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Research Article

Perturbation of a Period Annulus with a Unique Two-Saddle Cycle in Higher Order Hamiltonian

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In this paper, we study the number of limit cycles emerging from the period annulus by perturbing the Hamiltonian system $\dot{x} = y, \dot{y} = x(x^2-1)(x^2+1)(x^2+2)$. The period annulus has a heteroclinic cycle connecting two hyperbolic saddles as the outer boundary. It is proved that there exist at most 4 and at least 3 limit cycles emerging from the period annulus, and 3 limit cycles are near the boundaries.

1. Introduction

The maximal number of limit cycles of the n-degree polynomial system

$$\begin{aligned} \dot{x} &= P_n(x, y), \\ y &= Q_n(x, y) \end{aligned} \tag{1}$$

is the topic of second part of Hilbert's 16th problem [1]. It is rather difficult even for finding the limit cycles of a concrete system of degree 2. There exist several weaker versions, one of them is studying the limit cycles of the following system, which has simple form and potential applications in natural science,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{2}$$

Equation (2) includes many nonlinear oscillators such as the classical Van der Pol equation with $g(x) = \mu x$ and $f(x) = \rho(1 - x^2)$. When the degree of (2) is larger, it is usually called strongly nonlinear oscillator, which has rich application in material science; see a systematic study in the relatively new monograph [2]. System (2) has the following planar form after introducing $\dot{x} = y$ and taking new damping effect as $-\epsilon f(x)$, showing it is very weak.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) + \epsilon f(x) y. \end{aligned} \tag{3}$$

In fact, studying the maximal number of limit cycles of (2) is the intersection of weak Hilbert's 16th problem [3] and Smale's 13th problem [4]. The latter version is studying the limit cycles of classical Liénard system $\ddot{x} + f(x)\dot{x} + x = 0$, and the former one is studying the limit cycles of perturbed polynomial Hamiltonian system:

$$\begin{aligned} \dot{x} &= H_y(x, y) + \varepsilon p(x, y, \xi), \\ \dot{y} &= -H_x(x, y) + \varepsilon q(x, y, \xi), \end{aligned} \tag{4}$$

where $\max\{\deg(p), \deg(q)\} = \deg(H) - 1 = n$, $H(x, y) = h$ defines a family of closed curves Γ_h, ε is a sufficiently small parameter, and ξ is the coefficient vector of p and q. The problem is mainly restricted to first-order bifurcation and the train of thought is to investigate zeros of the integral; see Poincaré-Pontryagin Theorem [5].

$$M(h, \xi) = \oint_{\Gamma_h} q(x, y, \xi)\, dx - p(x, y, \xi)\, dy, \quad h \in J. \tag{5}$$

This integral is called Abelian integral; see the survey works [6, 7] for smooth systems and for nonsmooth cases see relatively new papers [8, 9].

System (3) will be called being of type (m,n) when $\deg\{g(x)\} = m$, $\deg\{f(x)\} = n$. Let $\mathbb{U}(m)$ denote the exactly maximal number of zeros of $M(h,\xi)$ for (3) of type $(m, m-1)$ in this paper. It is still very difficult even for the simpler systems (3) to get the exact bound of $\mathbb{U}(m)$. We recommend the work [10] and its introduction part for the related results with different m.

One more interesting restriction is that the period annulus is unique and bounded by a asymmetric heteroclinic cycle. For this case, it was proved that $\mathbb{U}(3) = 1$ and $\mathbb{U}(5) = 2$ in [11–13]. For $m = 7$, the general form is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \zeta x\left(x^2-1\right)\left(x^2+\alpha\right)\left(x^2+\beta\right) \\ &\quad + \epsilon\left(a_0 + a_1x^2 + a_2x^4 + a_3x^6\right)y. \end{aligned} \tag{6}$$

Kazemi et al. [14] and Sun [15] studied system (6) of the case $\zeta = 1, \alpha = \beta = 1$. They proved that $3 \le \mathbb{U}(7) \le 4$ corresponding system (6). However, it is still open that whether 3 or 4 is the sharp bound. Sun and Zhao [16] studied the case that $\zeta = -1, \alpha = 0, \beta = 1$, under which the unperturbed system has a heteroclinic cycle connecting two nilpotent cusps surrounding a nilpotent center. They prove a same result as [15]. For a period annulus of $(1.6)_{\epsilon=0}$ with a heteroclinic cycle, which connects two hyperbolic saddles and surrounds an elementary center, Sun [10] proved that it has at most 4 limit cycles and there indeed exist 4 limit cycles for some possible parameters; in other words, 4 is the sharp bound. It was proved that $\mathbb{U}(7) = 4$ for this case. It is interesting that the sharp bound is obtained for systems (6) without nilpotent singularities. Researchers usually believe that the perturbed systems has the same results on the number of limit cycles by first-order bifurcation when the unperturbed systems have the same topological portraits. One question is that does the nonexistence of the nilpotent singularity lead to the sharp bound 4, or is there any systems of the same kind may not has the sharp bound 4?

