\left|e_i(t-\tau)\right| + \sum_{i=1}^{n}\Bigg(P_i \\
&+ \left|\sum_{j=1}^{n} M_{ij}Q_j\right|\Bigg) \leq -\lambda_2 V(t) + \beta_2 V(t-\tau) + D_2.
\end{aligned} \tag{27}
$$

Then, by referring to (15)–(20) of the proof for Theorem 6, similarly we can know

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{n}\left|e_i(t)\right| = \sum_{i=1}^{n}\left|x_i(t) - \sum_{j=1}^{n} M_{ij}y_j(t)\right| \\
&\leq \widetilde{D}_2 + \varepsilon = \frac{D_2}{\lambda_2 - \beta_2} + \varepsilon \Longleftrightarrow \\
\|e(t)\|_1 &= \left|x(t) - My(t)\right|_1 \leq \frac{D_2}{\lambda_2 - \beta_2} + \varepsilon.
\end{aligned} \tag{28}
$$

So, quasi-inverse-matrix projective synchronization with error bound $D_2/(\lambda_2 - \beta_2) + \varepsilon$ between drive system (7) and response system (8) can be realized. This completes the proof. □

Remark 10. Substitute (24) into (8) and activate controller (24); response system (8) becomes

$$
\begin{aligned}
{}_0^C D_t^{\alpha} y_i(t) &= \sum_{j=1}^{n}\widehat{M}_{ij}\Bigg(a_j \sum_{i=1}^{n} M_{ji}y_i(t) \\
&- \sum_{i=1}^{n} b_{ji} f_i\left(\sum_{j=1}^{n} M_{ij}y_j(t)\right) \\
&- \sum_{i=1}^{n} c_{ji} g_i\left(\sum_{j=1}^{n} M_{ij}y_j(t-\tau)\right) + k_j e_j(t)\Bigg) + \widehat{d}_i\eta_i(t).
\end{aligned} \tag{29}
$$

Then, according to error system (25), drive system (7), and controlled response system (29), we can explore quasi-inverse-matrix projective synchronization behaviors between fractional order neural networks (7) and (8).

Remark 11. According to Theorems 6 and 9, choosing larger control parameter k, the error bound $D_i/(\lambda_i - \beta_i) + \varepsilon$, $(i = 1, 2)$ will become smaller. Therefore, by selecting appropriate control parameters, the synchronization error bound can be reduced to the required standard as small as what we need, which is of important and practical significance in nonlinear control and chaos synchronization for fractional order neural network.

Remark 12. When derivate order $\alpha = 1$, systems (7) and (8) are reduced to the integer order neural networks, from Theorems 6 and 9 and their proof; then we can obtain the quasi-matrix and quasi-inverse-matrix projective synchronization criteria for the disturbed and delayed integer order neural networks.

From the above, Theorems 6 and 9 and their proof process constitute the quasi-matrix and quasi-inverse-matrix projective synchronization method for synchronizing two disturbed and delayed fractional order neural networks. Additionally, it is particular to point out that the above two synchronization types are of general significance. Choosing different projective matrix and controller, they can be reduced to some special synchronization cases as in Remark 13.

Remark 13.

(1) Choosing projective matrix $\Lambda = M = I$, systems (7) and (8) can achieve the quasi-complete synchronization.

(2) Choosing projective matrix $\Lambda = M = -I$, they can achieve the quasi-antisynchronization.

(3) Choosing projective matrix $\Lambda = cI$ (*or* $M = cI$), (c = *const and* $c \neq \pm 1$), they can achieve the quasi-projective synchronization (or quasi-inverse projective synchronization).

(4) Choosing projective matrix $\Lambda = \mathrm{diag}(c_1, c_2, \ldots, c_n)$(*or* $M = \mathrm{diag}(c_1, c_2, \ldots, c_n)$), ($c_i$ = *const*, $i = 1, 2, \ldots, n$), they can achieve the quasi-modified projective synchronization (or quasi-inverse-modified projective synchronization).

(5) If external disturbances $d_i\xi_i(t)$, $d_i\eta_i(t) = 0$, they can achieve the complete matrix and inverse-matrix projective synchronization.

4. Some Applications

In this part, two numerical examples are presented to demonstrate the effectiveness and feasibility of the proposed theoretical results.

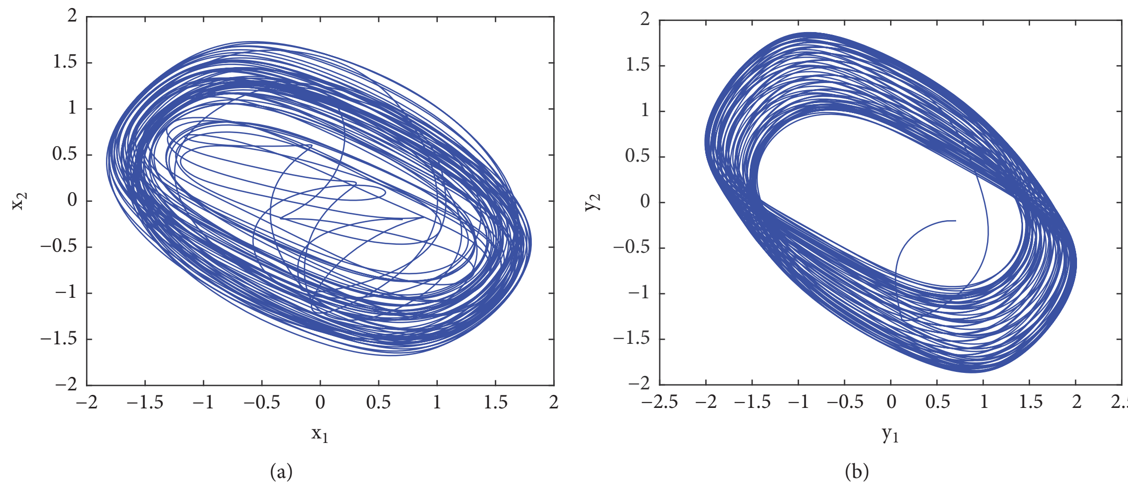


FIGURE 1: Chaotic attractors of systems (30) and (31) at $\alpha = 0.92$. (a) x_1-x_2 phase diagram. (b) y_1-y_2 phase diagram.

