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Research Article

Out Lag Synchronization of Fractional Order Delayed Complex Networks with Coupling Delay via Pinning Control

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This paper discusses the out lag synchronization of fractional order complex networks (FOCN) including both internal delay and coupling delay and with the employment of pinning control scheme. Using comparison theorem and constructing the auxiliary function, several synchronization criterions by linear feedback pinning control are presented. The model and the obtained results in this work are more general than the previous works. Correctness and effectiveness of the theoretical results are validated through numerical simulations.

1. Introduction

Complex networks (CN) have attracted great attentions due to their tremendous potentials in deferent fields, such as social organizations, World Wide Web, power grids, communication networks, and biological networks [1–4]. Among all the dynamical behaviours of CN, synchronization is an important and interesting behavior, which can explain many problems in the real world. Several control methods were employed to help synchronizing the CN, including intermittent control [5], impulsive control [6], state feedback control [7], adaptive control [8], sampled-data control [9], and pinning control [10, 11]. However, the existing results are mainly achieved on the case of "inner synchronization" where all nodes can gain a coherent dynamical behavior within a network. Unlike inner synchronization, synchronization can be performed between two or more CN, and this is referred to as "out synchronization." Some excellent conclusions about out synchronization in complex dynamical networks have been presented in the existing literatures [12–14]. In [12], authors discussed out synchronization between two uncertain general coupled CN. Ref. [13] studied out synchronization between master-slave dynamical networks by adaptive impulsive pinning control. Through the implementation of aperiodically adaptive intermittent pinning control, out synchronization in two hybrid-coupled delayed CN was investigated in [14].

However, integer order networks were considered in all previous works. Fractional calculus, which was first given by Leibniz in the 17th century, can act with differential and integrals of any arbitrary order. Compared with classical ordinary differential networks, fractional calculus provides an excellent instrument to depict memory and hereditary properties in processing. Taking these factors into account, many scientists have introduced fractional differential and integrals to complex dynamical networks, forming fractional-order complex networks (FOCN). Recently, the stability and synchronization analysis of FOCN have attracted considerable interest. Many interesting results were derived for FOCN such as hybrid synchronization of coupled FOCN that was studied in [15]. Ref. [16] investigated the pinning synchronization in delayed FOCN with nondelayed and delayed couplings. In [17], the outer synchronization between FOCN was considered using a nonfragile observer-based control scheme. Using centralized and decentralized data-sampling principles and the theory of fractional differential systems, out synchronization in FOCN was considered in [18].

Time delay is inevitable in real networks due to finite information transmission and processing speeds among the units. Therefore, both internal delay and coupling delay should be considered in dynamical networks. Synchronization of integer order CN with both internal delay and coupling delay was studied in [19, 20]. In general, many types of synchronization were considered in the literature, including lag synchronization [21], complete synchronization [22], projective synchronization [23], and robust synchronization [24]. Among all these types of synchronizations, lag synchronization, which requires the response system to synchronize the drive system with a propagation delay θ, has been extended to various fields, such as electronic circuits, lasers, and neural systems [25]. Lay synchronization has become an important concept in both theory and application. However, up to now, the out lag synchronization in FOCN was totally ignored in the literature.

Inspired by the above discussions, the goal of this work is to consider out lag synchronization of FOCN with both node delay and coupling delay through the use of pinning control. The main contributions of this paper are listed as follows. (1) In order to form a general model, node delay and coupling delay are incorporated into the fractional order model. (2)The theory of fractional order differential equations and some new analysis techniques are employed to obtain novel sufficient conditions. (3)Out lag synchronization scheme is adopted to perform synchronization between two complex networks.

2. Preliminaries

In this paper, the definition of Caputo fractional derivative is used.

Definition 1 (see [26]). The Caputo fractional derivative of order q for a function $\rho(t)$ is given as

$$D^q \rho(t) = \frac{1}{\Gamma(k-q)} \int_{t_0}^{t} (t-s)^{k-q-1} \rho^{(k)}(s)\, ds, \quad (1)$$

where $t \geq t_0$, $k \in Z$, $q \in (k-1, k)$, and $\Gamma(\cdot)$ is the Gamma function.

