Research Article

Dynamic Analysis of Stochastic Lotka–Volterra Predator–Prey Model with Discrete Delays and Feedback Control

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In this paper, a stochastic Lotka–Volterra predator–prey model with discrete delays and feedback control is studied. Firstly, the existence and uniqueness of global positive solution are proved. Further, we investigate the asymptotic property of stochastic system at the positive equilibrium point of the corresponding deterministic model and establish sufficient conditions for the persistence and extinction of the model. Finally, the correctness of the theoretical derivation is verified by numerical simulations.

1. Introduction

In nature, time delays exist in many ecosystems [1–5]. For example, maturity stage is a common phenomenon in biological population, and many diseases have a long incubation period. The mathematical model describing this phenomenon with time delay is called the delay differential equation. In 1999, Saito et al. [6] studied a Lotka–Volterra predator–prey model with discrete delays, which can be defined as follows:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[r_1 + ax(t) + \alpha x(t - \tau_1) - \beta y(t - \tau_2)], \\ \frac{dy(t)}{dt} = y(t)[r_2 + ay(t) + \beta x(t - \tau_1) + \alpha y(t - \tau_2)], \end{cases} \quad (1)$$

where $x(t)$ and $y(t)$ stand for the population density of prey and predator at time t , respectively. r_i ($i = 1, 2$) represent the intrinsic growth rate of corresponding population. τ_1 and τ_2 are discrete time delays. a ($a < 0$), α and β are constants.

Due to the environmental changes and increased human activities, many rare species are at risk of extinction. How to protect endangered species of floras and faunas and maintain the diversity of ecosystems is an important issue that needs to be solved urgently. In the process of marine fishery production, overfishing often results in the exhaustion of fishery resources. It is rewarding for humans to develop and utilize the ecological system of the population rationally, which also contributes to the sustainability of the system [7–14]. In 2003, Gopalsamy and Weng [15] studied the following population competition model with feedback control:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[b_1 - a_{11}x_1(t) - a_{12}x_2(t) - \alpha_1 u_1(t - \tau)], \\ \frac{dx_2(t)}{dt} = x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2(t) - \alpha_2 u_2(t - \tau)], \\ \frac{du_1(t)}{dt} = -\eta_1 u_1(t) + a_1 x_1(t - \tau), \\ \frac{du_2(t)}{dt} = -\eta_2 u_2(t) + a_2 x_2(t - \tau), \end{cases} \quad (2)$$

where $u_1(t)$ and $u_2(t)$ are the feedback control variables, $b_i > 0$, $a_{ij} > 0$, $\alpha_i > 0$, $\eta_i > 0$, and $a_i > 0$ ($i, j = 1, 2$). They also discussed the existence of positive equilibrium point and global attraction of the model. In 2013, Li et al. [16] introduced feedback control variables into the two-species competition system and discussed the extinction and global attraction of equilibrium points. They found that if the two-species competition model is globally stable, the system retains the stable property after adding feedback controls and the position of equilibrium point is changed. If the two-species competition model is extinct, by choosing the suitable values of feedback control variables, they can make extinct species become globally stable, or still keep the property of extinction. In 2017, Shi et al. [17] discussed a Lotka-Volterra predator-prey model with discrete delays and feedback control as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)], \\ \frac{dx_2(t)}{dt} = x_2(t)[r_2 + a_{21}x_1(t - \tau_2) - a_{22}x_2(t) - cu(t)], \\ \frac{du(t)}{dt} = -eu(t) + fx_2(t), \end{cases} \quad (3)$$

where $u(t)$ is the feedback control variable, e and f denote the feedback control coefficients, a_{ii} ($i = 1, 2$) denote the intraspecific competition rates, a_{ij} ($i \neq j, i, j = 1, 2$) stand for the capturing rates of the prey and predator populations, τ_1 is the time of catching prey, and τ_2 is maturation delay of predator. Shi et al. [17] show that

- (i) The solution $(x_1(t), x_2(t), u(t))$ of system (3) is ultimately bounded

- (ii) When the conditions $(r_1/r_2) > (a_{12}/(a_{22} + (cf/e)))$, $(a_{11}/a_{21}) > (a_{12}/a_{22})$ are established, system (3) has a unique globally asymptotically stable positive equilibrium point (x_1^*, x_2^*, u^*) , where $x_1^* = (e(r_1 a_{22} - r_2 a_{12}) + r_1 cf) / (e(a_{11} a_{22} + a_{12} a_{21}) + cf a_{11})$, $x_2^* = e(r_2 a_{11} + r_1 a_{21}) / (e(a_{11} a_{22} + a_{12} a_{21}) + cf a_{11})$, and $u^* = (f/e)x_2^*$

In fact, in nature, ecosystems are inevitably affected by various environmental noises [18–28]. Mathematical models with environmental disturbances can usually be described by stochastic differential equations. Stochastic noise can generally be divided into two categories: one type is a small number of strong interference, usually called colored noise or electrical noise, which can be described by the Markov chain [29–31]; the other type is the sum of many small, independent random interference, called white noise, which is usually represented by Brownian motion [32–35]. Assume that the population's intrinsic growth rate r_i is disturbed by white noise:

$$r_i \longrightarrow r_i + \sigma_i \dot{B}_i(t), \quad i = 1, 2. \quad (4)$$

Then, model (3) is transformed into

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)]dt \\ \quad + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)[r_2 + a_{21}x_1(t - \tau_2) - a_{22}x_2(t) - cu(t)]dt \\ \quad + \sigma_2 x_2(t)dB_2(t), \\ du(t) = [-eu(t) + fx_2(t)]dt, \end{cases} \quad (5)$$

and satisfies the initial conditions

$$\begin{aligned} x_i(\theta) &= \phi_i(\theta), \\ u(\theta) &= \psi(\theta), \\ \theta &\in [-\tau, 0], \\ i &= 1, 2, \end{aligned} \quad (6)$$

where $B_i(t)$ ($i = 1, 2$) denote the independent standard Brownian motion, σ_i^2 denote the intensity of white noise, $\tau = \max\{\tau_1, \tau_2\}$, $\phi_i(0) > 0$, $\psi(0) > 0$, and $\phi_i(\theta)$ and $\psi(\theta)$ are both nonnegative continuous functions on $[-\tau, 0]$.