4.1. Application to Fractional Quasi-Matrix Projective Synchronization. Two nonidentical drive and response systems are considered as

$$\begin{aligned} {}_0^C D_t^\alpha x_i(t) &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} \tanh\left(x_j(t)\right) \\ &\quad + \sum_{j=1}^{2} c_{ij} \tanh\left(x_j(t-\tau)\right) + d_i \sin(t), \end{aligned} \tag{30}$$

$$\begin{aligned} {}_0^C D_t^\alpha y_i(t) &= -\hat{a}_i y_i(t) + \sum_{j=1}^{2} \hat{b}_{ij} \tanh\left(y_j(t)\right) \\ &\quad + \sum_{j=1}^{2} \hat{c}_{ij} \tanh\left(y_j(t-\tau)\right) + \hat{d}_i \cos(t) \\ &\quad + u_i(t), \quad (i = 1, 2), \end{aligned} \tag{31}$$

where

$$\begin{gathered} a_1 = 3.7 \\ a_2 = 1.7, \\ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -0.4 & 2.7 \end{pmatrix}, \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} -3.7 & -2.6 \\ -1.7 & -3.4 \end{pmatrix}, \\ d_1 = 0.1 \\ d_2 = -0.2, \\ \hat{a}_1 = 3.7 \\ \hat{a}_2 = 1.7, \\ \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{pmatrix} = \begin{pmatrix} 2.1 & -2.3 \\ -0.35 & 2.75 \end{pmatrix}, \\ \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} \\ \hat{c}_{21} & \hat{c}_{22} \end{pmatrix} = \begin{pmatrix} -3.8 & -2.65 \\ -1.8 & -3.6 \end{pmatrix}, \\ \hat{d}_1 = 0.3 \\ \hat{d}_2 = -0.4, \\ \tau = 1 \end{gathered} \tag{32}$$

and $u_i(t)$ is the controller. In the following numerical analysis, in order to research the chaotic synchronization between systems (30) and (31), we will select derivative order as $\alpha = 0.92$, which can make the two systems generate chaotic attractors as shown in Figures 1(a) and 1(b) with initial conditions $[x_1(0), x_2(0)] = [y_1(0), y_2(0)] = [0.7, -0.2]$.

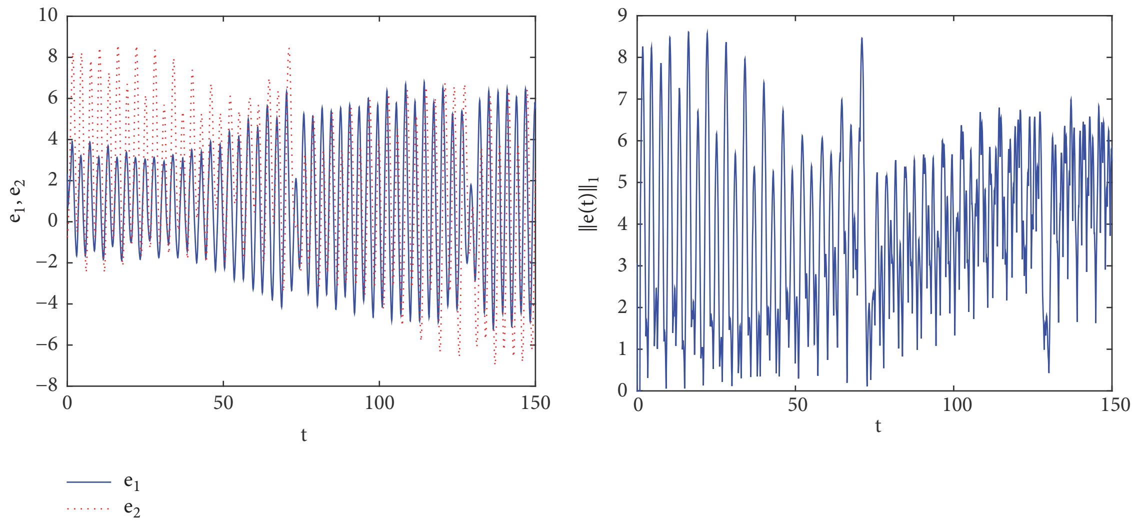
According to Definition 5, choosing projective matrix Λ as

$$\Lambda = \left(\Lambda_{ij}\right)_{2\times 2} = \begin{pmatrix} -2 & 1 \\ 0.5 & -1 \end{pmatrix} \tag{33}$$

then error function of quasi-matrix projective synchronization is computed as

$$\begin{aligned} e_1 &= y_1 - \sum_{j=1}^{2} \Lambda_{1j} x_j = y_1 - \left(-2x_1 + x_2\right), \\ e_2 &= y_2 - \sum_{j=1}^{2} \Lambda_{2j} x_j = y_2 - \left(0.5x_1 - x_2\right). \end{aligned} \tag{34}$$

Choosing $k_1 = 0$, $k_2 = 0$, one can easily obtain $\lambda = -3.35 < 0 < \beta = 6.25$ which obviously does not satisfy condition (13) of Theorem 6, so the time history of errors $e_1, e_2, \|e(t)\|_1$ cannot be limited to a bounded area as shown in Figures 2(a) and 2(b).



(a) (b)

Figure 2: Quasi-matrix projective synchronization errors e_1, e_2 and $\|e(t)\|_1$ without controller at $\alpha = 0.92$. (a) Trajectory of errors e_1, e_2. (b) Trajectory of error $\|e(t)\|_1$.

