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Research Article

Nonlocal Symmetry and Bäcklund Transformation of a Negative-Order Korteweg–de Vries Equation

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The residual symmetry of a negative-order Korteweg–de Vries (nKdV) equation is derived through its Lax pair. Such residual symmetry can be localized, and the original nKdV equation is extended into an enlarged system by introducing four new variables. By using Lie's first theorem, we obtain the finite transformation for the localized residual symmetry. Furthermore, we localize the linear superposition of multiple residual symmetries and construct n-th Bäcklund transformation for this nKdV equation in the form of the determinants.

1. Introduction

It is well known that infinitely many symmetries or flows appear for finding evolution equations, and a general method was proposed by Olver, which was applied to the Korteweg–de Vries (KdV), modified Korteweg–de Vries (mKdV), Burgers, and sine-Gordon equations [1, 2]. Then, Olver's concept of a recursion operator for symmetries of an evolution equation was extended to negative powers of the operator, and some negative-order Korteweg–de Vries (nKdV) equations were derived, including the following nKdV equation [3–12]:

$$\begin{aligned} u_x - v_t &= 0, \\ u_{xxx} + 4u_x v + 2uv_x &= 0. \end{aligned} \tag{1}$$

Several years later, Lou reobtained the negative KdV hierarchy from the conformal invariance of its Schwartz form [13]. This new method used for the KdV equation can be extended to get more symmetries from known ones for arbitrary partial differential equations without using a recursion operator. Also, the theory can be used as a new criterion to verify whether a model is integrable or not. Based on the regular KdV system, Qiao originally studied nKdV equations, particularly their Hamiltonian structures, Lax pairs, conservation laws, and explicit multisoliton and multikink wave solutions through bilinear Bäcklund transformations [14–17]. By using the simplified form of Hirota's direct method, Wazwaz developed the nKdV equation and negative-order Kadomtsev–Petvishvili (nKP) equation in $2+1$ dimensions and found multiple soliton solutions with free coefficients [18]. However, Kudryashov showed that existence of multisoliton solutions for the nonlinear evolution equation is the consequence of complete integrability [19, 20]. According to Theorem 1 in Reference [14], the nKdV equation (1) admits a Lax pair with the parameter λ as follows:

$$\begin{aligned} L\psi &\equiv \psi_{xx} + v\psi = \lambda\psi, \\ \psi_t &= -\frac{1}{2\lambda}u\psi_x + \frac{1}{4\lambda}u_x\psi, \end{aligned} \tag{2}$$

and also possesses a Lax pair without the parameter as follows:

$$\begin{aligned} L\psi &= \left(\partial_x^2 + v\right)\psi = 0, \\ M\psi &= \left(4\partial_x^2\partial_t + 4v\partial_t + 2u\partial_x + 3u_x\right)\psi = 0. \end{aligned} \tag{3}$$

This paper is organized as follows: In Section 2, with the aid of the Lax pair, the residual symmetry of the nKdV equation (1) is derived, and this nonlocal symmetry is localized by introducing four auxiliary variables. Subsequently, we can obtain the finite symmetry transformation by solving the initial value problem. In Section 3, through localizing the linear superposition of multiple residual symmetries and constructing the infinite transformation for the nKdV equation, multiple residual symmetries and n-th Bäcklund transformation are obtained. A direct result shows that one can derive special soliton solutions from some seed solutions. A brief summary is given in Section 4.

2. Nonlocal Symmetry and Finite Transformation of the nKdV Equation

First, under the infinitesimal transformations $u \longrightarrow u + \varepsilon\sigma_1$ and $v \longrightarrow v + \varepsilon\sigma_2$ with the infinitesimal parameter ε, the symmetries σ_1 and σ_2 of the nKdV equation (1) can be expressed as a solution of their linear equations:

$$\sigma_{1,x} - \sigma_{2,t} = 0, \tag{4a}$$

$$\sigma_{1,xxx} + 4v\sigma_{1,x} + 2v_x\sigma_1 + 4u_x\sigma_2 + 2u\sigma_{2,x} = 0, \tag{4b}$$

which means that equation (1) is form invariant. At the same time, we can obtain the linear form of equation (2) under the infinitesimal transformation as follows:

$$\psi \longrightarrow \psi + \varepsilon\sigma_3. \tag{5}$$

That is, the symmetric equations of the Lax pair (2) are

$$\sigma_{3,xx} - \lambda\sigma_3 + v\sigma_3 + \psi\sigma_2 = 0, \tag{6a}$$

$$\sigma_{3,t} + \frac{1}{2\lambda}\left(u\sigma_{3,x} + \psi_x\sigma_1\right) - \frac{1}{4\lambda}\left(u_x\sigma_3 + \psi\sigma_{1,x}\right) = 0. \tag{6b}$$

Second, supposing the symmetries σ_1 and σ_2 with the auxiliary variable ψ as

$$\begin{aligned} \sigma_1 &= \psi\psi_t, \\ \sigma_2 &= \psi\psi_x, \end{aligned} \tag{7}$$

the symmetry σ_3 can be derived from equations (1), (6a)-(6b), and (7) as follows:

$$\sigma_3 = -\frac{f\psi}{4}, \tag{8}$$

where the auxiliary variable ψ satisfies $\psi^2 = f_x$, and the corresponding symmetric equation is $2\psi\sigma_3 - \sigma_{4,x} = 0$, where f is a auxiliary function and its infinitesimal transformation is $f \longrightarrow f + \varepsilon\sigma_4$. The direct result is

$$\sigma_4 = -\frac{f^2}{4}. \tag{9}$$

So, equation (1) can be expressed by the auxiliary variable ψ through equation (2) as

$$u = -\frac{1}{2\psi\psi_x}\left(\psi\psi_{xxt} - \psi_{xx}\psi_t + 4\lambda\psi\psi_t\right), \tag{10a}$$

$$v = \frac{\lambda\psi - \psi_{xx}}{\psi}. \tag{10b}$$

The consistent condition of the auxiliary function f is derived from equation (10) as

$$f_t = \frac{1}{4\lambda\psi_x}\left(\psi^2\psi_{xxt} - \psi\psi_{xx}\psi_t + 4\lambda\psi^2\psi_t + 4\psi_x^2\psi_t - 4\psi\psi_x\psi_{xt}\right). \tag{11}$$

