

# Research Article Chaos in Duopoly Games via Furstenberg Family Couple

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Assume that  $H_1$  and  $H_2$  are two given closed subintervals of  $\mathbb{R}$  and that  $f_2 : H_1 \longrightarrow H_2$  and  $f_1 : H_2 \longrightarrow H_1$  are continuous maps. Let  $\Upsilon(h_1, h_2) = (f_1(h_2), f_2(h_1))$  be a Cournot map over the space  $H_1 \times H_2$ . In this paper, we study  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaos (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaos) of such a Cournot map. We will show that the following are true: (1)  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic). (2)  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1 \cup \Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic). (2)  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1 \cup \Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1 \cup \Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic). (3)  $f_1 \circ f_2$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if so is  $f_2 \circ f_1$ . *MR*(2000) *Subject Classification*: Primary 37D45, 54H20, and 37B40 and Secondary 26A18 and 28D20.

## 1. Introduction

Let  $H_1$  and  $H_2$  be closed subintervals of  $\mathbb{R}$ , and let  $f_1$ :  $H_2 \longrightarrow H_1$  and  $f_2: H_1 \longrightarrow H_2$  be continuous. In the whole paper,  $\Upsilon: H_1 \times H_2 \longrightarrow H_1 \times H_2$  is defined by  $\Upsilon(h_1, h_2) = (f_1(h_2), f_2(h_1))$  for any  $(h_1, h_2) \in H_1 \times H_2$ . Such a map has been investigated to give a mathematical analysis of Cournot duopoly (see [1]). Probably the first notion of chaos in a mathematically rigorous way was posed by Li and Yorke [2]. Since then, a lot of different notions of chaos have been posed. Akin and Kolyada gave the concept of Li-Yorke sensitivity for the first time [3]. They also gave the concept of spatiotemporal chaos. Schweizer and Smítal gave the concept of distributional chaos [4]. We know that distributional chaos is equivalent to positive topological entropy and some other chaotic properties for some particular spaces (see [4, 5]), and that this equivalence relationship will become invalid for some higher dimensional spaces [6] and some zero-dimensional spaces [7]. In [8], Wang et al. gave the definition of distributional chaos with respect to a sequence and got that such chaos is equivalent to Li-Yorke chaos for continuous maps over a closed subinterval. Over the past few decades, people have been paying very close attention to the chaotic properties of Cournot maps (see [1,9–13]). From [1, 12] one can see that there exist

Markov perfect equilibria processes. That is, two fixed players move alternatively and ensure that any of them chooses the best reply to the previous action of another player. Put  $\Lambda_1 = \{(f_1(h_2), h_2) : h_2 \in H_2\}, \Lambda_2 = \{(h_1, f_2) : h_2 \in H_2\}$  $(h_1)$ :  $h_1 \in H_1$ , and  $\Lambda_{12} = \Lambda_1 \cup \Lambda_2$ . Obviously,  $\Upsilon(Q_{12})$  $\subset \Lambda_{12}$ . The set  $\Lambda_{12}$  is said to be a MPE set for  $\Upsilon$  (see [9]). Moreover, in [9], the authors studied several kinds of chaos for Cournot maps and obtained that for any definition they considered in [9], and it does not satisfy the condition that  $\Upsilon$ is chaotic if and only if so is  $\left.\Upsilon\right|_{\Lambda_{12}}.$  It is well known that some chaotic properties of Cournot maps have been explored (see [1,12-17]). Recently, Lu and Zhu further studied some chaotic properties of Cournot maps and showed that some chaotic properties of  $\Upsilon|_{\Lambda_{12}}$ ,  $\Upsilon^2|_{\Lambda_1}$  and  $\Upsilon^2|_{\Lambda_2}$  are same. In this paper, it is shown that for any Cournot map  $\Upsilon(h_1, h_2) =$  $(f_1(h_2), f_2(h_1))$  over the product space  $H_1 \times H_2$ , the following properties are hold:

- (1)  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Gamma^2|_{\Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic).
- (2)  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if  $\Upsilon^2|_{\Lambda_1 \cup \Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic)

#### 2. Preliminaries

Let  $(H, \xi)$  be a compact metric space. A dynamic system (H, f) means that *f* is a continuous self-map over the space *H*.

Let  $f : H \longrightarrow H$  be a map on the space  $(H, \xi)$ . The map f is chaotic in the sense of Li–Yorke if there is an uncountable set  $\mathcal{C} \subset H$  satisfying that for any  $h_1, h_2 \in \mathcal{C}$  with  $h_1 \neq h_2$ :

$$\lim_{l \to \infty} \inf_{l \to \infty} \xi \left( f^{l}(h_{1}), f^{l}(h_{2}) \right) = 0,$$

$$\lim_{l \to \infty} \sup_{l \to \infty} \xi \left( f^{l}(h_{1}), f^{l}(h_{2}) \right) > 0.$$
(1)

This uncountable set  $\mathscr{C}$  is called a scrambled set of f.

An important generalization of Li–Yorke chaos is distributional chaos, which is given in 1994 by Puu and Sushko [1].

Let  $(H, \xi)$  be a metric space and  $f : H \longrightarrow H$  be continuous. For any  $h_1, h_2 \in H$ , the upper (lower) distribution function  $F_{h_1h_2}^*(t, f)$   $(F_{h_1h_2}(t, f))$  deduced by  $(h_1, h_2)$  and f is defined by

$$F_{h_{1}h_{2}}^{*}(t,f) = \limsup_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} \chi_{[0,t)} \left( \xi \left( f^{j}(h_{1}), f^{j}(h_{2}) \right) \right),$$
  

$$F_{h_{1}h_{2}}(t,f) = \liminf_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} \chi_{[0,t)} \left( \xi \left( f^{j}(h_{1}), f^{j}(h_{2}) \right) \right),$$
(2)

where  $\chi_{[0,t)}$  is the characteristic function of the set [0, t). The map f is distributional chaotic if there is an uncountable subset  $\mathscr{C} \subset H$  satisfying that for any  $h_1 \neq h_2 \in \mathscr{C}$ ,  $F_{h_1h_2}^*(t, f) = 1 \ (\forall t > 0)$ , and  $\exists t > 0$  such that  $F_{h_1h_2}(t, f) = 0$ . This uncountable subset  $\mathscr{C}$  is called a distributionally scrambled set of f. And this point pair  $(h_1, h_2)$  which satisfies the above two conditions is called a distributionally scrambled pair of f.

