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Research Article

Dynamics of a Stochastic Three-Species Food Web Model with Omnivory and Ratio-Dependent Functional Response

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This paper is concerned with a stochastic three-species food web model with omnivory which is defined as feeding on more than one trophic level. The model involves a prey, an intermediate predator, and an omnivorous top predator. First, by the stochastic comparison theorem, we show that there is a unique global positive solution to the model. Next, we investigate the asymptotic pathwise behavior of the model. Then, we conclude that the model is persistent in mean and extinct and discuss the stochastic persistence of the model. Further, by constructing a suitable Lyapunov function, we establish sufficient conditions for the existence of an ergodic stationary distribution to the model. Then, we present the application of the main results in some special models. Finally, we introduce some numerical simulations to support the main results obtained. The results in this paper generalize and improve the previous related results.

1. Introduction

The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. During the past one hundred years, there have been many investigations on predator-prey models. To the best of our knowledge, in the predator-prey interaction, the functional response plays an important role in the population dynamics, and most of the predator-prey models with the functional responses only depend on the prey. However, laboratory experiments show that the ratio-dependent response function is more reasonable in characterizing the relationship between predators and their preys [2]. Arditi and Ginzburg [3] first proposed a ratio-dependent functional response of form $(\alpha x)/(x+\beta y)$. Kuang and Beretta [4] investigated the following ratio-dependent type predator-prey model:

$$\begin{cases} dx_1(t) = x_1(t)\left[r - a_1 x_1(t) - \dfrac{\alpha_{12}x_2(t)}{x_1(t)+\beta_{12}x_2(t)}\right]dt, \\ dx_2(t) = x_2(t)\left[-d_2 + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t)+\beta_{12}x_2(t)}\right]dt, \end{cases} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ represent population sizes of prey and predator at time t, respectively. r, a_1, and d_2 stand for the prey intrinsic growth rate, the intraspecific competition rate of the prey, and the predator death rate, respectively; α_{12}, β_{12}, and e_{12} represent the encounter rate, half capturing saturation constant, and conversion rate, respectively, that predator x_2 preys on prey x_1.

Long-term ecological research studies show that three-species predator-prey models are fundamental building blocks of large scale ecosystems. However, it was only in the 1970s that some scholars began to study the dynamics of three-species predator-prey systems [5]. In particular, Hsu et al. [6] have classified all three-species predator-prey models into five types: two predators competing for one prey, one predator acting on two preys, food chain, food chain with omnivory, and food chain with cycle. Food chain architecture and strengths of species interactions are important determinants of trophic dynamics (see [7]). It is well known that tritrophic food chain model consists of one prey, one intermediate predator, and one top predator. Note that omnivory is a widespread mechanism in interacting populations. In [6], the authors investigated the following three-species predator-prey food chain model with an omnivory top predator:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 + a_{21}x_1(t) - a_{23}x_3(t)\right]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 + a_{31}x_1(t) + a_{32}x_2(t)\right]\mathrm{d}t, \end{cases} \tag{2}$$

where x_1, x_2, and x_3 denote the number of prey, intermediate predator, and omnivorous top predator, respectively, r_1 is the growth rate of prey, r_i is the death rate of species x_i $(i = 2, 3)$, a_{11} is the intraspecific competition rate of prey, a_{12}, a_{13}, and a_{23} are the capture rates, and a_{21}, a_{31}, and a_{32} denote the efficiency of food conversion. Model (2) describes that the intermediate predator preys on only the prey and the omnivorous top predator preys on both the prey and the intermediate predator. This is a general part of marine or terrestrial food web ecological systems. Based on model (2), Namba et al. [8] considered the intraspecific competition of the intermediate predator and the intraspecific competition of the top predator. Moreover, the authors demonstrated the stabilizing role of intraspecific competition among intermediate and top predators when the growth rate of prey species is adequate to support both the predator species. Furthermore, Sen et al. [9] investigated the following three-species Lotka–Volterra model with intraguild predation and mixed functional responses:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 + a_{21}x_1(t) - a_{22}x_2(t) - \dfrac{a_{23}x_3(t)}{1 + \beta x_2(t)}\right]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 + a_{31}x_1(t) + \dfrac{a_{32}x_2(t)}{1 + \beta x_2(t)} - a_{33}x_3(t)\right]\mathrm{d}t, \end{cases} \tag{3}$$

where x_1, x_2, and x_3 denote the number of prey, intermediate predator, and omnivorous top predator, respectively. Obviously, in [9], the authors considered Holling type-II functional response between the intermediate predator and top predator and other functional responses were assumed to be linear. All meanings of the parameters are exact to or similar as those for (2) except the following. Here, a_{ii} is the intraspecific competition rate of species x_i $(i = 2, 3)$ and β is the reciprocal of the half-saturation constant.

Note that the three-species food web models (2) and (3) with the functional responses only depend on prey density. However, in fact, the predator has to search and compete for food and the ratio-dependent function of the prey and the predator is more suitable to substitute for the model with complicated interaction between the prey and predator. Then, the ratio-dependent type three-species food web model with omnivory is expressed in the form:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_1x_1(t) - \dfrac{\alpha_{12}x_2(t)}{x_1(t) + \beta_{12}x_2(t)} - \dfrac{\alpha_{13}x_3(t)}{x_1(t) + \beta_{13}x_3(t)}\right]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 - a_2x_2(t) + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t) + \beta_{12}x_2(t)} - \dfrac{\alpha_{23}x_3(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 - a_3x_3(t) + \dfrac{e_{13}\alpha_{13}x_1(t)}{x_1(t) + \beta_{13}x_3(t)} + \dfrac{e_{23}\alpha_{23}x_2(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t, \end{cases} \tag{4}$$

where $x_1(t)$ stands for the total number of prey at time t, while $x_2(t)$ and $x_3(t)$ represent the total number of intermediate predators and omnivorous top predators at time t, respectively. Here, r is the intrinsic growth rate of prey; d_i represents the mortality rate of predator x_i $(i = 2, 3)$; a_i stands for the intraspecific competition rate of species x_i $(i = 1, 2, 3)$; α_{12}, β_{12}, and e_{12} are the encounter rate, half-saturation constant, and conversion rate, respectively, that x_2 preys on x_1; α_{13}, β_{13} and e_{13} stand for the same corresponding denotations that x_3 preys on x_1; and α_{23}, β_{23}, and e_{23} represent the same corresponding denotations that x_3 preys on x_2.

As mentioned above, we notice that population models (1)–(4) are described by the deterministic model. This is valid only at the macroscopic scale, that is, the stochastic effects can be neglected or averaged out, in view of the law of large numbers. However, in the real world, populations are actually subject to the environmental fluctuations. Generally speaking, such fluctuations could be modeled by a colored noise. It has been noted that if the colored noise is not strongly correlated, then we can approximate the colored noise by a white noise $\dot{w}(t)$, and the approximation works quite well (see [10]). It turns out that the white noise $\dot{w}(t)$ is formally regarded as the derivative of a Brownian motion $w(t)$, i.e., $\dot{w}(t) = \mathrm{d}w(t)/\mathrm{d}t$ (see [11]). As a result, the study of stochastic ecological dynamics model has already become one of the important subjects in biological mathematics.

After taking the effect of randomly fluctuating environment into account, many researchers introduced stochastic environmental variation described by the Brownian motion into parameters in the deterministic model to establish the stochastic population model (see [12–15]). Liu and Bai [12] considered the optimal harvesting problem of a stochastic logistic model with time delay. In [13–15], the authors investigated the dynamics of stochastic predator-prey models. Ji et al. [13] discussed a stochastic predator-prey model with modified Leslie–Gowerand Holling-type II schemes. Jovanović and Krstić [14] investigated the extinction of a stochastic predator-prey model with the Allee effect on the prey. Liu and Jiang [15] considered the periodic solution and stationary distribution of stochastic predator-prey models with higher-order perturbation. In [16], considering that fluctuations in the environment would manifest themselves mainly as fluctuations in the intrinsic growth rate of the prey population and in the death rate of the predator population (see [17]), Ji et al. supposed parameters r and d_2 in model (1) were perturbed with

$$\begin{aligned} r &\longrightarrow r + \sigma_1 \dot{w}_1(t), \\ -d_2 &\longrightarrow -d_2 + \sigma_2 \dot{w}_2(t), \end{aligned} \tag{5}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Brownian motions and σ_i^2 represents the intensity of white noise $\dot{w}_i(t)$ $(i = 1, 2)$. Moreover, they investigated the long time behavior of the following stochastic ratio-dependent prey-predator model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_1 x_1(t) - \dfrac{\alpha_{12} x_2(t)}{x_1(t) + \beta_{12} x_2(t)}\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 + \dfrac{e_{12}\alpha_{12} x_1(t)}{x_1(t) + \beta_{12} x_2(t)}\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t). \end{cases} \tag{6}$$

Based on (6), Wu et al. [18] considered the corresponding nonautonomous stochastic ratio-dependent model. Lv et al. [19] introduced the intraspecific competition of the predator population, denoted by a_2, into model (6). Nguyen and Ta [20] considered a corresponding nonautonomous stochastic ratio-dependent prey-predator model, in which the white noise makes the effect on both the growth rates of species and the intraspecific competition coefficient of the species.

For the study of stochastic three-species models, consult [21–26] and the references therein. Geng et al. [21] investigated the stability of a stochastic one-predator-two-prey population model with time delay, while Liu et al. [22] studied the stability of a stochastic two-predator one-prey population model with time delay. In [23, 24], the authors discussed the dynamical behaviors of stochastic tri-trophic food-chain models. Li et al. [23] investigated the persistence and nonpersistence of a stochastic food-chain model, while Liu and Bai [24] considered the optimal harvesting problem of a stochastic three species food-chain model. Furthermore, in [25, 26], the stochastic three-species food-chain models with omnivory are discussed. Qiu and Deng [25] investigated the stationary distribution and global asymptotic stability of a stochastic food-web model with omnivory and linear functional response, while R. Liu and G. Liu [26] discussed the persistence in mean and extinction of a stochastic food-web model with intraguild predation and mixed functional responses. In [26], the authors considered Holling type-II functional response between the intermediate predator and the top predator and other functional responses were assumed to be linear.

To the best of our knowledge, so far there is no investigation on the dynamics of the stochastic three-species food web model with omnivory and ratio-dependent functional response. The purpose of this paper is to make some contribution in this direction. Recall that parameters r, d_2, and d_3 in model (4) represent the intrinsic growth rate of the prey population, the death rate of the intermediate predator, and the death rate of the omnivorous top predator, respectively. As done in [16], in this paper, we may replace r, d_2, and d_3 in model (4), respectively, by

$$\begin{aligned} r &\longrightarrow r + \sigma_1 \dot{w}_1(t), \\ -d_2 &\longrightarrow -d_2 + \sigma_2 \dot{w}_2(t), \\ -d_3 &\longrightarrow -d_3 + \sigma_3 \dot{w}_3(t), \end{aligned} \tag{7}$$

where $\dot{w}_i(t)$ is the white noise and σ_i^2 is the intensity of white noise $\dot{w}_i(t)$ $(i = 1, 2, 3)$. Then, the stochastic three-species food web model with omnivory and ratio-dependent functional response took the following form:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_1 x_1(t) + \dfrac{\alpha_{12} x_2(t)}{x_1(t) + \beta_{12} x_2(t)} - \dfrac{\alpha_{13} x_3(t)}{x_1(t) + \beta_{13} x_3(t)}\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 - a_2 x_2(t) + \dfrac{e_{12}\alpha_{12} x_1(t)}{x_1(t) + \beta_{12} x_2(t)} - \dfrac{\alpha_{23} x_3(t)}{x_2(t) + \beta_{23} x_3(t)}\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 - a_3 x_3(t) + \dfrac{e_{13}\alpha_{13} x_1(t)}{x_1(t) + \beta_{13} x_3(t)} + \dfrac{e_{23}\alpha_{23} x_2(t)}{x_2(t) + \beta_{23} x_3(t)}\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{8}$$

with $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0\, i = 1, 2, 3\}$. All meanings of the parameters are exact to or similar as those for (4) except the following. Here, $w = \{w_1(t), w_2(t), w_3(t) : t \geq 0\}$ represents the three-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. σ_i^2 represents the intensity of noise $w_i(t)$ $(i = 1, 2, 3)$. Throughout this paper, unless otherwise specified, we would rather assume that $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq 0$, $\alpha_{13} \geq 0$, $\alpha_{23} \geq 0$, $\beta_{13} > 0$, $\beta_{23} > 0$, $e_{13} > 0$, and $e_{23} > 0$.