2. Main Result

The main aim of this paper is to report a different result of $\mathbb{U}(7)$. We pay attention mainly on (6) with fixed β, to study

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x\left(x^2-1\right)\left(x^2+1\right)\left(x^2+\beta\right), \end{aligned} \tag{7}$$

under the symmetric perturbations of $+\epsilon(\alpha_0 + \alpha_1x^2 + \alpha_2x^4 + \alpha_3x^6)y(\partial/\partial y)$; that is,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x\left(x^2-1\right)\left(x^2+1\right)\left(x^2+\beta\right) \\ &\quad + \epsilon\left(\alpha_0 + \alpha_1x^2 + \alpha_2x^4 + \alpha_3x^6\right)y, \end{aligned} \tag{8}$$

where $0 < |\epsilon| \ll 1$ and $\xi = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in R^4$. The Hamiltonian is

$$\mathbb{H}(x,y) = \frac{y^2}{2} + \frac{\beta}{2}x^2 + \frac{1}{4}x^4 - \frac{\beta}{6}x^6 - \frac{1}{8}x^8. \tag{9}$$

$\mathbb{H}(x,y)$ defines a family of ovals $\Gamma_h \subseteq \{(x,y) \mid \mathbb{H}(x,y) = h \in (0, (8\beta+3)/24)\}$, which are closed clockwise orbits of (7). The outer boundary of $\{\Gamma_h\}$ has the heteroclinic loop Γ_* as its outer boundary, connecting the hyperbolic saddles $\mathbb{S}_1(-1,0)$ and $\mathbb{S}_2(1,0)$, at which $\mathbb{H}(x,y) = (8\beta+3)/24$, and the elementary center Γ_0 at the origin, $\mathbb{H}(0,0) = 0$, is the inner boundary; see Figure 1.

Correspondingly, we have the Abelian integral defined on $\{\Gamma_h\}$,

$$\begin{aligned} \mathbb{I}(h,\xi) &= \oint_{\Gamma_h} \left(\alpha_0 + \alpha_1x^2 + \alpha_2x^4 + \alpha_3x^6\right) y\,dx \\ &= \alpha_0\mathbb{I}_0(h) + \alpha_1\mathbb{I}_1(h) + \alpha_2\mathbb{I}_2(h) + \alpha_3\mathbb{I}_3(h), \end{aligned} \tag{10}$$

where $\mathbb{I}_i(h) = \oint_{\Gamma_h} x^{2i} y\,dx$, $\xi = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in R^4$.

We will mainly pay attention to the least upper and lower bounds of number of zeros of $\mathbb{I}(h,\xi)$. It should be noted that there is no xy, x^3y and x^5y in the perturbation, their integration $\oint_{\Gamma_h} x^i y\,dx = 0$ on each closed orbit Γ_h will vanish. Symbolic computation will be the main tools to assist our analysis. We do not use the parameter β in our analysis because it induces rather complex computation. We fix $\beta = 2$ in our analysis and the same dynamical portraits are kept. However, we note that the same results are obtained when taking several values for β near 2 via the completely same analysis.

Theorem 1. *Considering the Liénard system (8), (i) there exist at most four limit cycles emerging from the period annulus $\{\Gamma_h\}$ for all $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in R^4$, and (ii) there exist some parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ such that system (8) can have exact 3 limit cycles bifurcated out near the boundary of the annulus.*

In Section 3, the parameter space is splitted into 4 parts, on each of which the upper bound of $\mathbb{U}(7)$ will be analyzed via an algebraic criteria. In Section 4, 3 zeros of $\mathbb{I}(h,\xi)$ will be detected via asymptotic analysis. Last, we present a discussion to compare the previous results as ours to stress our main purpose of this work.

3. Proof of Main Result (i)

First, we divide the parameter space $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ into the following four parts:

$$\begin{aligned} &\mathbb{U}_1 : \alpha_0 = \alpha_1 = 0, \\ &\mathbb{U}_2 : \alpha_0 \neq 0, \alpha_1 = 0, \\ &\mathbb{U}_3 : \alpha_0 = 0, \alpha_1 \neq 0, \\ &\mathbb{U}_4 : \alpha_0 \neq 0, \alpha_1 \neq 0. \end{aligned} \tag{11}$$

Defining $\Psi(x) = \mathscr{H}(x,y) - y^2/2$. First, we have the following.