In 1997, Furstenberg family is introduced by Akin [18]. Then, Xiong and Tan defined  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaos and described chaos via Furstenberg family couple. Also, they obtained some sufficient conditions of  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaos (see [19]).

Let  $\mathbb{N} = \{1, 2, 3, ...\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , and  $\mathscr{P}$  be the collection of all subsets of  $\mathbb{N}_0$ . A collection  $\mathscr{G} \subset \mathscr{P}$  is called a Furstenberg family (see [19]) if it satisfies that if  $G_1 \subset G_2$  and  $G_1 \in \mathscr{G}$  then  $G_2 \in \mathscr{G}$ . A family  $\mathscr{G}$  is said to be proper if it is a proper subset of  $\mathscr{P}$  (see [19]). In the whole paper, we suppose that all Furstenberg families are proper. Clearly, a family  $\mathscr{G}$  is proper if and only if  $\mathbb{N}_0 \in \mathscr{G}$  and  $\mathscr{O} \notin \mathscr{G}$  (see [19]).

For any Furstenberg families  $\mathscr{G}_1$  and  $\mathscr{G}_2$  and any map  $f: H \longrightarrow H, \mathscr{C} \subset H$  is called a  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of f (see [19]), if  $\forall h_1 \neq h_2 \in \mathscr{C}$ , the following two conditions are satisfied:

$$\begin{aligned} &(1) \ \forall t > 0, \ \left\{ m \in \mathbb{N} : \xi(f^m(h_1), f^m(h_2)) < t \right\} \in \mathcal{G}_1 \\ &(2) \ \exists \delta > 0, \ \left\{ m \in \mathbb{N} : \xi(f^m(h_1), f^m(h_2)) > \delta \right\} \in \mathcal{G}_2 \end{aligned}$$

This pair  $(h_1, h_2)$  is called a  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled pair of f. The map f is said to be  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic if there is an uncountable  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of f. When  $\mathscr{G}_1 = \mathscr{G}_2 = \mathscr{G}$ , the map f is said to be  $\mathscr{G}$ -chaotic and the pair  $(h_1, h_2)$  is a  $\mathscr{G}$ -scrambled pair. The map f is said to be strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic if one can find  $\delta > 0$  satisfying that for any  $h_1 \neq h_2 \in \mathscr{G}, \ \forall t > 0, \ \{m \in \mathbb{N} : \xi(f^m(h_1), f^m(h_2)) < t\} \in \mathscr{G}_1$  and  $\{m \in \mathbb{N} : \xi(f^m(h_1), f^m(h_2)) > \delta\} \in \mathscr{G}_2$ .

Similarly, one can give the concept of strong  $\mathscr{G}$ -chaos. Let  $G \subset \mathbb{N}$ . The upper density  $\overline{\mu}(G)$  and the lower density  $\mu(G)$  of *G* are defined by

$$\overline{\mu}(G) = \limsup_{m \to \infty} \frac{\#(G \cap \{1, 2, \dots, m\})}{m},$$

$$\underline{\mu}(G) = \liminf_{m \to \infty} \frac{\#(G \cap \{1, 2, \dots, m\})}{m},$$
(3)

where #(G) denotes the cardinality of the set *G*.

Let  $\mathscr{B}$  denotes the set of all infinite subsets of  $\mathbb{N}$ . For arbitrary  $t \in [0, 1]$ , put  $\overline{M}(t) = \{F \in \mathscr{B} : \overline{\mu}(F) \ge t\}$ . Then, a pair  $(h_1, h_2)$  is a  $(\overline{M}(0), \overline{M}(0))$ -scrambled pair if and only if it is a Li–Yorke scrambled pair (see [19]). A pair  $(h_1, h_2)$  is a  $(\overline{M}(1), \overline{M}(1))$ -scrambled pair if and only if it is a distributionally scrambled pair (see [19]). Hence,  $(\overline{M}(0), \overline{M}(0))$ -chaos is Li–Yorke chaos, and  $(\overline{M}(1), \overline{M}(1))$ -chaos is distributional chaos.

For  $i \in \mathbb{N}_0$  and  $G \in \mathcal{P}$ , let  $G + i = \{j + i : j \in G\} \cap \mathbb{N}_0$  and  $G - i = \{j - i : j \in G\} \cap \mathbb{N}_0$ . A Furstenberg family  $\mathcal{G}$  is said to be translation-invariant if for any  $G \in \mathcal{G}$  and any  $i \in \mathbb{N}_0$   $G + i \in \mathcal{G}$  and  $G - i \in \mathcal{G}$ . It is easily seen that  $\mathcal{B}$  is a proper and translation-invariant family (see [19]).

Clearly, for any  $t \in [0, 1]$ ,  $\overline{M}(t)$  is a translation-invariant Furstenberg family and  $\overline{M}(0) = \mathscr{B}$  (see [19]).

#### 3. Main Results

**Theorem 1.** Let the product metric  $\xi$  on the product space  $H_1 \times H_2$  be defined by  $\xi((a_1, b_1), (a_2, b_2)) = \max \{|a_2 - a_1|, |b_2 - b_1|\}$  and the product map  $\pi_1 \times \pi_2$  of  $\pi_1 : H_1 \longrightarrow H_1$  and  $\pi_2 : H_2 \longrightarrow H_2$  be defined by  $(\pi_1 \times \pi_2)(a,b) = (\pi_1(a), \pi_2(b))$  for any  $a \in H_1$  and any  $b \in H_2$ , where  $H_1, H_2 \subset \mathbb{R}$  are compact intervals, and let  $Y(a,b) = (f_1(b), f_2(a))$  be a Cournot map. If  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are two Furstenberg families such that  $\mathcal{G}_2$  is translation-invariant, then  $f_1 \circ f_2$  is  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic if and only if so is  $f_2 \circ f_1$ .