Choosing $k_1 = 6.35$, $k_2 = 11.35$, one can compute $\lambda_1 = \min_{1\le i\le 2}(\widehat{a}_i + k_i - \sum_{j=1}^{2} |\widehat{b}_{ji}|F_i) = \min\{7.6, 8\} = 7.6$, $\beta_1 = \max_{1\le i\le 2}(\sum_{j=1}^{2} |\widehat{c}_{ji}|G_i) = \max\{5.6, 6.25\} = 6.25$, $D_1 = \sum_{i=1}^{n}(Q_i + |\sum_{j=1}^{n} \Lambda_{ij}P_j|) = 1.35$. By Theorem 6, quasi-matrix projective synchronization with the estimated error bound $\delta = 1.001$ between systems (30) and (31) can be realized where $D_1/(\lambda_1 - \beta_1) = 1.35/(7.6 - 6.25) = 1$ and $\varepsilon = 0.001$. And there exists $T \ge t_0$ such that $|e_i(t)| \le \|e(t)\|_1 < 1.001$ for all $t \ge T$.

If a smaller error bound needs to be controlled to $\delta = 0.301$, then the larger control parameters should be designed as $k_1 = 9.5$, $k_2 = 14.35$. Next, one can calculate $\lambda_1 = \min_{1\le i\le 2}(\widehat{a}_i + k_i - \sum_{j=1}^{2} |\widehat{b}_{ji}|F_i) = \min\{10.75, 11\} = 10.75$. By Theorem 6, the quasi-matrix projective synchronization with the smaller error bound $\delta = 0.301$ between systems (30) and (31) can be realized where $D_1/(\lambda_1 - \beta_1) = 1.35/(10.75 - 6.25) = 0.3$ and $\varepsilon = 0.001$. And there exists $T \ge t_0$ such that $|e_i(t)| \le \|e(t)\|_1 < 0.301$ for all $t \ge T$.

Next, numerical simulation is given to verify the analysis of fractional quasi-matrix projective synchronization. The initial conditions for drive system, response system, and errors system are $[x_1(0), x_2(0)] = [y_1(0), y_2(0)] = [0.7, -0.2]$ and $[e_1(0), e_2(0), \|e(t)\|_1(0)] = [2.3, -0.75, 3.05]$. Figures 3(a)–3(d) show the time trajectories of quasi-matrix projective synchronization with projective matrix $\Lambda = \left(\begin{smallmatrix} -2 & 1 \\ 0.5 & -1 \end{smallmatrix}\right)$. Figures 4(a)-4(b) depict the time history of synchronization error with error bound $\delta = 1.001$ under control parameters $k_1 = 6.35$, $k_2 = 11.35$ and $|e_i(t)| \le \|e(t)\|_1 < 1.001$ can be acquired as $t > 5s$. Figures 5(a)-5(b) show the time evolution of synchronization error with a smaller error bound $\delta = 0.301$ under larger control parameters $k_1 = 9.5$, $k_2 = 14.35$ and $|e_i(t)| \le \|e(t)\|_1 < 0.301$ can be acquired as $t > 5s$, which verifies Remark 11 that the larger the control parameter k, the smaller the error bound $D_1/(\lambda_1 - \beta_1) + \varepsilon$.

4.2. Application to Fractional Quasi-Inverse-Matrix Projective Synchronization.

Two nonidentical 3-dimensional drive and response systems are constructed as

$$
\begin{aligned}
{}_{0}^{C}D_t^{\alpha} x_i(t) = {} & -a_i x_i(t) + \sum_{j=1}^{3} b_{ij} \tanh\left(x_j(t)\right) \\
& + \sum_{j=1}^{3} c_{ij} \tanh\left(x_j(t-\tau)\right) + d_i \sin(10t),
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
{}_{0}^{C}D_t^{\alpha} y_i(t) = {} & -\widehat{a}_i y_i(t) + \sum_{j=1}^{3} \widehat{b}_{ij} \tanh\left(y_j(t)\right) \\
& + \sum_{j=1}^{3} \widehat{c}_{ij} \tanh\left(y_j(t-\tau)\right) + \widehat{d}_i \cos(10t) \\
& + u_i(t), \quad (i = 1, 2, 3),
\end{aligned}
\tag{36}
$$

where

$$
\begin{gathered}
a_1 = 2.3 \\
a_2 = 1.3 \\
a_2 = 1.9, \\
\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 2.2 & -2.1 & 1.9 \\ -0.7 & 5.75 & 1.1 \\ -4.7 & -1 & 1.3 \end{pmatrix}, \\
\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} -4.1 & 2.6 & -3.1 \\ -1.6 & -3.5 & -2.5 \\ 0.3 & 1.9 & 1.1 \end{pmatrix},
\end{gathered}
$$