Consider the following FOCN consisting of N coupled identical nodes with linearly diffusive couplings, in which each node having an n−dimensional fractional order delayed dynamical system acts as a drive system; the whole network is described as follows:

$$\begin{aligned} D^q x_i(t) &= f(x_i(t), x_i(t-\tau_1)) + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 x_j(t) \\ &\quad + c_2 \sum_{j=1}^{N} b_{ij} \Upsilon_2 x_j(t-\tau_2), \quad i = 1, 2, \ldots, N, \end{aligned} \quad (2)$$

where $0 < q < 1$, $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^T \in R^n$ is the state variable of the ith node, c_1, c_2 are the coupling strength, τ_1 is the delay at each node, τ_2 is the coupling delay between nodes, Υ_1, Υ_2 represent the inner matrix connecting the coupled variables, and $A = (a_{ij})_{N\times N}$ and $B = (b_{ij})_{N\times N}$ are the topological structure of the drive networks at time t and $t - \tau_2$, respectively. The parameters a_{ij}, b_{ij} are given by the following: if there is a connection from node j to node i $(j \neq i)$, then $a_{ij} > 0, b_{ij} > 0$; otherwise $a_{ij} = 0, b_{ij} = 0$. The diagonal elements of A, B are defined as $a_{ii} + \sum_{j=1, j\neq i}^{N} a_{ij} = 0$, $b_{ii} + \sum_{j=1, j\neq i}^{N} b_{ij} = 0$, respectively. $f : R^n \times R^n \longrightarrow R^n$ is a smooth nonlinear vector function. The initial values of system (2) are defined as $x_i(s) = \psi_i(s) \in C([-\tau, 0], R^n)$.

Remark 2. In model (2), the symmetric and irreducible condition of coupling matrix is not necessary. Furthermore, the node delay τ_1 is different from the coupling delay τ_2.

Remark 3. Synchronizations of FOCN were discussed in previous works [15, 17, 18], but without considering the coupling delay in models. Model (2) is more practical in this paper.

Corresponding, the response system is given:

$$\begin{aligned} D^q y_i(t) &= f(y_i(t), y_i(t-\tau_1)) + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 y_j(t) \\ &\quad + c_2 \sum_{j=1}^{N} b_{ij} \Upsilon_2 y_j(t-\tau_2) + u_i(t), \\ &\qquad i = 1, 2, \ldots, N, \end{aligned} \quad (3)$$

where $y_i(t) = (y_{i1}(t), \ldots, y_{in}(t))^T \in R^n$ denotes the response state variable of the ith node and $u_i(t) (i = 1, 2, \ldots, N)$ are the controllers.

To proceed further, we give assumption and lemmas in the following.

Assumption 4. The function f satisfies the uniform semi-Lipschitz condition with Lipschitz constants; i.e., there exist positive constants L_1, L_2 such that

$$\begin{aligned} &(u-v)^T (f(u, \overline{u}) - f(v, \overline{v})) \\ &\quad \leq L_1 (u-v)^T (u-v) + L_2 (\overline{u} - \overline{v})^T (\overline{u} - \overline{v}), \end{aligned} \quad (4)$$

for all $u, v, \overline{u}, \overline{v} \in R^n$.

It has to be noted that Assumption 4 is very necessary and important. Many typical systems, such as the Lorenz system, L*ü* system, the unified chaotic system, and delayed Hopfield neural network, satisfy Assumption 4. The results in this paper are achieved, while satisfying Assumption 4.

Lemma 5 (see [27]). *Assume $x(t) \in R^n$ is a continuous and derivable function. Then, we can get*

$$\frac{1}{2} D^q x^2(t) \leq x(t) D^{\alpha} x(t), \quad 0 < \alpha < 1, \ t \geq 0. \quad (5)$$

Lemma 6 (see [28]). *Let $x, y \in R^n$. Then*

$$2x^T y \leq \varepsilon x^T x + \frac{1}{\varepsilon} y^T y, \quad (6)$$

for any $\varepsilon > 0$.