2. Existence and Uniqueness of Positive Solution

In this section, we consider the existence of the positive solution for all times. Typically, conditions assuring the nonexplosion of the solution involve local Lipschitz continuity and a linear growth condition. In our case, we miss this last condition, so it is necessary to prove that the solution does not explode at a finite time. To prove the solution is positive and does not explode at a finite time, we use the stochastic comparison theorem. For simplicity, we introduce the following notations:

$$\kappa_1 = r - \frac{\alpha_{12}}{\beta_{12}} - \frac{\alpha_{13}}{\beta_{13}};$$
$$\kappa_2 = e_{12}\alpha_{12} - d_2 - \frac{\alpha_{23}}{\beta_{23}}; \tag{9}$$
$$\kappa_3 = e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3.$$

Theorem 1. *For any given initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, model (8) has unique global positive solution $(x_1(t), x_2(t), x_3(t))$ for $t \geq 0$, that is, $(x_1(t), x_2(t), x_3(t)) \in \mathbb{R}_+^3$ with probability one for $t \in [0, \infty)$.*

Proof. Consider the following system:

$$\begin{cases} \mathrm{d}X_1(t) = \left[r - a_1 e^{X_1(t)} + \dfrac{\alpha_{12} e^{X_2(t)}}{e^{X_1(t)} + \beta_{12} e^{X_2(t)}} + \dfrac{\alpha_{13} e^{X_3(t)}}{e^{X_1(t)} + \beta_{13} e^{X_3(t)}} - \dfrac{\sigma_1^2}{2}\right]\mathrm{d}t + \sigma_1 \mathrm{d}w_1(t), \\ \mathrm{d}X_2(t) = \left[-d_2 - a_2 e^{X_2(t)} + \dfrac{e_{12}\alpha_{12} e^{X_1(t)}}{e^{X_1(t)} + \beta_{12} e^{X_2(t)}} - \dfrac{\alpha_{23} e^{X_3(t)}}{e^{X_2(t)} + \beta_{23} e^{X_3(t)}} - \dfrac{\sigma_2^2}{2}\right]\mathrm{d}t + \sigma_2 \mathrm{d}w_2(t), \\ \mathrm{d}X_3(t) = \left[-d_3 - a_3 e^{X_3(t)} + \dfrac{e_{13}\alpha_{13} e^{X_1(t)}}{e^{X_1(t)} + \beta_{13} e^{X_3(t)}} + \dfrac{e_{23}\alpha_{23} e^{X_2(t)}}{e^{X_2(t)} + \beta_{23} e^{X_3(t)}} - \dfrac{\sigma_3^2}{2}\right]\mathrm{d}t + \sigma_3 \mathrm{d}w_3(t), \end{cases} \tag{10}$$

with initial value $(X_1(0), X_2(0), X_3(0)) = (\ln x_{10}, \ln x_{20}, \ln x_{30})$. Obviously, the coefficients of (10) are locally Lipschitz continuous. Thus, there is a unique maximal local solution $(X_1(t), X_2(t), X_3(t))$ of (10) for $t \in [0, \tau_e)$, where τ_e denotes the explosion time. Let $x_i(t) = e^{X_i(t)}$ $(i = 1, 2, 3)$. Using Itô formula, it follows that $(x_1(t), x_2(t), x_3(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{X_3(t)})$ is the unique positive local solution of (8) with initial value (x_{10}, x_{20}, x_{30}) for $t \in [0, \tau_e)$.

Next, we show that $(X_1(t), X_2(t), X_3(t))$ is a global solution of (10), that is, $\tau_e = \infty$. Consider the following two stochastic differential systems:

$$\begin{cases} \mathrm{d}\Phi_1(t) = \Phi_1(t)[r - a_1\Phi_1(t)]\mathrm{d}t + \sigma_1\Phi_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\Phi_2(t) = \Phi_2(t)[e_{12}\alpha_{12} - a_2\Phi_2(t)]\mathrm{d}t + \sigma_2\Phi_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}\Phi_3(t) = \Phi_3(t)[e_{13}\alpha_{13} + e_{23}\alpha_{23} - a_3\Phi_3(t)]\mathrm{d}t + \sigma_3\Phi_3(t)\mathrm{d}w_3(t), \end{cases} \tag{11}$$

with initial value $(\Phi_1(0), \Phi_2(0), \Phi_3(0)) = (x_{10}, x_{20}, x_{30})$ and

$$\begin{cases} \mathrm{d}\phi_1(t) = \phi_1(t)[\kappa_1 - a_1\phi_1(t)]\mathrm{d}t + \sigma_1\phi_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\phi_2(t) = \phi_2(t)\left[\kappa_2 - \left(a_2 + \dfrac{e_{12}\alpha_{12}\beta_{12}}{\phi_1(t)}\right)\phi_2(t)\right]\mathrm{d}t + \sigma_2\phi_{23}(t)\mathrm{d}w_2(t), \\ \mathrm{d}\phi_3(t) = \phi_3(t)\left[\kappa_3 - \left(a_3 + \dfrac{e_{13}\alpha_{13}\beta_{13}}{\phi_1(t)} + \dfrac{e_{23}\alpha_{23}\beta_{23}}{\phi_2(t)}\right)\phi_3(t)\right]\mathrm{d}t + \sigma_3\phi_3(t)\mathrm{d}w_3(t), \end{cases} \tag{12}$$

with initial value $(\phi_1(0), \phi_2(0), \phi_3(0)) = (x_{10}, x_{20}, x_{30})$.

Thanks to Lemma 4.2 in [27], systems (11) and (12) can be explicitly solved as follows:

$$\begin{cases} \Phi_1(t) = \dfrac{\exp\{(r-(\sigma_1^2/2))t + \sigma_1 w_1(t)\}}{(1/x_{10}) + a_1 \int_0^t \exp\{(r-(\sigma_1^2/2))s + \sigma_1 w_1(s)\}ds}, \\ \Phi_2(t) = \dfrac{\exp\{(e_{12}\alpha_{12} - (\sigma_2^2/2))t + \sigma_2 w_2(t)\}}{(1/x_{20}) + a_2 \int_0^t \exp\{(e_{12}\alpha_{12} - (\sigma_2^2/2))s + \sigma_2 w_2(s)\}ds}, \\ \Phi_3(t) = \dfrac{\exp\{(e_{13}\alpha_{13} + e_{23}\alpha_{23} - (\sigma_3^2/2))t + \sigma_3 w_3(t)\}}{(1/x_{30}) + a_3 \int_0^t \exp\{(e_{13}\alpha_{13} + e_{23}\alpha_{23} - (\sigma_3^2/2))s + \sigma_3 w_3(s)\}ds}, \end{cases}$$
$$\begin{cases} \phi_1(t) = \dfrac{\exp\{(\kappa_1 - (\sigma_1^2/2))t + \sigma_1 w_1(t)\}}{(1/x_{10}) + a_1 \int_0^t \exp\{(\kappa_1 - (\sigma_1^2/2))s + \sigma_1 w_1(s)\}ds}, \\ \phi_2(t) = \dfrac{\exp\{(\kappa_2 - (\sigma_2^2/2))t + \sigma_2 w_2(t)\}}{(1/x_{20}) + \int_0^t (a_2 + ((e_{12}\alpha_{12}\beta_{12})/(\phi_1(s))))\exp\{(\kappa_2 - (\sigma_2^2/2))s + \sigma_2 w_2(s)\}ds}, \\ \phi_3(t) = \dfrac{\exp\{(\kappa_3 - (\sigma_3^2/2))t + \sigma_3 w_3(t)\}}{(1/x_{30}) + \int_0^t (a_3 + ((e_{13}\alpha_{13}\beta_{13})/(\phi_1(s))) + ((e_{23}\alpha_{23}\beta_{23})/(\phi_2(s))))\exp\{(\kappa_3 - (\sigma_3^2/2))s + \sigma_3 w_3(s)\}ds}. \end{cases} \tag{13}$$

Note that the local solution $(x_1(t), x_2(t), x_3(t))$ is positive on $[0, \tau_e)$. Then, from the comparison theorem of stochastic differential equations (see Theorem 3.1 in [28]), it follows that for $t \in [0, \tau_e)$

$$0 < \phi_i(t) \le x_i(t) \le \Phi_i(t), \quad \text{a.s.}, \ i = 1, 2, 3. \tag{14}$$

Thus, for $t \in [0, \tau_e)$

$$\ln \phi_i(t) \le \ln x_i(t) \le \ln \Phi_i(t), \quad \text{a.s.}, \ i = 1, 2, 3. \tag{15}$$

Since $\ln \phi_i(t)$ and $\ln \Phi_i(t)$ $(i = 1, 2, 3)$ exist for every $t \ge 0$, it follows that $\tau_e = \infty$. Thus, for any initial value $(X_1(0), X_2(0), X_3(0)) = (\ln x_{10}, \ln x_{20}, \ln x_{30}) \in \mathbb{R}^3$, and (10) has a unique global solution $(X_1(t), X_2(t), X_3(t))$ on $[0, \infty)$ a.s. Note that the coefficients of (8) are local Lipschitz continuous. Therefore, for any initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, model (8) has a unique global positive solution $(x_1(t), x_2(t), x_3(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{X_3(t)})$ on $[0, \infty)$ a.s. The proof is therefore complete. □

3. Asymptotic Behaviors

Lemma 1 (see [13]). *Consider one-dimensional stochastic differential equation:*

$$dx(t) = x(t)[a - bx(t)]dt + \sigma x(t)dw(t), \tag{16}$$

where a, b, and σ are positive constants and $w(t)$ is standard Brownian motion. For any $x_0 > 0$, let $x(t)$ be the solution of equation (16) with initial value x_0. If $a > (\sigma^2/2)$, then

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{a - (\sigma^2/2)}{b}, \quad \text{a.s.} \tag{17}$$

Theorem 2. *For any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), x_3(t))$ be the solution of model (8) with initial value (x_{10}, x_{20}, x_{30}). If $a_2 > 0$, $a_3 > 0$, and $\kappa_i - (\sigma_i^2/2) > 0$ $(i = 1, 2, 3)$, then*

$$\lim_{t\to\infty} \frac{\ln x_i(t)}{t} = 0, \ \text{a.s.}, \quad i = 1, 2, 3. \tag{18}$$

Proof. From Theorem 1, it follows that

$$\phi_i(t) \le x_i(t) \le \Phi_i(t), \ \text{a.s.}, \quad i = 1, 2, 3. \tag{19}$$

Note that $\phi_1(t)$ and $\Phi_1(t)$ are the solutions of the following stochastic equations, respectively,

$$\begin{aligned} d\phi_1(t) &= \phi_1(t)[\kappa_1 - a_1\phi_1(t)]dt + \sigma_1\phi_1(t)dw_1(t), \\ d\Phi_1(t) &= \Phi_1(t)[r - a_1\Phi_1(t)]dt + \sigma_1\Phi_1(t)dw_1(t). \end{aligned} \tag{20}$$

with initial value $x_{10} > 0$. Obviously, from Lemma 1, it follows that if $\kappa_1 - (\sigma_1^2/2) > 0$:

$$\lim_{t\to\infty} \frac{\ln \phi_1(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\ln \Phi_1(t)}{t} = 0 \ \text{a.s.} \tag{21}$$

This, together with (19), yields

$$\lim_{t\longrightarrow\infty}\frac{\ln x_1(t)}{t}=0\text{ a.s.}\tag{22}$$

Now, we show $\lim_{t\longrightarrow\infty}((\ln x_2(t))/t)=0$ a.s. Note that, $x_2(t)\le\Phi_2(t)$ a.s. and $\Phi_2(t)$ is the solution of equation

$$d\Phi_2(t)=\Phi_2(t)\left[e_{12}\alpha_{12}-a_2\Phi_2(t)\right]dt+\sigma_2\Phi_2(t)dw_2(t),\tag{23}$$

with initial $x_{20}>0$. Thus, from Lemma 1 and $\kappa_2-(\sigma_2^2/2)>0$, it follows that

$$\lim_{t\longrightarrow\infty}\frac{\ln\Phi_2(t)}{t}=0\text{ a.s.}\tag{24}$$

Note that $\lim_{t\longrightarrow\infty}((\ln\phi_1(t))/t)=0$, a.s. Thus, for any $\varepsilon>0$, there exists $T_1>0$ such that

$$e^{-\varepsilon t}\le\phi_1(t)\le e^{\varepsilon t},\quad\text{for } t\ge T_1.\tag{25}$$

By the strong law of large numbers of local martingales (see Theorem 1.3.4 in [11]), it follows that $\lim_{t\longrightarrow\infty}((\sigma_2 w_2(t))/t)=0$ a.s. Thus, for any $\varepsilon>0$, there exists $T_2>0$ such that

$$-\varepsilon t\le\sigma_2 w_2(t)\le\varepsilon t,\quad\text{for } t\ge T_2.\tag{26}$$

From the expression of $\phi_2(t)$ that for any $t\ge T=T_1\vee T_2$, we have

$$\begin{aligned}\frac{1}{\phi_2(t)}&=\frac{1}{x_2(T)}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-T)-\sigma_2\left(w_2(t)-w_2(T)\right)\right]}+a_2\int_T^t e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)-\sigma_2\left(w_2(t)-w_2(s)\right)\right]}ds\\&\quad+\int_T^t\frac{e_{12}\alpha_{12}\beta_{12}}{\phi_1(s)}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)-\sigma_2\left(w_2(t)-w_2(s)\right)\right]}ds\\&\le\frac{1}{x_2(T)}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-T)+\varepsilon(t+T)\right]}+a_2\int_T^t e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)+\varepsilon(t+s)\right]}ds\\&\quad+e_{12}\alpha_{12}\beta_{12}\int_T^t e^{\varepsilon s}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)+\varepsilon(t+s)\right]}ds.\end{aligned}\tag{27}$$