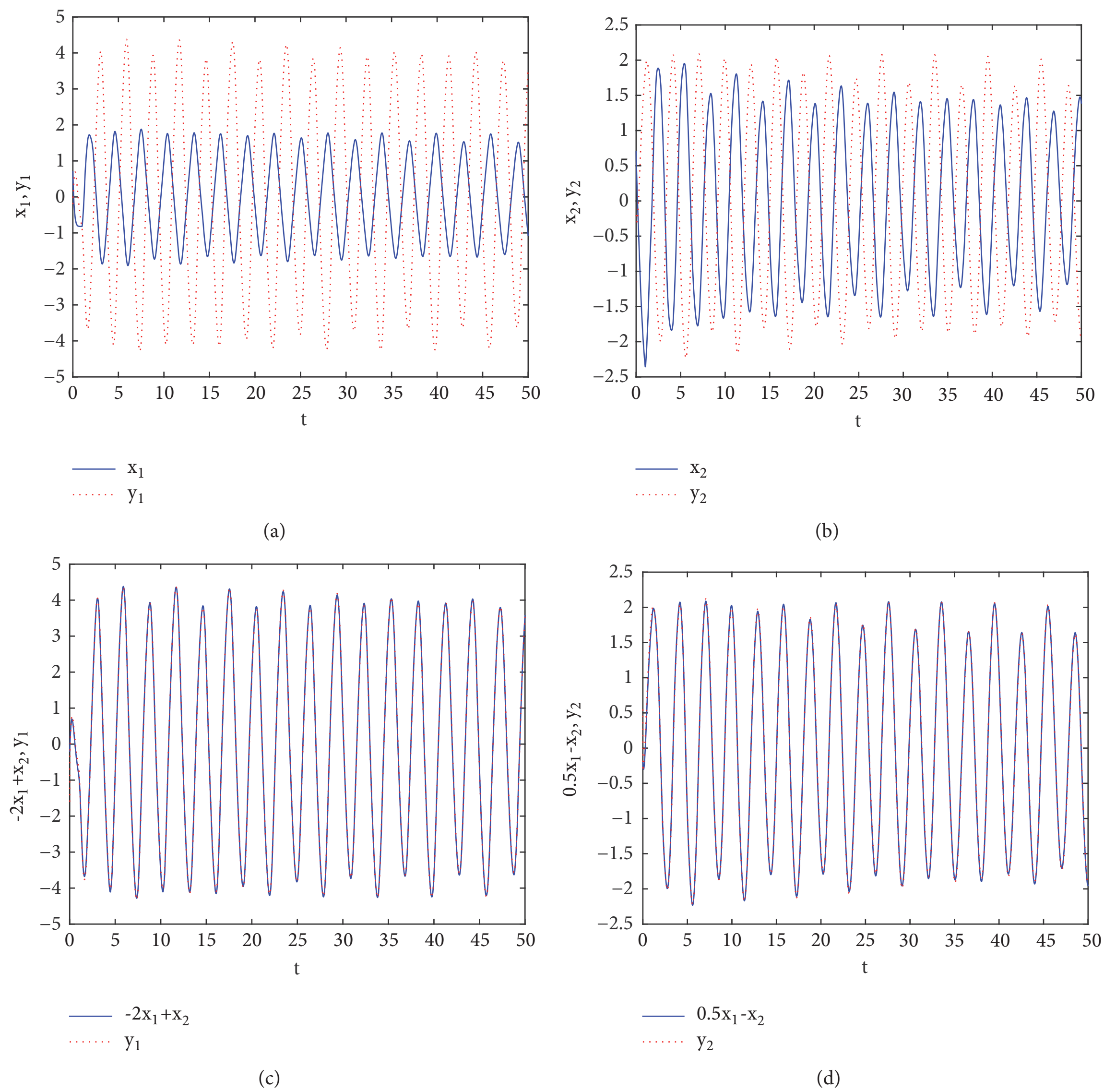


FIGURE 3: Quasi-matrix projective synchronization trajectories of state variables x, y with control parameters $k_1 = 6.35$, $k_2 = 11.35$ at $\alpha = 0.92$. (a) Time history of x_1, y_1. (b) Time history of x_2, y_2. (c) Time history of $-2x_1 + x_2$, y_1. (d) Time history of $0.5x_1 - x_2$, y_1.

$$\begin{gathered} d_1 = 0.2 \\ d_2 = -0.1 \\ d_3 = 0.1, \\ \hat{a}_1 = 2.3 \\ \hat{a}_2 = 1.3 \\ \hat{a}_3 = 1.9, \\ \begin{pmatrix} \hat{b}_{11} & \hat{b}_{12} & \hat{b}_{13} \\ \hat{b}_{21} & \hat{b}_{22} & \hat{b}_{23} \\ \hat{b}_{31} & \hat{b}_{32} & \hat{b}_{33} \end{pmatrix} = \begin{pmatrix} 2.1 & -2.1 & 1.8 \\ -0.6 & 5.7 & 1 \\ -4.5 & -0.9 & 1.2 \end{pmatrix}, \\ \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{21} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{31} & \hat{c}_{32} & \hat{c}_{33} \end{pmatrix} = \begin{pmatrix} -3.8 & 2.5 & -2.8 \\ -1.5 & -3.5 & -2.5 \\ 0.5 & 1.8 & 1.3 \end{pmatrix}, \\ \hat{d}_1 = 0.1 \\ \hat{d}_2 = -0.1 \\ \hat{d}_3 = 0.2, \\ \tau = 0.8 \end{gathered} \tag{37}$$

and $u_i(t)$ is the controller. In the following numerical analysis, in order to research the chaotic synchronization between