Lemma 7 (see [29]). *Consider the following fractional differential inequality with multiple delays*

$$D^q V(\omega(t)) \leq -\alpha V(\omega(t)) + \sum_{j=1}^{n} \beta_j V(\omega(t-\tau_j)), \quad 0 < \alpha < 1, \tag{7}$$
$$V(\omega(t)) = m(t) \geq 0, \quad t \in [-\tau, 0], \ 1 \leq j \leq n,$$

and the linear fractional differential equation with multiple delays

$$D^q V(\eta(t)) = -\alpha V(\eta(t)) + \sum_{j=1}^{n} \beta_j V(\eta(t-\tau_j)), \quad 0 < \alpha < 1, \tag{8}$$
$$V(\eta(t)) = m(t) \geq 0, \quad t \in [-\tau, 0], \ 1 \leq j \leq n,$$

where $\omega(t), \eta(t) \in R^n$ are continuous in $t \in (0, \infty)$, and $m(t)$ is continuous in $[-\tau, 0]$, $\tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$. If $\alpha > 0$, $\beta_j > 0$, and $\tau_j > 0$ for all $1 \leq j \leq n$, then $V(\omega(t)) \leq V(\eta(t))$, $\forall t \in [0, \infty)$.

3. Out Lag Synchronization

This section investigates the out lag synchronization by employing the linear feedback pinning control. Without loss of generality, let the first l nodes be controlled. The controllers $u_i(t)$ are chosen as follows:

$$\begin{aligned} u_i(t) &= -k_i(y_i(t) - x_i(t-\sigma)), \quad i = 1, 2, \ldots, l, \\ u_i(t) &= 0, \quad i = l+1, \ l+2, \ldots, N, \end{aligned} \tag{9}$$

where the control gains $k_i > 0$ $(i = 1, 2, \ldots, N)$.

The error vector is defined as

$$e_i(t) = y_i(t) - x_i(t-\sigma), \quad i = 1, 2, \ldots, N \tag{10}$$

where σ is the propagation delay.

Therefore, one gets the error dynamical system:

$$\begin{aligned} D^q e_i(t) &= f(y_i(t), y_i(t-\tau_1)) \\ &\quad - f(x_i(t-\sigma), x_i(t-\tau_1-\sigma)) \\ &\quad + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 e_j(t) + c_2 \sum_{j=1}^{n} b_{ij} \Upsilon_2 e_j(t-\tau_2) \\ &\quad - k_i e_i(t), \quad 1 \leq i \leq l, \\ D^q e_i(t) &= f(y_i(t), y_i(t-\tau_1)) \\ &\quad - f(x_i(t-\sigma), x_i(t-\tau_1-\sigma)) \\ &\quad + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 e_j(t) + c_2 \sum_{j=1}^{n} b_{ij} \Upsilon_2 e_j(t-\tau_2), \\ &\qquad l+1 \leq i \leq N. \end{aligned} \tag{11}$$

Let $\lambda_{min}, \lambda_{max}$ be the minimum eigenvalue and the maximum eigenvalue of the matrix, respectively. $\otimes$ denotes the Kronecker product. I_N is the N dimensional identity matrix. Note that $\mu_0 = \|\Upsilon_1\|, \mu = \lambda_{min}((\Upsilon_1 + \Upsilon_1^T)/2)$, $\overline{A} = \operatorname{diag}\{a_{11}, a_{22}, \ldots, a_{NN}\}$. $P = (2L_1 I_N - 2K) + 2c_1((\mu - \mu_0)\overline{A} + \mu_0((A + A^T)/2)) + \varepsilon c_2 \lambda_{max}(BB^T)\lambda_{max}(\Upsilon_2 \Upsilon_2^T) I_N$, $K = \operatorname{diag}\{\underbrace{k_1, k_2, \ldots, k_l}_{l}, \underbrace{0, \ldots, 0}_{N-l}\}$.

Theorem 8. *Suppose that Assumption 4 and condition (9) hold; if there exists $k_i > 0$ such that*

$$\lambda > 2L_2 + \frac{c_2}{\varepsilon}, \tag{12}$$

then system (2) can synchronize system (3), where $\varepsilon > 0, -\lambda = \lambda_{max}(P)$, $i = 1, 2, \ldots, N$.