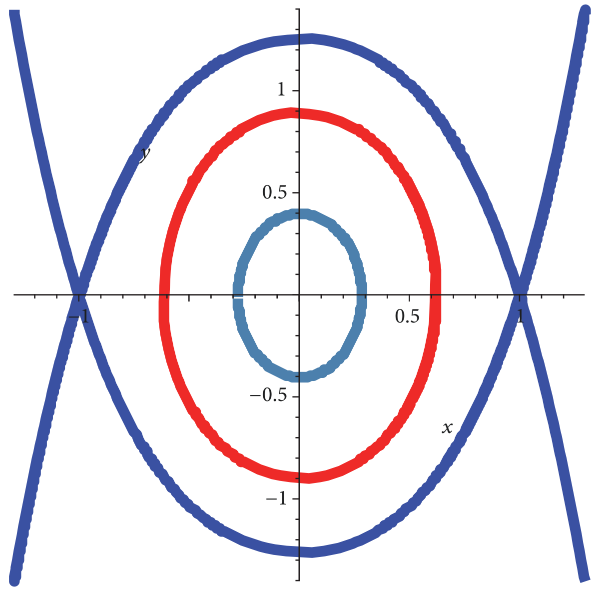

FIGURE 1: The level set of $\mathbb{H}(x, y)$.

Lemma 2.

$$8h^3 \mathbb{I}_i(h) = \oint_{\Gamma_h} \chi_i(x)\, y^7 dx \equiv \widetilde{I}_i(h), \tag{12}$$

for $i = 0, 1, 2, 3$, *where* $\chi_i(x) = x^{2i} f_i^*(x)/181440(x-1)^6(x+1)^6(x^2+2)^6(x^2+1)^6$ *and* $f_i^*(x)$ *is a polynomial.*

Proof. On each closed curves $\Gamma_h = \{\mathbb{H}(x, y) = h\}$,

$$\frac{2\Psi(x) + y^2}{2h} = 1. \tag{13}$$

Then

$$\begin{aligned} \mathbb{I}_i(h) &= \frac{1}{2h} \int_{\Gamma_h} \left(2\Psi(x) + y^2\right) x^{2i} y dx \\ &= \frac{1}{2h} \left(\int_{\Gamma_h} 2x^{2i}\Psi(x)\, y dx + \int_{\Gamma_h} x^{2i} y^3 dx \right), \end{aligned} \tag{14}$$

$$i = 0, 1, 2, 3.$$

By Lemma 4.1 of [17], taking $k = 3$ and $F(x) = 2x^{2i}\Psi(x)$, we obtain

$$\int_{\Gamma_h} 2x^{2i}\Psi(x)\, y dx = \int_{\Gamma_h} \mathcal{G}_i(x)\, y^3 dx, \tag{15}$$

where

$$\begin{aligned} \mathcal{G}_i(x) &= \frac{d}{3dx}\left(\frac{2x^{2i}\Psi(x)}{\Psi'(x)}\right) \\ &= \frac{x^{2i}\left(6ix^{12} + 3x^{12} + 28ix^{10} + 10x^{10} + 14ix^8 + 19x^8 - 100ix^6 + 66x^6 - 116ix^4 + 70x^4 + 72ix^2 + 12x^2 + 96i + 48\right)}{36\left(x^2-1\right)^2\left(x^2+2\right)^2\left(x^2+1\right)^2}. \end{aligned} \tag{16}$$

Therefore

$$\begin{aligned} \mathbb{I}_i(h) &= \frac{1}{2h}\int_{\Gamma_h} \left(x^{2i} + \mathcal{G}_i(x)\right) y^3 dx \\ &= \frac{1}{4h^2}\int_{\Gamma_h} \left(2\Psi(x) + y^2\right)\left(x^{2i} + \mathcal{G}_i(x)\right) y^3 dx \\ &= \frac{1}{4h^2}\int_{\Gamma_h} 2\Psi(x)\left(x^{2i} + \mathcal{G}_i(x)\right) y^3 dx \\ &\quad + \frac{1}{4h^2}\int_{\Gamma_h} \left(x^{2i} + \mathcal{G}_i(x)\right) y^5 dx. \end{aligned} \tag{17}$$

Applying Lemma 4.1 of [17] again, then

$$\int_{\Gamma_h} 2\Psi(x)\left(x^{2i} + \mathcal{G}_i(x)\right) y^3 dx = \int_{\Gamma_h} \widetilde{\mathcal{G}}_i(x)\, y^5 dx, \tag{18}$$

where $\widetilde{\mathcal{G}}_i(x) = (d/5dx)(2\Psi(x)(x^{2i} + \mathcal{G}_i(x))/\Psi'(x)) = x^{2i}g_i^*(x)/2160(x^2-1)^4(x^2+2)^4(x^2+1)^4$ with

$$\begin{aligned} g_i^*(x) = {} & 36i^2x^{24} + 252ix^{24} + 336i^2x^{22} + 117x^{24} \\ & + 2136ix^{22} + 952i^2x^{20} + 864x^{22} + 5800ix^{20} \\ & - 416i^2x^{18} + 2406x^{20} + 1216ix^{18} \\ & - 6796i^2x^{16} + 3656x^{18} - 20188ix^{16} \\ & - 8432i^2x^{14} + 4597x^{16} - 26984ix^{14} \\ & + 11936i^2x^{12} + 4144x^{14} + 11096ix^{12} \\ & + 30592i^2x^{10} - 76x^{12} + 35920ix^{10} \\ & + 1744i^2x^8 + 2904x^{10} - 4064ix^8 \\ & - 35904i^2x^6 + 28764x^8 - 43392ix^6 \\ & - 17088i^2x^4 + 56016x^6 - 15936ix^4 \\ & + 13824i^2x^2 + 37008x^4 + 31104ix^2 \\ & + 9216i^2 + 6336x^2 + 23040i + 9216. \end{aligned} \tag{19}$$