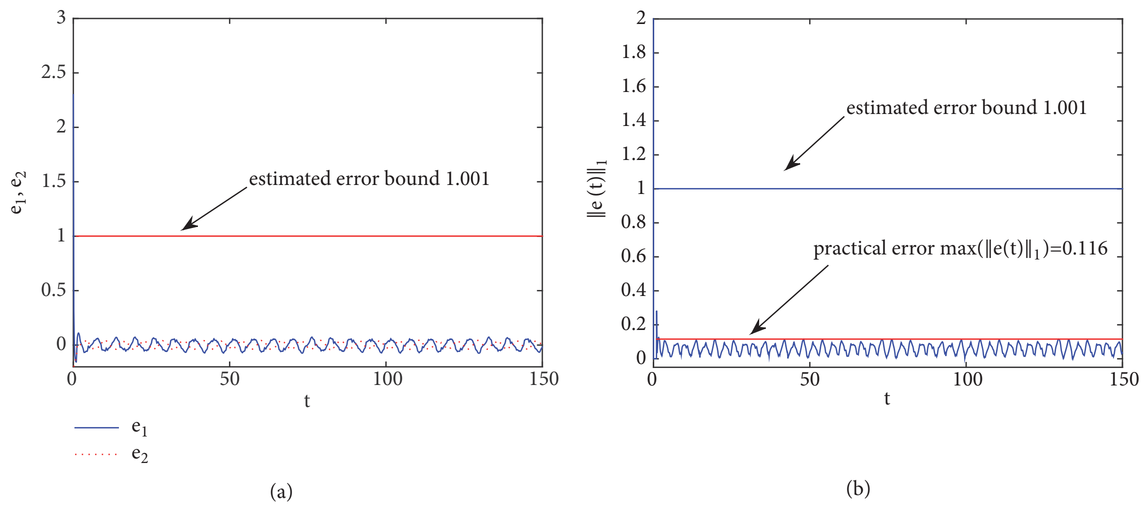


FIGURE 4: Quasi-matrix projective synchronization errors e_1, e_2 and $\|e(t)\|_1$ with control parameters $k_1 = 6.35$, $k_2 = 11.35$ at $\alpha = 0.92$. (a) Trajectory of errors e_1, e_2. (b) Trajectory of error $\|e(t)\|_1$.

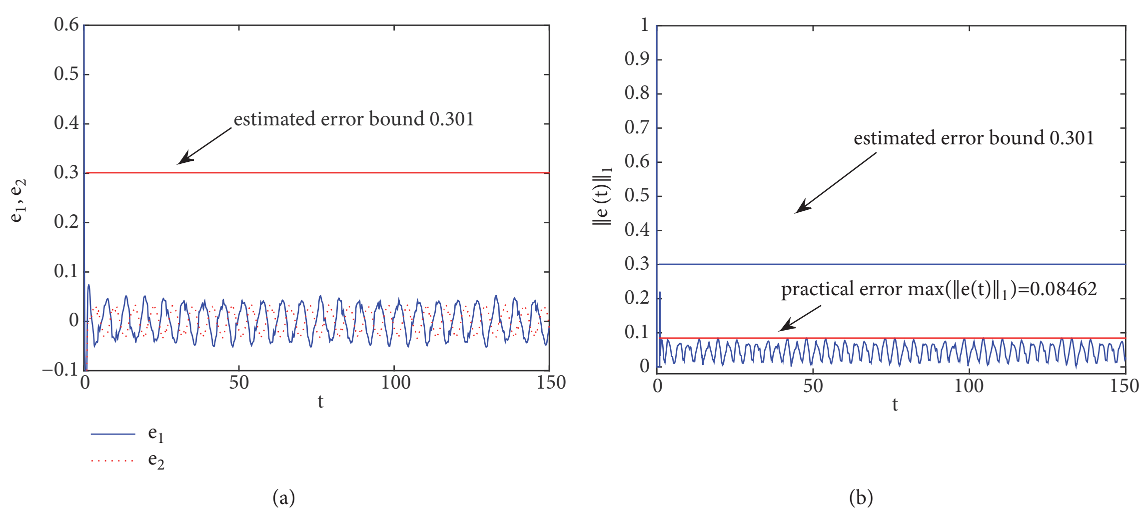


FIGURE 5: Quasi-matrix projective synchronization errors e_1, e_2 and $\|e(t)\|_1$ with control parameters $k_1 = 9.5$, $k_2 = 14.35$ at $\alpha = 0.92$. (a) Trajectory of errors e_1, e_2. (b) Trajectory of error $\|e(t)\|_1$.

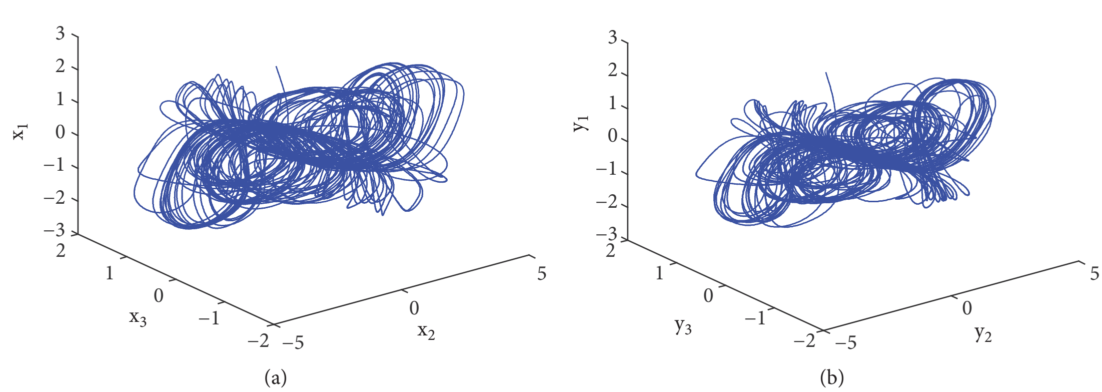


FIGURE 6: Chaotic attractors of systems (35) and (36) at $\alpha = 0.98$. (a) x_2-x_3-x_1 phase diagram. (b) y_2-y_3-y_1 phase diagram.