Proof. Constructing the auxiliary function:

$$V(e(t)) = \frac{1}{2} e^T(t) e(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t), \tag{13}$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_N(t))^T$.

From Lemma 5 and Assumption 4, we can obtain

$$\begin{aligned} D^q V(e(t)) &= \frac{1}{2} D^q \sum_{i=1}^{N} e_i^T(t) e_i(t) \leq \sum_{i=1}^{N} e_i^T(t) D^\alpha e_i(t) \\ &= \sum_{i=1}^{N} e_i^T(t) \Bigg[f(y_i(t), y_i(t-\tau_1)) \\ &\quad - f(x_i(t-\sigma), x_i(t-\tau_1-\sigma)) + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 e_j(t) \\ &\quad + c_2 \sum_{j=1}^{N} b_{ij} \Upsilon_2 e_j(t-\tau_2) \Bigg] - \sum_{i=1}^{l} e_i^T(t) k_i e_i(t) \\ &\leq \sum_{i=1}^{N} \left[L_1 e_i^T(t) e_i(t) + L_2 e_i^T(t-\tau_1) e_i(t-\tau_1) \right] \\ &\quad + c_1 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \Upsilon_1 e_i^T(t) e_j(t) \\ &\quad + c_2 \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \Upsilon_2 e_i^T(t) e_j(t-\tau_2) - \sum_{i=1}^{l} e_i^T k_i e_i(t). \end{aligned} \tag{14}$$

Then we can obtain

$$\begin{aligned} & c_1 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \Upsilon_1 e_i^T(t) e_j(t) \\ &\quad = c_1 \sum_{i=1}^{N} a_{ii} \Upsilon_1 e_i^T(t) e_i(t) + c_1 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} a_{ij} \Upsilon_1 e_i^T(t) e_j(t) \end{aligned}$$

$$\leq c_1 \sum_{i=1}^{N} \mu a_{ii} e_i^T(t) e_i(t) + c_1 \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \mu_0 a_{ij} \left\| e_i^T(t) \right\| \left\| e_j(t) \right\| = c_1 \sum_{i=1}^{N} e_i^T \left[(\mu - \mu_0) \overline{A} + \mu_0 \frac{A + A^T}{2} \right] e_i(t). \tag{15}$$

Based on Lemma 6, we have the estimations:

$$\begin{aligned} c_2 \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \Upsilon_2 e_i^T(t) e_j(t-\tau_2) &= c_2 e^T(t) (B \otimes \Upsilon_2) e(t-\tau_2) \\ &\leq \frac{\varepsilon}{2} c_2 e^T(t) (B \otimes \Upsilon_2)(B \otimes \Upsilon_2)^T e(t) + \frac{c_2}{2\varepsilon} e^T(t-\tau_2)(I_N \otimes I_n) e(t-\tau_2) \\ &= \frac{\varepsilon}{2} c_2 e^T(t) \left(BB^T\right) \otimes \left(\Upsilon_2 \Upsilon_2^T\right) e(t) + \frac{c_2}{2\varepsilon} e^T(t-\tau_2)(I_N \otimes I_n) e(t-\tau_2) \\ &\leq \frac{\varepsilon}{2} c_2 e^T(t) \lambda_{max}\left(BB^T\right) \lambda_{max}\left(\Upsilon_2 \Upsilon_2^T\right) e(t) + \frac{c_2}{2\varepsilon} e^T(t-\tau_2) e(t-\tau_2) \\ &= \frac{\varepsilon c_2}{2} \lambda_{max}\left(BB^T\right) \lambda_{max}\left(\Upsilon_2 \Upsilon_2^T\right) \sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{c_2}{2\varepsilon} \sum_{i=1}^{N} e_i^T(t-\tau_2) e_i(t-\tau_2). \end{aligned} \tag{16}$$