Due to the interference of stochastic noise, system (5) does not possess an equilibrium point. An interesting question is: Does model (5) still have stability? What is the influence of white noise on system (5)? This paper mainly studies the dynamical properties of stochastic systems (5) also satisfying initial conditions (6). The second part proves the suitability of the system. The third part discusses the oscillation of the stochastic model near the positive equilibrium point (x_1^*, x_2^*, u^*) of the corresponding

deterministic model. The fourth and fifth parts, respectively, obtain the conditions for the persistence and extinction of the stochastic system. Finally, the correctness of the theoretical derivation is verified by numerical simulation.

2. Existence and Uniqueness of Global Positive Solutions

The stochastic differential equation is expressed as

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), x \in R^n. \quad (7)$$

If the Lyapunov function $V(x, t) \in C^{2,1}[R^n \times R_+, R]$, the stochastic differential equation of $V(x, t)$ along system (7) is defined as [36]

$$\begin{aligned} dV(x(t), t) = & \left[V_t(x(t), t) + V_x(x(t), t)f(x(t), t) \right. \\ & \left. + \frac{1}{2} \text{tr}(g^T(x(t), t)V_{xx}(x(t), t)g(x(t), t)) \right] dt \\ & + V_x(x(t), t)g(x(t), t)dB(t) \text{ a.s.} \end{aligned} \quad (8)$$

where $LV = V_t(x, t) + V_x(x, t)f(x, t) + (1/2)\text{tr}(g^T(x, t)V_{xx}(x, t)g(x, t))$ represent diffusion operator.

Theorem 1. *For any given initial condition (6), model (5) has a unique global positive solution $(x_1(t), x_2(t), u(t))$, and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. Since the coefficients of system (5) satisfy the locally Lipschitz condition, for any given initial condition (6), model (5) has a unique local positive solution $(x_1(t), x_2(t), u(t))$ in interval $t \in [0, \tau_e)$, where τ_e is the explosion time.

To prove that this solution is global, we only need to prove $\tau_e = \infty$ a.s. Let $k_0 > 0$ be a sufficiently large constant for any initial value $x_1(0), x_2(0)$, and $u(0)$ lying within the interval $[(1/k_0), k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\begin{aligned} \tau_k = \inf \left\{ t \in [0, \tau_e) : \min\{x_1(t), x_2(t), u(t)\} \right. \\ \left. \leq \frac{1}{k} \text{ or } \max\{x_1(t), x_2(t), u(t)\} \geq k \right\}. \end{aligned} \quad (9)$$

Obviously, τ_k is increasing as $k \rightarrow \infty$. Let $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$; therefore, $\tau_\infty \leq \tau_e$ a.s. Now, we need to verify $\tau_\infty = \infty$ a.s. Otherwise, there are two constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. So, there is a positive integer $k_1 \geq k_0$, such that

$$P\{\tau_k \leq T\} > \epsilon, \quad k \geq k_1. \quad (10)$$

Define a C^2 -function $V: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ by

$$\begin{aligned} V(x_1, x_2, u) = & x_1 - 1 - \ln x_1 + \frac{a_{11}a_{22}}{a_{21}^2} (x_2 - 1 - \ln x_2) \\ & + \frac{1}{2}a_{11} \int_{t-\tau_2}^t x_1^2(s)ds \\ & + a_{12} \int_{t-\tau_1}^t x_2(s)ds + \frac{a_{11}a_{22}c}{2fa_{21}^2} u^2 + u - 1 - \ln u \\ = & V_1(x_1, x_2) + V_2(u), \end{aligned} \quad (11)$$

where

$$\begin{aligned} V_1(x_1, x_2) = & x_1 - 1 - \ln x_1 + \frac{a_{11}a_{22}}{a_{21}^2} (x_2 - 1 - \ln x_2) \\ & + \frac{1}{2}a_{11} \int_{t-\tau_2}^t x_1^2(s)ds + a_{12} \int_{t-\tau_1}^t x_2(s)ds, \\ V_2(u) = & \frac{a_{11}a_{22}c}{2fa_{21}^2} u^2 + u - 1 - \ln u. \end{aligned} \quad (12)$$

The nonnegativity of this function can be obtained from

$$x - 1 - \ln x \geq 0, \quad x > 0. \quad (13)$$

Applying Itô's formula yields

$$\begin{aligned} dV(x_1, x_2, u) = & LV(x_1, x_2, u)dt + \sigma_1(x_1 - 1)dB_1(t) \\ & + \frac{a_{11}a_{22}}{a_{21}^2} \sigma_2(x_2 - 1)dB_2(t), \end{aligned} \quad (14)$$