Hence, from $\kappa_2-(\sigma_2^2/2)>0$, $\varepsilon>0$ and $t\ge T$, it follows that

$$\begin{aligned}\frac{e^{-3\varepsilon(t+T)}}{\phi_2(t)}&\le\frac{1}{x_2(T)}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-T)-2\varepsilon(t+T)\right]}+a_2\int_T^t e^{-\varepsilon(t-s)}e^{-\varepsilon t}e^{-3\varepsilon T}e^{-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)}ds\\&\quad+e_{12}\alpha_{12}\beta_{12}e^{-3\varepsilon T}\int_T^t e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-s)-2\varepsilon(t-s)\right]}ds\\&=\frac{1}{x_2(T)}e^{\left[-\left(\kappa_2-(\sigma_2^2/2)\right)(t-T)-2\varepsilon(t+T)\right]}+\frac{a_2}{\kappa_2+\varepsilon-(\sigma_2^2/2)}e^{-\varepsilon t}e^{-3\varepsilon T}e^{-\left(\kappa_2+\varepsilon-(\sigma_2^2/2)\right)T}\\&\quad+\frac{e_{12}\alpha_{12}\beta_{12}}{\kappa_2+2\varepsilon-(\sigma_2^2/2)}e^{-3\varepsilon T}e^{-\left(\kappa_2+2\varepsilon-(\sigma_2^2/2)\right)T}\le\frac{1}{x_2(T)}+\frac{a_2}{\kappa_2-(\sigma_2^2/2)}+\frac{e_{12}\alpha_{12}\beta_{12}}{\kappa_2-(\sigma_2^2/2)}\doteq K_1.\end{aligned}\tag{28}$$

That is $(1/(\phi_2(t)))\le K_1e^{3\varepsilon(t+T)}$ a.s., for $t\ge T$. Then, $-\ln\phi_2(t)\le\ln K_1+3\varepsilon(t+T)$. Thus, for any $\varepsilon>0$,

$$\liminf_{t\longrightarrow\infty}\frac{\ln\phi_2(t)}{t}\ge 0\text{ a.s.}\tag{29}$$

Consequently,

$$\begin{aligned}0&\le\liminf_{t\longrightarrow\infty}\frac{\ln\phi_2(t)}{t}\le\liminf_{t\longrightarrow\infty}\frac{\ln x_2(t)}{t}\le\limsup_{t\longrightarrow\infty}\frac{\ln x_2(t)}{t}\\&\le\lim_{t\longrightarrow\infty}\frac{\ln\Phi_2(t)}{t}=0\text{ a.s.}\end{aligned}\tag{30}$$

In addition,

$$\limsup_{t\to\infty}\frac{\ln\phi_2(t)}{t}\le\limsup_{t\to\infty}\frac{\ln x_2(t)}{t}\le 0 \text{ a.s.} \quad (31)$$

Thus,

$$\lim_{t\to\infty}\frac{\ln\phi_2(t)}{t}=0,$$
$$\lim_{t\to\infty}\frac{\ln x_2(t)}{t}=0, \text{ a.s.} \quad (32)$$

Similarly, if $\kappa_1-(\sigma_1^2/2)>0$, $\kappa_2-(\sigma_2^2/2)>0$, and $\kappa_3-(\sigma_3^2/2)>0$, then

$$\lim_{t\to\infty}\frac{\ln x_3(t)}{t}=0 \text{ a.s.} \quad (33)$$

The proof is therefore complete. □

4. Persistence in Mean and Extinction

In this section, we show that under some conditions, model (8) is persistent in mean and extinct.

Theorem 3. *Suppose that $a_2>0$, $a_3>0$, and $\kappa_i-(\sigma_i^2/2)>0$ $(i=1,2,3)$. Then, for any $(x_{10},x_{20},x_{30})\in\mathbb{R}_+^3$, the solution $(x_1(t),x_2(t),x_3(t))$ of model (8) with initial value (x_{10},x_{20},x_{30}) obeys*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds\ge\frac{\kappa_1-(\sigma_1^2/2)}{a_1} \text{ a.s.},$$
$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left[a_2x_2(s)+\frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)}\right]ds\ge\kappa_2-\frac{\sigma_2^2}{2} \text{ a.s.},$$
$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left[a_3x_3(s)+\frac{e_{13}\alpha_{13}\beta_{13}x_3(s)}{x_1(s)}+\frac{e_{23}\alpha_{23}\beta_{23}x_3(s)}{x_2(s)}\right]ds$$
$$\ge\kappa_3-\frac{\sigma_3^2}{2} \text{ a.s.} \quad (34)$$

Proof. For prey x_1, from Theorem 1, it follows that

$$\phi_1(t)\le x_1(t) \text{ a.s.}, \quad (35)$$

and $\phi_1(t)$ is the solution of the following stochastic equation:

$$d\phi_1(t)=\phi_1(t)[\kappa_1-a_1\phi_1(t)]dt+\sigma_1\phi_1(t)dw_1(t), \quad (36)$$

with initial value $x_{10}>0$. Obviously, from Lemma 1, it follows that if $\kappa_1-(\sigma_1^2/2)>0$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\phi_1(s)ds=\frac{\kappa_1-(\sigma_1^2/2)}{a_1} \text{ a.s.} \quad (37)$$

This, together with (35), yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds\ge\lim_{t\to\infty}\frac{1}{t}\int_0^t\phi_1(s)ds$$
$$=\frac{\kappa_1-(\sigma_1^2/2)}{a_1}>0 \text{ a.s.} \quad (38)$$

For intermediate predator x_2, using Itô formula, it follows that

$$\ln x_2(t)=\int_0^t\left[-d_2-a_2x_2(s)+\frac{e_{12}\alpha_{12}x_1(s)}{x_1(s)+\beta_{12}x_2(s)}\right.$$
$$\left.-\frac{\alpha_{23}x_3(s)}{x_2(s)+\beta_{23}x_3(s)}-\frac{\sigma_2^2}{2}\right]ds+\sigma_2w_2(t)+\ln x_{20}$$
$$\ge\left[\kappa_2-\frac{\sigma_2^2}{2}\right]t-a_2\int_0^t x_2(s)ds$$
$$-\int_0^t\frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)+\beta_{12}x_2(s)}ds+\sigma_2w_2(t)+\ln x_{20}. \quad (39)$$

Hence,

$$\frac{1}{t}\int_0^t\left[a_2x_2(s)+\frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)}\right]ds\ge\left[\kappa_2-\frac{\sigma_2^2}{2}\right]$$
$$+\frac{\sigma_2w_2(t)}{t}+\frac{\ln x_{20}}{t}-\frac{\ln x_2(t)}{t}. \quad (40)$$

By the strong law of numbers of local martingales and Theorem 2, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left[a_2x_2(s)+\frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)}\right]ds\ge\kappa_2-\frac{\sigma_2^2}{2} \text{ a.s.} \quad (41)$$

For the omnivorous top predator x_3, it follows from Itô formula that

$$\ln x_3(t)=\int_0^t\left[-d_3-a_3x_3(s)+\frac{e_{13}\alpha_{13}x_1(s)}{x_1(s)+\beta_{13}x_3(s)}\right.$$
$$\left.+\frac{e_{23}\alpha_{23}x_2(s)}{x_2(s)+\beta_{23}x_3(s)}-\frac{\sigma_3^2}{2}\right]ds+\sigma_3w_3(t)+\ln x_{30}$$
$$\ge\left[\kappa_3-\frac{\sigma_3^2}{2}\right]t-\int_0^t\left[a_3x_3(s)+\frac{e_{13}\alpha_{13}\beta_{13}x_3(s)}{x_1(s)+\beta_{13}x_3(s)}\right.$$
$$\left.+\frac{e_{23}\alpha_{23}\beta_{23}x_3(s)}{x_2(s)+\beta_{23}x_3(s)}\right]ds+\sigma_3w_3(t)+\ln x_{30}. \quad (42)$$

Hence,

$$\frac{1}{t}\int_0^t\left[a_3x_3(s)+\frac{e_{13}\alpha_{13}\beta_{13}x_3(s)}{x_1(s)}+\frac{e_{23}\alpha_{23}\beta_{23}x_3(s)}{x_2(s)}\right]ds$$
$$\ge\left[\kappa_3-\frac{\sigma_3^2}{2}\right]+\frac{\sigma_3w_3(t)}{t}+\frac{\ln x_{30}}{t}-\frac{\ln x_3(t)}{t}. \quad (43)$$

By the strong law of numbers of local martingales and Theorem 2, we get

$$\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t\left[a_3x_3(s)+\frac{e_{13}\alpha_{13}\beta_{13}x_3(s)}{x_1(s)}+\frac{e_{23}\alpha_{23}\beta_{23}x_3(s)}{x_2(s)}\right]ds \geq \kappa_3-\frac{\sigma_3^2}{2}\ \text{a.s.} \tag{44}$$

The proof is therefore complete. □

Theorem 4. *Suppose that* $r-(\sigma_1^2/2)<0$, $e_{12}\alpha_{12}-d_2-(\sigma_2^2/2)<0$ *and* $e_{13}\alpha_{13}+e_{23}\alpha_{23}-d_3-(\sigma_3^2/2)<0$. *Then, for any initial value* $(x_{10},x_{20},x_{30})\in\mathbb{R}_+^3$, *model (8) is extinct exponentially with probability one.*

Proof. From Itô formula, it follows that

$$\begin{aligned}\ln x_1(t)&\leq\left[r-\frac{\sigma_1^2}{2}\right]t+\sigma_1w_1(t)+\ln x_{10},\\ \ln x_2(t)&\leq\left[e_{12}\alpha_{12}-d_2-\frac{\sigma_2^2}{2}\right]t+\sigma_2w_2(t)+\ln x_{20},\\ \ln x_3(t)&\leq\left[e_{13}\alpha_{13}+e_{23}\alpha_{23}-d_3-\frac{\sigma_3^2}{2}\right]t+\sigma_3w_3(t)+\ln x_{30}.\end{aligned} \tag{45}$$

Note that $\lim_{t\longrightarrow\infty}[((\sigma_iw_i(t))/t)+((\ln x_{i0})/t)]=0$ $(i=1,2,3)$ and $r-(\sigma_1^2/2)<0$, $e_{12}\alpha_{12}-d_2-(\sigma_2^2/2)<0$, and $e_{13}\alpha_{13}+e_{23}\alpha_{23}-d_3-(\sigma_3^2/2)<0$. Then,

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\frac{\ln x_1(t)}{t}&\leq r-\frac{\sigma_1^2}{2}<0\ \text{a.s.},\\ \limsup_{t\longrightarrow\infty}\frac{\ln x_2(t)}{t}&\leq e_{12}\alpha_{12}-d_2-\frac{\sigma_2^2}{2}<0\ \text{a.s.},\\ \limsup_{t\longrightarrow\infty}\frac{\ln x_3(t)}{t}&\leq e_{13}\alpha_{13}+e_{23}\alpha_{23}-d_3-\frac{\sigma_3^2}{2}<0\ \text{a.s.}\end{aligned} \tag{46}$$

Therefore, model (8) is extinct exponentially. The proof is complete. □

5. Stochastic Permanence

In this section, we discuss the stochastic permanence of model (8). The definition of stochastic permanence and stochastically ultimately boundness of model (8) were introduced in the literature [29, 30] as follows.

Definition 1 (see [29, 30]). Model (8) is called stochastically ultimate bounded, if for any $\varepsilon\in(0,1)$, there exist three positive constants $H_1=H_1(\varepsilon)$, $H_2=H_2(\varepsilon)$, and $H_3=H_3(\varepsilon)$ such that the solution $(x_1(t),x_2(t),x_3(t))$ of model (8) with any initial value $(x_{10},x_{20},x_{30})\in\mathbb{R}_+^3$ has the property that

$$\limsup_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)>H_i\}<\varepsilon,\quad i=1,2,3. \tag{47}$$

Definition 2 (see [29, 30]). Model (8) is said to be stochastically permanent, if for any $\varepsilon\in(0,1)$, there exist positive constants $\delta_i=\delta_i(\varepsilon)$, $H_i=H_i(\varepsilon)$, and $\delta_i<H_i$ $(i=1,2,3)$, such that the solution $(x_1(t),x_2(t),x_3(t))$ of model (8) with any initial value $(x_{10},x_{20},x_{30})\in\mathbb{R}_+^3$ has the property that

$$\begin{aligned}&\liminf_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)\leq H_i\}\geq 1-\varepsilon,\\ &\liminf_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)\geq\delta_i\}\geq 1-\varepsilon,\quad i=1,2,3.\end{aligned} \tag{48}$$

It is obvious that if stochastic model (8) is stochastically permanent, its solutions must be stochastically ultimately bounded.

5.1. Boundness. In this subsection, we investigate the stochastically ultimate boundness of model (8) in two different ways.

Lemma 2 (see [31]). *For any positive constants p, m, and n, the Bernoulli equation*

$$\frac{dx(t)}{dt}=pmx(t)-pnx^{1+(1/p)}(t), \tag{49}$$

with the initial value $x(0)=x_0>0$, *has the solution*

$$x(t)=\left[\frac{m}{n\left(1-e^{-mt}+(m/n)x_0^{-(1/p)}e^{-mt}\right)}\right]^p. \tag{50}$$

Theorem 5. *For any* $(x_{10},x_{20},x_{30})\in\mathbb{R}_+^3$, *let* $(x_1(t),x_2(t),x_3(t))$ *be the solution of model (8) with initial value* (x_{10},x_{20},x_{30}). *If* $a_2>0$ *and* $a_3>0$, *then for any* $p\geq 0$,

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_1^p(t)\right]&\leq\left[\frac{r+(p/2)\sigma_1^2}{a_1}\right]^p,\\ \limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_2^p(t)\right]&\leq\left[\frac{e_{12}\alpha_{12}+(p/2)\sigma_2^2}{a_2}\right]^p,\\ \limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_3^p(t)\right]&\leq\left[\frac{e_{13}\alpha_{13}+e_{23}\alpha_{23}+(p/2)\sigma_3^2}{a_3}\right]^p.\end{aligned} \tag{51}$$

That is, the solution of model (8) is uniformly bounded in the pth moment.