Substituting (15), (16) into (14) gives

$$\begin{aligned} D^q V(e(t)) &\leq \sum_{i=1}^{N} L_1 e_i^T(t) e_i(t) + \sum_{i=1}^{N} L_2 e_i^T(t-\tau_1) e_i(t-\tau_1) + c_1 \sum_{i=1}^{N} e_i^T \left[(\mu - \mu_0) \overline{A} + \mu_0 \frac{A + A^T}{2} \right] e_i(t) \\ &+ \frac{\varepsilon c_2}{2} \lambda_{max}\left(BB^T\right) \lambda_{max}\left(\Upsilon_2 \Upsilon_2^T\right) \sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{c_2}{2\varepsilon} \cdot \sum_{i=1}^{N} e_i^T(t-\tau_2) e_i(t-\tau_2) - \sum_{i=1}^{l} e_i^T k_i e_i(t) = e^T(t) \\ &\cdot \left[\left(L_1 + \frac{\varepsilon c_2}{2} \lambda_{max}\left(BB^T\right) \lambda_{max}\left(\Upsilon_2 \Upsilon_2^T\right) \right) I_N + c_1 \left((\mu - \mu_0) \overline{A} + \mu_0 \frac{A + A^T}{2} \right) - K \right] e(t) + e^T(t \\ &- \tau_1)(L_2 I_N) e(t-\tau_1) + \frac{c_2}{2\varepsilon} e^T(t-\tau_2) I_N e(t-\tau_2) \\ &\leq \frac{1}{2} \cdot \lambda_{max} \left[\left(2L_1 + \varepsilon c_2 \lambda_{max}\left(BB^T\right) \lambda_{max}\left(\Upsilon_2 \Upsilon_2^T\right) \right) I_N - 2K + 2c_1 \left((\mu - \mu_0) \overline{A} + \mu_0 \frac{A + A^T}{2} \right) \right] e^T(t) e(t) \\ &+ 2L_2 \frac{1}{2} e^T(t-\tau_1) e(t-\tau_1) + \frac{c_2}{\varepsilon} \frac{1}{2} e^T(t-\tau_2) I_N e(t-\tau_2) = -\lambda V(e(t)) + 2L_2 V(e(t-\tau_1)) + \frac{c_2}{\varepsilon} \cdot V(e(t-\tau_2)). \end{aligned} \tag{17}$$

Next, we consider the linear fractional order systems:

$$\begin{aligned} D^q V(w(t)) &= -\lambda V(w(t)) + 2L_2 V(w(t-\tau_1)) + \frac{c_2}{\varepsilon} V(w(t-\tau_2)), \\ V(w(t)) &= \varrho(t) \geq 0, \quad t \in [-\tau, 0], \end{aligned} \tag{18}$$

and giving the initial values $\varrho(t)$ is the same as in system (17).

According to Corollary 3 in Ref. [30], when the characteristic equation of (18)

$$s^q + \lambda - 2L_2 e^{-s\tau_1} - \frac{c_2}{\varepsilon} e^{-s\tau_2} = 0. \tag{19}$$

has no purely imaginary roots and $\lambda > 2L_2 + c_2/\varepsilon$, then we can get that the zero solution of (18) is asymptotically stable.

In case of having a pure imaginary root $s = \omega i = |\omega|(\cos(\pi/2) + i\sin(\pm\pi/2))$ of (19), in which ω is a real number. When $\omega > 0$, $s = |\omega|(\cos(\pi/2) + i\sin(\pi/2))$, while $\omega < 0$, $s = |\omega|(\cos(\pi/2) - i\sin(\pi/2))$. We can lead to

$$(\omega i)^q + \lambda - 2L_2 e^{-\tau_1 \omega i} - \frac{c_2}{\varepsilon} e^{\tau_2 \omega i} = 0. \tag{20}$$

Equivalently,

$$\left| (\omega i)^q + \lambda \right|^2 = \left(2L_2 \cos\omega\tau_1 + \frac{c_2}{\varepsilon} \cos\omega\tau_2 \right)^2 + \left(2L_2 \sin\omega\tau_1 + \frac{c_2}{\varepsilon} \sin\omega\tau_2 \right)^2, \tag{21}$$

that is,

$$\begin{aligned} |\omega|^{2q} + 2L_1 \cos\left(\frac{q\pi}{2}\right) |\omega|^q + \lambda^2 &= 4L_2^2 + \left(\frac{c_2}{\varepsilon}\right)^2 + \frac{4L_2 c_2}{\varepsilon} \cos\omega(\tau_1 - \tau_2). \\ &\leq \left(2L_2 + \frac{c_2}{\varepsilon} \right)^2. \end{aligned} \tag{22}$$