Indeed, equation (11) has a typical Schwarzian form of the nKdV equation defined as follows:

$$\begin{aligned} &S_t + 4\lambda C_x = 0, \\ &C = \frac{f_t}{f_x}, \\ &S = \frac{f_{xxx}}{f_x} - \frac{3f_{xx}^2}{2f_x^2}, \end{aligned} \tag{12}$$

which is invariant under the Möbius transformation

$$f \longrightarrow \frac{a + bf}{c + df}, \quad (ad - bc \neq 0). \tag{13}$$

The corresponding symmetric equation of equation (11) is

$$\begin{aligned} &\left(\psi_{xx}\psi_t + 4\psi_x\psi_{xt} - 2\psi\psi_{xxt} - 8\lambda\psi\psi_t\right)\sigma_3 \\ &\quad + 4\left(\lambda f_t + \psi\psi_{xt} - 2\psi_x\psi_t\right)\sigma_{3,x} \\ &\quad + \left(\psi\psi_{xx} - 4\psi_x^2 - 4\lambda\psi^2\right)\sigma_{3,t} + \psi\psi_t\sigma_{3,xx} \\ &\quad + 4\psi\psi_x\sigma_{3,xt} - \psi^2\sigma_{3,xxt} + 4\lambda\psi_x\sigma_{4,t} = 0. \end{aligned} \tag{14}$$

Since the nonlocal symmetry could not be used to construct the explicit solutions of a partial differential equation (PDE) directly, we need to transform these components to local ones. In this part, we will seek an enlarged system which possesses a Lie point symmetry for the nonlocal symmetry. For this purpose, we further introduce four auxiliary variables g, h, p, and q, which need to obey the rule

$$\begin{aligned} \psi_t &= g, \\ \psi_x &= h, \\ f_t &= p, \\ h_t &= q. \end{aligned} \tag{15}$$

The related symmetries are

$$\begin{aligned} \sigma_{3,t} - \sigma_5 &= 0, \\ \sigma_{3,x} - \sigma_6 &= 0, \\ \sigma_{4,t} - \sigma_7 &= 0, \\ \sigma_{6,t} - \sigma_8 &= 0, \end{aligned} \tag{16}$$

under the infinitesimal transformations $g \longrightarrow g + \varepsilon\sigma_5$, $h \longrightarrow h + \varepsilon\sigma_6$, $p \longrightarrow p + \varepsilon\sigma_7$, and $q \longrightarrow q + \varepsilon\sigma_8$.

It can be verified that equations (7)–(9), (14), and (16) have the following solution:

$$\sigma \equiv \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\} = \left\{ g\psi, h\psi, -\frac{f\psi}{4}, -\frac{f^2}{4}, -\frac{1}{4}(p\psi + fg), -\frac{1}{4}\left(\psi^3 + fh\right), -\frac{fp}{2}, -\frac{1}{4}\left(3g\psi^2 + ph + fq\right) \right\}. \quad (17)$$

This is a local Lie point symmetry of the prolonged equations (1), (2), and (15) with $\psi^2 = f_x$.

Correspondingly, the initial value problem can be written as follows:

$$\frac{\mathrm{d}U(\varepsilon)}{\mathrm{d}\varepsilon} = G(\varepsilon)\Psi(\varepsilon), \quad U(0) = u, \qquad \frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}\varepsilon} = H(\varepsilon)\Psi(\varepsilon), \quad V(0) = v, \quad (18a)$$

$$\frac{\mathrm{d}\Psi(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}F(\varepsilon)\Psi(\varepsilon), \quad \Psi(0) = \psi, \qquad \frac{\mathrm{d}F(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}F^2(\varepsilon), \quad F(0) = f, \quad (18b)$$

$$\frac{\mathrm{d}G(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}[P(\varepsilon)\Psi(\varepsilon) + F(\varepsilon)G(\varepsilon)], \quad G(0) = g, \qquad \frac{\mathrm{d}H(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}\left[\Psi^3(\varepsilon) + F(\varepsilon)H(\varepsilon)\right], \quad H(0) = h, \quad (18c)$$

$$\frac{\mathrm{d}P(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{2}F(\varepsilon)P(\varepsilon), \quad P(0) = p, \qquad \frac{\mathrm{d}Q(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}\left[3G(\varepsilon)\Psi^2(\varepsilon) + P(\varepsilon)H(\varepsilon) + F(\varepsilon)Q(\varepsilon)\right], \quad Q(0) = q. \quad (18d)$$

By solving this initial value problem, one can obtain the following finite transformation theorem.

Theorem 1. *If $\{u, v, \psi, f, g, h, p, q\}$ is a solution of the extended system (1), (2), and (15), so is*

$$U(\varepsilon) = u + \frac{4g\psi\varepsilon}{4 + f\varepsilon} - \frac{2p\psi^2\varepsilon^2}{(4 + f\varepsilon)^2}, \qquad V(\varepsilon) = v + \frac{4h\psi\varepsilon}{4 + f\varepsilon} - \frac{2\psi^4\varepsilon^2}{(4 + f\varepsilon)^2}, \quad (19a)$$

$$\Psi(\varepsilon) = \frac{4\psi}{4 + f\varepsilon}, \qquad F(\varepsilon) = \frac{4f}{4 + f\varepsilon}, \quad (19b)$$

$$G(\varepsilon) = \frac{4g}{4 + f\varepsilon} - \frac{4p\psi\varepsilon}{(4 + f\varepsilon)^2}, \qquad H(\varepsilon) = \frac{h}{4 + f\varepsilon} - \frac{\psi^3\varepsilon}{(4 + f\varepsilon)^2}, \quad (19c)$$

$$P(\varepsilon) = \frac{16p}{(4 + f\varepsilon)^2}, \qquad Q(\varepsilon) = \frac{4q}{4 + f\varepsilon} - \frac{4(ph + 3g\psi^2)\varepsilon}{(4 + f\varepsilon)^2} + \frac{8p\psi^3\varepsilon^2}{(4 + f\varepsilon)^3}, \quad (19d)$$

where ε is an arbitrary group parameter.