Proof. For Φ_1 in system (11), applying Itô formula to Φ_1^p leads to

$$\Phi_1^p(t)=x_{10}^p+\int_0^t p\Phi_1^p(s)\left[r+\frac{p-1}{2}\sigma_1^2-a_1\Phi_1(s)\right]ds+\int_0^t p\sigma_1\Phi_1^p(s)dw_1(s). \tag{52}$$

Taking the expectation on both sides of the above equation, we have

$$\mathbb{E}\left[\Phi_1^p(t)\right] = x_{10}^p + \mathbb{E}\int_0^t p\Phi_1^p(s)\left[r + \frac{p-1}{2}\sigma_1^2 - a_1\Phi_1(s)\right]\mathrm{d}s. \tag{53}$$

Then, using the Höder inequality, it follows that

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}\left[\Phi_1^p(t)\right]}{\mathrm{d}t} &= p\left(r + \frac{p-1}{2}\sigma_1^2\right)\mathbb{E}\left[\Phi_1^p(t)\right] - pa_1\mathbb{E}\left[\Phi_1^{p+1}(t)\right] \\ &\le p\left(r + \frac{p}{2}\sigma_1^2\right)\mathbb{E}\left[\Phi_1^p(t)\right] - pa_1\left(\mathbb{E}\left[\Phi_1^p(t)\right]\right)^{1+(1/p)} \\ &\doteq pb_1\mathbb{E}\left[\Phi_1^p(t)\right] - pa_1\left(\mathbb{E}\left[\Phi_1^p(t)\right]\right)^{1+(1/p)}.\end{aligned} \tag{54}$$

From Lemma 2 and the comparison theorem, it follows that

$$\mathbb{E}\left[\Phi_1^p(t)\right] \le \left[\frac{b_1}{a_1\left(1 - e^{-b_1 t} + (b_1/a_1)x_{10}^{-1}e^{-b_1 t}\right)}\right]^p. \tag{55}$$

Note that $b_1 = r + (p/2)\sigma_1^2 > 0$. Thus,

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\Phi_1^p(t)\right] \le \left[\frac{r + (p/2)\sigma_1^2}{a_1}\right]^p. \tag{56}$$

By a similar the discussion as in $\Phi_1(t)$, we also know that

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\Phi_2^p(t)\right] &\le \left[\frac{e_{12}\alpha_{12} + ((p-1)/2)\sigma_2^2}{a_2}\right]^p, \\ \limsup_{t\longrightarrow\infty}\mathbb{E}\left[\Phi_3^p(t)\right] &\le \left[\frac{e_{13}\alpha_{13} + e_{23}\alpha_{23} + ((p-1)/2)\sigma_3^2}{a_3}\right]^p.\end{aligned} \tag{57}$$

From Theorem 1, it follows that $0 < x_i(t) \le \Phi_i(t)$ a.s. $i = 1, 2, 3$. Then, for any $p \ge 0$, we have

$$0 < \mathbb{E}\left[x_i^p(t)\right] \le \mathbb{E}\left[\Phi_i^p(t)\right], \quad i = 1, 2, 3. \tag{58}$$

Now Theorem 5 follows immediately from the above analysis. The proof is complete. □

Theorem 6. *For any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), x_3(t))$ be the solution of model (8) with initial value (x_{10}, x_{20}, x_{30}). Then,*

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_1(t)\right] &\le \frac{K_2}{d^L(e_{12}e_{23} + e_{13})}, \\ \limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_2(t)\right] &\le \frac{K_2}{d^L e_{23}}, \\ \limsup_{t\longrightarrow\infty}\mathbb{E}\left[x_3(t)\right] &\le \frac{K_2}{d^L},\end{aligned} \tag{59}$$

where $d^L = d_2 \wedge d_3$ and $K_2 = ((e_{12}e_{23} + e_{13})(r + d^L)^2)/(4a_1)$.

Proof. Define $H(t) = (e_{12}e_{23} + e_{13})x_1(t) + e_{23}x_2(t) + x_3(t)$. By Itô formula, we have

$$\begin{aligned}\mathrm{d}H(t) &= (e_{12}e_{23} + e_{13})\left[\left(rx_1 - a_1x_1^2 - \frac{\alpha_{12}x_1x_2}{x_1 + \beta_{12}x_2} - \frac{\alpha_{13}x_1x_3}{x_1 + \beta_{13}x_3}\right)\mathrm{d}t\right. \\ &\quad \left.+ \sigma_1x_1\mathrm{d}w_1(t)\right] + e_{23}\left[\left(-d_2x_2 - a_2x_2^2 + \frac{e_{12}\alpha_{12}x_1x_2}{x_1 + \beta_{12}x_2}\right.\right. \\ &\quad \left.\left.- \frac{\alpha_{23}x_2x_3}{x_2 + \beta_{23}x_3}\right)\mathrm{d}t + \sigma_2x_2\mathrm{d}w_2(t)\right] + \left[\left(-d_3x_3 - a_3x_3^2\right.\right. \\ &\quad \left.\left.+ \frac{e_{13}\alpha_{13}x_1x_3}{x_1 + \beta_{13}x_3} + \frac{e_{23}\alpha_{23}x_2x_3}{x_2 + \beta_{23}x_3}\right)\mathrm{d}t + \sigma_3x_3\mathrm{d}w_3(t)\right] \\ &= \left[-e_{23}d_2x_2 - d_3x_3 + (e_{12}e_{23} + e_{13})rx_1\right. \\ &\quad - (e_{12}e_{23} + e_{13})a_1x_1^2 - e_{23}a_2x_2^2 - a_3x_3^2) \\ &\quad \left.- \frac{e_{12}e_{23}\alpha_{13}x_1x_3}{x_1 + \beta_{13}x_3} - \frac{e_{13}\alpha_{12}x_1x_2}{x_1 + \beta_{12}x_2}\right]\mathrm{d}t \\ &\quad + (e_{12}e_{23} + e_{13})\sigma_1x_1\mathrm{d}w_1(t) + e_{23}\sigma_2x_2\mathrm{d}w_2(t) \\ &\quad + \sigma_3x_3\mathrm{d}w_3(t).\end{aligned} \tag{60}$$

Integrating it from 0 to t and taking expectation yields

$$\begin{aligned}\mathbb{E}[H(t)] = H(0) + \mathbb{E}\int_0^t &\left[-e_{23}d_2x_2 - d_3x_3 + (e_{12}e_{23} + e_{13})rx_1\right. \\ &- (e_{12}e_{23} + e_{13})a_1x_1^2 - e_{23}a_2x_2^2 - a_3x_3^2 \\ &\left.- \frac{e_{12}e_{23}\alpha_{13}x_1x_3}{x_1 + \beta_{13}x_3} - \frac{e_{13}\alpha_{12}x_1x_2}{x_1 + \beta_{12}x_2}\right]\mathrm{d}s.\end{aligned} \tag{61}$$

Thus, using the Hölder inequality yields

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}[H(t)]}{\mathrm{d}t} &= -e_{23}d_2\mathbb{E}\left[x_2(t)\right] - d_3\mathbb{E}\left[x_3(t)\right] \\ &\quad + (e_{12}e_{23} + e_{13})r\mathbb{E}\left[x_1(t)\right] \\ &\quad - (e_{12}e_{23} + e_{13})a_1\mathbb{E}\left[x_1^2(t)\right] - e_{23}a_2\mathbb{E}\left[x_2^2(t)\right] \\ &\quad - a_3\mathbb{E}\left[x_3^2(t)\right] \\ &\quad - e_{12}e_{23}\alpha_{13}\mathbb{E}\left[\frac{x_1(t)x_3(t)}{x_1(t) + \beta_{13}x_3(t)}\right] \\ &\quad - e_{13}\alpha_{12}\mathbb{E}\left[\frac{x_1(t)x_2(t)}{x_1(t) + \beta_{12}x_2(t)}\right] \\ &\le -d^L(e_{12}e_{23} + e_{13})\mathbb{E}\left[x_1(t)\right] - d^Le_{23}\mathbb{E}\left[x_2(t)\right] \\ &\quad - d^L\mathbb{E}\left[x_3(t)\right] \\ &\quad + (e_{12}e_{23} + e_{13})\left(r + d^L\right)\mathbb{E}\left[x_1(t)\right] \\ &\quad - (e_{12}e_{23} + e_{13})a_1\left(\mathbb{E}\left[x_1(t)\right]\right)^2 \\ &= (e_{12}e_{23} + e_{13})\left[\left(r + d^L\right)\mathbb{E}\left[x_1(t)\right] - a_1\left(\mathbb{E}\left[x_1(t)\right]\right)^2\right] \\ &\quad - d^L\mathbb{E}[H(t)].\end{aligned} \tag{62}$$

It is clear that quadratic function $g(x) = (e_{12}e_{23} + e_{13})[(r + d^L)x - a_1 x^2]$ reaches its maximum value at $x = ((r + d^L)/(2a_1)) > 0$. Thus, $g_{\max} = (((e_{12}e_{23} + e_{13})(r + d^L)^2)/(4a_1)) \doteq K_2$. Therefore, we have

$$\frac{\mathrm{d}\mathbb{E}[H(t)]}{\mathrm{d}t} \le K_2 - d^L \mathbb{E}[H(t)]. \tag{63}$$

Then, by the comparison theorem, we have

$$0 \le \limsup_{t\longrightarrow\infty} \mathbb{E}[H(t)] \le \frac{K_2}{d^L}. \tag{64}$$

Note that the solution of model (8) is positive. Thus,

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty} \mathbb{E}[x_1(t)] \le \frac{K_2}{d^L(e_{12}e_{23} + e_{13})},\\
&\limsup_{t\longrightarrow\infty} \mathbb{E}[x_2(t)] \le \frac{K_2}{d^L e_{23}},\\
&\limsup_{t\longrightarrow\infty} \mathbb{E}[x_3(t)] \le \frac{K_2}{d^L}.
\end{aligned} \tag{65}$$

The proof is therefore complete. □

Theorem 7. *Model (8) is stochastically ultimate bounded.*

Proof. Let $(x_1(t), x_2(t), x_3(t))$ be solution of (8) with any initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$. For any $\varepsilon \in (0,1)$, let $H_1 = (K_2/(d^L(e_{12}e_{23} + e_{13})\varepsilon)) + 1$, $H_2 = (K_2/d^L e_{23}\varepsilon) + 1$, and $H_3 = (K_2/d^L\varepsilon) + 1$. Then, by Chebyshev's inequality

$$\mathbb{P}\{x_i(t) > H_i\} \le \frac{\mathbb{E}[x_i(t)]}{H_i}, \quad i = 1,2,3. \tag{66}$$

Hence, from Theorem 6

$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{x_i(t) > H_i\} \le \limsup_{t\longrightarrow\infty} \frac{\mathbb{E}[x_i(t)]}{H_i} < \varepsilon, \quad i = 1,2,3. \tag{67}$$

The proof is therefore complete. □

5.2. Stochastic Permanence. In this section, we give some sufficient conditions to guarantee that model (8) is stochastically permanent. Denote $\gamma_i \doteq \kappa_i - \sigma_i^2$ $(i = 1,2,3)$. Define

$$\begin{aligned}
u_1(t) &= \frac{1}{\phi_1(t)},\\
u_2(t) &= \frac{1}{\phi_2(t)},\\
u_3(t) &= \frac{1}{\phi_3(t)}.
\end{aligned} \tag{68}$$

By the Itô formula, we have

$$\begin{cases}
\mathrm{d}u_1(t) = [a_1 - \gamma_1 u_1(t)]\mathrm{d}t - \sigma_1 u_1(t)\mathrm{d}w_1(t),\\
\mathrm{d}u_2(t) = [a_2 - \gamma_2 u_2(t) + e_{12}\alpha_{12}\beta_{12}u_1(t)]\mathrm{d}t\\
\quad - \sigma_2 u_2(t)\mathrm{d}w_2(t),\\
\mathrm{d}u_3(t) = [a_3 - \gamma_3 u_3(t) + e_{13}\alpha_{13}\beta_{13}u_1(t)\\
\quad + e_{23}\alpha_{23}\beta_{23}u_2(t)]\mathrm{d}t - \sigma_3 u_3(t)\mathrm{d}w_3(t),
\end{cases} \tag{69}$$

with initial value $(u_1(0), u_2(0), u_3(0)) = (1/x_{10}, 1/x_{20}, 1/x_{30}) \in \mathbb{R}^3_+$.

Lemma 3. *Let $(x_1(t), x_2(t), x_3(t))$ be the solution of model (8) with any initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$. If $\gamma_i > 0$ $(i = 1,2,3)$, then*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\frac{1}{x_i(t)}\right] \le M_i, \quad i = 1,2,3, \tag{70}$$

where $M_1 = (a_1/\gamma_1)$, $M_2 = (a_2/\gamma_2) + ((a_1 e_{12}\alpha_{12}\beta_{12})/(\gamma_1\gamma_2))$, and $M_3 = (a_3/\gamma_3) + ((a_1 e_{13}\alpha_{13}\beta_{13})/\gamma_1\gamma_3) + +((e_{23}\alpha_{23}\beta_{23})/\gamma_3)[(a_2/\gamma_2) + ((a_1 e_{12}\alpha_{12}\beta_{12})/\gamma_1\gamma_2)]$.