*Proof.* Suppose that  $f_1 \circ f_2$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic. By the definition, there is an uncountable  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set  $D \in H_1$  of  $f_1 \circ f_2$ . By the definition, for any given b > 0 and any  $h_1, h_2 \in H_1$  with  $h_1 \neq h_2$  one has that

$$\left\{ m \in \mathbb{N} : \left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| < b \right\} \in \mathcal{G}_1.$$
 (4)

As  $f_2$  is uniformly continuous, for any a > 0 there is b > 0such that  $|p_1 - p_2| < b$  and  $p_1, p_2 \in H_1$  imply that  $|f_2(p_1) - f_2(p_2)| < a$ . So, if

$$\left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| < b, \tag{5}$$

then

$$\left| \left( f_2 \circ f_1 \right)^m \left( f_2(h_1) \right) - \left( f_2 \circ f_1 \right)^m \left( f_2(h_2) \right) \right| < a.$$
 (6)

Consequently, by

$$\left\{ m \in \mathbb{N} : \left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| < b \right\} \in \mathscr{G}_1,$$

$$\left\{ m \in \mathbb{N} : \left| \left( f_2 \circ f_1 \right)^m (f_2 (h_1)) - \left( f_2 \circ f_1 \right)^m (f_2 (h_2)) \right| < a \right\} \in \mathscr{G}_1.$$

$$(7)$$

By the definition, for any  $h_1, h_2 \in H_1$  with  $h_1 \neq h_2$  there is  $\delta > 0$  such that

$$\left\{m \in \mathbb{N} : \left| \left(f_1 \circ f_2\right)^m (h_1) - \left(f_1 \circ f_2\right)^m (h_2) \right| > \delta \right\} \in \mathscr{G}_2.$$
(8)

As  $f_1$  is uniformly continuous, for the above  $\delta > 0$  there is  $\delta' > 0$  such that  $|p_1 - p_2| \le \delta'$  and  $p_1, p_2 \in H_2$  imply that  $|f_1(p_1) - f_1(p_2)| \le \delta$ . So, if

$$\left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| > \delta, \tag{9}$$

then

$$\left| \left( f_2 \circ f_1 \right)^{m-1} \left( f_2(h_1) \right) - \left( f_2 \circ f_1 \right)^{m-1} \left( f_2(h_2) \right) \right| > \delta'.$$
 (10)

As  $\mathscr{G}_2$  is translation-invariant, by

$$\left\{ m \in \mathbb{N} : \left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| > \delta \right\} \in \mathscr{G}_2,$$
  
$$\left\{ m - 1 \in \mathbb{N} : \left| \left( f_2 \circ f_1 \right)^{m-1} (f_2 (h_1)) - \left( f_2 \circ f_1 \right)^{m-1} (f_2 (h_2)) \right| > \delta' \right\} \in \mathscr{G}_2.$$
  
(11)

This means that

$$\left\{m \in \mathbb{N} : \left| (f_2 \circ f_1)^m (f_2(h_1)) - (f_2 \circ f_1)^m (f_2(h_2)) \right| > \delta' \right\} \in \mathscr{G}_2.$$
(12)

Thus, Theorem 1 is true.

**Theorem 2.** Let the product metric  $\xi$  on the product space  $H_1 \times H_2$  be defined by  $\xi((a_1, b_1), (a_2, b_2)) = \max\{|a_2 - a_1|, |b_2 - b_1|\}$  and the product map  $\pi_1 \times \pi_2$  of  $\pi_1 : H_1 \longrightarrow H_1$  and  $\pi_2 : H_2 \longrightarrow H_2$  be defined by  $(\pi_1 \times \pi_2)(a,b) = (\pi_1(a), \pi_2(b))$  for any  $a \in H_1$  and any  $b \in H_2$ , where  $H_1, H_2 \subset \mathbb{R}$  are compact intervals, and let  $Y(a,b) = (f_1(b), f_2(a))$  be a Cournot map. If  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are two Furstenberg families such that  $\mathcal{G}_2$  is translation-invariant, then  $f_1 \circ f_2$  is strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic if and only if so is  $f_2 \circ f_1$ .

*Proof.* Suppose that  $f_1 \circ f_2$  is strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic. By the definition, there is an uncountable strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set  $D \in H_1$  of  $f_1 \circ f_2$ . By the definition, for any given b > 0 and any  $h_1, h_2 \in H_1$  with  $h_1 \neq h_2$  one has that

$$\left\{m \in \mathbb{N} : \left| \left(f_1 \circ f_2\right)^m (h_1) - \left(f_1 \circ f_2\right)^m (h_2) \right| < b \right\} \in \mathscr{G}_1.$$
(13)

As  $f_2$  is uniformly continuous, for any a > 0 there is b > 0such that  $|p_1 - p_2| < b$  and  $p_1, p_2 \in H_1$  imply that  $|f_2(p_1) - f_2(p_2)| < a$ . So, if

$$\left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| < b, \tag{14}$$

then

$$\left| \left( f_2 \circ f_1 \right)^m \left( f_2 \left( h_1 \right) \right) - \left( f_2 \circ f_1 \right)^m \left( f_2 \left( h_2 \right) \right) \right| < a.$$
 (15)

Consequently, by

$$\left\{ m \in \mathbb{N} : \left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| < b \right\} \in \mathcal{G}_1,$$

$$\left\{ m \in \mathbb{N} : \left| \left( f_2 \circ f_1 \right)^m (f_2 (h_1)) - \left( f_2 \circ f_1 \right)^m (f_2 (h_2)) \right| < a \right\} \in \mathcal{G}_1.$$

$$(16)$$

By the definition, for any  $h_1, h_2 \in H_1$  with  $h_1 \neq h_2$  there is  $\delta > 0$  satisfying that for any  $h_1, h_2 \in H_1$  with  $h_1 \neq h_2$ , one has that

$$\left\{ m \in \mathbb{N} : \left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| > \delta \right\} \in \mathscr{G}_2.$$
(17)