where

$$LV = LV_1 + LV_2,$$

$$\begin{aligned}
LV_1 &= (x_1(t) - 1)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)] + \frac{1}{2}\sigma_1^2 + \frac{a_{11}a_{22}}{a_{21}^2}(x_2(t) - 1)[r_2 + a_{21}x_1(t - \tau_2) - a_{22}x_2(t) - cu(t)] \\
&\quad + \frac{a_{11}a_{22}}{2a_{21}^2}\sigma_2^2 + \frac{1}{2}a_{11}x_1^2(t) - \frac{1}{2}a_{11}x_1^2(t - \tau_2) + a_{12}x_2(t) - a_{12}x_2(t - \tau_1) \\
&= (r_1 + a_{11})x_1(t) - \frac{1}{2}a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t - \tau_1) - r_1 + \frac{1}{2}\sigma_1^2 + \frac{a_{11}a_{22}}{a_{21}^2}[(r_2 + a_{22})x_2(t) - a_{22}x_2^2(t) - cx_2(t)u(t) + cu(t)] \\
&\quad + \frac{a_{11}a_{22}}{a_{21}}(x_2(t) - 1)x_1(t - \tau_2) + \frac{a_{11}a_{22}}{a_{21}^2}\left(-r_2 + \frac{1}{2}\sigma_2^2\right) - \frac{1}{2}a_{11}x_1^2(t - \tau_2) + a_{12}x_2(t) \\
&\leq (r_1 + a_{11})x_1(t) - \frac{1}{2}a_{11}x_1^2(t) + \frac{a_{11}a_{22}}{a_{21}^2}\left[r_2x_2(t) - \frac{1}{2}a_{22}x_2^2(t) - cx_2(t)u(t) + cu(t)\right] + a_{12}x_2(t) - r_1 + \frac{1}{2}\sigma_1^2 \\
&\quad + \frac{a_{11}a_{22}}{a_{21}^2}\left(-r_2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}a_{22}\right), \\
LV_2 &= \frac{a_{11}a_{22}c}{fa_{21}^2}u(t)(-eu(t) + fx_2(t)) + \left(1 - \frac{1}{u(t)}\right)(-eu(t) + fx_2(t)) \\
&\leq -\frac{a_{11}a_{22}ce}{fa_{21}^2}u^2(t) + \frac{a_{11}a_{22}c}{a_{21}^2}u(t)x_2(t) - eu(t) + fx_2(t) + e.
\end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}
LV &= LV_1 + LV_2 \\
&\leq (r_1 + a_{11})x_1(t) - \frac{1}{2}a_{11}x_1^2(t) + \frac{a_{11}a_{22}}{a_{21}^2}\left[r_2x_2(t) - \frac{1}{2}a_{22}x_2^2(t) - cx_2(t)u(t) + cu(t)\right] + a_{12}x_2(t) - r_1 \\
&\quad + \frac{1}{2}\sigma_1^2 + \frac{a_{11}a_{22}}{a_{21}^2}\left(-r_2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}a_{22}\right) - \frac{a_{11}a_{22}ce}{fa_{21}^2}u^2(t) + \frac{a_{11}a_{22}c}{a_{21}^2}u(t)x_2(t) - eu(t) + fx_2(t) + e \\
&\leq \max\left\{-\frac{1}{2}a_{11}x_1^2(t) + (r_1 + a_{11})x_1(t)\right\} + \frac{a_{11}a_{22}}{a_{21}^2}\max\left\{-\frac{1}{2}a_{22}x_2^2(t) + \left(r_2 + \frac{a_{21}^2(f + a_{12})}{a_{11}a_{22}}\right)x_2(t)\right\} \\
&\quad + \frac{a_{11}a_{22}}{a_{21}^2}\max\left\{-\frac{ce}{f}u^2(t) + \left(c - \frac{a_{21}^2e}{a_{11}a_{22}}\right)u(t)\right\} + \frac{a_{11}a_{22}}{a_{21}^2}\left(\frac{1}{2}a_{22} + \frac{1}{2}\sigma_2^2 - r_2\right) + \frac{1}{2}\sigma_1^2 + e - r_1 \\
&\leq K,
\end{aligned} \tag{16}$$

where K is a positive constant. So, we get

$$dV \leq Kdt + \sigma_1(x_1 - 1)dB_1(t) + \frac{a_{11}a_{22}}{a_{21}^2}\sigma_2(x_2 - 1)dB_2(t). \tag{17}$$

Integrating (17) from 0 to $\tau_k \wedge T$ and taking expectation on both sides, we have

$$\begin{aligned}
&EV(x_1(\tau_k \wedge T), x_2(\tau_k \wedge T), u(\tau_k \wedge T)) \\
&\leq EV(x_1(0), x_2(0), u(0)) + KT.
\end{aligned} \tag{18}$$

Set $\Omega_k = \{\tau_k \leq T\}$, and from inequality (10), we have $P(\Omega_k) \geq \epsilon$. Note that, for every $\omega \in \Omega_k$, there is at least one of $x_1(\tau_k, \omega)$, $x_2(\tau_k, \omega)$, or $u(\tau_k, \omega)$ equaling either k or $(1/k)$, and then, we have

$$\begin{aligned}
& V(x_1(\tau_k \wedge T), x_2(\tau_k \wedge T), u(\tau_k \wedge T)) \\
& \geq (k-1-\ln k) \wedge \left(\frac{1}{k} - 1 - \ln \frac{1}{k}\right).
\end{aligned} \tag{19}$$

It can be obtained by (18)

$$\begin{aligned}
EV(x_1(0), x_2(0), u(0)) + KT & \geq E[1_{\Omega_k(\omega)} V(x_1(\tau_k, \omega), x_2(\tau_k, \omega), u(\tau_k, \omega))] \\
& \geq \varepsilon(k-1-\ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right),
\end{aligned} \tag{20}$$

where 1_{Ω_k} is the indicator function of Ω_k , and letting $k \rightarrow \infty$ yields

$$\infty > EV(x_1(0), x_2(0), u(0)) + KT = \infty. \tag{21}$$

This is a contradiction; we must have $\tau_\infty = \infty$, and we have completed the proof. $\square$

3. Asymptotic Property

Due to the interference of white noise, the solution of system (5) will have stochastic oscillation. Next, we discuss the asymptotic property of stochastic system at the positive equilibrium point of the corresponding deterministic model. In order to study the problem conveniently, the hypothesis is

$$(A_1) : \frac{r_1}{r_2} > \frac{a_{12}}{a_{22} + (cf/e)}, \frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}}. \tag{22}$$

Theorem 2. For any given initial condition (6), if hypothesis (A_1) is established, the solution $(x_1(t), x_2(t), u(t))$ of system (5) has the property that

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t \left[(x_1(\theta) - x_1^*)^2 + (x_2(\theta) - x_2^*)^2 \right. \\
& \left. + (u(\theta) - u^*)^2 \right] d\theta \leq \frac{n}{m},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
m &= \min \left\{ \frac{a_{11}f(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{22} + a_{12}a_{21}}, \frac{a_{12}f(a_{11}a_{22} - a_{12}a_{21})}{2a_{11}a_{21}}, \frac{a_{12}ce}{a_{21}} \right\}, \\
n &= \frac{fx_1^*\sigma_1^2}{2} + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}},
\end{aligned} \tag{24}$$

where (x_1^*, x_2^*, u^*) is the positive equilibrium point of the corresponding deterministic model (3).