Proof. First, integrating both sides of the first equation of (69) from 0 to t yields

$$u_1(t) = \frac{1}{x_{10}} + \int_0^t [a_1 - \gamma_1 u_1(s)]\mathrm{d}s - \int_0^t \sigma_1 u_1(s)\mathrm{d}w_1(s). \tag{71}$$

Taking the expectation on both sides of the above equation, we have

$$\mathbb{E}[u_1(t)] = \frac{1}{x_{10}} + \mathbb{E}\int_0^t [a_1 - \gamma_1 u_1(s)]\mathrm{d}s. \tag{72}$$

Thus,

$$\frac{\mathrm{d}\mathbb{E}[u_1(t)]}{\mathrm{d}t} = a_1 - \gamma_1 \mathbb{E}[u_1(t)], \tag{73}$$

with initial value $\mathbb{E}[u_1(0)] = 1/x_{10}$. By a simple computation, we can get

$$\mathbb{E}[u_1(t)] = \frac{1}{x_{10}} e^{-\gamma_1 t} + \frac{a_1}{\gamma_1}\left[1 - e^{-\kappa_1 t}\right]. \tag{74}$$

This, together with $\gamma_1 > 0$, yields

$$\lim_{t\longrightarrow\infty} \mathbb{E}[u_1(t)] = \frac{a_1}{\gamma_1} = M_1. \tag{75}$$

Next, integrating both sides of the second equation of system (69) from 0 to t yields

$$\begin{aligned}
u_2(t) = \frac{1}{x_{20}} &+ \int_0^t [a_2 - \gamma_2 u_2(s) + e_{12}\alpha_{12}\beta_{12}u_1(s)]\mathrm{d}s\\
&- \int_0^t \sigma_2 u_2(s)\mathrm{d}w_2(s).
\end{aligned} \tag{76}$$

Taking the expectation on both sides of the above equation, we have

$$\mathbb{E}[u_2(t)] = \frac{1}{x_{20}} + \mathbb{E}\int_0^t [a_2 - \gamma_2 u_2(s) + e_{12}\alpha_{12}\beta_{12}u_1(s)]\mathrm{d}s. \tag{77}$$

Thus,

$$\frac{\mathrm{d}\mathbb{E}[u_2(t)]}{\mathrm{d}t} = a_2 - \gamma_2\mathbb{E}[u_2(t)] + e_{12}\alpha_{12}\beta_{12}\mathbb{E}[u_1(t)], \tag{78}$$

with initial value $\mathbb{E}[u_2(0)] = (1/x_{20})$. By a simple computation, we can get

$$\begin{aligned}\mathbb{E}[u_2(t)] &= \frac{1}{x_{20}}e^{-\gamma_2 t} + \frac{a_2}{\gamma_2}\left[1 - e^{-\gamma_2 t}\right] \\ &\quad + e_{12}\alpha_{12}\beta_{12}\int_0^t e^{-\gamma_2(t-s)}\mathbb{E}[u_1(s)]\mathrm{d}s.\end{aligned} \tag{79}$$

From (75), it follows that

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_0^t e^{-\gamma_2(t-s)}\mathbb{E}[u_1(s)]\mathrm{d}s &= \lim_{t\longrightarrow\infty}\frac{\int_0^t e^{\gamma_2 s}\mathbb{E}[u_1(s)]\mathrm{d}s}{e^{\gamma_2 t}} \\ &= \lim_{t\longrightarrow\infty}\frac{\mathbb{E}[u_1(t)]}{\gamma_2} = \frac{a_1}{\gamma_1\gamma_2}.\end{aligned} \tag{80}$$

This, together with (79) yields

$$\lim_{t\longrightarrow\infty}\mathbb{E}[u_2(t)] \le \frac{a_2}{\gamma_2} + \frac{a_1 e_{12}\alpha_{12}\beta_{12}}{\gamma_1\gamma_2}. \tag{81}$$

At last, integrating both sides of the third equation of system (69) from 0 to t and taking the expectation, we have

$$\begin{aligned}\mathbb{E}[u_3(t)] &= \frac{1}{x_{30}} + \mathbb{E}\int_0^t [a_3 - \gamma_3 u_3(s) + e_{13}\alpha_{13}\beta_{13}u_1(s) \\ &\quad + e_{23}\alpha_{23}\beta_{23}u_2(s)]\mathrm{d}s.\end{aligned} \tag{82}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}[u_3(t)]}{\mathrm{d}t} &= a_3 - \gamma_3\mathbb{E}[u_3(t)] + e_{13}\alpha_{13}\beta_{13}\mathbb{E}[u_1(t)] \\ &\quad + e_{23}\alpha_{23}\beta_{23}\mathbb{E}[u_2(t)],\end{aligned} \tag{83}$$

with initial value $\mathbb{E}[u_3(0)] = (1/x_{30})$. By a simple computation, we can get

$$\begin{aligned}\mathbb{E}[u_3(t)] &= \frac{1}{x_{30}}e^{-\gamma_3 t} + \frac{a_3}{\gamma_3}\left[1 - e^{-\gamma_3 t}\right] \\ &\quad + e_{13}\alpha_{13}\beta_{13}\int_0^t e^{-\gamma_3(t-s)}\mathbb{E}[u_1(s)]\mathrm{d}s + e_{23}\alpha_{23}\beta_{23} \\ &\quad \cdot \int_0^t e^{-\gamma_3(t-s)}\mathbb{E}[u_2(t)]\mathrm{d}s.\end{aligned} \tag{84}$$

It follows from (75), (81), and (84) that

$$\begin{aligned}\lim_{t\longrightarrow\infty}\mathbb{E}[u_3(t)] &= \frac{a_3}{\gamma_3} + e_{13}\alpha_{13}\beta_{13}\lim_{t\longrightarrow\infty}\int_0^t e^{-\gamma_3(t-s)}\mathbb{E}[u_1(s)]\mathrm{d}s \\ &\quad + e_{23}\alpha_{23}\beta_{23}\lim_{t\longrightarrow\infty}\int_0^t e^{-\gamma_3(t-s)}\mathbb{E}[u_2(t)]\mathrm{d}s \\ &= \frac{a_3}{\gamma_3} + \frac{a_1 e_{13}\alpha_{13}\beta_{13}}{\gamma_1\gamma_3} + \frac{e_{23}\alpha_{23}\beta_{23}}{\gamma_3} \\ &\quad \cdot \left[\frac{a_2}{\gamma_2} + \frac{a_1 e_{12}\alpha_{12}\beta_{12}}{\gamma_1\gamma_2}\right].\end{aligned} \tag{85}$$

From the comparison theorem of stochastic differential equations, it follows that

$$\frac{1}{x_i(t)} \le \frac{1}{\phi_i(t)} = u_i(t), \quad i = 1, 2, 3. \tag{86}$$

Now, Lemma 3 follows immediately from the above analysis. The proof is complete. □

Theorem 8. *If $\gamma_i > 0$ $(i = 1, 2, 3)$, then model (8) is stochastically permanent.*

Proof. Let $(x_1(t), x_2(t), x_3(t))$ be solution of (8) with initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$. For any $\varepsilon \in (0, 1)$, let $\delta_i = (\varepsilon/M_i)$ $(i = 1, 2, 3)$, then

$$\begin{aligned}\mathbb{P}\{x_i(t) < \delta_i\} &= \mathbb{P}\left\{\frac{1}{x_i(t)} > \frac{1}{\delta_i}\right\} \le \frac{\mathbb{E}[1/x_i(t)]}{1/\delta_i} \\ &= \delta_i\mathbb{E}\left[\frac{1}{x_i(t)}\right], \quad i = 1, 2, 3.\end{aligned} \tag{87}$$

Thus, from Lemma 3, it follows that

$$\limsup_{t\longrightarrow\infty}\mathbb{P}\{x_i(t) < \delta_i\} \le \limsup_{t\longrightarrow\infty}\delta_i\mathbb{E}\left[\frac{1}{x_i(t)}\right] \le \varepsilon, \quad i = 1, 2, 3. \tag{88}$$

This implies

$$\liminf_{t\longrightarrow\infty}\mathbb{P}\{x_i(t) \ge \delta_i\} \ge 1 - \varepsilon, \quad i = 1, 2, 3. \tag{89}$$

Let $\varepsilon \in (0, 1)$ be sufficiently small such that $\delta_i < H_i$. From (67) and Definition 2, model (8) is stochastically permanent. The proof is therefore complete. □

6. Stationary Distribution and Ergodicity

In this section, we will show that there is an ergodic stationary distribution for the solution of (8). For the completeness of the paper, in this section, we list some theories about stationary distribution (see [32]). Let $X(t)$ be a homogeneous Markov process in E_d (denotes d-dimensional Euclidean space), described by the following stochastic differential equation:

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), X(0) = X_0. \tag{90}$$

The diffusion matrix of the process $X(t)$ is defined as $J(X) = g(X)g^{\mathbb{T}}(X) = (a_{ij}(X))$.

Definition 3 (see [32]). Let $\mathbb{P}(t, X, \cdot)$ be the probability measure induced by $X(t)$ with initial value $X(0) = X_0$. That is, $\mathbb{P}(t, X_0, A) = \mathbb{P}(X(t) \in A \mid X(0) = X_0)$, for any Borel set $A \in \mathscr{B}(\mathbb{R}_+^d)$. If there exists a probability measure $\mu(\cdot)$ such that $\lim_{t \longrightarrow \infty} \mathbb{P}(t, X_0, A) = \mu(A)$ for all $X_0 \in \mathbb{R}_+^d$ and $A \in \mathscr{B}(\mathbb{R}_+^d)$, then we say that stochastic differential equation (90) has a stationary distribution $\mu(\cdot)$.

Lemma 4 (see [33, 34]). *Assume that there exists a bounded domain $D \subset E_d$ with regular boundary Γ and*

(i) (A1) There is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(X)\xi_i\xi_j \geq M|\xi|^2$, $X \in D$, and $\xi \in \mathbb{R}^d$;

(ii) (A2) There exists a nonnegative C^2-function V such that there exists a positive constant C, such that

$$LV \leq -C \text{ for any } X \in \frac{E_d}{D}. \tag{91}$$

Then, the Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$. Moreover, if $f(\cdot)$ is a function integrable with respect to the measure μ, then

$$\mathbb{P}\left\{\lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T f(X(t))\mathrm{d}t = \int_{E_d} f(x)\mu(\mathrm{d}x)\right\} = 1. \tag{92}$$

Let $X(t) = (x_1(t), x_2(t), x_3(t))^{\mathbb{T}}$, $g(X) = \operatorname{diag}(\sigma_1 x_1, \sigma_2 x_2, \sigma_3 x_3)$, $W(t) = (w_1(t), w_2(t), w_3(t))^{\mathbb{T}}$, and

$$b(X) = \begin{pmatrix} x_1\left(r - a_1x_1 - \dfrac{\alpha_{12}x_2}{x_1 + \beta_{12}x_2} - \dfrac{\alpha_{13}x_3}{x_1 + \beta_{13}x_3}\right) \\ x_2\left(-d_2 - a_2x_2 + \dfrac{e_{12}\alpha_{12}x_1}{x_1 + \beta_{12}x_2} - \dfrac{\alpha_{23}x_3}{x_2 + \beta_{23}x_3}\right) \\ x_3\left(-d_3 - a_3x_3 + \dfrac{e_{13}\alpha_{13}x_1}{x_1 + \beta_{13}x_3} + \dfrac{e_{23}\alpha_{23}x_2}{x_2 + \beta_{23}x_3}\right) \end{pmatrix}. \tag{93}$$

Then, system (90) reduces to model (8) with diffusion matrix $J(X) = \operatorname{diag}(\sigma_1^2x_1^2, \sigma_2^2x_2^2, \sigma_3^2x_3^2)$.