3. Bäcklund Transformation Related to Multiple Residual Symmetries

Considering the intrusion of the spectral parameter λ in the nonlocal symmetry of equation (7), we can derive infinitely many residual symmetries of the fields u and v, that is,

$$\sigma_1 = \sum_{i=1}^{n} c_i \psi_i \psi_{i,t}, \qquad \sigma_2 = \sum_{i=1}^{n} c_i \psi_i \psi_{i,x}, \quad (20)$$

where $\psi_i\,(i = 1, 2, \ldots, n)$ are spectral functions of the Lax pair in equation (2) with different spectral parameters $\lambda_i \neq \lambda_j\,(i \neq j)$.

Just as the case of $n = 1$, to find the finite transformation of equation (20), we have to introduce a suitable prolonged system such that the symmetry can be localized to a Lie point symmetry. The corresponding finite transformation can be summarized as the following theorem.

Theorem 2. *If* $\{u, v, \psi_i, f_i, g_i, h_i, p_i, q_i, i = 1, 2, \ldots, n\}$ *is a solution of the enlarged system*

$$u_x - v_t = 0, \quad u_{xxx} + 4u_x v + 2uv_x = 0, \tag{21a}$$

$$\psi_{i,xx} + v\psi_i - \lambda_i \psi_i = 0, \tag{21b}$$

$$\psi_{i,t} + \frac{1}{2\lambda_i} u\psi_{i,x} - \frac{1}{4\lambda_i} u_x \psi_i = 0, \tag{21c}$$

$$\begin{gathered} \psi_i^2 - f_{i,x} = 0, \\ 4\lambda_i f_{i,t}\psi_{i,x} - \psi_i^2 \psi_{i,xxt} + \psi_i \psi_{i,xx}\psi_{i,t} - 4\lambda_i \psi_i^2 \psi_{i,t} \\ - 4\psi_{i,x}^2 \psi_{i,t} + 4\psi_i \psi_{i,x}\psi_{i,xt} = 0, \end{gathered} \tag{21d}$$

$$\begin{gathered} \psi_{i,t} - g_i = 0, \\ \psi_{i,x} - h_i = 0, \\ f_{i,t} - p_i = 0, \\ h_{i,t} - q_i = 0, \end{gathered} \tag{21e}$$

then the symmetry (20) is localized to a Lie point symmetry as follows:

$$\begin{gathered} \sigma_1 = \sum_{i=1}^{n} c_i g_i \psi_i, \\ \sigma_2 = \sum_{i=1}^{n} c_i h_i \psi_i, \end{gathered} \tag{22a}$$

$$\begin{gathered} \sigma_{3_i} = -\frac{1}{4} c_i f_i \psi_i - \frac{1}{4} \sum_{j \neq i} \frac{c_j \psi_j (h_i \psi_j - h_j \psi_i)}{\lambda_i - \lambda_j}, \\ \sigma_{4_i} = -\frac{1}{4} c_i f_i^2 - \frac{1}{4} \sum_{j \neq i} \frac{c_j (h_i \psi_j - h_j \psi_i)^2}{(\lambda_i - \lambda_j)^2}, \end{gathered} \tag{22b}$$

$$\sigma_{5_i} = -\frac{1}{4} c_i (p_i \psi_i - f_i g_i) - \frac{1}{4} \sum_{j \neq i} \frac{c_j [g_j (h_i \psi_j - h_j \psi_i) + \psi_j (q_i \psi_j - q_j \psi_i) + \psi_j (g_j h_i - g_i h_j)]}{\lambda_i - \lambda_j}, \tag{22c}$$

$$\sigma_{6_i} = -\frac{1}{4} c_i (\psi_i^3 + f_i h_i) - \frac{1}{4} \sum_{j \neq i} c_j \left[\frac{h_j (h_i \psi_j - h_j \psi_i)}{\lambda_i - \lambda_j} + \psi_i \psi_j^2 \right], \tag{22d}$$

$$\sigma_{7_i} = -\frac{1}{2} c_i f_i p_i - \frac{1}{2} \sum_{j \neq i} \frac{c_j (h_i \psi_j - h_j \psi_i)(q_i \psi_j - q_j \psi_i + g_j h_i - g_i h_j)}{(\lambda_i - \lambda_j)^2}, \tag{22e}$$

$$\sigma_{8_i} = -\frac{1}{4} c_i (3g_i \psi_i^2 + p_i h_i + f_i q_i) - \frac{1}{4} \sum_{j \neq i} c_j \left[\frac{q_j (h_i \psi_j - h_j \psi_i) + h_j (g_j h_i - g_i h_j) + h_j (q_i \psi_j - q_j \psi_i)}{\lambda_i - \lambda_j} + \psi_j^2 g_i + 2\psi_j \psi_i g_j \right]. \tag{22f}$$