Inserting (18) into (17),

$$\begin{aligned} \mathbb{I}_i(h) &= \frac{1}{4h^2}\oint_{\Gamma_h}\left(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x)\right)y^5dx = \frac{1}{8h^3} \\ &\cdot \int_{\Gamma_h}\left(2\Psi(x) + y^2\right)\left(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x)\right)y^5dx \\ &= \frac{1}{8h^3}\int_{\Gamma_h} 2\Psi(x)\left(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x)\right)y^5dx \\ &+ \frac{1}{8h^3}\int_{\Gamma_h}\left(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x)\right)y^7dx. \end{aligned} \tag{20}$$

Final applying Lemma 4.1 of [17] gives

$$\begin{aligned} &\int_{\Gamma_h} 2\Psi(x)\left(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x)\right)y^5dx \\ &\quad = \int_{\Gamma_h}\overline{\mathcal{G}}_i(x)\,y^7dx, \end{aligned} \tag{21}$$

where $\overline{\mathcal{G}}_i(x) = (d/7dx)(2\Psi(x)(x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x))/\Psi'(x)) = x^{2i}g_i^{**}(x)/181440(x-1)^6(x+1)^6(x^2+2)^6(x^2+1)^6$, and $g_i^{**}(x)$ is some polynomial of degree 39.

Combining (20) and (21) gives

$$8h^3\mathbb{I}_i(h) = \oint_{\Gamma_h}\chi_i(x)\,y^7dx \equiv \widetilde{I}_i(h), \tag{22}$$

where $\chi_i(x) = x^{2i} + \mathcal{G}_i(x) + \widetilde{\mathcal{G}}_i(x) + \overline{\mathcal{G}}_i(x)$. □

Further, we have the following.

Lemma 3. *Let*

$$\mathcal{I}_1 = \oint_{\Gamma_h}\left(1 + \frac{1}{\gamma}x^2\right)dx, \quad \lambda \neq 0, \tag{23}$$

and then

$$8h^3\mathcal{I}_1(h)\oint_{\Gamma_h}\widetilde{\mathcal{F}}_1(x)\,y^7dx \equiv \widetilde{\mathcal{I}}_1(h) \tag{24}$$

where $\widetilde{\mathcal{F}}_1(x) = \chi_0(x) + (1/\gamma)\chi_1(x)$.

Next, we will discuss the Chebyshev property of $\{\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3\}$ instead of $\{\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3\}$. Lemma 3 reveals that the system $\{\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3\}$ can be deduced into $\{\widetilde{\mathcal{F}}_1, \widetilde{I}_2, \widetilde{I}_3\}$ with a parameter. The combination will simplify the computation and lead to a better result (see Remark in Section 4).

It is ready to set the determining functions and then checking if the integral system has Chebyshev property by the algebraic criteria [17, 18]. We set

$$\begin{aligned} l_i(x) &= \left(\frac{\chi_i}{\Psi'}\right)(x) - \left(\frac{\chi_i}{\Psi'}\right)(z(x)), \\ \mathcal{L}_1(x) &= \left(\frac{\widetilde{\mathcal{F}}_1}{\Psi'}\right)(x) - \left(\frac{\widetilde{\mathcal{F}}_1}{\Psi'}\right)(z(x)), \end{aligned} \tag{25}$$

where the symmetry of $\Psi(x)$ reveals that $\Psi(x) - \Psi(z) = 0$ defines $z(x) = -x$. Then, $z \in (-1, 0)$ if restricting $x \in (0, 1)$. Therefore,

$$\begin{aligned} \frac{d}{dx}l_i(x) &= \frac{d}{dx}\frac{\chi_i}{\Psi'}(x) - \frac{d}{dz}\left(\frac{\chi_i}{\Psi'}\right)(z) \times \frac{dz}{dx} \\ &= \frac{d}{dx}\frac{\chi_i}{\Psi'}(x) - \frac{d}{dz}\left(\frac{\chi_i}{\Psi'}\right)(z) \times (-1), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{d}{dx}\mathcal{L}_1(x) &= \frac{d}{dx}\frac{\widetilde{\mathcal{F}}_1}{\Psi'}(x) - \frac{d}{dz}\left(\frac{\widetilde{\mathcal{F}}_1}{\Psi'}\right)(z) \times \frac{dz}{dx} \\ &= \frac{d}{dx}\frac{\widetilde{\mathcal{F}}_1}{\Psi'}(x) - \frac{d}{dz}\left(\frac{\widetilde{\mathcal{F}}_1}{\Psi'}\right)(z) \times (-1). \end{aligned} \tag{27}$$