Theorem 9. *If $a_2 > 0$, $a_3 > 0$, $\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2 > 0$, $\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2 > 0$, and $\kappa_3 - \sigma_3^2 > 0$, then for any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, model (8) has a stationary distribution and the solutions have ergodic property.*

Proof. Define C^2-function V_1: $\mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ by

$$V_1(X) = x_1 + x_2 + x_3, \tag{94}$$

for $X = (x_1, x_2, x_3) \in \mathbb{R}_+^3$. By Itô formula, we have

$$\begin{aligned} LV_1(X) &= x_1\left(r - a_1x_1 - \frac{\alpha_{12}x_2}{x_1 + \beta_{12}x_2} - \frac{\alpha_{13}x_3}{x_1 + \beta_{13}x_3}\right) \\ &\quad + x_2\left(-d_2 - a_2x_2 + \frac{e_{12}\alpha_{12}x_1}{x_1 + \beta_{12}x_2} - \frac{\alpha_{23}x_3}{x_2 + \beta_{23}x_3}\right) \\ &\quad + x_3\left(-d_3 - a_3x_3 + \frac{e_{13}\alpha_{13}x_1}{x_1 + \beta_{13}x_3} + \frac{e_{23}\alpha_{23}x_2}{x_2 + \beta_{23}x_3}\right) \\ &\leq -a_1x_1^2 + rx_1 - a_2x_2^2 + (e_{12}\alpha_{12} - d_2)x_2 - a_3x_3^2 \\ &\quad + (e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3)x_3. \end{aligned} \tag{95}$$

Define C^2-function V_2: $\mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ by

$$V_2(X) = x_1^{-1} + x_2^{-1} + x_3^{-1}, \tag{96}$$

for $X = (x_1, x_2, x_3) \in \mathbb{R}_+^3$. By Itô formula, we have

$$\begin{aligned} LV_2(X) &= -x_1^{-1}\left(r - a_1x_1 - \frac{\alpha_{12}x_2}{x_1 + \beta_{12}x_2} - \frac{\alpha_{13}x_3}{x_1 + \beta_{13}x_3}\right) + \sigma_1^2x_1^{-1} \\ &\quad - x_2^{-1}\left(-d_2 - a_2x_2 + \frac{e_{12}\alpha_{12}x_1}{x_1 + \beta_{12}x_2} - \frac{\alpha_{23}x_3}{x_2 + \beta_{23}x_3}\right) \\ &\quad + \sigma_2^2x_2^{-1} \\ &\quad - x_3^{-1}\left(-d_3 - a_3x_3 + \frac{e_{13}\alpha_{13}x_1}{x_1 + \beta_{13}x_3} + \frac{e_{23}\alpha_{23}x_2}{x_2 + \beta_{23}x_3}\right) \\ &\quad + \sigma_3^2x_3^{-1} \\ &\leq -x_1^{-1}(\kappa_1 - a_1x_1) + \sigma_1^2x_1^{-1} \\ &\quad - x_2^{-1}\left(\kappa_2 - a_2x_2 - \frac{e_{12}\alpha_{12}\beta_{12}x_2}{x_1 + \beta_{12}x_2}\right) + \sigma_2^2x_2^{-1} \\ &\quad - x_3^{-1}\left(\kappa_3 - a_3x_3 - \frac{e_{13}\alpha_{13}\beta_{13}x_3}{x_1 + \beta_{13}x_3} - \frac{e_{23}\alpha_{23}\beta_{23}x_3}{x_2 + \beta_{23}x_3}\right) \\ &\quad + \sigma_3^2x_3^{-1} \\ &\leq -x_1^{-1}(\kappa_1 - a_1x_1) + \sigma_1^2x_1^{-1} \\ &\quad - x_2^{-1}\left(\kappa_2 - a_2x_2 - \frac{e_{12}\alpha_{12}\beta_{12}x_2}{x_1}\right) + \sigma_2^2x_2^{-1} \\ &\quad - x_3^{-1}\left(\kappa_3 - a_3x_3 - \frac{e_{13}\alpha_{13}\beta_{13}x_3}{x_1} - \frac{e_{23}\alpha_{23}\beta_{23}x_3}{x_2}\right) + \sigma_3^2x_3^{-1} \\ &= -\left(\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2\right)x_1^{-1} + a_1 \\ &\quad - \left(\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2\right)x_2^{-1} + a_2 - \left(\kappa_3 - \sigma_3^2\right)x_3^{-1} + a_3. \end{aligned} \tag{97}$$

Let $V(X) = V_1(X) + V_2(X)$. Then,

$$
\begin{aligned}
LV(X) \leq & -a_1x_1^2 + rx_1 - \left(\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2\right)x_1^{-1} \\
& + a_1 - a_2x_2^2 + (e_{12}\alpha_{12} - d_2)x_2 \\
& - \left(\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2\right)x_2^{-1} + a_2 - a_3x_3^2 \\
& + (e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3)x_3 - \left(\kappa_3 - \sigma_3^2\right)x_3^{-1} + a_3 \\
= & f(x_1) + g(x_2) + h(x_3),
\end{aligned} \tag{98}
$$

where

$$
\begin{aligned}
f(x_1) &= -a_1x_1^2 + rx_1 - \left(\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2\right)x_1^{-1} \\
&\quad + a_1, \\
g(x_2) &= -a_2x_2^2 + (e_{12}\alpha_{12} - d_2)x_2 - \left(\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2\right)x_2^{-1} \\
&\quad + a_2, \\
h(x_3) &= -a_3x_3^2 + (e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3)x_3 - \left(\kappa_3 - \sigma_3^2\right)x_3^{-1} \\
&\quad + a_3.
\end{aligned} \tag{99}
$$

Clearly, $f(x_1)$, $g(x_2)$, and $h(x_3)$ have upper bound on $\mathbb{R}_+$. Denote

$$
\begin{aligned}
f^u &= \sup_{x_1 \in \mathbb{R}_+} \{f(x_1)\}, \\
g^u &= \sup_{x_2 \in \mathbb{R}_+} \{g(x_2)\}, \\
h^u &= \sup_{x_3 \in \mathbb{R}_+} \{h(x_3)\}.
\end{aligned} \tag{100}
$$

From $\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2 > 0$, it follows that

$$
\begin{aligned}
LV(X) &\leq f(x_1) + g(x_2) + h(x_3) \leq f(x_1) + g^u \\
&\quad + h^u \longrightarrow -\infty, \quad \text{a.s. } x_1 \longrightarrow 0^+ \text{ or } x_1 \longrightarrow +\infty.
\end{aligned} \tag{101}
$$

Similarly, from $\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2 > 0$ and $\kappa_3 - \sigma_3^2 > 0$, we have

$$
\begin{aligned}
LV(X) &\leq f(x_1) + g(x_2) + h(x_3) \leq f^u + g(x_2) \\
&\quad + h^u \longrightarrow -\infty, \quad \text{a.s. } x_2 \longrightarrow 0^+ \text{ or } x_2 \longrightarrow +\infty, \\
LV(X) &\leq f(x_1) + g(x_2) + h(x_3) \leq f^u + g^u \\
&\quad + h(x_3) \longrightarrow -\infty, \quad \text{a.s. } x_3 \longrightarrow 0^+ \text{ or } x_3 \longrightarrow +\infty.
\end{aligned} \tag{102}
$$

Consequently, there exists $\rho > 0$ (sufficiently small) such that

$$
LV(X) \leq -1, \quad \text{for all } (x_1, x_2, x_3) \in \frac{\mathbb{R}_+^3}{D}, \tag{103}
$$

where

$$
D = \left\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid \rho < x_1 < \frac{1}{\rho}, \rho < x_2 < \frac{1}{\rho}, \rho < x_3 < \frac{1}{\rho}\right\} \subset \mathbb{R}_+^3. \tag{104}
$$

Hence, (A2) in Lemma 4 is satisfied.

Denote $\sigma^2 = \sigma_1^2 \wedge \sigma_2^2 \wedge \sigma_3^2$. Then, for any $X = (x_1, x_2, x_3) \in D$ and $\xi = (\xi_1, \xi_2, \xi_2) \in \mathbb{R}^3$, we have

$$
\sum_{i,j=1}^{3} a_{ij}(X)\xi_i\xi_j = \sigma_1^2x_1^2\xi_1^2 + \sigma_2^2x_2^2\xi_2^2 + \sigma_3^2x_3^2\xi_3^2 \geq M|\xi|^2, \tag{105}
$$

where $M = \rho^2\sigma^2$. Thus, condition (A1) of Lemma 4 holds. According to Lemma 4, model (8) is ergodic and admits a unique stationary distribution. The proof is therefore complete. □

7. Application of Main Results

In this section, we present the application of the main results in some special models.

7.1. Two Species Predator-Prey Model. Let $\alpha_{13} = \alpha_{23} = 0$. Then, the first two equations of (8) form the following closed two-population system:

$$
\begin{cases}
\mathrm{d}x_1(t) = x_1(t)\left[r - a_1x_1(t) + \dfrac{\alpha_{12}x_2(t)}{x_1(t) + \beta_{12}x_1(t)}\right]\mathrm{d}t \\
\quad + \sigma_1x_1(t)\mathrm{d}w_1(t), \\
\mathrm{d}x_2(t) = x_2(t)\left[-d_2 - a_2x_2(t) + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t) + \beta_{12}x_2(t)}\right]\mathrm{d}t \\
\quad + \sigma_2x_2(t)\mathrm{d}w_2(t),
\end{cases} \tag{106}
$$

with initial value $(x_{10}, x_{20}) \in \mathbb{R}_+^2$. This is also the stochastic predator-prey model discussed in [19]. From Theorems 3 and 4, we have the following result.

Corollary 1. *Let $(x_1(t), x_2(t))$ be solution of model (106) with initial value $(x_{10}, x_{20}) \in \mathbb{R}_+^2$.*

(i) If $a_2 > 0$, $r - (\alpha_{12}/\beta_{12}) - (\sigma_1^2/2) > 0$, and $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) > 0$, then

$$
\begin{aligned}
&\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t x_1(s)\mathrm{d}s \geq \frac{r - (\alpha_{12}/\beta_{12}) - (\sigma_1^2/2)}{a_1} > 0 \ \text{ a.s.}, \\
&\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \left[a_2x_2(s) + \frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)}\right]\mathrm{d}s \\
&\quad \geq e_{12}\alpha_{12} - d_2 - \frac{\sigma_2^2}{2} > 0 \ \text{ a.s.}
\end{aligned} \tag{107}
$$

(ii) If $r - (\sigma_1^2/2) < 0$ and $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) < 0$, then (106) is extinct exponentially with probability one.

Remark 1. It is clear that Corollary 1 is consistent with Theorems 7 and 8 in [19]. Moreover, from Theorems 3 and 4, the persistence in mean and extinction conditions of the three-species model (8) are more complicated. Thus, our work can be seen as the extension of [19].

For model (106), similar to the proof of Theorem 6 (denote $H = e_{12}x_1 + x_2$), we have

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[x_1(t)\right] \le \frac{K_2'}{e_{12}d_2}, \qquad \limsup_{t\longrightarrow\infty} \mathbb{E}\left[x_2(t)\right] \le \frac{K_2'}{d_2}, \tag{108}$$

where $K_2' = e_{12}(r+d_2)^2/4a_1$. Furthermore, from Theorems 7–9, for model (106), we have the following result.

Corollary 2

(i) *Model (106) is stochastically ultimate bounded*

(ii) *If* $r - (\alpha_{12}/\beta_{12}) - \sigma_1^2 > 0$ *and* $e_{12}\alpha_{12} - d_2 - \sigma_2^2 > 0$, *then model (106) is stochastically permanent*

(iii) *If* $a_2 > 0$, $r - (\alpha_{12}/\beta_{12}) - e_{12}\alpha_{12}\beta_{12} - \sigma_1^2 > 0$, *and* $e_{12}\alpha_{12} - d_2 - \sigma_2^2 > 0$, *then for any* $(x_{10}, x_{20}) \in \mathbb{R}_+^2$, *model (106) has a stationary distribution and the solutions have ergodic property*

If we do not consider the intraspecific competition of the predator, i.e., $a_2 = 0$ in model (106), then model (6) is available. From Theorems 4, 7, and 8, for model (6), we have the following result.

Corollary 3

(i) *If* $r - (\sigma_1^2/2) < 0$ *and* $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) < 0$, *then model* (6) *is extinct exponentially with probability one*

(ii) *Model* (6) *is stochastically ultimate bounded*

(iii) *If* $r - (\alpha_{12}/\beta_{12}) - \sigma_1^2 > 0$ *and* $e_{12}\alpha_{12} - d_2 - \sigma_2^2 > 0$, *then model* (6) *is stochastically permanent*

Remark 2. From Theorem 4.11 in [18], it follows that if $r - (\alpha_{12}/\beta_{12}) - (3/2)\sigma_1^2 > 0$ and $e_{12}\alpha_{12} - d_2 - (3/2)\sigma_3^2 > 0$, then model (6) is stochastically permanent. Obviously, if conditions of Theorem 4.11 in [18] hold, then conditions in Corollary 3 hold. On the contrary, it is not set up. Therefore, Corollary 3 generalizes and improves Theorem 4.11 in [18].

Remark 3. If $r - (\alpha_{12}/\beta_{12}) - \sigma_1^2 > 0$ and $e_{12}\alpha_{12} - d_2 - \sigma_3^2 > 0$, then by Theorem 3.3 in [16], model (6) is persistent in mean, but by Corollary 3, model (6) is stochastically permanent.

7.2. Three-Species Food-Chain Model. Let $\alpha_{13} = 0$. Then, we can get the following stochastic three-species food chain model:

$$\begin{cases} dx_1(t) = x_1(t)\left[r - a_1x_1(t) + \dfrac{\alpha_{12}x_2(t)}{x_1(t)+\beta_{12}x_2(t)}\right]dt \\ \quad + \sigma_1x_1(t)dw_1(t), \\ dx_2(t) = x_2(t)\left[-d_2 - a_2x_2(t) + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t)+\beta_{12}x_2(t)} \right.\\ \quad \left. - \dfrac{\alpha_{23}x_3(t)}{x_2(t)+\beta_{23}x_3(t)}\right]dt + \sigma_2x_2(t)dw_2(t), \\ dx_3(t) = x_3(t)\left[-d_3 - a_3x_3(t) + \dfrac{e_{23}\alpha_{23}x_2(t)}{x_2(t)+\beta_{23}x_3(t)}\right]dt \\ \quad + \sigma_3x_3(t)dw_3(t), \end{cases} \tag{109}$$

with initial value $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$. Denote

$$\kappa_1' = r - \frac{\alpha_{12}}{\beta_{12}}; \quad \kappa_2' = e_{12}\alpha_{12} - d_2 - \frac{\alpha_{23}}{\beta_{23}}; \quad \kappa_3' = e_{23}\alpha_{23} - d_3; \quad \gamma_i' = \kappa_i' - \sigma_i^2, \quad i = 1, 2, 3. \tag{110}$$

For model (109), we have the following results.

Corollary 4. *For any* $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, *let* $(x_1(t), x_2(t), x_3(t))$ *be the solution of model* (109) *with initial value* (x_{10}, x_{20}, x_{30}).