Proof. The enlarged system (21a)–(21e) has the following linearized form:

$$\sigma_{1,x} - \sigma_{2,t} = 0, \quad \sigma_{1,xxx} + 4v\sigma_{1,x} + 2v_x \sigma_1 + 4u_x \sigma_2 + 2u\sigma_{2,x} = 0, \tag{23a}$$

$$\sigma_{3_i,xx} - \lambda_i \sigma_{3_i} + v\sigma_{3_i} + \psi_i \sigma_2 = 0, \tag{23b}$$

$$\sigma_{3_i,t} + \frac{1}{2\lambda_i} (u\sigma_{3_i,x} + \psi_{i,x}\sigma_1) - \frac{1}{4\lambda_i} (u_x \sigma_{3_i} + \psi_i \sigma_{1,x}) = 0, \tag{23c}$$

$$\begin{gathered} (\psi_{i,xx}\psi_{i,t} + 4\psi_{i,x}\psi_{i,xt} - 2\psi_i \psi_{i,xxt} - 8\lambda_i \psi_i \psi_{i,t}) \sigma_{3_i} \\ + 4(\lambda_i f_{i,t} + \psi_i \psi_{i,xt} - 2\psi_{i,x}\psi_{i,t}) \sigma_{3_i,x} \\ + (\psi_i \psi_{i,xx} - 4\psi_{i,x}^2 - 4\lambda_i \psi_i^2) \sigma_{3_i,t} + \psi_i \psi_{i,t} \sigma_{3_i,xx} \\ + 4\psi_i \psi_{i,x} \sigma_{3_i,xt} - \psi_i^2 \sigma_{3_i,xxt} + 4\lambda_i \psi_{i,x} \sigma_{4_i,t} = 0, \end{gathered} \tag{23d}$$

$$\begin{aligned} &2\psi_i\sigma_{3_i} - \sigma_{4_i,x} = 0,\\ &\sigma_{3_i,t} - \sigma_{5_i} = 0,\\ &\sigma_{3_i,x} - \sigma_{6_i} = 0,\\ &\sigma_{4_i,t} - \sigma_{7_i} = 0,\\ &\sigma_{6_i,t} - \sigma_{8_i} = 0,\\ &i = 1, 2, \ldots, n. \end{aligned} \tag{23e}$$

We first consider the special case, i.e., for any fixed $i = k, c_k \neq 0$ while $c_j = 0\,(j \neq k)$ in equations (22a)–(22f). In this case, we obtain the localized symmetry for $u, v, \psi_k, f_k, g_k, h_k, p_k$, and q_k from equation (17) as follows:

$$\begin{aligned} \sigma &\equiv \left\{\sigma_1, \sigma_2, \sigma_{3_k}, \sigma_{4_k}, \sigma_{5_k}, \sigma_{6_k}, \sigma_{7_k}, \sigma_{8_k}\right\} \\ &= \left\{c_k g_k \psi_k, c_k h_k \psi_k, -\frac{c_k f_k \psi_k}{4}, -\frac{c_k f_k^2}{4}, -\frac{c_k}{4}\left(p_k\psi_k + f_k g_k\right), -\frac{c_k}{4}\left(\psi_k^3 + f_k h_k\right), -\frac{c_k f_k p_k}{2}, -\frac{c_k}{4}\left(3 g_k \psi_k^2 + p_k h_k + f_k q_k\right)\right\}. \end{aligned} \tag{24}$$

For $j \neq k$, we eliminate v through equation (21b) by taking $i = k$ and $i = j$, respectively. Then, we have

$$\psi_{j,xx} = \frac{\psi_j \psi_{k,xx}}{\psi_k} - \left(\lambda_k - \lambda_j\right)\psi_j. \tag{25}$$

Substituting $\sigma_2 = c_k h_k \psi_k$, into equation (23b) with $i = j$ and vanishing $\psi_{j,xx}$ through equation (25), we have

$$\left(\lambda_k - \lambda_j + \frac{\psi_{j,xx}}{\psi_j}\right)\sigma_{3_j} - \sigma_{3_j,xx} - c_j \psi_k \psi_j \psi_{j,x} = 0. \tag{26}$$

It can be easily verified that equation (26) has the following solution:

$$\sigma_{3_j} = -\frac{c_j \psi_j \left(h_k \psi_j - h_j \psi_k\right)}{4\left(\lambda_k - \lambda_j\right)}. \tag{27}$$

The symmetry for $\sigma_{4_j}, \sigma_{5_j}, \sigma_{6_j}, \sigma_{7_j}$, and σ_{8_j} can be easily obtained from equation (23e) with $i = j$:

$$\begin{aligned} \sigma_{4_j} &= -\frac{c_j \left(h_j \psi_k - h_k \psi_j\right)^2}{4\left(\lambda_k - \lambda_j\right)^2},\\ \sigma_{5_j} &= -\frac{c_j \left[g_j\left(h_k\psi_j - h_j\psi_k\right) + \psi_j\left(q_k\psi_j - q_j\psi_k\right) + \psi_j\left(g_j h_k - g_k h_j\right)\right]}{4\left(\lambda_k - \lambda_j\right)}, \end{aligned} \tag{28a}$$

$$\begin{aligned} \sigma_{6_j} &= -\frac{1}{4} c_j \left[\frac{h_j\left(h_k\psi_j - h_j\psi_k\right)}{\lambda_k - \lambda_j} + \psi_k \psi_j^2\right],\\ \sigma_{7_j} &= -\frac{1}{2}\frac{c_j\left(h_k\psi_j - h_j\psi_k\right)\left(q_k\psi_j - q_j\psi_k + g_j h_k - g_k h_j\right)}{\left(\lambda_i - \lambda_j\right)^2}, \end{aligned} \tag{28b}$$

$$\sigma_{8_j} = -\frac{1}{4} c_j \left[\frac{q_j\left(h_k\psi_j - h_j\psi_k\right) + h_j\left(g_j h_k - g_k h_j\right) + h_j\left(q_k\psi_j - q_j\psi_k\right)}{\lambda_k - \lambda_j} + \psi_j^2 g_k + 2\psi_j\psi_k g_j\right]. \tag{28c}$$

After taking the linear combination of the above results for all $k = 1, 2, \ldots, n$, Theorem 2 is proved.