Computing the related Wronskions obtains

$$W[l_2](x) = \frac{-xq_1(x)}{90720(x-1)^{14}(x+1)^{14}(x^2+2)^{14}(x^2+1)^{14}},$$
$$W[l_2,l_3](x) = \frac{x^7 q_2(x)}{4115059200(x^2+1)^{13}(x^2+2)^{13}(x-1)^{13}(x+1)^{13}},$$
$$W[l_2,l_3,l_0](x) = \frac{x^4 q_3(x)}{3110984755200(x^2+1)^{18}(x^2+2)^{18}(x+1)^{18}(x-1)^{18}}, \quad (28)$$
$$W[l_2,l_3,l_1](x) = \frac{x^6 q_4(x)}{3110984755200(x^2+1)^{18}(x^2+2)^{18}(x+1)^{18}(x-1)^{18}},$$
$$W[l_2,l_3,\mathscr{L}_1](x) = \frac{-x^4 q_5(x,\gamma)}{3110984755200(x^2+1)^{18}(x^2+2)^{18}(x+1)^{18}(x-1)^{18}},$$

The long expression of the polynomials $q_1(x)$, $q_2(x)$, $q_3(x)$, $q_4(x)$, and $q_5(x)$ has the degrees of 36, 66, 90, 90, and 92, respectively. In particular, $q_5(x,\gamma) = \gamma S_{12}(x) - S_{11}(x)$, and S_{11} and S_{12} have the degrees 92 and 90, respectively.

Lemma 4. *$W[l_2](x)$, $W[l_2,l_3](x)$, and $W[l_2,l_3,l_0](x)$ do not vanish on $(0,1)$, and $W[l_2,l_3,l_1](x)$ has a uniqueroot in $(0,1)$, which is simple one.*

Proof. The existence and nonexistence of zeros are verified by Sturm's Theorem to $q_1(x)$, $q_2(x)$, q_3, and $q_4(x)$, respectively. The remainder is to prove the zero of $q_4(x)$ is of multiplicity 1. We have

$$p_4'(x) = \frac{dq_4(x)}{dx} = 2x\mu(x). \quad (29)$$

where $\mu(x)$ do not vanish in $(0,1)$, and this reveals that x_* is a simple zero. □

Based on Lemma 4, the Chebyshev criteria in [17, 18] reveal the following result.

Proposition 5. *Both of $\{\mathbb{l}_2,\mathbb{l}_3\}$ and $\{\mathbb{l}_2,\mathbb{l}_3,\mathbb{l}_0\}$ are Chebyshev systems. $\{\mathbb{l}_2,\mathbb{l}_3,\mathbb{l}_1\}$ forms a Chebyshev system with accuracy 1. Hence, $\mathbb{I}(h,\xi)$ has at most i zero(s) when $(\alpha_0,\alpha_1,\alpha_2,\alpha_3) \in \mathbb{U}_i$ for $i = 1,2,3$, respectively.*

The nonvanishing of $S_{12}(x)$ in $(0,1)$ implies that

$$\gamma(x) = \frac{S_{11}(x)}{S_{12}(x)}, \quad (30)$$

is well defined, Further, we have the following.

Lemma 6. *$W[l_1,l_3,\mathscr{L}_2](x)$ has 2, 1, and 0 roots in $(0,1)$, counting multiplicities, when $\gamma \in [\gamma_1^*,0) \cup (\gamma_2^*,21/73]$, $[0,21/73)$, and $(-\infty,\gamma_1^*) \cup (21/73,+\infty)$, respectively, where $\gamma_1^* = \gamma(x_1^*)$, $\gamma_2^* = \gamma(x_2^*)$, and $\gamma(1) = 21/73$, and x_1^* and x_2^* are given below.*

Proof. We define a curve

$$\Theta: \{(x,\gamma(x)) \mid ,x \in (0,1)\}. \quad (31)$$

The points on the curve satisfy $q_5(x,\gamma(x)) = 0$.

Further,

$$\gamma'(x) = \frac{d\gamma(x)}{dx} = \frac{\zeta_1(x)}{\zeta_2(x)}, \quad (32)$$

where $\deg(\zeta_1(x)) = 186$ and $\deg(\zeta_2(x)) = 185$.

$\zeta_2(x)$ does not vanish in $(0,1)$, $\zeta_1(x)$ has 2 roots: denoted by x_1^*, x_2^* and isolated in $[108659480905/137438953472, 54329740453/68719476736]$ and $[68710347613/68719476736, 137420695227/137438953472]$, respectively. (Numerical approximation shows $x_1^* \approx 0.7906017774$ and $x_2^* \approx 0.9998671538$).