Proof. Define the function

$$\begin{aligned}
V(x_1, x_2, u) &= f \left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + \frac{a_{12}f}{a_{21}} \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) + \frac{a_{12}c}{2a_{21}} (u - u^*)^2 \\
&+ \frac{a_{12}f}{2\omega_2} \int_{t-\tau_2}^t (x_1(s) - x_1^*)^2 ds + \frac{a_{12}f\omega_1}{2} \int_{t-\tau_1}^t (x_2(s) - x_2^*)^2 ds,
\end{aligned} \tag{25}$$

where ω_1 and ω_2 are positive constants. Define

$$\begin{aligned}
V_1 &= x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*}, \\
V_2 &= x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*}, \\
V_3 &= (u - u^*)^2, \\
V_4 &= \frac{a_{12}f}{2\omega_2} \int_{t-\tau_2}^t (x_1(s) - x_1^*)^2 ds + \frac{a_{12}f\omega_1}{2} \int_{t-\tau_1}^t (x_2(s) - x_2^*)^2 ds.
\end{aligned} \tag{26}$$

By Itô's formula, we obtain

$$dV_1 = LV_1 dt + (x_1 - x_1^*)\sigma_1 dB_1(t), \tag{27}$$

where

$$\begin{aligned}
LV_1 &= (x_1(t) - x_1^*)[r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_1)] + \frac{1}{2}x_1^*\sigma_1^2 \\
&= (x_1(t) - x_1^*)[-a_{11}(x_1(t) - x_1^*) - a_{12}(x_2(t - \tau_1) - x_2^*)] + \frac{1}{2}x_1^*\sigma_1^2 \\
&\leq -a_{11}(x_1(t) - x_1^*)^2 + \frac{a_{12}}{2\omega_1}(x_1(t) - x_1^*)^2 + \frac{a_{12}\omega_1}{2}(x_2(t - \tau_1) - x_2^*)^2 + \frac{1}{2}x_1^*\sigma_1^2.
\end{aligned} \tag{28}$$

Similarly,

$$dV_2 = LV_2 dt + (x_2 - x_2^*)\sigma_2 dB_2(t), \tag{29}$$

where

$$\begin{aligned}
LV_2 &= (x_2(t) - x_2^*)[r_2 + a_{21}x_1(t - \tau_2) - a_{22}x_2(t) - cu(t)] + \frac{1}{2}x_2^*\sigma_2^2 \\
&= (x_2(t) - x_2^*)[a_{21}(x_1(t - \tau_2) - x_1^*) - a_{22}(x_2(t) - x_2^*) - c(u(t) - u^*)] + \frac{1}{2}x_2^*\sigma_2^2 \\
&\leq \frac{a_{21}}{2\omega_2}(x_1(t - \tau_2) - x_1^*)^2 + \frac{a_{21}\omega_2}{2}(x_2(t) - x_2^*)^2 - a_{22}(x_2(t) - x_2^*)^2 - c(x_2(t) - x_2^*)(u(t) - u^*) + \frac{1}{2}x_2^*\sigma_2^2.
\end{aligned} \tag{30}$$

In the same way,

$$\begin{aligned}
dV_3 &= 2(u(t) - u^*)(-eu(t) + fx_2(t))dt \\
&= [-2e(u(t) - u^*)^2 + 2f(u(t) - u^*)(x_2(t) - x_2^*)]dt,
\end{aligned} \tag{31}$$

$$dV_4 = \left[\frac{a_{12}f}{2\omega_2}(x_1(t) - x_1^*)^2 - \frac{a_{12}f}{2\omega_2}(x_1(t - \tau_2) - x_1^*)^2 + \frac{a_{12}f\omega_1}{2}(x_2(t) - x_2^*)^2 - \frac{a_{12}f\omega_1}{2}(x_2(t - \tau_1) - x_2^*)^2 \right] dt.$$

Therefore, we have

$$\begin{aligned}
LV &= fLV_1 + \frac{a_{12}f}{a_{21}}LV_2 + \frac{a_{12}c}{2a_{21}}\frac{dV_3}{dt} + \frac{dV_4}{dt} \\
&\leq -a_{11}f(x_1(t) - x_1^*)^2 + \frac{a_{12}f}{2\omega_1}(x_1(t) - x_1^*)^2 + \frac{a_{12}f\omega_1}{2}(x_2(t - \tau_1) - x_2^*)^2 + \frac{1}{2}fx_1^*\sigma_1^2 \\
&\quad + \frac{a_{12}f}{2\omega_2}(x_1(t - \tau_2) - x_1^*)^2 + \frac{a_{12}f\omega_2}{2}(x_2(t) - x_2^*)^2 - \frac{a_{12}a_{22}f}{a_{21}}(x_2(t) - x_2^*)^2 - \frac{a_{12}cf}{a_{21}}(x_2(t) - x_2^*)(u(t) - u^*) \\
&\quad + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}} - \frac{a_{12}ce}{a_{21}}(u(t) - u^*)^2 + \frac{a_{12}cf}{a_{21}}(u(t) - u^*)(x_2(t) - x_2^*) + \frac{a_{12}f}{2\omega_2}(x_1(t) - x_1^*)^2 - \frac{a_{12}f}{2\omega_2}(x_1(t - \tau_2) - x_1^*)^2 \\
&\quad + \frac{a_{12}f\omega_1}{2}(x_2(t) - x_2^*)^2 - \frac{a_{12}f\omega_1}{2}(x_2(t - \tau_1) - x_2^*)^2 \\
&= -f\left(a_{11} - \frac{a_{12}}{2\omega_1} - \frac{a_{12}}{2\omega_2}\right)(x_1(t) - x_1^*)^2 - f\left(\frac{a_{12}a_{22}}{a_{21}} - \frac{a_{12}\omega_2}{2} - \frac{a_{12}\omega_1}{2}\right)(x_2(t) - x_2^*)^2 - \frac{a_{12}ce}{a_{21}}(u(t) - u^*)^2 + \frac{fx_1^*\sigma_1^2}{2} \\
&\quad + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}}.
\end{aligned} \tag{32}$$

Let $\omega_1 = \omega_2 = ((a_{11}a_{22} + a_{12}a_{21})/2a_{11}a_{21})$, we obtain

$$\begin{aligned}
LV &\leq -\frac{a_{11}f(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{22} + a_{12}a_{21}}(x_1(t) - x_1^*)^2 - \frac{a_{12}f(a_{11}a_{22} - a_{12}a_{21})}{2a_{11}a_{21}}(x_2(t) - x_2^*)^2 \\
&\quad - \frac{a_{12}ce}{a_{21}}(u(t) - u^*)^2 + \frac{fx_1^*\sigma_1^2}{2} + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}}.
\end{aligned} \tag{33}$$

Therefore,

$$\begin{aligned}
dV(x_1, x_2, u) &= LVdt + f(x_1 - x_1^*)\sigma_1 dB_1(t) \\
&\quad + \frac{a_{12}f}{a_{21}}(x_2 - x_2^*)\sigma_2 dB_2(t).
\end{aligned} \tag{34}$$