When a nonlocal symmetry is localized to a Lie point symmetry, searching for its finite transformation is inevitable according to Lie's first principle. For the Lie point symmetry (22a)–(22f), its initial value problem has

$$\frac{\mathrm{d}U(\varepsilon)}{\mathrm{d}\varepsilon} = \sum_{i=1}^{n} c_i G_i(\varepsilon)\Psi_i(\varepsilon), \quad U(0) = u,$$
$$\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}\varepsilon} = \sum_{i=1}^{n} c_i H_i(\varepsilon)\Psi_i(\varepsilon), \quad V(0) = v, \tag{29a}$$

$$\frac{\mathrm{d}\Psi_i(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}c_i F_i(\varepsilon)\Psi_i(\varepsilon) - \frac{1}{4}\sum_{j\neq i}\frac{c_j\Psi_j(\varepsilon)\left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right]}{\lambda_i - \lambda_j}, \quad \Psi_i(0) = \psi_i, \tag{29b}$$

$$\frac{\mathrm{d}F_i(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}c_i F_i^2(\varepsilon) - \frac{1}{4}\sum_{j\neq i}\frac{c_j\left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right]^2}{\left(\lambda_i - \lambda_j\right)^2}, \quad F_i(0) = f_i, \tag{29c}$$

$$\begin{aligned}\frac{\mathrm{d}G_i(\varepsilon)}{\mathrm{d}\varepsilon} = {}& -\frac{1}{4}c_i\left[P_i(\varepsilon)\Psi_i(\varepsilon) - F_i(\varepsilon)G_i(\varepsilon)\right] - \frac{1}{4}\sum_{j\neq i}\frac{c_j}{\lambda_i - \lambda_i}\Big\{G_j(\varepsilon)\left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right] \\ & + \Psi_j(\varepsilon)\left[Q_i(\varepsilon)\Psi_j(\varepsilon) - Q_j(\varepsilon)\Psi_i(\varepsilon)\right] + \Psi_j(\varepsilon)\left[G_j(\varepsilon)H_i(\varepsilon) - G_i(\varepsilon)H_j(\varepsilon)\right]\Big\}, \quad G_i(0) = g_i,\end{aligned} \tag{29d}$$

$$\frac{\mathrm{d}H_i(\varepsilon)}{\mathrm{d}\varepsilon} = -\frac{1}{4}c_i\left[\Psi_i^3(\varepsilon) + F_i(\varepsilon)H_i(\varepsilon)\right] - \frac{1}{4}\sum_{j\neq i}c_j\left\{\frac{H_j(\varepsilon)\left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right]}{\lambda_i - \lambda_j} + \Psi_i(\varepsilon)\Psi_j^2(\varepsilon)\right\}, \quad H_i(0) = h_i, \tag{29e}$$

$$\begin{aligned}\frac{\mathrm{d}P_i(\varepsilon)}{\mathrm{d}\varepsilon} = {}& -\frac{1}{2}c_i F_i(\varepsilon)P_i(\varepsilon) - \frac{1}{2}\sum_{j\neq i}\frac{c_j}{\left(\lambda_i - \lambda_j\right)^2} \cdot \left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right]\%\left[Q_i(\varepsilon)\Psi_j(\varepsilon)\right. \\ & \left. - Q_j(\varepsilon)\Psi_i(\varepsilon) + G_j(\varepsilon)H_i(\varepsilon) - G_i(\varepsilon)H_j(\varepsilon)\%\right], \quad P_i(0) = p_i,\end{aligned} \tag{29f}$$

$$\begin{aligned}\frac{\mathrm{d}Q_i(\varepsilon)}{\mathrm{d}\varepsilon} = {}& -\frac{1}{4}c_i\left[3G_i(\varepsilon)\Psi_i^2(\varepsilon) + P_i(\varepsilon)H_i(\varepsilon) + F_i(\varepsilon)Q_i(\varepsilon)\right] - \frac{1}{4}\sum_{j\neq i}c_j\left[\frac{1}{\lambda_i - \lambda_j}\%\left(Q_j(\varepsilon)\left(H_i(\varepsilon)\Psi_j(\varepsilon)\right.\right.\right. \\ & \left.\left. - H_j(\varepsilon)\Psi_i(\varepsilon)\right) + H_j(\varepsilon)\left(G_j(\varepsilon)H_i(\varepsilon) - G_i(\varepsilon)H_j(\varepsilon)\right) \right. \\ & \left. + H_j(\varepsilon)\left(Q_i(\varepsilon)\Psi_j(\varepsilon) - Q_j(\varepsilon)\Psi_i(\varepsilon)\right)\%\right)\Big],\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}Q_i(\varepsilon)}{\mathrm{d}\varepsilon} = {}& -\frac{1}{4}c_i\left[3G_i(\varepsilon)\Psi_i^2(\varepsilon) + P_i(\varepsilon)H_i(\varepsilon) + F_i(\varepsilon)Q_i(\varepsilon)\right] - \frac{1}{4}\sum_{j\neq i}c_j\left\{\frac{Q_j(\varepsilon)\left[H_i(\varepsilon)\Psi_j(\varepsilon) - H_j(\varepsilon)\Psi_i(\varepsilon)\right]}{\lambda_i - \lambda_j}\right. \\ & + \frac{H_j(\varepsilon)\left[G_j(\varepsilon)H_i(\varepsilon) - G_i(\varepsilon)H_j(\varepsilon)\right]}{\lambda_i - \lambda_j} + \frac{H_j(\varepsilon)\left[Q_i(\varepsilon)\Psi_j(\varepsilon) - Q_j(\varepsilon)\Psi_i(\varepsilon)\right]}{\lambda_i - \lambda_j} \\ & \left. + \Psi_j^2(\varepsilon)G_i(\varepsilon) + 2\Psi_j(\varepsilon)\Psi_i(\varepsilon)G_j(\varepsilon)\right\}, \quad Q_i(0) = q_i.\end{aligned} \tag{29g}$$