As  $f_1$  is uniformly continuous, for the above  $\delta > 0$  there is  $\delta' > 0$  such that  $|p_1 - p_2| \le \delta'$  and  $p_1, p_2 \in H_2$  imply that  $|f_1(p_1) - f_1(p_2)| \le \delta$ . So, if

$$\left| \left( f_1 \circ f_2 \right)^m (h_1) - \left( f_1 \circ f_2 \right)^m (h_2) \right| > \delta,$$
 (18)

then

$$\left| \left( f_2 \circ f_1 \right)^{m-1} \left( f_2 \left( h_1 \right) \right) - \left( f_2 \circ f_1 \right)^{m-1} \left( f_2 \left( h_2 \right) \right) \right| > \delta'.$$
 (19)

As  $\mathscr{G}_2$  is translation-invariant, by

$$\left\{m \in \mathbb{N} : \left| \left(f_{1} \circ f_{2}\right)^{m} \left(h_{1}\right) - \left(f_{1} \circ f_{2}\right)^{m} \left(h_{2}\right) \right| > \delta \right\} \in \mathscr{G}_{2}$$

$$\left\{m - 1 \in \mathbb{N} : \left| \left(f_{2} \circ f_{1}\right)^{m-1} \left(f_{2} \left(h_{1}\right)\right) - \left(f_{2} \circ f_{1}\right)^{m-1} \left(f_{2} \left(h_{2}\right)\right) \right| > \delta' \right\} \in \mathscr{G}_{2}$$

$$(20)$$

This means that

$$\left\{ m \in \mathbb{N} : \left| (f_2 \circ f_1)^m (f_2(h_1)) - (f_2 \circ f_1)^m (f_2(h_2)) \right| > \delta' \right\} \in \mathcal{G}_2.$$
(21)

Thus, Theorem 2 is true.  $\Box$ 

**Corollary 1.** Let  $\Upsilon(a,b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . Then, for any  $a, b \in [0,1]$ ,  $\underline{f_1} \circ f_2$  is  $(\overline{M}(a), \overline{M}(b))$ -chaotic (resp. strong  $(\overline{M}(a), \overline{M}(b))$ -chaotic) if and only if so is  $f_2 \circ f_1$ .

*Proof.* As  $\overline{M}(t)$  is a translation-invariant Furstenberg family for any  $t \in [0, 1]$ , by Theorems 1 and 2 one can see that Corollary 1 holds.

**Theorem 3.** Let  $\Upsilon(a, b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . If  $\mathscr{G}_1$  and  $\mathscr{G}_2$  are two Furstenberg families such that  $\mathscr{G}_2$  is translation-invariant and satisfy that for any  $k \in \{1, 2\}$  and any  $G \in \mathscr{G}_k$ , there is  $j \in \{0, 1\}$  satisfying that  $G_{2,j} \coloneqq \{i \in \{0, 1, \ldots\} : 2j + i \in G\}$  $\in \mathscr{G}_k$ , and that for any  $k \in \{1, 2\}$  and any  $G \in \mathscr{G}_k$ ,

$$G_2 \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathcal{G}_k,$$
(22)

then  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic if and only if so is  $\Upsilon^2|_{\Lambda_2}$ .

Claim 1.  $\Upsilon^2$  is  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic.

The Proof of Claim 1. Assume that  $D \in H_1 \times H_2$  is a  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of the system  $(H_1 \times H_2, \Upsilon)$ . As  $\Upsilon$ and  $\Upsilon^2$  are uniformly continuous, for any t > 0 there is t' > 0satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) < t'$  imply  $\xi(\Upsilon(h_1), \Upsilon(h_2)) < t$  and  $\xi(\Upsilon^2(h_1), \Upsilon^2(h_2)) < t$ . By the hypothesis and the definition, for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ , one has that

$$G = \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^{m}(d_{1}), \Upsilon^{m}(d_{2})) < t'\} \in \mathscr{G}_{1}.$$
(23)

As  $\mathscr{G}_1$  satisfies that for any  $G \in \mathscr{G}_1$ , there is  $j \in \{0, 1\}$ satisfying that  $G_{2,j} := \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathcal{G}_1$ , by the definition there is  $j \in \{0, 1\}$  satisfying that

$$G_{2,j} \coloneqq \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathcal{G}_1.$$
(24)

By the above argument, one has that

$$G_{2,j} \subset \left\{ m \in \{0, 1, \ldots\} : \xi \left( \Upsilon^{2m+j+2-j} \left( d_1 \right), \Upsilon^{2m+j+2-j} \left( d_2 \right) \right) < t \right\}.$$
(25)

That is,

$$G_{2,j} \subset \left\{ m \in \{0, 1, \ldots\} : \xi \big( \Upsilon^{2m+2} (d_1), \Upsilon^{2m+2} (d_2) \big) < t \right\}.$$
(26)

So,

$$\left\{m \in \{0, 1, \ldots\} : \xi \left(\Upsilon^{2m+2}(d_1), \Upsilon^{2m+2}(d_2)\right) < t\right\} \in \mathcal{G}_1.$$
(27)

As  $\mathscr{G}_1$  is translation-invariant,

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2)\right) < t\right\} \in \mathscr{G}_1.$$
 (28)

By the hypothesis and the definition, for any given  $d_1, d_2 \in D$  with  $d_1 \neq d_2$  there is  $\delta > 0$  satisfying that

$$G' = \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) > \delta\} \in \mathscr{G}_2.$$
(29)

As  $\mathscr{G}_2$  satisfies that for any  $G \in \mathscr{G}_2$ , there is  $j \in \{0, 1\}$ satisfying that  $G_{2,i} := \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathcal{G}_2$ , by the definition there is  $j \in \{0, 1\}$  satisfying that

$$G'_{2,j} \coloneqq \left\{ i \in \{0, 1, \ldots\} : 2j + i \in G' \right\} \in \mathcal{G}_2. \tag{30}$$

As  $\Upsilon$  and  $\Upsilon^2$  are uniformly continuous, for the above  $\delta > 0$ , there is  $\delta' > 0$  satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) \le \delta'$  imply  $\xi(\Upsilon(h_1), \Upsilon(h_2)) \le \delta$  and  $\xi(\Upsilon^2(h_1), \chi(h_2)) \le \delta$  $\Upsilon^2(h_2)) \leq \delta$ . Clearly,

$$G'_{2,j} \in \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2)) > \delta'\},$$
 (31)

which means that

$$\left\{m \in \{0, 1, \ldots\} : \xi(\Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2)) > \delta'\right\} \in \mathscr{G}_2.$$
 (32)

Thus, Claim 1 holds.