Integrate both sides of (18) from 0 to t and take the expectation, and then we get

$$\begin{aligned}
EV(t) - EV(0) &\leq -E \int_0^t \frac{a_{11}f(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{22} + a_{12}a_{21}}(x_1(\theta) - x_1^*)^2 d\theta \\
&\quad - E \int_0^t \frac{a_{12}f(a_{11}a_{22} - a_{12}a_{21})}{2a_{11}a_{21}}(x_2(\theta) - x_2^*)^2 d\theta - E \int_0^t \frac{a_{12}ce}{a_{21}}(u(\theta) - u^*)^2 d\theta + \frac{fx_1^*\sigma_1^2 t}{2} + \frac{a_{12}fx_2^*\sigma_2^2 t}{2a_{21}}.
\end{aligned} \tag{35}$$

Divide both sides by t and take the limit superior, and then we have

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t \frac{a_{11}f(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{22} + a_{12}a_{21}} (x_1(\theta) - x_1^*)^2 d\theta \\
& + \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t \frac{a_{12}f(a_{11}a_{22} - a_{12}a_{21})}{2a_{11}a_{21}} (x_2(\theta) - x_2^*)^2 d\theta \\
& + \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t \frac{a_{12}ce}{a_{21}} (u(\theta) - u^*)^2 d\theta \leq \frac{fx_1^*\sigma_1^2}{2} + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}}.
\end{aligned} \tag{36}$$

Obviously,

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t \left[(x_1(\theta) - x_1^*)^2 + (x_2(\theta) - x_2^*)^2 + (u(\theta) - u^*)^2 \right] d\theta \leq \frac{n}{m}, \tag{37}$$

where

$$\begin{aligned}
m &= \min \left\{ \frac{a_{11}f(a_{11}a_{22} - a_{12}a_{21})}{a_{11}a_{22} + a_{12}a_{21}}, \frac{a_{12}f(a_{11}a_{22} - a_{12}a_{21})}{2a_{11}a_{21}}, \frac{a_{12}ce}{a_{21}} \right\}, \\
n &= \frac{fx_1^*\sigma_1^2}{2} + \frac{a_{12}fx_2^*\sigma_2^2}{2a_{21}}.
\end{aligned} \tag{38}$$

We have completed the proof.

Theorem 2 shows that if the condition (A_1) holds, the solution oscillates around the equilibrium point x^* , and the amplitude of oscillation is positively correlated with the intensity σ_1^2 and σ_2^2 of environmental noise. In particular, if $\sigma_1^2 = \sigma_2^2 = 0$, the influence of environmental noise is not taken into account:

$$x_1(t) \longrightarrow x_1^*, x_2(t) \longrightarrow x_2^*, u(t) \longrightarrow u^* \quad \text{for } t \longrightarrow +\infty. \tag{39}$$

The equilibrium point x^* is globally asymptotically stable. This is the conclusion of reference [17]. $\square$

4. Persistence

In nature, whether ecosystems can survive or not is our main concern. Before discussing the persistence of stochastic system, we give the following assumption:

$$(A_2) : \mu = \max(\sigma_1, \sigma_2) < \min \left\{ x_1^* \sqrt{\frac{m}{n_0}}, x_2^* \sqrt{\frac{m}{n_0}}, u^* \sqrt{\frac{m}{n_0}} \right\}, \quad n_0 = \frac{fx_1^*}{2} + \frac{a_{12}fx_2^*}{2a_{21}}. \tag{40}$$

Theorem 3. For any given initial condition (6), if assumptions (A_1) and (A_2) hold at the same time, the solution $(x_1(t), x_2(t), u(t))$ of system (5) is persistent that

$$\begin{aligned}
& \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t x_1(\theta) d\theta > 0, \\
& \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t x_2(\theta) d\theta > 0, \\
& \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t u(\theta) d\theta > 0.
\end{aligned} \tag{41}$$

Proof. According to (37), we have

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t [(x_1(\theta) - x_1^*)^2] d\theta \leq \frac{n}{m}, \\
& \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t [(x_2(\theta) - x_2^*)^2] d\theta \leq \frac{n}{m}, \\
& \limsup_{t \rightarrow +\infty} \frac{1}{t} E \int_0^t [(u(\theta) - u^*)^2] d\theta \leq \frac{n}{m}.
\end{aligned} \tag{42}$$

As we know, $x_1(t) \geq 0$ and $x_1^* > 0$, from $2x_1(t)x_1^* \geq (x_1^*)^2 - (x_1(t) - x_1^*)^2$, one can get

$$x_1(t) \geq \frac{x_1^*}{2} - \frac{(x_1(t) - x_1^*)^2}{2x_1^*}. \quad (43)$$

By the condition $\mu < x_1^* \sqrt{m/n_0}$, we have

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t x_1(\theta) d\theta \\ \geq \frac{x_1^*}{2} - \limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \frac{(x_1(\theta) - x_1^*)^2}{2x_1^*} d\theta \\ \geq \frac{x_1^*}{2} - \frac{n}{2mx_1^*} \geq \frac{x_1^*}{2} - \frac{\mu^2 n_0}{2mx_1^*} > 0. \end{aligned} \quad (44)$$

Similarly, when $\mu < \min\{x_2^* \sqrt{m/n_0}, u^* \sqrt{m/n_0}\}$,

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t x_2(\theta) d\theta > 0 \\ \liminf_{t \rightarrow \infty} \frac{1}{t} E \int_0^t u(\theta) d\theta > 0. \end{aligned} \quad (45)$$

□

5. Extinction

Define

$$\begin{aligned} \Delta &= a_{21} \left(r_1 - \frac{1}{2} \sigma_1^2 \right) + a_{11} \left(r_2 - \frac{1}{2} \sigma_2^2 \right), \\ \Delta_1 &= e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf, \\ \Delta_2 &= a_{12}\Delta [e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf] \\ &\quad - \left(r_1 - \frac{1}{2} \sigma_1^2 \right) e(a_{11}a_{22} + a_{12}a_{21})^2. \end{aligned} \quad (46)$$

For the extinction of system (5), we have the following conclusions.