Since $|\omega|^q > 0, \cos(q\pi/2) > 0$ and $\lambda > 0$, if $\lambda > 2L_2 + c_2/\varepsilon > 0$, (21) has no real roots. According to Lemma 7, one can obtain $V(e(t)) \le V(w(t))$, indicating that the zero solution of (17) is asymptotically stable. Therefore, the system (2) can synchronize the system (3). □

Remark 9. Refs. [31, 32] discussed the out synchronization in CN with both internal delay and coupling delay, while considering integer order. In comparison with integer order networks, our proposed system can be applied to some CN in the real world. Moreover, the lag synchronization is considered, which is more general in Refs. [31, 32].

Remark 10. Unlike LMI method [33] for discussing the synchronization of coupled-delay FOCN, we here apply the comparison theorem via pinning control to establish the synchronization criteria for the FOCN with delayed couplings.

If the coupling matrix $A = 0$, then model (2) can be rewritten as follows:

$$D^q x_i(t) = f(x_i(t), x_i(t-\tau_1)) + c_2 \sum_{j=1}^{N} b_{ij} \Upsilon_2 x_j(t-\tau_2), \quad i = 1, 2, \ldots, N. \tag{23}$$

For this case, we can get the result below.

Corollary 11. *Let Assumption 4 and condition (9) hold; if there exists $k_i > 0$ such that*

$$\lambda > 2L_2 + \frac{c_2}{\varepsilon}, \tag{24}$$

then system (2) synchronizes system (3), where $\varepsilon > 0, -\lambda = \lambda_{max}(P), P = (2L_1 I_N - 2K) + \varepsilon c_2 \lambda_{max}(BB^T)\lambda_{max}(\Upsilon_2 \Upsilon_2^T) I_N$.

If the coupling matrix $B = 0$, then the fractional order complex networks only contain inner delay; i.e., coupling delay is not considered here, and accordingly model (2) will change to

$$D^q x_i(t) = f(x_i(t), x_i(t-\tau_1)) + c_1 \sum_{j=1}^{N} a_{ij} \Upsilon_1 x_j(t), \quad i = 1, 2, \ldots, N. \tag{25}$$

For this case, we can get the result below.

Corollary 12. *Let Assumption 4 and condition (9) hold; if there exists $k_i > 0$ such that*

$$\lambda > 2L_2, \tag{26}$$

then system (2) synchronizes system (3), where $-\lambda = \lambda_{max}(P), P = (2L_1 I_N - 2K) + 2c_1((\mu - \mu_0)\overline{A} + \mu_0((A + A^T)/2))$.

Remark 13. In complex dynamical systems, the challenging problems lie on the types of pinning nodes and their possible minimum number. In [34], it is reported that nodes whose out-degrees ($\sum_{j=1, j\neq i}^{N} a_{ji}$) are bigger than their in-degrees ($\sum_{j=1, j\neq i}^{N} a_{ij}$) should be pinned candidates. Here, according to the difference between out-degrees and in-degrees, we could rearrange nodes in the light of descending sort and select the first l nodes as pinned candidates which satisfy Theorem 8.

4. Numerical Example

The following 2-D CN model is considered as a drive system:

$$D^q x_i(t) = A_1 x_i(t) + B_1 g(x_i(t)) + C_1 g(x_i(t-\tau_1)) + c_1 \sum_{j=1}^{2} a_{ij} \Upsilon_1 x_j(t) + c_2 \sum_{j=1}^{2} b_{ij} \Upsilon_2 x_j(t-\tau_2), \tag{27}$$

where $q = 0.97$, $\tau_1 = 1, x_i(t) = (x_{i1}(t), x_{i2}(t))^T$ $(i = 1, 2, \ldots, 10)$, $g(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T$. $A_1 = \begin{pmatrix} -1 & -0 \\ 0 & -1 \end{pmatrix}$, $B_1 = \begin{pmatrix} -2 & -0.1 \\ -5 & 2 \end{pmatrix}$. $C_1 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix}$. The coupling strengths are $c_1 = 10, c_2 = 1$, the inner connecting matrix is $\Upsilon_1 = \Upsilon_2 = I_2$, and coupling time delay is $\tau_2 = 1.5$. The weight configuration coupling matrices are taken as

$$A = (a_{ij})_{10\times10} = B = (b_{ij})_{10\times10} = \begin{pmatrix} -5 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & -5 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -5 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & -5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & -5 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -5 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & -5 \end{pmatrix}. \tag{28}$$