As  $\Upsilon^2 = (f_1 \circ f_2) \times (f_2 \circ f_1)$ , by hypothesis, Claim 1, the definition of  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaos, and Theorem 1 and its proof, one can easily verify that  $f_1 \circ f_2$  and  $f_2 \circ f_1$  are  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic.

Assume that  $f_1 \circ f_2$  is  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic. By the definition, there is an uncountable subset  $C \in H_1$  which is  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set of  $f_1 \circ f_2$ . By the proof of Theorem 1,  $f_2(C)$  is an uncountable  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of  $f_2 \circ f_1$ . Set  $A = \{(a, g_2(a)) : a \in C\}$ . Then, A is uncountable. By the above argument, the definition of  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaos and the proof of Theorem 1, it is easy to prove that A is a  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of  $\Upsilon^2|_{\Lambda_2}$ . Now, we assume that  $\Upsilon^2|_{\Lambda_2}$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.

Claim 2.  $\Upsilon$  is  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic.

The Proof of Claim 2. By the hypothesis and the definitions,  $\Upsilon^2$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic. Assume that D is a  $(\mathscr{G}_1, \mathscr{G}_2)$  $\mathscr{G}_2$ )-scrambled set of the system  $(H_1 \times H_2, \Upsilon^2)$ . As  $\Upsilon^n$  is uniformly continuous for any  $n \in \{0, 1\}$ , for any t > 0 there is t' > 0 satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) < t'$ imply  $\xi(\Upsilon^n(h_1), \Upsilon^n(h_2)) < t$  for any  $n \in \{0, 1\}$ . By hypothesis and the definition, for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ , one has that

$$G = \left\{ m \in \{0, 1, \ldots\} : \xi \left( \Upsilon^{2m} \left( d_1 \right), \Upsilon^{2m} \left( d_2 \right) \right) < t' \right\} \in \mathscr{G}_1.$$
(33)

So, for any  $m \in G$  and any  $n \in \{0, 1\}$  we have that

$$\xi(\Upsilon^{2m+n}(d_1),\Upsilon^{2m+n}(d_2)) < t.$$
 (34)

As 
$$\mathscr{G}_1$$
 satisfies that for any  $G \in \mathscr{G}_1$ ,  

$$G_2 \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathscr{G}_1, \quad (35)$$

by the definition we have

$$G_2 := \{2m + n : m \in G, n \in \{0, 1\}\} \in \mathcal{G}_1.$$
(36)

Clearly,

$$G_2 \in \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) < t\}.$$
(37)

This means that

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^m\left(d_1\right), \Upsilon^m\left(d_2\right)\right) < t\right\} \in \mathscr{G}_1.$$
(38)

By the hypothesis and the definition, for any given  $d_1, d_2 \in D$  with  $d_1 \neq d_2$  there is  $\delta' > 0$  satisfying that

$$G' = \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) > \delta'\} \in \mathscr{G}_2.$$
(39)

As  $\Upsilon^n$  is uniformly continuous for any  $n \in \{0, 1\}$ , for the above  $\delta' > 0$  there is  $\delta > 0$  satisfying that  $h_1, h_2 \in H_1 \times H_2$ and  $\xi(\Upsilon^2(h_1),\Upsilon^2(h_2)) > \delta$  imply  $\xi(\Upsilon^n(h_1),\Upsilon^n(h_2)) > \delta$  for any  $n \in \{0, 1\}$ . So, for any  $m \in G'$  and any  $n \in \{0, 1\}$  we have that

$$\xi \Big( \Upsilon^{2(m-1)+n}(d_1), \Upsilon^{2(m-1)+n}(d_2) \Big) > \delta.$$
(40)

As  $\mathscr{G}_2$  is translation-invariant,  $G' - 1 \in \mathscr{G}_2$ . As  $\mathscr{G}_2$  satisfies that for any  $G \in \mathscr{G}_2$ ,

$$G_{2} \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathscr{G}_{2},$$
$$(G' - 1)_{2} \coloneqq \{2(m - 1) + n : m - 1 \in G' - 1, n \in \{0, 1\}\} \in \mathscr{G}_{2}.$$
(41)

Clearly,

$$(G'-1)_2 \subset \{m \in \{0,1,\ldots\} : \xi(\Upsilon^m(d_1),\Upsilon^m(d_2)) > \delta\},$$
  
(42)

which means that

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^m(d_1), \Upsilon^m(d_2)\right) > \delta\right\} \in \mathscr{G}_2.$$
(43)

Thus, Claim 2 holds. Consequently, Theorem 3 is true.

**Theorem 4.** Let  $\Upsilon(a, b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . If  $\mathscr{G}_1$  and  $\mathscr{G}_2$  are two Furstenberg families such that  $\mathscr{G}_2$  is translation-invariant and satisfy that for any  $k \in \{1, 2\}$  and any  $G \in \mathscr{G}_k$ , there is  $j \in \{0, 1\}$  satisfying that  $G_{2,j} \coloneqq \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathscr{G}_k$ , and that for any  $k \in \{1, 2\}$  and any  $G \in \mathscr{G}_k$ ,

$$G_2 \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathcal{G}_k, \quad (44)$$

then  $\Upsilon$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic if and only if so is  $\Upsilon^2|_{\Lambda_2}$ .

*Proof.* We assume that  $\Upsilon$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.  $\Box$ 

Claim 3.  $\Upsilon^2$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.