(i) *If* $a_2 > 0$, $a_3 > 0$, *and* $\kappa_I' - (\sigma_i^2/2) > 0$ $(i = 1, 2, 3)$, *then*

$$\lim_{t\longrightarrow\infty} \frac{\ln x_1(t)}{t} = 0, \quad \lim_{t\longrightarrow\infty} \frac{\ln x_2(t)}{t} = 0, \quad \lim_{t\longrightarrow\infty} \frac{\ln x_3(t)}{t} = 0 \text{ a.s.} \tag{111}$$

(ii) *If* $a_2 > 0$, $a_3 > 0$, *and* $\kappa_i' - (\sigma_i^2/2) > 0$ $(i = 1, 2, 3)$, *then*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\mathrm{d}s \geq \frac{\kappa_1' - (\sigma_1^2/2)}{a_1} \text{ a.s.},$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \left[a_2x_2(s) + \frac{e_{12}\alpha_{12}\beta_{12}x_2(s)}{x_1(s)}\right]\mathrm{d}s \geq \kappa_2' - \frac{\sigma_2^2}{2} \text{ a.s.},$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \left[a_3x_3(s) + \frac{e_{23}\alpha_{23}\beta_{23}x_3(s)}{x_2(s)}\right]\mathrm{d}s \geq \kappa_3' - \frac{\sigma_3^2}{2} \text{ a.s.} \tag{112}$$

(iii) *If* $r - (\sigma_1^2/2) < 0$, $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) < 0$, *and* $e_{23}\alpha_{23} - d_3 - (\sigma_3^2/2) < 0$, *then model (109) is extinct exponentially with probability one.*

Corollary 5. *Model (109) is stochastically ultimate bounded. Furthermore, if* $\gamma_i' > 0$ $(i = 1, 2, 3)$, *then model (109) is stochastically permanent.*

Corollary 6. *If* $a_2 > 0$, $a_3 > 0$, $\kappa_1' - e_{12}\alpha_{12}\beta_{12} - \sigma_1^2 > 0$, $\kappa_2' - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2 > 0$, *and* $\kappa_3' - \sigma_3^2 > 0$, *then for any* $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, *model (109) has a stationary distribution and the solutions have ergodic property.*

If we do not consider the intraspecific competition of the predator, i.e., $a_2 = a_3 = 0$ in model (109), then we obtain the following stochastic three-species food chain model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_2x_1(t) - \dfrac{\alpha_{12}x_2(t)}{x_1(t) + \beta_{12}x_2(t)}\right]\mathrm{d}t + \sigma_1x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t) + \beta_{12}x_2(t)} - \dfrac{\alpha_{23}x_3(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t + \sigma_2x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 + \dfrac{e_{23}\alpha_{23}x_2(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t + \sigma_3x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{113}$$

with initial value $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$. From Theorems 4, 7, and 8, for model (113), we have the following result.

Corollary 7

(i) *If* $r - (\sigma_1^2/2) < 0$, $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) < 0$, *and* $e_{23}\alpha_{23} - d_3 - (\sigma_3^2/2) < 0$, *then model (113) is extinct exponentially with probability one*

(ii) *Model* (113) *is stochastically ultimate bounded*

(iii) *If* $\gamma_i' > 0$ $(i = 1, 2, 3)$, *then model* (113) *is stochastically permanent*

7.3. Food-Web Model without Intraspecific Competition of Predators. If we do not consider the intraspecific competition of the predator, i.e., $a_2 = a_3 = 0$ in model (8), then we obtain the following stochastic three-species model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r - a_1x_1(t) - \dfrac{\alpha_{12}x_2(t)}{x_1(t) + \beta_{12}x_2(t)} - \dfrac{\alpha_{13}x_3(t)}{x_1(t) + \beta_{13}x_3(t)}\right]\mathrm{d}t + \sigma_1x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-d_2 + \dfrac{e_{12}\alpha_{12}x_1(t)}{x_1(t) + \beta_{12}x_2(t)} - \dfrac{\alpha_{23}x_3(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t + \sigma_2x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-d_3 + \dfrac{e_{13}\alpha_{13}x_1(t)}{x_1(t) + \beta_{13}x_3(t)} - \dfrac{e_{23}\alpha_{23}x_2(t)}{x_2(t) + \beta_{23}x_3(t)}\right]\mathrm{d}t + \sigma_3x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{114}$$

with initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$. For model (114), we have the following result.

Corollary 8

(i) *If* $r - (\sigma_1^2/2) < 0$, $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) < 0$, *and* $e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3 - (\sigma_3^2/2) < 0$, *then model (114) is extinct exponentially with probability one*

(ii) *Model (114) is stochastically ultimate bounded*

(iii) *If* $\gamma_i > 0$ $(i = 1, 2, 3)$, *then model (114) is stochastically permanent*

8. Numerical Simulations

In this section, we use the Milstein method (see [35]) to substantiate our main results. The numerical simulations of population dynamics are carried out for the academic tests with the arbitrary values of the vital rates and other parameters which do not correspond to some specific biological populations and exhibit only the theoretical properties of numerical solutions of the considered model. To illustrate the theoretical results, we take the parameter values as following with different noise intensities:

$$\begin{cases} r = 0.62, \\ d_2 = 0.0005, \\ d_3 = 0.0006, \\ a_1 = 0.0005, \\ a_2 = 0.0002, \\ a_3 = 0.0003, \\ \alpha_{12} = 0.2, \\ \alpha_{13} = 0.15, \\ \alpha_{23} = 0.08, \\ e_{12} = 0.8, \\ e_{13} = 0.5, \\ e_{23} = 0.6, \\ \beta_{12} = \beta_{13} = \beta_{23} = 1, \\ x_{10} = 800, \\ x_{20} = 300, \\ x_{30} = 200. \end{cases} \tag{115}$$

In Figure 1, we choose $\sigma_i = 0$ $(i = 1, 2, 3)$ and get the solutions of the corresponding deterministic model.

(i) Assume that $\sigma_1^2 = 1.4$, $\sigma_2^2 = 0.4$, and $\sigma_3^2 = 0.3$. By a simple computation, $r - (\sigma_1^2/2) = -0.08 < 0$, $e_{12}\alpha_{12} - d_2 - (\sigma_2^2/2) = -0.0405 < 0$, and $e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3 - (\sigma_3^2/2) = -0.0276 < 0$. Thus, the condition of Theorem 4 holds. From Theorem 4, model (8) will become extinct with probability one. As can be seen from Figure 2, all the population becomes extinct.

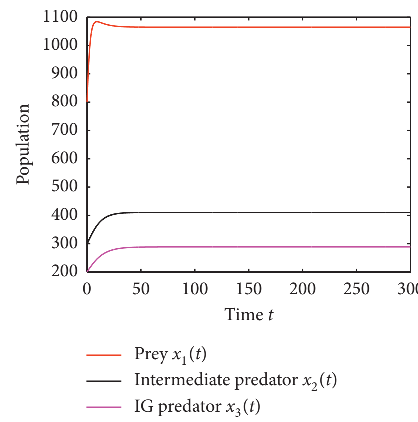

Figure 1: The trajectories of model (8) with $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

(ii) Assume that $\sigma_1^2 = 0.44$, $\sigma_2^2 = 0.08$, and $\sigma_3^2 = 0.08$. Then, $\kappa_1 = r - (\alpha_{12}/\beta_{12}) - (\alpha_{13}/\beta_{13}) = 0.27$, $\kappa_2 = e_{12}\alpha_{12} - d_2 - (\alpha_{23}/\beta_{23}) = 0.0795$, and $\kappa_3 = e_{13}\alpha_{13} + e_{23}\alpha_{23} - d_3 = 0.1224$. Thus, $\kappa_1 - (\sigma_1^2/2) = 0.05 > 0$, $\kappa_2 - (\sigma_2^2/2) = 0.0395 > 0$, and $\kappa_3 - (\sigma_3^2/2) = 0.0824 > 0$. That is, the conditions of Theorem 3 hold. In view of Theorem 3, model (8) is persistent in mean. As can be seen from Figure 3, all the populations are permanent in mean. This is consistent to Theorem 3.

(iii) Assume that $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.01$, $\sigma_3^2 = 0.01$. From (ii), it follows that $\kappa_1 = 0.27$, $\kappa_2 = 0.0795$ and $\kappa_3 = 0.1224$. Thus, $\kappa_1 - \sigma_1^2 = 0.25 > 0$, $\kappa_2 - \sigma_2^2 = 0.0695 > 0$ and $\kappa_3 - \sigma_3^2 = 0.1124 > 0$. Hence, the conditions of Theorem 8 hold. In view of Theorem 8, model (8) is stochastically permanent. From Figure 4 that all the populations are stochastically permanent. This is consistent to Theorem 8.

(iv) Assume that $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.01$, and $\sigma_3^2 = 0.01$. From (ii), it follows that $\kappa_1 = 0.27$, $\kappa_2 = 0.0795$, and $\kappa_3 = 0.1224$. Furthermore, $\kappa_1 - e_{12}\alpha_{12}\beta_{12} - e_{13}\alpha_{13}\beta_{13} - \sigma_1^2 = 0.015 > 0$, $\kappa_2 - e_{23}\alpha_{23}\beta_{23} - \sigma_2^2 = 0.0215 > 0$, and $\kappa_3 - \sigma_3^2 = 0.1244 > 0$. Thus, the conditions of Theorem 9 hold. Therefore, model (8) has a stationary distribution according to Theorem 9 (see Figures 5 and 6).

9. Conclusions and Discussions

This paper is concerned with a stochastic three-species predator-prey food web model with omnivory and ratio-dependent functional response. First, by the comparison theorem of stochastic differential equations, we prove the existence and uniqueness of global positive solution of the model. Next, we investigate an important asymptotic property of the solution, which is crucial to the study of the dynamic behavior of the model. Then, under some conditions, we conclude that the model is persistent in mean and extinct. Moreover, we discuss the stochastic persistence of the model. Furthermore, by constructing a suitable Lyapunov function, we establish sufficient conditions for the

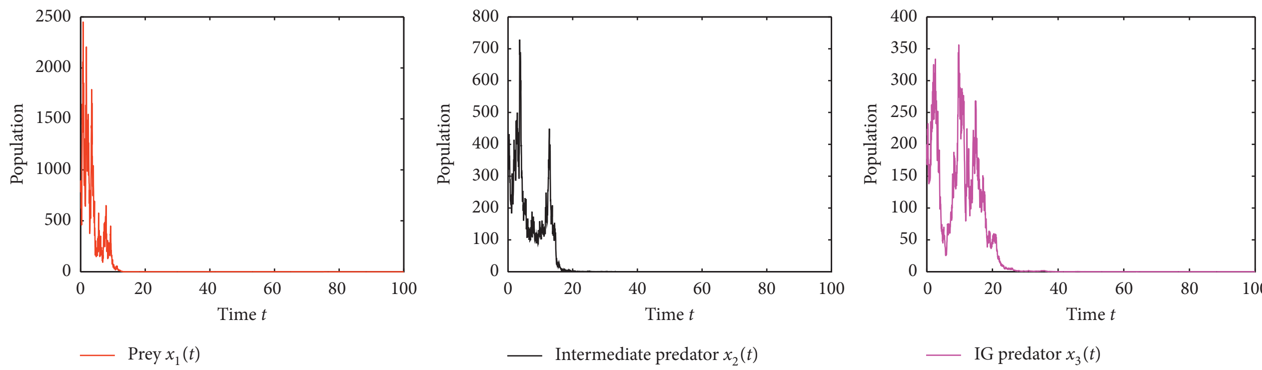

FIGURE 2: The trajectories of stochastic model (8) with $\sigma_1^2 = 1.4$, $\sigma_2^2 = 0.4$, and $\sigma_3^2 = 0.3$.

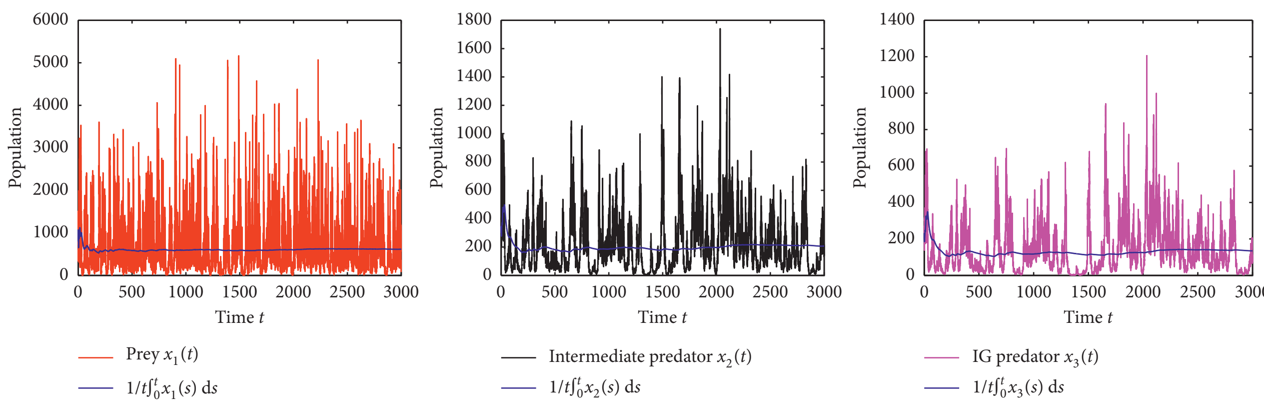

FIGURE 3: The trajectories of stochastic model (8) with $\sigma_1 = 0.44$, $\sigma_2 = 0.08$, and $\sigma_3 = 0.08$.