Then, one can get the following n-th Bäcklund theorem for the enlarged system (21a)–(21e) by solving (29a)–(29g). □

Theorem 3. *If* $\{u, v, \psi_i, f_i, g_i, h_i, p_i, q_i, i = 1, 2, \ldots, n\}$ *is a solution of the prolonged nKdV equations (21a)–(21e), so is* $\{U(\varepsilon), V(\varepsilon), \Psi_i(\varepsilon), F_i(\varepsilon), G_i(\varepsilon), H_i(\varepsilon), P_i(\varepsilon), Q_i(\varepsilon), i = 1, 2, \ldots, n\}$, *where*

$$U(\varepsilon) = u + 2(\ln\Delta)_{xt},$$
$$V(\varepsilon) = v + 2(\ln\Delta)_{xx}, \tag{30a}$$

$$\Psi_i(\varepsilon) = \frac{2\Gamma_i}{\Delta},$$
$$F_i(\varepsilon) = \frac{2\Delta_i}{\Delta}, \tag{30b}$$
$$G_i(\varepsilon) = \Psi_{i,t}(\varepsilon),$$

$$H_i(\varepsilon) = \Psi_{i,x}(\varepsilon),$$
$$P_i(\varepsilon) = F_{i,x}(\varepsilon), \quad (30c)$$
$$Q_i(\varepsilon) = \Psi_{i,xt}(\varepsilon),$$

where Δ, Δ_i, and Γ_i are three determinants of the matrices D, D_i, and E_i, respectively, which are defined as follows:

$$D = \begin{pmatrix} c_1\varepsilon f_1 + 4 & c_1\varepsilon\mu_{1,2} & \cdots & c_1\varepsilon\mu_{1,j} & \cdots & c_1\varepsilon\mu_{1,n} \\ c_2\varepsilon\mu_{1,2} & c_2\varepsilon f_2 + 4 & \cdots & c_2\varepsilon\mu_{2,j} & \cdots & c_2\varepsilon\mu_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_j\varepsilon\mu_{1,j} & c_j\varepsilon\mu_{2,j} & \cdots & c_j\varepsilon f_j + 4 & \cdots & c_j\varepsilon\mu_{j,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n\varepsilon\mu_{1,n} & c_n\varepsilon\mu_{2,n} & \cdots & c_n\varepsilon\mu_{j,n} & \cdots & c_n\varepsilon f_n + 4 \end{pmatrix}, \quad \mu_{i,j} = \frac{h_j\psi_i - h_i\psi_j}{\lambda_i - \lambda_j},$$

$$D_i = \begin{pmatrix} c_1\varepsilon f_1 + 4 & c_1\varepsilon\mu_{1,2} & \cdots & c_1\varepsilon\mu_{1,i-1} & c_1\varepsilon\mu_{1,i} & c_1\varepsilon\mu_{1,i+1} & \cdots & c_1\varepsilon\mu_{1,n} \\ c_2\varepsilon\mu_{1,2} & c_2\varepsilon f_2 + 4 & \cdots & c_2\varepsilon\mu_{2,i-1} & c_2\varepsilon\mu_{2,i} & c_2\varepsilon\mu_{2,i+1} & \cdots & c_2\varepsilon\mu_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i-1}\varepsilon\mu_{1,i-1} & c_{i-1}\varepsilon\mu_{2,i-1} & \cdots & c_{i-1}\varepsilon f_{i-1} + 4 & c_{i-1}\varepsilon\mu_{i-1,i} & c_{i-1}\varepsilon\mu_{i-1,i+1} & \cdots & c_{i-1}\varepsilon\mu_{i-1,n} \\ \mu_{1,i} & \mu_{2,i} & \cdots & \mu_{i,i-1} & f_i & \mu_{i,i+1} & \cdots & \mu_{i,n} \\ c_{i+1}\varepsilon\mu_{1,i+1} & c_{i+1}\varepsilon\mu_{2,i+1} & \cdots & c_{i+1}\varepsilon\mu_{i-1,i+1} & c_{i+1}\varepsilon\mu_{i,i+1} & c_{i+1}\varepsilon f_{i+1} + 4 & \cdots & c_{i+1}\varepsilon\mu_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n\varepsilon\mu_{1,n} & c_n\varepsilon\mu_{2,n} & \cdots & c_n\varepsilon\mu_{i-1,n} & c_n\varepsilon\mu_{i,n} & c_n\varepsilon\mu_{i+1,n} & \cdots & c_n\varepsilon f_n + 4 \end{pmatrix},$$