The Proof of Claim 3. Assume that  $D \in H_1 \times H_2$  is a strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set of the system  $(H_1 \times H_2, \Upsilon)$ . As  $\Upsilon$  and  $\Upsilon^2$  are uniformly continuous, for any t > 0 there is t' > 0 satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) < t'$  imply  $\xi(\Upsilon(h_1), \Upsilon(h_2)) < t$  and  $\xi(\Upsilon^2(h_1), \Upsilon^2(h_2)) < t$ . By hypothesis and the definition, for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ , one has that

$$G = \{m \in \{0, 1, ...\} : \xi(\Upsilon^{m}(d_{1}), \Upsilon^{m}(d_{2})) < t'\} \in \mathscr{G}_{1}.$$
(45)

As 
$$\mathscr{G}_1$$
 satisfies that for any  $G \in \mathscr{G}_1$ ,  
 $G_2 := \{2i + j \in \{0, 1, ...\} : j \in \{0, 1\}, i \in G\} \in \mathscr{G}$ 

by the definition there is  $j \in \{0, 1\}$  satisfying that

$$G_{2,j} \coloneqq \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathcal{G}_1.$$
(47)

(46)

1,

By the above argument, one has that

$$G_{2,j} \in \left\{ m \in \{0, 1, \ldots\} : \xi \left( \Upsilon^{2m+j+2-j} \left( d_1 \right), \Upsilon^{2m+j+2-j} \left( d_2 \right) \right) < t \right\}.$$
(48)

That is,

$$G_{2,j} \in \left\{ m \in \{0, 1, \ldots\} : \xi \Big( \Upsilon^{2m+2} (d_1), \Upsilon^{2m+2} (d_2) \Big) < t \right\}.$$
(49)

So,

$$\left\{m \in \{0, 1, \ldots\} : \xi(\Upsilon^{2m+2}(d_1), \Upsilon^{2m+2}(d_2)) < t\right\} \in \mathcal{G}_1.$$
(50)

As  $\mathcal{G}_1$  is translation-invariant,

$$\left\{m \in \{0, 1, \ldots\} : \xi(\Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2)) < t\right\} \in \mathscr{G}_1.$$
 (51)

By the hypothesis and the definition, there is  $\delta > 0$  such that for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ ,

$$G' = \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) > \delta\} \in \mathscr{G}_2.$$
(52)

As  $\mathscr{G}_2$  satisfies that for any  $G \in \mathscr{G}_2$ , there is  $j \in \{0, 1\}$  satisfying that  $G_{2,j} := \{i \in \{0, 1, \ldots\} : 2j + i \in G\} \in \mathscr{G}_2$ , by the definition there is  $j \in \{0, 1\}$  satisfying that

$$G'_{2,j} \coloneqq \{i \in \{0, 1, \ldots\} : 2j + i \in G'\} \in \mathscr{G}_2.$$
(53)

As  $\Upsilon$  and  $\Upsilon^2$  are uniformly continuous, for the above  $\delta > 0$  there is  $\delta' > 0$  satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) \leq \delta'$  imply  $\xi(\Upsilon(h_1), \Upsilon(h_2)) \leq \delta$  and  $\xi(\Upsilon^2(h_1), \Upsilon^2(h_2)) \leq \delta$ . Clearly,

$$G'_{2,j} \in \left\{ m \in \{0, 1, \ldots\} : \xi \left( \Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2) \right) > \delta' \right\},$$
 (54)

which means that

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^{2m}(d_1), \Upsilon^{2m}(d_2)\right) > \delta'\right\} \in \mathscr{G}_2.$$
 (55)

Thus, Claim 3 holds.

As  $\Upsilon^2 = (f_1 \circ f_2) \times (f_2 \circ f_1)$ , by hypothesis, Claim 3, the definition of strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaos, and Theorem 2 and its proof, one can easily verify that  $f_1 \circ f_2$  and  $f_2 \circ f_1$  are strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.

Assume that  $f_1 \circ f_2$  is strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaotic. By the definition, there is an uncountable subset  $C \subset H_1$  which is strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set of  $f_1 \circ f_2$ . By the proof of Theorem 2,  $f_2(C)$  is an uncountable and strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set of  $f_2 \circ f_1$ . Set  $A = \{(a, g_2(a)) : a \in C\}$ . Then, A is uncountable. By the above argument, the definition of strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -chaos and the proof of Theorem 2, it is easy to prove that A is a strong  $(\mathcal{G}_1, \mathcal{G}_2)$ -scrambled set of  $\Upsilon^2|_{\Lambda_2}$ .

Now, we assume that  $\Upsilon^2|_{\Lambda_2}$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.

Claim 4.  $\Upsilon$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic.

The Proof of Claim 4. By the hypothesis and the definitions,  $\Upsilon^2$  is strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic. Assume that *D* is a strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -scrambled set of the system  $(H_1 \times H_2, \Upsilon^2)$ . As  $\Upsilon^n$ is uniformly continuous for any  $n \in \{0, 1\}$ , for any t > 0 there is t' > 0 satisfying that  $h_1, h_2 \in H_1 \times H_2$  and  $\xi(h_1, h_2) < t'$ imply  $\xi(\Upsilon^n(h_1), \Upsilon^n(h_2)) < t$  for any  $n \in \{0, 1\}$ . By the hypothesis and the definition, for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ , one has that

$$G = \left\{ m \in \{0, 1, \ldots\} : \xi \big( \Upsilon^{2m} (d_1), \Upsilon^{2m} (d_2) \big) < t' \right\} \in \mathcal{G}_1.$$
(56)

So, for any  $m \in G$  and any  $n \in \{0, 1\}$ , we have that

$$\xi(\Upsilon^{2m+n}(d_1),\Upsilon^{2m+n}(d_2)) < t.$$
 (57)

As  $\mathscr{G}_1$  satisfies that for any  $G \in \mathscr{G}_1$ ,

$$G_2 \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathcal{G}_1,$$
(58)

by the definition we have

$$G_2 := \{2m + n : m \in G, n \in \{0, 1\}\} \in \mathcal{G}_1.$$
(59)

Clearly,

$$G_2 \subset \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) < t\}.$$
(60)

This means that

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^m\left(d_1\right), \Upsilon^m\left(d_2\right)\right) < t\right\} \in \mathscr{G}_1.$$
 (61)