Direct computation reveals that

$$\begin{aligned} \gamma(0^+) &= 0, \\ \gamma'(0^+) &< 0, \\ \gamma(1^-) &= \frac{21}{73}, \\ \gamma'(1^-) &< 0. \end{aligned} \quad (33)$$

Then, the graph of the function $\gamma(x)$ is clear: it arises from $(0,0)$, decreasing to $(x_1^*,\gamma(x_1^*))$ and then increasing to $(x_2^*,\gamma(x_2^*))$, last decreasing to $(1,21/73)$; see Figure 2.

From the above discussion, we have

(i) for each $\gamma \in [\gamma_1^*,0) \cup (21/73,\gamma_2^*]$, there exist two points $(x_\gamma^1,\gamma(x_\gamma^1))$, $(x_\gamma^2,\gamma(x_\gamma^2)) \in \Theta$, which reveals that $q_5(x,\gamma)$ has two zeros for x in $(0,1)$ when fixing γ in this interval.

(ii) For each $\gamma \in (0,21/73)$, there exist one point $(x_\gamma,\gamma(x_\gamma)) \in \Theta$, which reveals that $q_5(x,\gamma)$ has one zero for $x \in (0,1)$.

(iii) For each $\gamma \in (-\infty,\gamma_1^*) \cup (\gamma_2^*,+\infty)$, there exist no point $(x,\gamma(x)) \in \Theta$, showing $q_5(x,\gamma)$ does not vanish in $(0,1)$.

Direct computation gives

$$\text{resultant}\left(q_5,q_5',\gamma\right) = -\zeta_1(x), \quad (34)$$

where $\zeta_1(x)$ has only two zeros at x_1^* and x_2^* as shown before. Further, $q_5'' = S_{11}''(x)\gamma - S_{12}''(x)$ has no roots on $[108659480905/137438953472, 54329740453/68719476736]$ or $[68710347613/68719476736, 137420695227/137438953472]$ (the intervals x_i^* exists). Hence, $q_5(x,\gamma)$ has two roots (x_i^*, $i = 1,2$) with multiply multiplicities 2 if and only if taking $\gamma = \gamma_i$, $i = 1,2$, respectively. And more, $q_5(x,\gamma)$ has two simple roots when

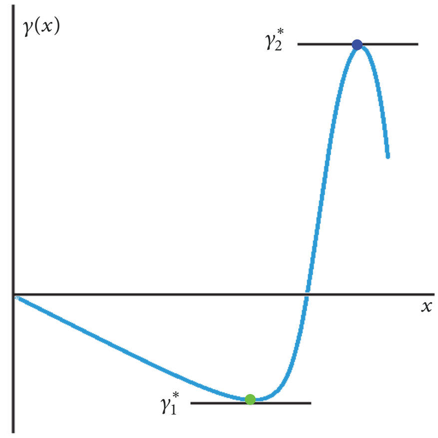

Figure 2: The graph of $y(x)$.

$y \in (y_1^*, 0) \cup (21/73, y_2^*)$, $p(x, y)$ has unique simple root for $y \in [0, 21/73)$.

When the parameters in $\mathbb{U}_4$, we fix $\alpha_1 = 1$ and redenote $\alpha_0 = y$. Then,

$$\begin{aligned} \mathbb{I}(h,\xi) &= \oint_{\Gamma_h} \left(\alpha_2 x^4 + \alpha_3 x^6 + y\left(1 + \frac{1}{y}x^2\right)\right) y\,dx \\ &= \alpha_2 I_1 + \alpha_3 I_3 + y\mathscr{I}_1. \end{aligned} \tag{35}$$

□

Based on the above discussion, we apply the Chebyshev criteria to the wronskians $W[l_2](x)$ and $W[l_2, l_3](x)$ and have the following.

Proposition 7. *$\{\mathbb{I}_2, \mathbb{I}_3, \mathscr{I}_1\}$ forms one Chebyshev system with accuracy 2. Hence, when $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{U}_4$, $\mathbb{I}(h, \xi)$ has, at most, four zeros in $(0, 19/24)$.*

Propositions 5 and 7 prove the part (i) of the main result.

Remark 8. In the above discussion, we did not prove the four generating elements $\mathbb{I}_0$, $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ form a Chebyshev system with some accuracy directly, by which one can verify that $\{\mathbb{I}_1, \mathbb{I}_3, \mathbb{I}_0, \mathbb{I}_2\}$ forms one Chebyshev system with accuracy 2, showing $\mathbb{I}(h, \xi)$ has at most five roots in $(0, 19/24)$, even changing any order of $\mathbb{I}_i$ in $\{\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3\}$. This stresses that the combination between $\mathbb{I}_0$ and $\mathbb{I}_1$ is rather crucial for a smaller upper bound compared the direct analysis.