Theorem 4. For any given initial condition (6), the solution $(x_1(t), x_2(t), u(t))$ of system (5) has the following properties:

- (i) when $r_1 - (1/2)\sigma_1^2 < 0$ and $r_2 - (1/2)\sigma_2^2 < 0$, the population is extinct
- (ii) when $r_1 - (1/2)\sigma_1^2 > 0$, and $\Delta < 0$, population x_1 is persistent and x_2 and u are extinct
- (iii) when $r_1 - (1/2)\sigma_1^2 > 0$, $\Delta > 0$, $\Delta_1 > 0$, and $\Delta_2 > 0$, population x_1 is extinct and x_2 and u are persistent

Before proving Theorem 4, consider the following auxiliary system [37]:

$$\begin{cases} dz_1(t) = z_1(t)[r_1 - a_{11}z_1(t)]dt + \sigma_1 z_1(t)dB_1(t), \\ dz_2(t) = z_2(t)[r_2 + a_{21}z_1(t - \tau_2) - a_{22}z_2(t)]dt \\ \quad + \sigma_2 z_2(t)dB_2(t), \end{cases} \quad (47)$$

and it satisfies the initial condition: $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta)) \in C([- \tau, 0], R_+^2)$.

Lemma 1. If $r_1 - (1/2)\sigma_1^2 > 0$, the solution $z(t)$ of system (47) has the following properties:

- (i) $\lim_{t \rightarrow +\infty} t^{-1} \int_0^t z_1(s)ds = (r_1 - ((1/2)\sigma_1^2))/a_{11}$
- (ii) if $\Delta < 0$, $\lim_{t \rightarrow +\infty} z_2(t) = 0$
- (iii) if $\Delta > 0$, $\lim_{t \rightarrow +\infty} t^{-1} \int_0^t z_2(s)ds = (\Delta/a_{11}a_{22})$, a.s

Proof. By Itô's formula, we obtain

$$\ln z_1(t) - \ln z_1(0) = \left(r_1 - \frac{1}{2} \sigma_1^2 \right) t - a_{11} \int_0^t z_1(s)ds + \sigma_1 B_1(t), \quad (48)$$

$$\begin{aligned} \ln z_2(t) - \ln z_2(0) &= \left(r_2 - \frac{1}{2} \sigma_2^2 \right) t + a_{21} \int_0^t z_1(s - \tau_2)ds - a_{22} \int_0^t z_2(s)ds + \sigma_2 B_2(t) \\ &= \left(r_2 - \frac{1}{2} \sigma_2^2 \right) t + a_{21} \int_0^t z_1(s)ds - a_{21} \left[\int_{t-\tau_2}^t z_1(s)ds - \int_{-\tau_2}^0 z_1(s)ds \right] - a_{22} \int_0^t z_2(s)ds + \sigma_2 B_2(t). \end{aligned} \quad (49)$$

Dividing both sides of (48) and (49) by t , we have

$$t^{-1} \ln \frac{z_1(t)}{z_1(0)} = \left(r_1 - \frac{1}{2} \sigma_1^2 \right) - a_{11} t^{-1} \int_0^t z_1(s)ds + t^{-1} \sigma_1 B_1(t), \quad (50)$$

$$t^{-1} \ln \frac{z_2(t)}{z_2(0)} + t^{-1} a_{21} \left[\int_{t-\tau_2}^t z_1(s)ds - \int_{-\tau_2}^0 z_1(s)ds \right] = \left(r_2 - \frac{1}{2} \sigma_2^2 \right) + a_{21} t^{-1} \int_0^t z_1(s)ds - a_{22} t^{-1} \int_0^t z_2(s)ds + t^{-1} \sigma_2 B_2(t). \quad (51)$$

By using Lemma 2 of [37], from equation (48), it follows that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} t^{-1} \int_0^t z_1(s) ds &\leq \frac{r_1 - (1/2)\sigma_1^2}{a_{11}}, \\ \liminf_{t \rightarrow +\infty} t^{-1} \int_0^t z_1(s) ds &\geq \frac{r_1 - (1/2)\sigma_1^2}{a_{11}}, \end{aligned} \quad (52)$$

and therefore,

$$\lim_{t \rightarrow +\infty} t^{-1} \int_0^t z_1(s) ds = \frac{r_1 - (1/2)\sigma_1^2}{a_{11}}. \quad (53)$$

Substitute (53) into (50), and from $\lim_{t \rightarrow +\infty} t^{-1} B_i(t) = 0$ ($i = 1, 2$), we get

$$\lim_{t \rightarrow +\infty} t^{-1} \ln z_1(t) = 0. \quad (54)$$

On the other hand, computing (50) $\times a_{21} + (51) \times a_{11}$, we have

$$\begin{aligned} &a_{21} t^{-1} \ln \frac{z_1(t)}{z_1(0)} + a_{11} t^{-1} \ln \frac{z_2(t)}{z_2(0)} + t^{-1} a_{11} a_{21} \left[\int_{t-\tau_2}^t z_1(s) ds - \int_{-\tau_2}^0 z_1(s) ds \right] \\ &= a_{21} \left(r_1 - \frac{1}{2} \sigma_1^2 \right) + a_{11} \left(r_2 - \frac{1}{2} \sigma_2^2 \right) - a_{11} a_{22} t^{-1} \int_0^t z_2(s) ds + t^{-1} [a_{21} \sigma_1 B_1(t) + a_{11} \sigma_2 B_2(t)]. \end{aligned} \quad (55)$$

By Lemma 2 in literature [37], when $\Delta < 0$,

$$\lim_{t \rightarrow +\infty} z_2(t) = 0. \quad (56)$$

If $\Delta > 0$,

$$\lim_{t \rightarrow +\infty} t^{-1} \int_0^t z_2(s) ds = \frac{\Delta}{a_{11} a_{22}} \text{ a.s.} \quad (57)$$

Proof of lemma is completed. $\square$

This is where we prove Theorem 4.