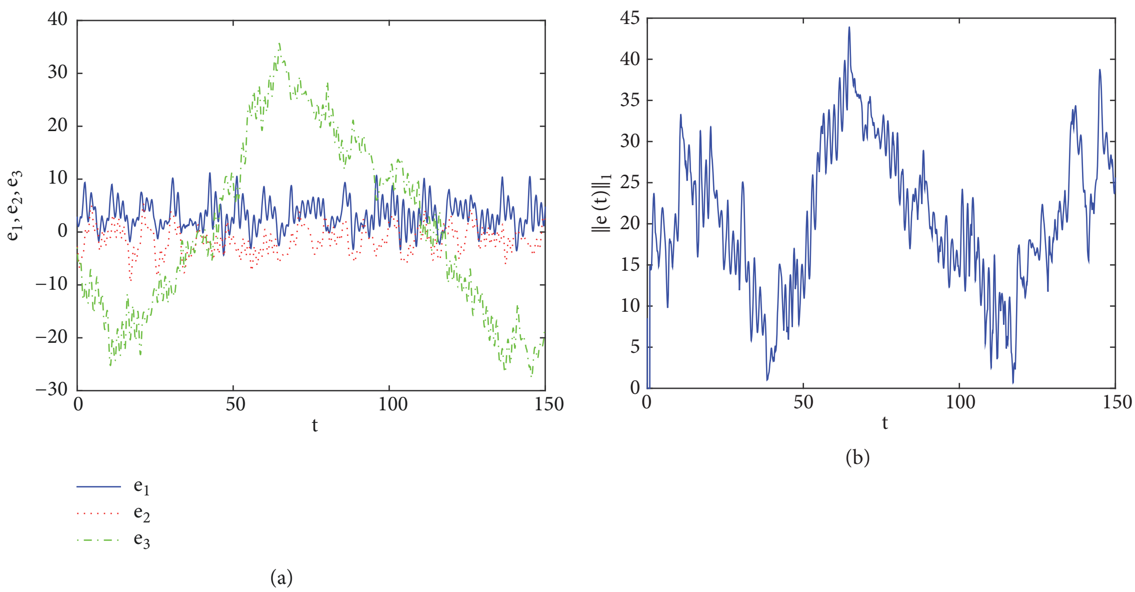


FIGURE 7: Quasi-inverse-matrix projective synchronization errors e_1, e_2, e_3 and $\|e(t)\|_1$ without controller at $\alpha = 0.98$. (a) Trajectory of errors e_1, e_2, e_3. (b) Trajectory of error $\|e(t)\|_1$.

systems (35) and (36), we will select derivative order as $\alpha = 0.98$, which can make the two systems exhibit the chaotic attractors as depicted in Figures 6(a) and 6(b) with initial conditions $[x_1(0), x_2(0), x_3(0)] = [y_1(0), y_2(0), y_3(0)] = [2.2, -0.3, 0.4]$.

According to Definition 8, selecting the projective matrix M and its inverse-matrix M^{-1} as

$$M = \left(M_{ij}\right)_{3\times 3} = \begin{pmatrix} -0.5 & 1 & 2 \\ 1 & 0.5 & 1 \\ 1 & -1 & 2 \end{pmatrix},$$
$$M^{-1} = \begin{pmatrix} -0.4 & 0.8 & 0 \\ 0.2 & 0.6 & -0.5 \\ 0.3 & -0.1 & 0.25 \end{pmatrix}, \quad (38)$$

then error function of quasi-inverse-matrix projective synchronization is computed as

$$\begin{aligned} e_1 &= x_1 - \sum_{j=1}^{3} M_{1j} y_j = x_1 - \left(-0.5y_1 + y_2 + 2y_3\right), \\ e_2 &= x_2 - \sum_{j=1}^{3} M_{2j} y_j = x_2 - \left(y_1 + 0.5y_2 + y_3\right), \\ e_3 &= x_3 - \sum_{j=1}^{3} M_{3j} y_j = x_3 - \left(y_1 - y_2 + 2y_3\right). \end{aligned} \quad (39)$$

Choosing $k_1 = 0$, $k_2 = 0$, $k_3 = 0$, one can easily obtain $\lambda = -7.55 < 0 < \beta = 8$ which does not satisfy condition (26) of Theorem 9, so the time history of error $\|e(t)\|_1$ cannot be limited to a bounded area as shown in Figures 7(a) and 7(b).

Choosing $k_1 = 16.3$, $k_2 = 19.55$, $k_3 = 15.4$, one can compute $\lambda_2 = \min_{1\le i\le 3}(\widehat{a}_i + k_i - \sum_{j=1}^{3} |\widehat{b}_{ji}| F_i) = \min(11, 12, 13) = 11$, $\beta_2 = \max_{1\le i\le 3}(\sum_{j=1}^{3} |\widehat{c}_{ji}| G_i) = \max(6, 8, 6.7) = 8$, $D_2 = \sum_{i=1}^{n} (P_i + |\sum_{j=1}^{n} M_{ij} Q_j|) = 1.5$. By Theorem 9, the quasi-inverse-matrix projective synchronization with the estimated error bound $\delta = 0.501$ between systems (35) and (36) can be realized where $D_2/(\lambda_2 - \beta_2) = 1.5/(11-8) = 0.5$ and $\varepsilon = 0.001$. And there exists $T \ge t_0$ such that $|e_i(t)| \le \|e(t)\|_1 < 0.501$ for all $t \ge T$.

If a smaller error bound needs to be controlled to $\delta = 0.301$, then larger control parameters should be designed as $k_1 = 18.3$, $k_2 = 32.55$, $k_3 = 25.4$. Next, one can calculate $\lambda_2 = \min_{1\le i\le 3}(\widehat{a}_i + k_i - \sum_{j=1}^{3} |\widehat{b}_{ji}| F_i) = \min\{13, 25, 23\} = 13$. By Theorem 9, the quasi-inverse-matrix projective synchronization with the smaller estimated error bound $\delta = 0.301$ between systems (35) and (36) is realized where $D_2/(\lambda_2 - \beta_2) = 1.5/(13 - 8) = 0.3$ and $\varepsilon = 0.001$. And there exists $T \ge t_0$ such that $|e_i(t)| \le \|e(t)\|_1 < 0.301$ for all $t \ge T$.