By the hypothesis and the definition, there is  $\delta' > 0$ satisfying that for any  $d_1, d_2 \in D$  with  $d_1 \neq d_2$ ,

$$G' = \{m \in \{0, 1, \ldots\} : \xi(\Upsilon^m(d_1), \Upsilon^m(d_2)) > \delta'\} \in \mathscr{G}_2.$$
(62)

As  $\Upsilon^n$  is uniformly continuous for any  $n \in \{0, 1\}$ , for the above  $\delta' > 0$  there is  $\delta > 0$  satisfying that  $h_1, h_2 \in H_1 \times H_2$ and  $\xi(\Upsilon^2(h_1),\Upsilon^2(h_2)) > \delta$  imply  $\xi(\Upsilon^n(h_1),\Upsilon^n(h_2)) > \delta$  for any  $n \in \{0, 1\}$ . So, for any  $m \in G'$  and any  $n \in \{0, 1\}$ , we have that

$$\xi \Big( \Upsilon^{2(m-1)+n}(d_1), \Upsilon^{2(m-1)+n}(d_2) \Big) > \delta.$$
 (63)

As  $\mathscr{G}_2$  is translation-invariant,  $G' - 1 \in \mathscr{G}_2$ . As  $\mathscr{G}_2$  satisfies that for any  $G \in \mathcal{G}_2$ ,

$$G_{2} \coloneqq \{2i+j \in \{0,1,\ldots\} : j \in \{0,1\}, i \in G\} \in \mathscr{G}_{2}, (G'-1)_{2} \coloneqq \{2(m-1)+n : m-1 \in G'-1, n \in \{0,1\}\} \in \mathscr{G}_{2}.$$
(64)

Clearly,

$$(G'-1)_2 \subset \{m \in \{0,1,\ldots\} : \xi(\Upsilon^m(d_1),\Upsilon^m(d_2)) > \delta\},$$
  
(65)

which means that

$$\left\{m \in \{0, 1, \ldots\} : \xi\left(\Upsilon^{m}\left(d_{1}\right), \Upsilon^{m}\left(d_{2}\right)\right) > \delta\right\} \in \mathscr{G}_{2}.$$
 (66)

Thus, Claim 4 holds. Consequently, Theorem 4 is true.

**Corollary 2.** Let  $\Upsilon(a,b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . Then, for any  $a, b \in [0, 1]$ , Y is  $(\overline{M}(a), \overline{M}(b))$ -chaotic (resp. strong  $(\overline{M}(a),$ M(b))-chaotic) if and only if so is  $\Upsilon^2|_{\Lambda_2}$ .

*Proof.* We have the following two claims.

Claim 5. For any  $t \in [0,1]$ ,  $\overline{M}(t)$  satisfies that for any  $G \in \mathcal{G}$ , there is  $j \in \{0, 1\}$  such that  $G_{2,j} := \{i \in \{0, 1, \dots, j\}$  $\ldots$ } : 2 $j + i \in G$ }  $\in \mathscr{G}$ .

The Proof of Claim 5. It is clear that  $\overline{M}(0) = \mathscr{B}$  and that if  $G \in \mathcal{B}$ , then there is  $j \in \{0, 1\}$  satisfying that  $G_{2,j} \in \mathcal{B}$ . Assume that there is  $t \in (0, 1]$  such that  $\overline{M}(t)$  does not have the property P(2). By this assumption and the definition, there is  $G \in M(t)$  such that for any  $j \in \{0, 1\}, \overline{\mu}(G_{2,j})$  $= e_j < t$ . Choose  $\delta_j \in (0, t - e_j)$  for any  $j \in \{0, 1\}$ . As  $\overline{\mu}(G_{2,j}) = e_j < t$  for any  $j \in \{0, 1\}$ , by the definition there is an integer M > 0 such that for any  $j \in \{0, 1\}$  and any integer  $m \ge M, \ \#(G_{2,i} \cap \{1, 2, \dots, m\}) < (t - \delta_i)m$ . This implies that

$$\#\left(\left(\{1,2,\ldots\}\backslash G_{2,j}\right)\cap\{1,2,\ldots,m\}\right)>m-\left(t-\delta_{j}\right)m.$$
 (67)

Let  $n \in \{2M + 1, 2M + 1, \dots\}$  and write  $n = 2[n/2] + l_n$ , where [n/2] is the integral part of n/2 and  $l_n \in \{0, 1\}$ . By the definition,  $i \notin G_{2,j}$  implies  $2i + j \notin G$  for any  $j \in \{0, 1\}$ . Obviously, if  $j_1, j_2 \in \{0, 1\}$  and  $j_1 \neq j_2$  then  $2i_1 + j_1 \neq 2i_2 + j_2$ for any  $i_1, i_2 \in \{0, 1, ..., m\}$ . So, F 14 7

$$\sharp \left( \left( \{1, 2, \ldots\} \setminus G \right) \cap \{1, 2, \ldots, n\} \right) \ge \sharp \left( A_0 \right) + \sharp \left( A_1 \right) > \left\lfloor \frac{n}{2} \right\rfloor \\
- \left( t - e_0 \right) \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left( t - e_1 \right) \left\lfloor \frac{n}{2} \right\rfloor,$$
(68)

where

$$A_{0} = \left(\{1, 2, \ldots\} \setminus G_{2,0}\right) \cap \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\},$$

$$A_{1} = \left(\{1, 2, \ldots\} \setminus G_{2,1}\right) \cap \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}.$$
(69)

This implies that

$$\# (G \cap \{1, 2, \dots, n\}) < n - \left( \left[ \frac{n}{2} \right] - (t - e_0) \left[ \frac{n}{2} \right] + \left[ \frac{n}{2} \right] - (t - e_1) \left[ \frac{n}{2} \right] \right).$$
(70)

Set 
$$\delta = \min\{\delta_0, \delta_1\}$$
. As  

$$\overline{\mu}(G) = \limsup_{n \to \infty} \frac{1}{n} \# (G \cap \{1, 2, \dots, n\}) \leq \limsup_{n \to \infty} \frac{1}{n}$$

$$\cdot \left(n - \left(\left[\frac{n}{2}\right] - (t - e_0)\left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] - (t - e_1)\left[\frac{n}{2}\right]\right)\right),$$

$$\overline{\mu}(G) \leq \frac{1}{n} \left(n - 2\left(\left[\frac{n}{2}\right] - (t - \delta)\left[\frac{n}{2}\right]\right)\right).$$
(71)

By 
$$n = 2[n/2] + l_n$$
,  
 $\overline{\mu}(G) \le t - \delta.$  (72)

This is a contraction. Consequently, Claim 5 holds.