4. Proof of Main Result (ii)

We will detect zeros of $\mathbb{I}(h, \xi)$ by the expansion theory of the Abelian integrals, see [19]. The Abelian integrals can be expanded near the outer and inner boundaries of the period annulus. One can verify the coefficients are linear dependent and take them as free parameters to detect zeros. According to the type of the outer boundary and inner one, we can apply the theories and computation methods in [20, 21] to establish the asymptotic expansions of $\mathbb{I}(h, \xi)$ near $h = 19/24$ and $h = 0$. Here, the normal forms for application the methods in [20, 21] are obtained by the transforms $x = u/\sqrt{12} - 1$, $y = v$ and $x = u/\sqrt{2} - 1$, $y = v$. We omit the computation and analysis for briefness and have

$$\begin{aligned} \mathbb{I}(h,\xi) = {} & c_0(\xi) + c_1(\xi)\left(h - \frac{19}{24}\right)\ln\left|h - \frac{19}{24}\right| \\ & + c_2(\xi)\left(h - \frac{19}{24}\right) \\ & + c_3(\xi)\left(h - \frac{19}{24}\right)^2 \ln\left|h - \frac{19}{24}\right| + \cdots, \end{aligned} \tag{36}$$

for $0 < -(h - 19/24) \ll 1$, and

$$\mathbb{I}(h,\xi) = \sum_{j \geq 0} b_j(\xi)\, h^{j+1} \tag{37}$$

for $0 < h \ll 1$. The coefficients are as follows:

$$\begin{aligned} c_0(\xi) &= K_0\alpha_0 + K_1\alpha_1 + K_2\alpha_2 + K_3\alpha_3 \\ c_1(\xi) &= -\frac{\sqrt{3}}{3}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3), \\ c_2(\xi) &= J_1\alpha_1 + J_2\alpha_2 + J_3\alpha_3, \quad \text{if } c_1(\xi) = 0, \\ c_3(\xi) &= -\frac{\sqrt{3}}{54}(4\alpha_1 + 5\alpha_2 + 3\alpha_3) + O(c_1), \end{aligned} \tag{38}$$

and

$$\begin{aligned} b_0 &= \sqrt{2}\pi\alpha_0, \\ b_1 &= -\frac{\sqrt{2}}{32}\pi(3\alpha_0 - 8\alpha_1), \\ b_2 &= \frac{\sqrt{2}\pi}{3072}(425\alpha_0 - 240\alpha_1 + 384\alpha_2), \end{aligned} \tag{39}$$

where $K_i = 2\int_{-1}^{1} x^{2i}\sqrt{19/12 - 2\Psi(x)}dx$ and $J_i = 2\int_{-1}^{1}((x^{2i} - 1)/\sqrt{19/12 - 2\Psi(x)})dx$; see the Appendix for their exact values in detail.

Note

$$\begin{vmatrix} K_0 & K_1 & K_2 \\ -\dfrac{\sqrt{3}}{3} & -\dfrac{\sqrt{3}}{3} & -\dfrac{\sqrt{3}}{3} \\ \sqrt{2}\pi & 0 & 0 \end{vmatrix} = \frac{\sqrt{6}}{3}\pi\left(K_2 - K_1\right) \neq 0, \tag{40}$$

which implies

$$\operatorname{rank}\frac{\partial\left(c_0, c_1, b_0\right)}{\partial\left(\alpha_0, \alpha_1, \alpha_2\right)} = 3. \tag{41}$$

Solving the equations $c_0 = c_1 = b_0 = 0$ reveals

$$\begin{aligned} \alpha_0 &= 0, \\ \alpha_1 &= \frac{K_2 - K_3}{K_1 - K_2}\alpha_3, \\ \alpha_2 &= \frac{K_3 - K_1}{K_1 - K_2}\alpha_3 \end{aligned} \tag{42}$$

Taking $\xi_1^* = (0, (K_2 - K_3)/(K_1 - K_2), (K_3 - K_1)/(K_1 - K_2), 1)$ and substituting it into $c_2(\xi)$ and $b_1(\xi)$ reveal

$$\begin{aligned} b_1\left(\xi_1^*\right) &= \frac{\pi\sqrt{2}\left(K_3 - K_2\right)}{4K_1 - 4K_2} \approx 0.3852173169 > 0, \\ c_2\left(\xi_1^*\right) &= \frac{J_1K_2 - J_1K_3 - J_2K_1 + J_2K_3 + J_3K_1 - J_3K_2}{K_1 - K_2} \\ &\approx -0.201496254 < 0 \end{aligned} \tag{43}$$

Taking ε_1 and ε_2 positive and sufficiently small, $h_1 = 0 + \varepsilon_1$ and $h_2 = 19/24 - \varepsilon_2$, and then

$$\frac{1 - \operatorname{sgn}\left(\mathbb{I}\left(h_1, \xi_1^*\right)\mathbb{I}\left(h_2, \xi_1^*\right)\right)}{2} = 0. \tag{44}$$

This shows that $\mathbb{I}(h, \xi_1^*)$ has no zero between h_1 and h_2.