$$E_i = \begin{pmatrix} c_1\varepsilon f_1 + 4 & c_1\varepsilon\mu_{1,2} & \cdots & c_1\varepsilon\mu_{1,i-1} & c_1\varepsilon\mu_{1,i} & c_1\varepsilon\mu_{1,i+1} & \cdots & c_1\varepsilon\mu_{1,n} \\ c_2\varepsilon\mu_{1,2} & c_2\varepsilon f_2 + 4 & \cdots & c_2\varepsilon\mu_{2,i-1} & c_2\varepsilon\mu_{2,i} & c_2\varepsilon\mu_{2,i+1} & \cdots & c_2\varepsilon\mu_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i-1}\varepsilon\mu_{1,i-1} & c_{i-1}\varepsilon\mu_{2,i-1} & \cdots & c_{i-1}\varepsilon f_{i-1} + 4 & c_{i-1}\varepsilon\mu_{i-1,i} & c_{i-1}\varepsilon\mu_{i-1,i+1} & \cdots & c_{i-1}\varepsilon\mu_{i-1,n} \\ \psi_1 & \psi_2 & \cdots & \psi_{i-1} & \psi_i & \psi_{i+1} & \cdots & \psi_n \\ c_{i+1}\varepsilon\mu_{1,i+1} & c_{i+1}\varepsilon\mu_{2,i+1} & \cdots & c_{i+1}\varepsilon\mu_{i-1,i+1} & c_{i+1}\varepsilon\mu_{i,i+1} & c_{i+1}\varepsilon f_{i,i+1} + 4 & \cdots & c_{i+1}\varepsilon\mu_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n\varepsilon\mu_{1,n} & c_n\varepsilon\mu_{2,n} & \cdots & c_n\varepsilon\mu_{i-1,n} & c_n\varepsilon\mu_{i,n} & c_n\varepsilon\mu_{i+1,n} & \cdots & c_n\varepsilon f_n + 4 \end{pmatrix}. \quad (31)$$

From Theorem 3, we can obtain an infinite number of new solutions from a suitable seed solution of the nKdV equation (1). Especially, one can obtain recursive soliton solutions of this system from the known one. For example, taking the seed solution $u = \alpha$ and $v = \beta$ for equation (1), it is not difficult to verify that equations (21a)–(21e) possess the following solution:

$$\psi_i = \exp\left(\frac{\xi_i}{2}\right),$$
$$f_i = \frac{1}{2k_i}\exp(\xi_i),$$
$$\lambda_i = k_i^2 + \beta, \quad (32)$$
$$\mu_{i,j} = \frac{\exp((\xi_i + \xi_j)/2)}{k_i + k_j},$$
$$\xi_i = 2k_i x - \frac{\alpha k_i t}{k_i^2 + \beta},$$
$$i, j = 1, 2, \ldots, n.$$

The corresponding first three multiple wave solutions for equation (1) are

$$\Delta_1 \equiv \Delta = c_1\varepsilon f_1 + 4, \quad (33)$$

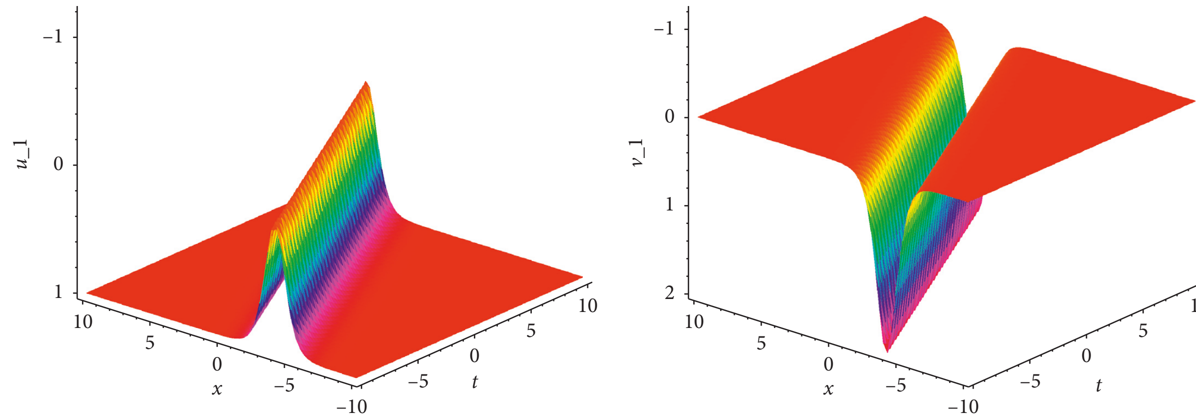
$$u = \alpha + 2(\ln\Delta_1)_{xt} = \alpha - \frac{32c_1k_1^3\alpha\varepsilon\exp(\xi_1)}{(k_1^2 + \beta)[8k_1 + c_1\varepsilon\exp(\xi_1)]^2}, \quad (34a)$$

$$v = \beta + 2(\ln\Delta_1)_{xx} = \beta + \frac{64c_1k_1^3\varepsilon\exp(\xi_1)}{[8k_1 + c_1\varepsilon\exp(\xi_1)]^2}, \quad (34b)$$

for the line soliton solution:

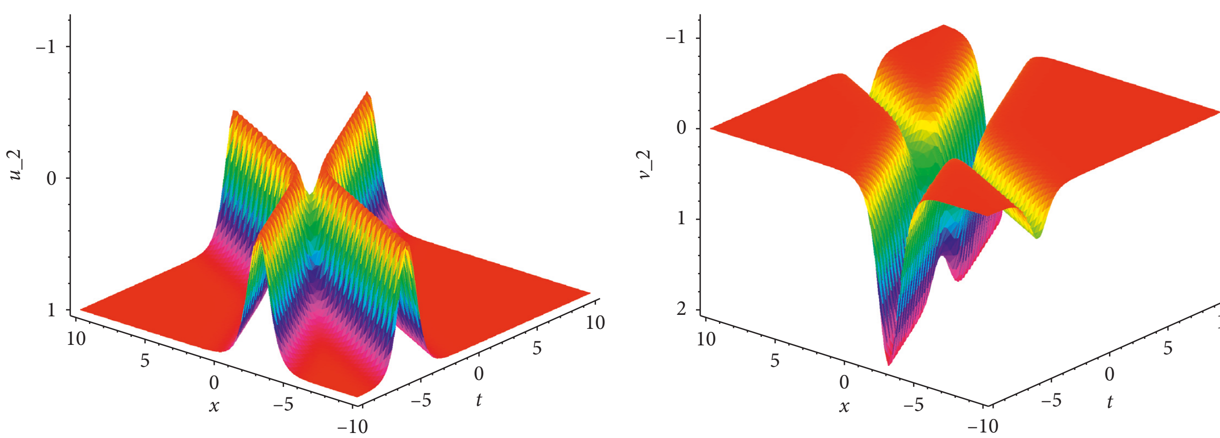
$$\Delta_2 \equiv \Delta = (c_1\varepsilon f_1 + 4)(c_2\varepsilon f_2 + 4) - c_1c_2\varepsilon^2\mu_{1,2}^2$$
$$= 16 + 4c_1\varepsilon f_1 + 4c_2\varepsilon f_2 + \frac{c_1c_2\varepsilon^2(k_1 - k_2)^2 f_1 f_2}{(k_1 + k_2)^2}, \quad (35)$$

$$u = \alpha + 2(\ln\Delta_2)_{xt},$$
$$v = \beta + 2(\ln\Delta_2)_{xx}, \quad (36)$$