Claim 6. For any  $t \in [0,1]$ ,  $\overline{M}(t)$  satisfies that for any  $G \in \mathcal{G}$ :

$$G_2 \coloneqq \{2i + j \in \{0, 1, \ldots\} : j \in \{0, 1\}, i \in G\} \in \mathscr{G}.$$
(73)

The Proof of Claim 6. It is obvious that  $\overline{M}(0) = \mathscr{B}$ , and that if  $G \in \mathscr{B}$ , then  $G_2 \in \mathscr{B}$ . Assume that there is  $t \in (0, 1]$  such that  $\overline{M}(t)$  does not have the property Q(2). By this assumption and the definition, there is  $G \in \overline{M}(t)$  such that  $\overline{\mu}(G_2) = e < t$ . Choose  $\delta \in (0, t - e)$ . As  $\overline{\mu}(G_2) = e < t$ , by the definition there is an integer M > 0 such that for any integer  $m \ge M$ ,  $\sharp (G_2 \cap \{1, 2, \dots, m\}) < (t - \delta)m$ . Let  $n \in \{2M + 1, 2$  $M + 1, \dots\}$  and write  $n = 2[n/2] + l_n$ , where [n/2] is the integral part of n/2 and  $l_n \in \{0, 1\}$ . By the definition,  $i \in G$ implies  $2i + j \in G_2$  for any  $j \in \{0, 1\}$ . Obviously, if  $i_1, i_2 \in G$ and  $j_1 \neq j_2$ , then  $2i_1 + j_1 \neq 2i_2 + j_2$  for any  $j_1, j_2 \in \{0, 1\}$ . So,  $2\sharp \left(G \cap \{1, 2, \dots, [\frac{n}{2}]\}\right) \le \sharp \left(G_2 \cap \{1, 2, \dots, n\}\right) < n(t - \delta)$ .

$$2\#\left(G \cap \left\{1, 2, \dots, \left\lfloor\frac{n}{2}\right\rfloor\right\}\right) \leq \#\left(G_2 \cap \{1, 2, \dots, n\}\right) < n(t - \delta).$$
(74)

This implies that

$$\#\left(G \cap \left\{1, 2, \dots, \left[\frac{n}{2}\right]\right\}\right) < \frac{1}{2}\left(2\left[\frac{n}{2}\right] + l_n\right)(t - \delta).$$
(75)

Consequently,

$$\overline{\mu}(G) \le \lim_{n \to \infty} \frac{1}{2[n/2]} \left( 2 \left[ \frac{n}{2} \right] + l_n \right) (t - \delta) = t - \delta < t.$$
(76)

This is a contraction. Consequently, Claim 5 is true.

From the above two claims we know that  $\overline{M}(t)$  satisfies the conditions of Theorems 3 and 4 for any  $t \in [0, 1]$ . Thus, by these two theorems one can see that Corollary 2 holds.

**Theorem 5.** Let  $\Upsilon(a, b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . If  $\mathscr{G}_1$  and  $\mathscr{G}_2$  are two Furstenberg families such that  $\mathscr{G}_2$  is translation-invariant, then  $\Upsilon$ is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if so is  $\Upsilon^2|_{\Lambda_1}$ .

*Proof.* The proof is similar to those of Theorems 3 and 4 and is omitted.  $\Box$ 

**Corollary 3.** Let  $\Upsilon(a,b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . Then, for any  $a, b \in [0,1]$ ,  $\Upsilon$  is  $(\overline{M}(a), \overline{M}(b))$ -chaotic (resp. strong  $(\overline{M}(a), \overline{M}(b))$ -chaotic) if and only if so is  $\Upsilon^2|_{\Lambda_1}$ .

*Proof.* By Theorem 5 and the proof of Corollary 2 one can easily see that Corollary 3 holds.  $\Box$ 

**Theorem 6.** Let  $\Upsilon(a, b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . If  $\mathscr{C}_1$  and  $\mathscr{C}_2$  are two Furstenberg families such that  $\mathscr{C}_2$  is translation-invariant, then  $\Upsilon$ is  $(\mathscr{C}_1, \mathscr{C}_2)$ -chaotic (resp. strong  $(\mathscr{C}_1, \mathscr{C}_2)$ -chaotic) if and only if so is  $\Upsilon^2|_{\Lambda, \cup \Lambda_2}$ .

*Proof.* By hypothesis, the definitions of  $\Lambda_1$ ,  $\Lambda_2$ , and  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaos (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaos) and Theorems 3 and 4, it is easily seen that  $\Upsilon$  is  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic (resp. strong  $(\mathscr{G}_1, \mathscr{G}_2)$ -chaotic) if and only if so is  $\Upsilon^2|_{\Lambda_1 \cup \Lambda_2}$ .  $\Box$ 

**Corollary 4.** Let  $\Upsilon(a,b) = (f_1(b), f_2(a))$  be a Cournot map on the product space  $H_1 \times H_2$ . Then, for any  $a, b \in [0,1]$ ,  $\Upsilon$  is  $(\overline{M}(a), \overline{M}(b))$ -chaotic (resp. strong  $(\overline{M}(a), \overline{M}(b))$ -chaotic) if and only if so is  $\Upsilon^2|_{A, \cup A}$ .

*Proof.* By Theorem 6 and the proof of Corollary 2 one can easily see that Corollary 4 holds.  $\Box$ 

#### **Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

#### **Conflicts of Interest**

The authors declare that they have no conflicts of interest.

### **Authors' Contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

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