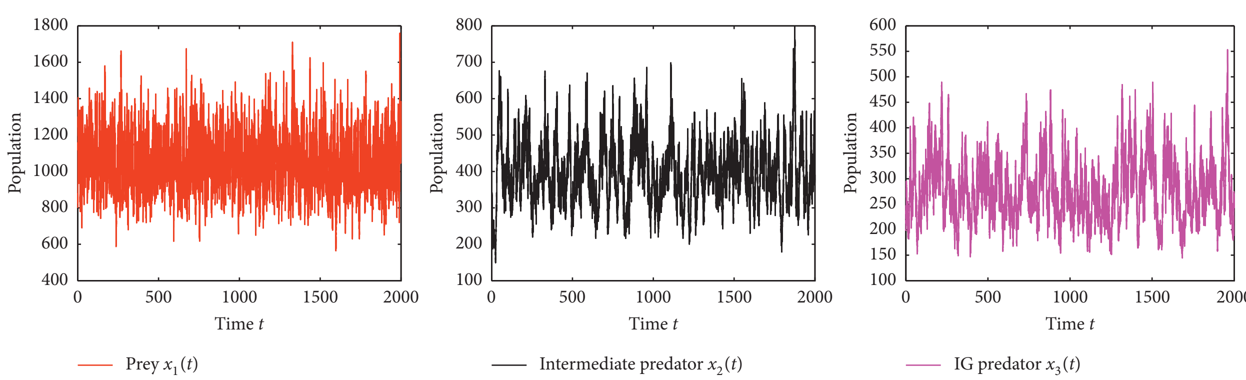

FIGURE 4: The trajectories of stochastic model (8) with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, and $\sigma_3 = 0.01$.

existence of an ergodic stationary distribution to the model. Then, we present the application of the main results in some special models. Finally, some numerical simulations are introduced to support the main results.

In Section 4, we prove that there are two typical phenomena arising in accordance with the relative values of the parameters of the model. In Theorem 3, we give the conditions on the parameters that informally can be stated by saying that the noise intensities σ_i^2 $(i = 1, 2, 3)$ are small compared to the other parameters, such that the species in model (8) are persistent in mean. From Theorem 4, it follows that in the case that the noise intensities σ_i^2 $(i = 1, 2, 3)$ are large with respect to the other parameters, then the solution of model (8) tends to extinction almost surely.

Later, in Section 5, we discuss on the stochastic permanence of the solution. This concept, which can be paraphrased by saying that the species in model (8) will survive forever, is one of the most important and interesting topics in the analysis of the model. From Theorem 8, if the

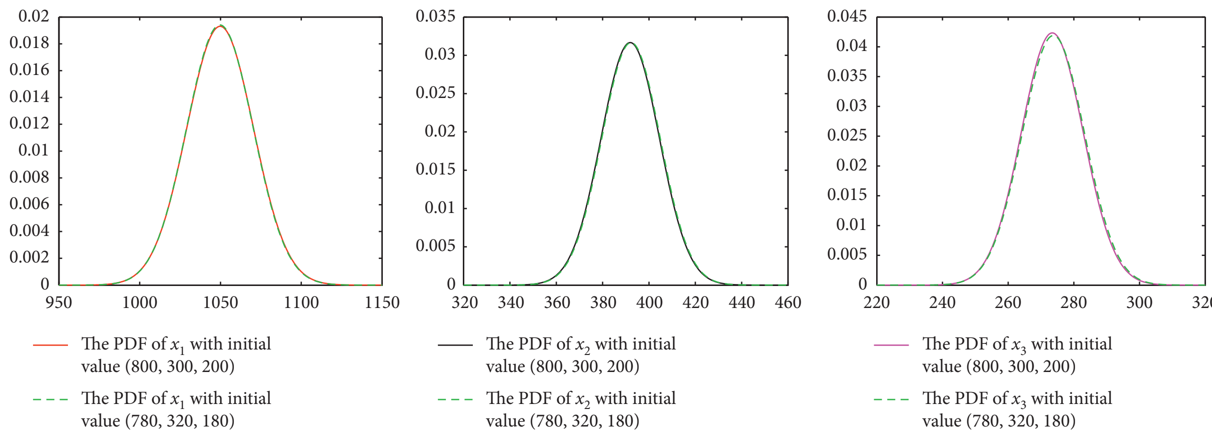

Figure 5: The density for each population at time $t = 10000$ with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, and $\sigma_3 = 0.01$.

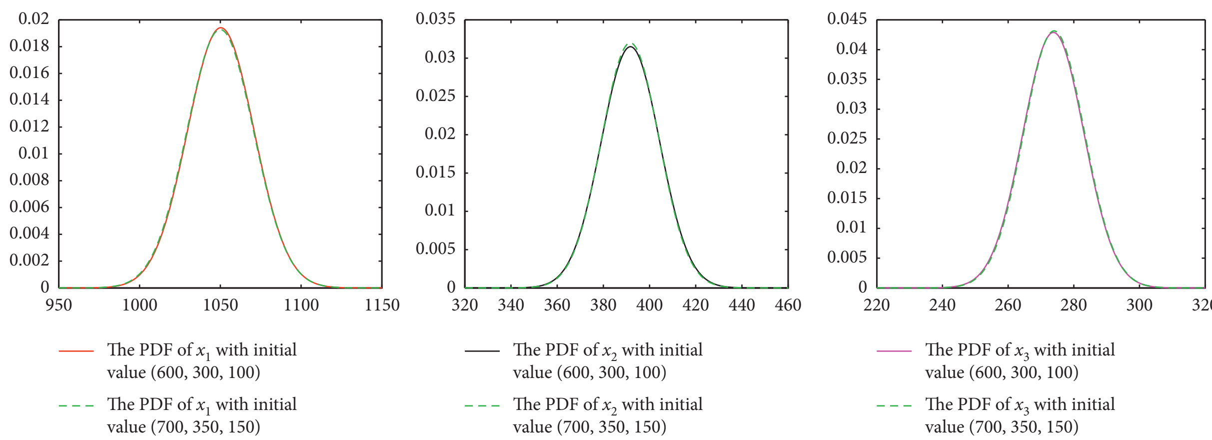

Figure 6: The density for each population at time $t = 50000$ with $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, and $\sigma_3 = 0.01$.

noise intensities σ_i^2 $(i = 1, 2, 3)$ are small compared to the other parameters, such that $\kappa_i - \sigma_i^2 > 0$ $(i = 1, 2, 3)$, then model (8) is stochastically permanent.

Moreover, in Section 6, by constructing a suitable Lyapunov function, we show that there is an ergodic stationary distribution for the solution of model (8). In Theorem 9, we give the conditions on the parameters that can be stated by saying that the intensity σ_i^2 of white noise $\dot{w}_i(t)$ is sufficiently small, such that the solution model (8) has an ergodic stationary distribution.

The results in this paper generalize and improve the previous related results. From Remark 1, we know that our work can be seen as the extension of [19]. From Remark 2, we know that Theorem 8 generalizes and improves Theorem 4.11 in [18].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

[1] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, NY, USA, 1980.

[2] M. Ouyang and X. Li, "Permanence and asymptotical behavior of stochastic prey-predator system with Markovian switching," *Applied Mathematics and Computation*, vol. 266, pp. 539–559, 2015.

[3] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-Dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.

[4] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 36, no. 4, pp. 389–406, 1998.

[5] H. I. Freedman and J. W.-H. So, "Global stability and persistence of simple food chains," *Mathematical Biosciences*, vol. 76, no. 1, pp. 69–86, 1985.

[6] S.-B. Hsu, S. Ruan, and T.-H. Yang, "Analysis of three species Lotka-Volterra food web models with omnivory," *Journal of Mathematical Analysis and Applications*, vol. 426, no. 2, pp. 659–687, 2015.

[7] R. J. Williams and N. D. Martinez, "Simple rules yield complex food webs," *Nature*, vol. 404, no. 6774, pp. 180–183, 2000.

[8] T. Namba, Y. Takeuchi, and M. Banerjee, "Stabilizing effect of intra-specific competition on prey-predator dynamics with intraguild predation," *Mathematical Modelling of Natural Phenomena*, vol. 13, no. 3, Aritcle Number 29, 41 pages, 2018.

[9] D. Sen, S. Ghorai, and M. Banerjee, "Complex dynamics of a three species prey-predator model with intraguild predation," *Ecological Complexity*, vol. 34, pp. 9–22, 2018.

[10] M. Liu, K. Wang, and Q. Wu, "Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle," *Bulletin of Mathematical Biology*, vol. 73, no. 9, pp. 1969–2012, 2011.

[11] X. Mao, *Stochsatic Differential Equations and Applications*, Horwood, Chichester, UK, 2007.

[12] M. Liu and C. Bai, "Optimal harvesting of a stochastic Logistic model with time delay," *Journal of Nonlinear Science*, vol. 25, no. 2, pp. 277–289, 2015.

[13] C. Ji, D. Jiang, and N. Shi, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 2, pp. 482–498, 2009.

[14] M. Jovanović and M. Krstić, "Extinction in stochastic predator-prey population model with Allee effect on prey," *Discrete and Continuous Dynamical Systems Series B*, vol. 22, no. 7, pp. 2651–2667, 2017.

[15] Q. Liu and D. Jiang, "Periodic solution and stationary distribution of stochastic predator-prey models with higher-order perturbation," *Journal of Nonlinear Science*, vol. 28, no. 2, pp. 423–442, 2018.

[16] C. Ji, D. Jiang, and X. Li, "Qualitative analysis of a stochastic ratio-dependent predator-prey system," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1326–1341, 2011.

[17] P. K. Tapaswi and A. Mukhopadhyay, "Effects of environmental fluctuation on plankton allelopathy," *Journal of Mathematical Biology*, vol. 39, no. 1, pp. 39–58, 1999.

[18] Z. Wu, H. Huang, and L. L. Wang, "Dynamical behavior of a stochastic ratio-dependent predator-prey system," *Journal of Applied Mathematics*, vol. 2012, Article ID 857134, 17 pages, 2012.

[19] J. Lv, K. Wang, and D. Chen, "Analysis on a stochastic two-species ratio-dependent predator-prey model," *Methodology and Computing in Applied Probability*, vol. 17, no. 2, pp. 403–418, 2015.

[20] T. H. L. Nguyen and V. T. Ta, "Dynamics of a stochastic ratio-dependent predator-prey model," *Analysis & Applications*, vol. 9, no. 3, pp. 329–344, 2011.

[21] J. Geng, M. Liu, and Y. Zhang, "Stability of a stochastic one-predator-two-prey population model with time delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 53, pp. 65–82, 2017.

[22] M. Liu, C. Bai, and Y. Jin, "Population dynamical behavior of a two-predator one-prey stochastic model with time delay," *Discrete & Continuous Dynamical Systems—A*, vol. 37, no. 5, pp. 2513–2538, 2017.

[23] H. Li, F. Cong, D. Jiang, and H. Hua, "Persistence and non-persistence of a food chain model with stochastic perturbation," *Abstract and Applied Analysis*, vol. 2013, Article ID 125089, 9 pages, 2013.

[24] M. Liu and C. Bai, "Analysis of a stochastic tri-trophic food-chain model with harvesting," *Journal of Mathematical Biology*, vol. 73, no. 3, pp. 597–625, 2016.

[25] H. Qiu and W. Deng, "Stationary distribution and global asymptotic stability of a three-species stochastic food-chain system," *Turkish Journal of Mathematics*, vol. 41, pp. 1292–1307, 2017.

[26] R. Liu and G. Liu, "Analysis on stochastic food-web model with intraguild predation and mixed functional responses," *Physica A: Statistical Mechanics and Its Applications*, vol. 531, Article ID 121782, 2019.

[27] J. Bao, X. Mao, G. Yin, and C. Yuan, "Competitive Lotka-Volterra population dynamics with jumps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 6601–6616, 2011.

[28] S. Peng and X. Zhu, "Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations," *Stochastic Processes and their Applications*, vol. 116, no. 3, pp. 370–380, 2006.

[29] X. Li, A. Gray, D. Jiang, and X. Mao, "Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 11–28, 2011.

[30] M. Liu and M. Fan, "Permanence of stochastic Lotka-Volterra systems," *Journal of Nonlinear Science*, vol. 27, no. 2, pp. 425–452, 2017.

[31] R. Liu and G. Liu, "Asymptotic behavior of a stochastict two-species competition model under the effect of disease," *Complexity*, vol. 2018, no. 15, Article ID 3127404, 2018.

[32] T. Caraballo, M. El Fatini, R. Pettersson, and R. Taki, "A stochastic SIRI epidemic model with relapse and media coverage," *Discrete & Continuous Dynamical Systems—B*, vol. 23, no. 8, pp. 3483–3501, 2018.

[33] Q. Liu, D. Jiang, T. Hayat, and B. Ahmad, "Stationary distribution and extinction of a stochastic predator-prey model with additional food and nonlinear perturbation," *Applied Mathematics and Computation*, vol. 320, pp. 226–239, 2018.

[34] Q. Liu, D. Jiang, T. Hayat, and B. Ahmad, "Stationary distribution and extinction of a stochastic SIRI epidemic model with relapse," *Stochastic Analysis and Applications*, vol. 36, no. 1, pp. 138–151, 2018.

[35] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *SIAM Review*, vol. 43, no. 3, pp. 525–546, 2001.