Next, numerical simulation is given to verify the analysis of fractional quasi-inverse-matrix projective synchronization. The initial conditions for drive-response system and error system are $[x_1(0), x_2(0), x_3(0)] = [y_1(0), y_2(0), y_3(0)] = [2.2, -0.3, 0.4]$ and $[e_1(0), e_2(0), e_3(0), \|e(t)\|_1(0)] = [2.8, -2.75, -2.9, 8.45]$. Figures 8(a)–8(f) show the time trajectories of synchronization for systems (35) and (36) with projective matrix $M = \begin{pmatrix} -0.5 & 1 & 2 \\ 1 & 0.5 & 1 \\ 1 & -1 & 2 \end{pmatrix}$. Figures 9(a)-9(b) depict the time history of synchronization error with error bound $\delta = 0.501$ under control parameters $k_1 = 16.3$, $k_2 = 19.55$, $k_3 = 15.4$ and $|e_i(t)| \le \|e(t)\|_1 < 0.501$ can be acquired as $t > 5s$. Figures 10(a)-10(b) show the time evolution of synchronization error with a smaller error bound $\delta = 0.301$ under larger control parameters $k_1 = 18.3$, $k_2 = 32.55$, and $k_3 = 25.4$ and $|e_i(t)| \le \|e(t)\|_1 < 0.301$ can be acquired as $t > 5s$, which also verifies Remark 11.

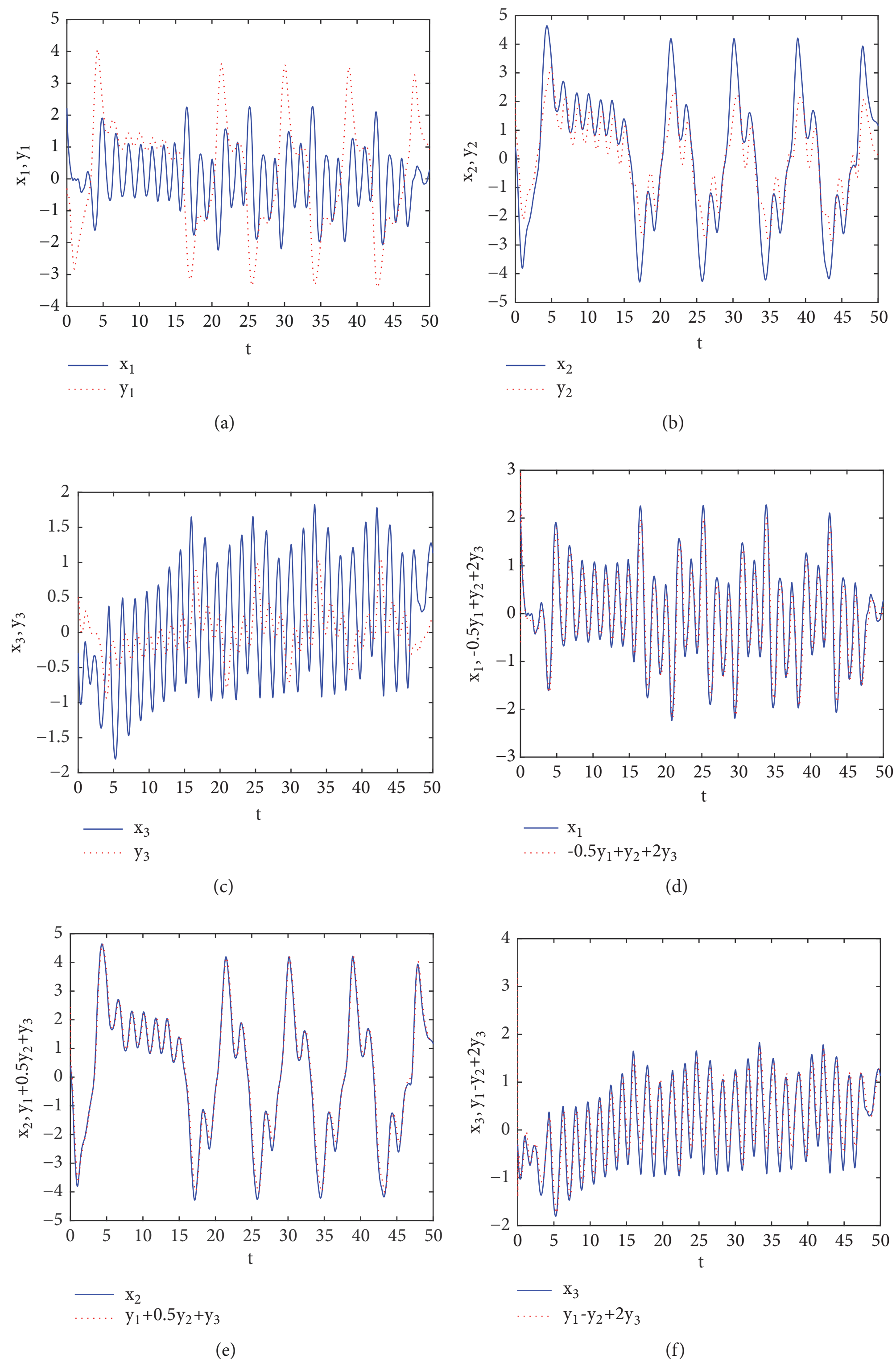


Figure 8: Quasi-inverse-matrix projective synchronization trajectories of state variables x, y with control parameters $k_1 = 16.3$, $k_2 = 19.55$, and $k_3 = 15.4$ at $\alpha = 0.98$. (a) Time history of x_1, y_1. (b) Time history of x_2, y_2. (c) Time history of x_3, y_3. (d) Time history of $x_1, -0.5y_1 + y_2 + 2y_3$. (e) Time history of x_2, $y_1 + 0.5y_2 + y_3$. (f) Time history of x_3, $y_1 - y_2 + 2y_3$.