By (41), we take c_0, c_1, and b_0 as free parameters and change them in turn satisfying

$$\begin{aligned} \left|c_0\right| &\ll \left|c_1\right| \ll \left|c_2\left(\xi_1^*\right)\right|, \\ c_2\left(\xi_1^*\right)c_1 &> 0, \\ c_1c_0 &< 0, \\ \left|b_0\right| &\ll \left|b_1\left(\xi_1^*\right)\right|, \\ b_0b_1\left(\xi_1^*\right) &< 0. \end{aligned} \tag{45}$$

Then, there exist 2 zeros of $\mathbb{I}(h, \xi)$ near $h = 19/24$ in $(h_2, 19/24)$ and 1 zero in $(0, h_1)$ near $h = 0$.

Let $U(\xi_1^*) = \{\xi \mid (45)\text{ holds}\}$, a subset of some small neighborhood of ξ_1^*. Then, the following assertion is clear.

Theorem 9. *$\mathbb{I}(h, \xi)$ has, for $\xi \in U(\xi_1^*)$, 2 zeros in $(h_2, 19/24)$ and 1 zero in $(0, h_1)$, totally 3 zeros.*

Similarly, we can have the following.

Theorem 10. *$\mathbb{I}(h, \xi)$ can have 1 zero in $(h_2, 19/24)$ and 2 zeros in $(0, h_1)$.*

Theorem 11. *$\mathbb{I}(h, \xi)$ can have 3 zeros in $(0, h_1)$.*

Theorem 12. *$\mathbb{I}(h, \xi)$ can have 3 zeros in $(h_2, 19/24)$.*

Theorems 9, 10, 11, and 12 prove part (ii) of the main result.

Remark 13. System (8) has the same dynamical portraits as that investigated in [10], however, different results can be obtained. Therefore, the non-existence of nilpotent singularity may not be the key condition of existence of sharp bound 4. The combination analysis of $\mathbb{I}_0$ and $\mathbb{I}_1$ is rather important to get a smaller upper bound (comparing 4 as 5), while it is only to prove that the upper bound is 5 by studying four generating elements directly.

Appendix

In this Appendix, we present the exact values of coefficients in the expansions of $\mathbb{I}(h, \xi)$.

$$\begin{aligned} K_0 &= -\frac{52\left(s + t\right)E}{27\sqrt{57} + 81} + \frac{\left(196t + 548s\right)F}{135\sqrt{57} + 405} \\ &\quad + \frac{-48\sqrt{19} + 472\sqrt{3}}{45\sqrt{57} + 135}, \\ K_1 &= -\frac{4\left(s + t\right)E}{81\sqrt{57} + 243} + \frac{\left(-692t - 4756s\right)F}{2835\sqrt{57} + 8505} \\ &\quad + \frac{2616\sqrt{19} + 2896\sqrt{3}}{945\sqrt{57} + 2835}, \\ K_2 &= \frac{1828\left(t + s\right)E}{729\sqrt{57} + 2187} + \frac{\left(-28492t - 9836s\right)F}{25515\sqrt{57} + 76545} \\ &\quad + \frac{-2544\sqrt{19} - 130504\sqrt{3}}{8505\sqrt{57} + 25515}, \\ K_3 &= -\frac{17420\left(t + s\right)E}{2187\sqrt{57} + 6561} + \frac{\left(707900t + 906460s\right)F}{168399\sqrt{57} + 505197} \\ &\quad + \frac{2510864\sqrt{3} - 171816\sqrt{19}}{56133\sqrt{57} + 168399}, \\ J_1 &= \frac{2tF}{19}, \\ J_2 &= 4\frac{\left(s + t\right)E}{\sqrt{57} + 3} - 4\frac{\left(19s + 11t\right)F}{19\sqrt{57} + 57} - 24\frac{\sqrt{3}}{\sqrt{57} + 3}, \end{aligned}$$

$$J_3 = -76\frac{(s+t)\,E}{9\sqrt{57}+27} + \frac{(782t+1102s)\,F}{171\sqrt{57}+513} + \frac{-24\sqrt{19}+128\sqrt{3}}{3\sqrt{57}+9}, \tag{A.1}$$

with

$$\begin{aligned} s &= 3^{3/4}19^{1/4},\\ t &= 3^{1/4}19^{3/4},\\ E &= EllipticE\left(\frac{\left(\sqrt{57}-3\right)s}{24}, \frac{\sqrt{6498-798\sqrt{57}}}{114}\right),\\ F &= EllipticF\left(\frac{\left(\sqrt{57}-3\right)s}{24}, \frac{\sqrt{6498-798\sqrt{57}}}{114}\right). \end{aligned} \tag{A.2}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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