The response system is given by

$$D^q y_i(t) = A_1 y_i(t) + B_1 g(y_i(t)) + C_1 g(y_i(t-\tau_1)) + c_1 \sum_{j=1}^{2} a_{ij} \Upsilon_1 y_j(t) + c_2 \sum_{j=1}^{2} b_{ij} \Upsilon_2 y_j(t-\tau_2) + u_i(t), \tag{29}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t))^T$, $u_i(t)$ are the control functions.

By simple computing, one gets $L_1 = 8.5402, L_2 = 1.2613, \mu = 1, \mu_0 = 1, \lambda_{max}(BB^T) = 47.1782, \lambda_{max}(\Upsilon_2 \Upsilon_2^T) = 1$. In the simulation, we choose $\sigma = 0.5, \varepsilon = 1, K_i = 40$ $(i = 1, 2, 3)$, and the conditions of Theorem 8 are satisfied. Initial conditions are chosen as $x_i(0) = (1 + 0.1i, 2 + 0.1i)^T (i = 1, 2, \ldots, 10)$. Based on the pinned node selection strategy in Remark 13 and Theorem 8, the nodes 1, 2, 3 are chosen as pinned candidates. The synchronization errors e_{i1} and e_{i2} $(i = 1, 2, \ldots, 10)$ are shown in Figures 1 and 2.

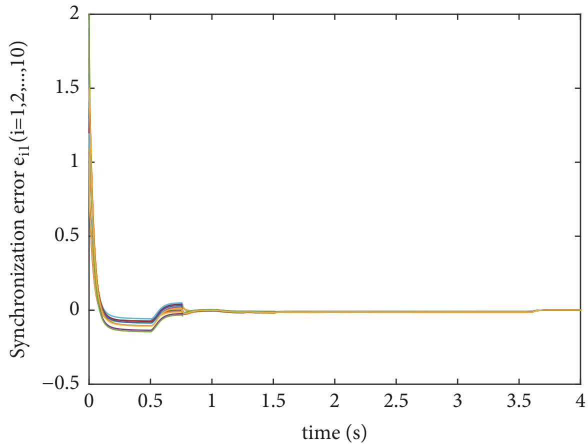

FIGURE 1: The synchronization error e_{i1} $(i = 1, 2, \ldots, 10)$ state.

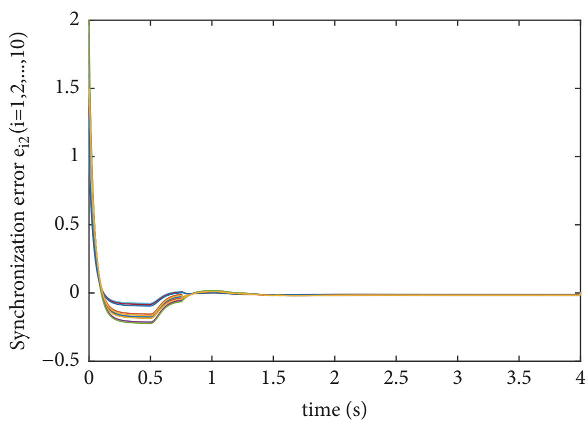

FIGURE 2: The synchronization error e_{i2} $(i = 1, 2, \ldots, 10)$ state.

5. Conclusions

Recently, synchronization of CN has been studied extensively. However, for fractional order dynamical networks, it is more appropriate to choose fractional order stability theory to realise the synchronization. In this paper, the stability theory of fractional order systems and comparison theorem are implemented. Synchronization criterions are achieved by pinning control strategy. A numerical example is elaborated to verify the feasibility of the obtained results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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