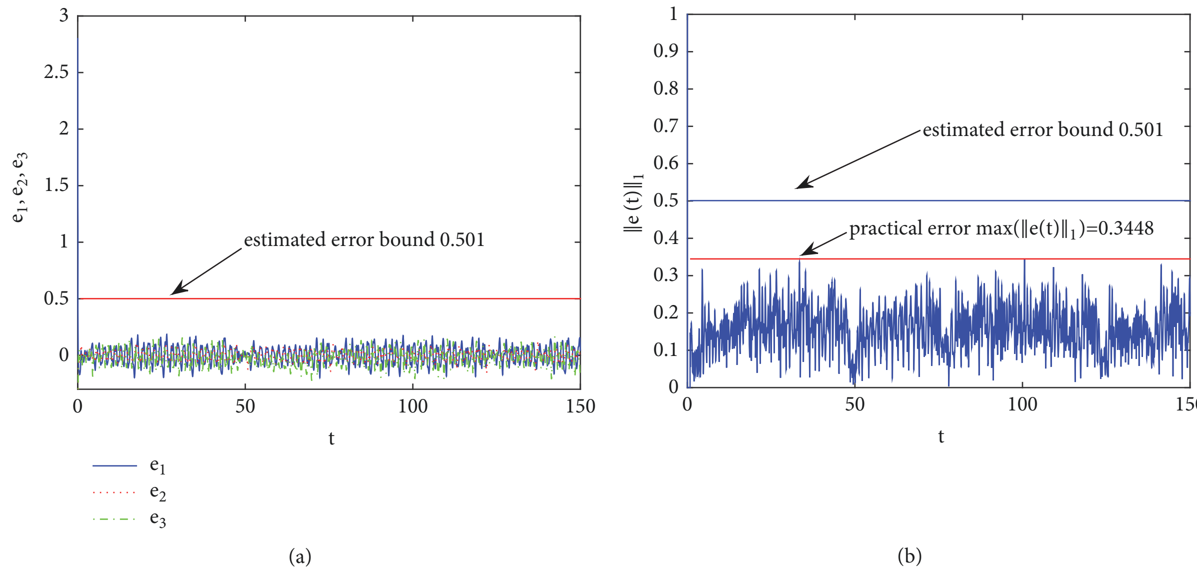


FIGURE 9: Quasi-inverse-matrix projective synchronization errors e_1, e_2, e_3 and $\|e(t)\|_1$ with control parameters $k_1 = 16.3$, $k_2 = 19.55$, and $k_3 = 15.4$ at $\alpha = 0.98$. (a) Trajectory of errors e_1, e_2, e_3. (b) Trajectory of error $\|e(t)\|_1$.

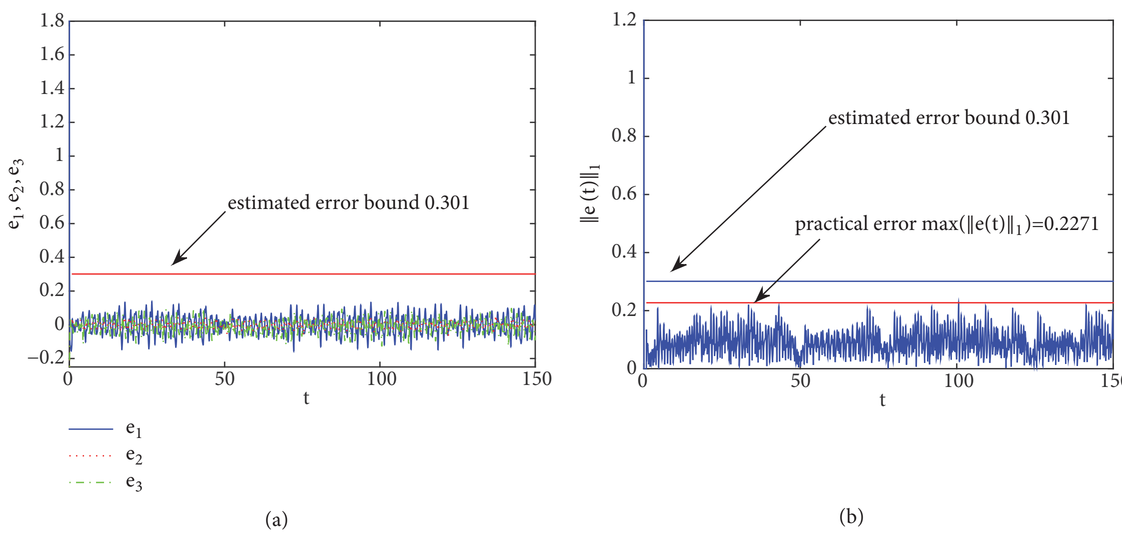


FIGURE 10: Quasi-inverse-matrix projective synchronization errors e_1, e_2, e_3 and $\|e(t)\|_1$ with control parameters $k_1 = 18.3$, $k_2 = 32.55$, and $k_3 = 25.4$ at $\alpha = 0.98$. (a) Trajectory of errors e_1, e_2, e_3. (b) Trajectory of error $\|e(t)\|_1$.

5. Conclusions

For synchronizing two nonidentical delayed and disturbed fractional order neural networks, the paper proposes the quasi-matrix and quasi-inverse-matrix projective synchronization and establishes their synchronization criteria. By selecting appropriate control parameters, the synchronization error bound is obtained and can be reduced to the required standard as small as what we need, which is of important significance to practical problem. Two numerical examples verify the feasibility of synchronization analysis. Simply put, Definitions 5 and 8, Theorems 6 and 9 and their proofs, and synchronization analysis of two numerical examples are all new work.

This research extends the projective scaling factor to an arbitrary constant matrix and offers a general approach for synchronizing the delayed and disturbed fractional order neural network. Evidently, fractional quasi-matrix and quasi-inverse-matrix projective synchronization can guarantee faster and safer image encryption and text encryption in communication and provide new insights for researching the fractional order neural network, which is a meaningful work. The complex-valued neural network with time delay, which is a more general neural network system, is becoming more and more popular, and its finite time stability, boundedness, global robust stability, and global exponential stability have been well researched in [22, 28, 40, 47–51]. So, in future works, by using adaptive control method and sliding mode control method, we will generalize our main results to quasi-matrix and quasi-inverse-matrix projective synchronization for delayed fractional order complex-variable neural networks, fractional order memristor-based neural network, and fractional order neural network with unknown parameters.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11872201, 11572148, 11632008, and 11772148).

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