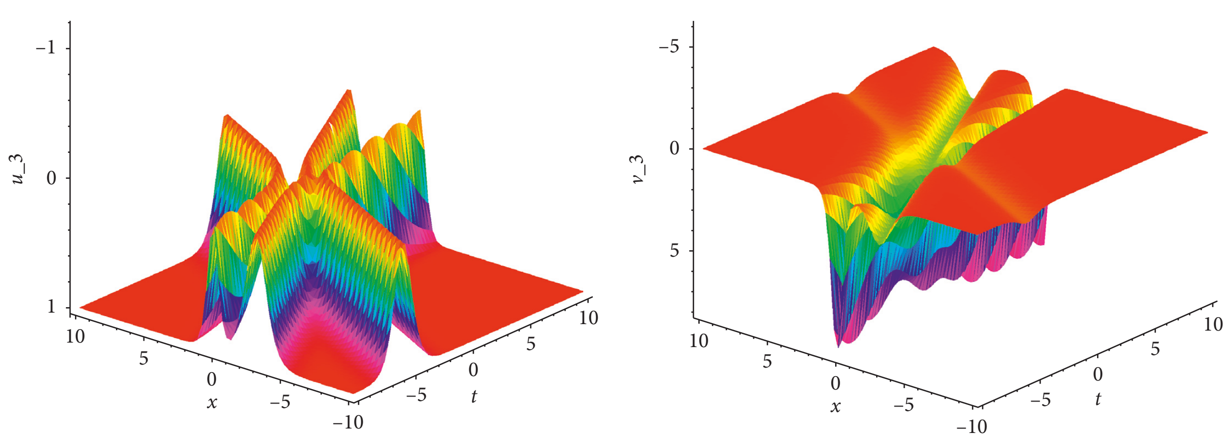
(a) (b)

Figure 1: Plots of two line solitons expressed by equations (34a)–(34b) for solution of the nKdV equation (1) with the parameters $a = \varepsilon = c_1 = k_1 = 1$.



(a) (b)

Figure 2: Interactions of two-resonant solitons expressed by equation (36) for solution of the nKdV equation (1) with the parameters $a = \varepsilon = c_1 = c_2 = k_1 = 1$ and $k_2 = 1/2$.



(a) (b)

Figure 3: Interactions of three-resonant solitons expressed by equation (37) for solution of the nKdV equation (1) with the parameters $a = \varepsilon = c_1 = c_2 = c_3 = k_1 = 1, k_2 = 1/2$, and $k_3 = 2$.

$$u = \alpha + 2(\ln \Delta_3)_{xt},$$
$$v_3 = \beta + 2(\ln \Delta_3)_{xx}, \quad (37)$$

$$\Delta_3 = \begin{vmatrix} c_1\varepsilon f_1 + 4 & c_1\varepsilon\mu_{1,2} & c_1\varepsilon\mu_{1,3} \\ c_2\varepsilon\mu_{1,2} & c_2\varepsilon f_2 + 4 & c_2\varepsilon\mu_{2,3} \\ c_3\varepsilon\mu_{1,3} & c_3\varepsilon\mu_{2,3} & c_3\varepsilon f_3 + 4 \end{vmatrix}$$
$$= 64 + 16c_1\varepsilon f_1 + 16c_2\varepsilon f_2 + 16c_3\varepsilon f_3 + \frac{4c_1c_2\varepsilon^2(k_1-k_2)^2 f_1 f_2}{(k_1+k_2)^2} + \frac{4c_1c_3\varepsilon^2(k_1-k_3)^2 f_1 f_3}{(k_1+k_3)^2} + \frac{4c_2c_3\varepsilon^2(k_2-k_3)^2 f_2 f_3}{(k_2+k_3)^2} + \frac{c_1c_2c_3\varepsilon^3(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2 f_1 f_2 f_3}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}. \quad (38)$$

For illustrating more details, the parameters are set as follows: $a = \varepsilon = c_1 = c_2 = c_3 = k_1 = 1, k_2 = 1/2$, and $k_3 = 2$. Figure 1 displays the bell-like bright and dark solitons for the above condition of equations (34a)–(34b). Figure 1(a) shows a line dark soliton for u_1 with the amplitude 1, while Figure 1(b) shows a bright one for v_1 with the amplitude 2. Similarly, Figure 2 shows the collision of two-resonant solitons in equation (36), and Figure 3 shows interactions of three-resonant solitons expressed in equation (37).

4. Summary

The nonlocal symmetry of the nKdV equation is obtained with the aid of its Lax pair. After introducing four auxiliary variables g, h, p, and q, an enlarged system which possesses a Lie point symmetry for the nonlocal symmetry is taken. By applying Lie's first theorem for the localized point symmetries, we obtain the corresponding finite transformation. Furthermore, we can localize the linear superposition of multiple residual symmetries and construct the infinite transformation for the nKdV equation. From Theorem 3, the n-th Bäcklund transformation can be expressed in a compact way of determinants. According to this conclusion, one can derive special soliton solutions from some seed solutions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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