Proof. By using Itô's formula for system (5), we have

$$\begin{aligned} \ln x_1(t) - \ln x_1(0) &= \left(r_1 - \frac{1}{2} \sigma_1^2 \right) t - a_{11} \int_0^t x_1(s) ds - a_{12} \int_0^t x_2(s - \tau_1) ds + \sigma_1 B_1(t) \\ &= \left(r_1 - \frac{1}{2} \sigma_1^2 \right) t - a_{11} \int_0^t x_1(s) ds - a_{12} \int_0^t x_2(s) ds + a_{12} \left[\int_{t-\tau_1}^t x_2(s) ds - \int_{-\tau_1}^0 x_2(s) ds \right] + \sigma_1 B_1(t), \end{aligned} \quad (58)$$

$$\begin{aligned} \ln x_2(t) - \ln x_2(0) &= \left(r_2 - \frac{1}{2} \sigma_2^2 \right) t + a_{21} \int_0^t x_1(s - \tau_2) ds - a_{22} \int_0^t x_2(s) ds - c \int_0^t u(s) ds + \sigma_2 B_2(t) \\ &= \left(r_2 - \frac{1}{2} \sigma_2^2 \right) t + a_{21} \int_0^t x_1(s) ds - a_{21} \left[\int_{t-\tau_2}^t x_1(s) ds - \int_{-\tau_2}^0 x_1(s) ds \right] \\ &\quad - a_{22} \int_0^t x_2(s) ds - c \int_0^t u(s) ds + \sigma_2 B_2(t). \end{aligned} \quad (59)$$

We first prove (i): from equation (58), we can get

$$\ln x_1(t) - \ln x_1(0) \leq \left(r_1 - \frac{1}{2} \sigma_1^2 \right) t - a_{11} \int_0^t x_1(s) ds + \sigma_1 B_1(t). \quad (60)$$

By the condition $r_1 - (1/2)\sigma_1^2 < 0$ and Lemma 2 in literature [37],

$$\lim_{t \rightarrow +\infty} x_1(t) = 0 \text{ a.s.} \quad (61)$$

From (57), (61), and the condition $r_2 - (1/2)\sigma_2^2 < 0$, we have

$$\lim_{t \rightarrow +\infty} x_2(t) = 0 \text{ a.s.} \quad (62)$$

Further, from the third equation of model (5), we can get

$$\lim_{t \rightarrow +\infty} u(t) = 0 \text{ a.s.} \quad (63)$$

Secondly, it proves (ii): comparing model (5) with auxiliary system (37), one can get $x_1(t) \leq z_1(t)$ and $x_2(t) \leq z_2(t)$. By Lemma 1, when $r_1 - (1/2)\sigma_1^2 > 0$ and $\Delta < 0$, $\lim_{t \rightarrow +\infty} z_2(t) = 0$ a.s.; therefore, $\lim_{t \rightarrow +\infty} x_2(t) = 0$ a.s. So, we have $\lim_{t \rightarrow +\infty} u(t) = 0$. Then, the limit system of model (5) is as follows:

$$dx_1(t) = x_1(t) [r_1 - a_{11} x_1(t)] dt + \sigma_1 x_1(t) dB_1(t). \quad (64)$$

From Lemma 1, we can conclude that

FIGURE 1: (a) Time series diagram of $(x_1(t), x_2(t), u(t))$ and (b) its phase diagram.

FIGURE 2: Time series diagram of $(x_1(t), x_2(t), u(t))$.

FIGURE 3: Time series diagram of $(x_1(t), x_2(t), u(t))$.

FIGURE 4: (a) Time series diagram of $(x_1(t), x_2(t), u(t))$. (b) Phase diagram of $(x_2(t), u(t))$.

$$\lim_{t \rightarrow +\infty} t^{-1} \int_0^t x_1(s) ds = \frac{r_1 - (1/2)\sigma_1^2}{a_{11}} \text{ a.s.} \quad (65)$$

Therefore, population x_1 is persistent and x_2 and u are extinct.

Finally, we prove (iii): by Lemma 1, when $r_1 - (1/2)\sigma_1^2 > 0$ and $\Delta > 0$, $\lim_{t \rightarrow +\infty} t^{-1} \int_0^t z_2(s) ds = (\Delta/a_{11}a_{22})$, a.s. And, by (54),

$$\limsup_{t \rightarrow +\infty} t^{-1} \ln x_1(t) \leq \lim_{t \rightarrow +\infty} t^{-1} \ln z_1(t) = 0. \quad (66)$$

Computing $(58) \times a_{21} + (59) \times a_{11}$,

$$\begin{aligned} & a_{21}t^{-1} \ln \frac{x_1(t)}{x_1(0)} + a_{11}t^{-1} \ln \frac{x_2(t)}{x_2(0)} + t^{-1}a_{11}a_{21} \left[\int_{t-\tau_2}^t x_1(s) ds - \int_{-\tau_2}^0 x_1(s) ds \right] \\ &= t^{-1}a_{12}a_{21} \left[\int_{t-\tau_1}^t x_2(s) ds - \int_{-\tau_1}^0 x_2(s) ds \right] + a_{21} \left(r_1 - \frac{1}{2}\sigma_1^2 \right) \\ &+ a_{11} \left(r_2 - \frac{1}{2}\sigma_2^2 \right) - a_{11}ct^{-1} \int_0^t u(s) ds - (a_{11}a_{22} + a_{12}a_{21})t^{-1} \int_0^t x_2(s) ds + t^{-1} [a_{21}\sigma_1 B_1(t) + a_{11}\sigma_2 B_2(t)]. \end{aligned} \quad (67)$$

Therefore,

$$\begin{aligned} & a_{21}t^{-1} \ln \frac{x_1(t)}{x_1(0)} + a_{11}t^{-1} \ln \frac{x_2(t)}{x_2(0)} + t^{-1}a_{11}a_{21} \left[\int_{t-\tau_2}^t x_1(s) ds - \int_{-\tau_2}^0 x_1(s) ds \right] - t^{-1}a_{12}a_{21} \left[\int_{t-\tau_1}^t x_2(s) ds - \int_{-\tau_1}^0 x_2(s) ds \right] \\ &\leq a_{21} \left(r_1 - \frac{1}{2}\sigma_1^2 \right) + a_{11} \left(r_2 - \frac{1}{2}\sigma_2^2 \right) - (a_{11}a_{22} + a_{12}a_{21})t^{-1} \int_0^t x_2(s) ds + t^{-1} [a_{21}\sigma_1 B_1(t) + a_{11}\sigma_2 B_2(t)]. \end{aligned} \quad (68)$$

Let $t \rightarrow +\infty$ and the condition $\Delta = a_{21}(r_1 - (1/2)\sigma_1^2) + a_{11}(r_2 - (1/2)\sigma_2^2) > 0$ is satisfied, so

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t x_2(s) ds \leq \frac{\Delta}{a_{11}a_{22} + a_{12}a_{21}} \text{ a.s.} \quad (69)$$

Considering the third equation of model (5),

$$\frac{u(t) - u(0)}{t} = -et^{-1} \int_0^t u(s) ds + ft^{-1} \int_0^t x_2(s) ds; \quad (70)$$

hence,

$$\limsup_{t \rightarrow +\infty} t^{-1} \int_0^t u(s) ds \leq \frac{f\Delta}{e(a_{11}a_{22} + a_{12}a_{21})} \text{ a.s.} \quad (71)$$

Substitute (71) into equation (67), and by the condition $\Delta_1 = e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf > 0$, we have

$$\begin{aligned} \liminf_{t \rightarrow +\infty} t^{-1} \int_0^t x_2(s) ds &\geq \frac{\Delta - (a_{11}cf\Delta/e(a_{11}a_{22} + a_{12}a_{21}))}{a_{11}a_{22} + a_{12}a_{21}} \\ &= \frac{(e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf)\Delta}{e(a_{11}a_{22} + a_{12}a_{21})^2} > 0 \text{ a.s.} \end{aligned} \quad (72)$$

It is concluded that predator $x_2(t)$ is persistent.

When $t \rightarrow +\infty$ and substituting (72) into equation (70), one can see that

$$\liminf_{t \rightarrow +\infty} t^{-1} \int_0^t u(s) ds \geq \frac{f[e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf]\Delta}{e^2(a_{11}a_{22} + a_{12}a_{21})^2} > 0 \text{ a.s.} \quad (73)$$

So, $u(t)$ is persistent.

The condition $\Delta_2 > 0$ means that

$$r_1 - \frac{1}{2}\sigma_1^2 < \frac{a_{12}[e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf]\Delta}{e(a_{11}a_{22} + a_{12}a_{21})^2}. \quad (74)$$

Substituting (72) into equation (58) and by the condition $\Delta_2 > 0$, we have

$$\lim_{t \rightarrow +\infty} x_1(t) = 0 \text{ a.s.} \quad (75)$$

So, $x_1(t)$ is extinction. The proof of Theorem 4 is complete. $\square$

6. Conclusions and Numerical Simulations

This paper proposes a stochastic Lotka–Volterra predator–prey model with discrete delays and feedback control. We firstly study the existence and uniqueness of global positive solution. By constructing appropriate Lyapunov functions and applying Itô's formula, we discuss the asymptotic behavior of stochastic system at the positive equilibrium point of the corresponding deterministic model. Finally, this paper gives the conditions for the persistence and extinction of stochastic system. Theorem 3 shows that the system is persistent if the intensity σ_i ($i = 1, 2$) of random disturbance and the coefficient c of feedback control variable satisfy the condition (A_2) . Theorem 4 indicates that

- (1) If the coefficient c of the feedback control $cu(t)x_2(t)$ remains unchanged and the intensity σ_i ($i = 1, 2$) of random disturbance increases, the population cannot resist the disturbance of the external environment and extinct
- (2) If the intensity of random disturbance σ_i ($i = 1, 2$) remains unchanged and the coefficient c of feedback control $cu(t)x_2(t)$ is small, $\Delta_1 > 0$ and $\Delta_2 > 0$ are satisfied, which will cause the continuous increase of

predator number for a period of time, thus leading to the extinction of the prey population.

Therefore, utilizing the feedback control measures to limit the predator quantity within a certain range is beneficial for the sustained existence of the population.

In order to verify the correctness of the theoretical analysis, we carry out the following numerical simulations. Choose the parameters in system (5) as follows:

$$\begin{aligned} r_1 &= 1.5, \\ r_2 &= 1.2, \\ e &= 0.5, \\ f &= 0.2, \\ a_{11} &= 0.2, \\ a_{21} &= 0.2, \\ a_{22} &= 0.4, \\ \tau_1 &= 1, \\ \tau_2 &= 1. \end{aligned} \quad (76)$$

Let $\Delta t = 0.01$ and the initial value $x_1(0) = 7$, $x_2(0) = 6$, and $u(0) = 5$.

- (i) Set $c = 0.4$, $a_{12} = 0.25$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.05$, and the positive equilibrium point of the corresponding deterministic model is $(x_1^*, x_2^*, u^*) = (10/3, 10/3, 4/3)$. It is proved that conditions (A_1) and (A_2) are satisfied and Theorem 2 and Theorem 3 are valid. System (5) oscillates slightly near the point $(10/3, 10/3, 4/3)$ and persistently (see Figure 1).
- (ii) Set $c = 0.1$, $a_{12} = 0.25$, $\sigma_1 = 1.8$, and $\sigma_2 = 1.6$, one can get $r_1 - (1/2)\sigma_1^2 = -0.12 < 0$ and $r_2 - (1/2)\sigma_2^2 = -0.08 < 0$. According to Theorem 4 (i), system (5) is extinct (see Figure 2).
- (iii) Set $c = 0.1$, $a_{12} = 0.25$, $\sigma_1 = 0.05$, and $\sigma_2 = 2.5$, we have $r_1 - (1/2)\sigma_1^2 = 1.4988 > 0$ and $\Delta = a_{21}(r_1 - (1/2)\sigma_1^2) + a_{11}(r_2 - (1/2)\sigma_2^2) = -0.0852 < 0$. According to Theorem 4 (ii), population x_1 is persistent and x_2 and u are extinct (see Figure 3).
- (iv) Set $c = 0.1$, $a_{12} = 0.55$, $\sigma_1 = 0.05$ and $\sigma_2 = 0.05$, we obtain $r_1 - (1/2)\sigma_1^2 = 1.4988 > 0$, $\Delta = a_{21}(r_1 - (1/2)\sigma_1^2) + a_{11}(r_2 - (1/2)\sigma_2^2) = 0.5395 > 0$, $\Delta_1 = e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf = 0.091 > 0$, and $\Delta_2 = a_{12}\Delta[e(a_{11}a_{22} + a_{12}a_{21}) - a_{11}cf] - (r_1 - (1/2)\sigma_1^2)e(a_{11}a_{22} + a_{12}a_{21})^2 = 0.027 > 0$. According to Theorem 4 (iii), population x_1 is extinct and x_2 and u are persistent (see Figure 4).

Data Availability

All data sets used in this study are hypothetical.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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