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Research Article

Spatial and Temporal Dynamics of a Viral Infection Model with Two Nonlocal Effects

Xiaoyan Wang,[1] **Yuming Chen**,[2] **and Junyuan Yang**[3,4]

[1] *School of Information Management, Shanxi University of Finance and Economics, Taiyuan, Shanxi 030051, China*
[2] *Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada*
[3] *Complex Systems Research Center, Shanxi University, Taiyuan 030006, Shanxi, China*
[4] *Shanxi Key Laboratory of Mathematical Techniques and Big Data Analysis on Disease Control and Prevention, Shanxi University, Taiyuan 030006, China*

Correspondence should be addressed to Junyuan Yang; yangjunyuan00@126.com

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We propose and study a viral infection model with two nonlocal effects and a general incidence rate. First, the semigroup theory and the classical renewal process are adopted to compute the basic reproduction number $\mathscr{R}_0$ as the spectral radius of the next-generation operator. It is shown that $\mathscr{R}_0$ equals the principal eigenvalue of a linear operator associated with a positive eigenfunction. Then we obtain the existence of endemic steady states by Shauder fixed point theorem. A threshold dynamics is established by the approach of Lyapunov functionals. Roughly speaking, if $\mathscr{R}_0 < 1$, then the virus-free steady state is globally asymptotically stable; if $\mathscr{R}_0 > 1$, then the endemic steady state is globally attractive under some additional conditions on the incidence rate. Finally, the theoretical results are illustrated by numerical simulations based on a backward Euler method.

1. Introduction

Viruses are very tiny germs. Their presence in the body causes not only familiar infectious diseases (such as the common cold, flu, and warts) but also severe illnesses (such as HIV/AIDS [1–3], hepatitis B [4–7], hepatitis C [6, 7], and human T cell leukemia [8]). Because of the long infectious periods and difficulties in treating them, viral infections have been regarded as serious health problems and have brought heavy economic burden worldwide. The mechanism on a viral infection is quite complicated. Roughly, viruses invade target cells (healthy cells) and replicate in them and then replicated viruses are released. Mathematical modeling has been a very important and efficient way to better understand the evolution of viral infections and to evaluate antiviral drug therapies. In a typical compartmental viral infection model, there are three compartments for uninfected target cells (T), infected cells (I), and free virions (v).

In recent years, spatial-structured models have played a crucial role in exploring viral dynamics. In most of the study (see, e.g., [5, 9–11]), uninfected cells and infected cells are assumed to be motionless, while virions can move freely. Obtained results include the existence of travelling waves [5] and asymptotical behavior [10, 11]. To the best of our knowledge, the spatial domain Ω is either one-dimensional or infinitely dimensional. Usually a Laplacian operator with a diffusion coefficient is used to describe the random diffusion of each virion in the adjacent habitat (position). The diffusion term follows Fick's law and this leads to systems coupling two ordinary differential equations with one parabolic equation.

Such mentioned diffusion is assumed such that the diffusive habitat is small. In fact, the motions of virions are always free and thus the limitations for the short diffusion are not reasonable. Thus, nonlocal diffusion accounting for the long-range diffusion effect has been proposed and extensively investigated. The convolution diffusion operator takes the form

$$L(v) := d \int_{\Omega} J(x-y)\left[v(y)-v(x)\right] dy, \quad x, y \in \Omega, \quad (1)$$

where $v \in B$ and B is a suitable Banach space. This means that a virion at position y can affect another virion at position

x at the probability of $J(x-y)$. In [12], Garcia-Melian and Rossi assumed that $J(x-y)$ represents the probability of skipping from location y to x. Then virions arrive at location x from other places at rate $\int_\Omega J(x-y)v(y)dy$. Based on peculiar features of the nonlocal diffusion operators, models in ecology [13–15], in epidemiology [16–19], and even in materials science [20, 21] have been investigated.

It is well known that incidence rates play a key role in understanding intrinsic mechanisms of viral infections. Though commonly used, the bilinear incidence (or called mass action) may not completely capture the viral dynamics. To overcome this deficiency, several forms of incidence rates have been proposed, which include the saturated incidence rate $\beta Tv/(1+\alpha v)(\alpha \geq 0)$ [22], Crowley-Martin functional response $\beta Tv/(1+\gamma T)(1+\alpha v)(\alpha, \gamma \geq 0)$ [23], and a general form of $f(T,v)$ [8, 24].

Motivated by the aforementioned works, we propose a nonlocal diffusive viral infection model with a general incidence rate. To build the model, we still assume that there are three compartments involved in the viral infection for uninfected target cells, infected cells, and free virions. Their densities at time t and position $x \in \Omega$ are denoted by $T(t,x)$, $I(t,x)$, and $v(t,x)$, respectively. Here $\Omega \subset \mathbb{R}^n$ is compact and connected with a smooth boundary and satisfies $\Omega = \overline{\operatorname{int}(\Omega)}$. The model to be studied is

$$
\begin{aligned}
\frac{\partial T(t,x)}{\partial t} &= \lambda(x) - \mu(x) T(t,x) - \int_\Omega f(x,y,T(t,x),v(t,y))\,dy, \quad x \in \Omega,\\
\frac{\partial I(t,x)}{\partial t} &= \int_\Omega f(x,y,T(t,x),v(t,y))\,dy - \delta(x) I(t,x), \quad x \in \Omega,\\
\frac{\partial v(t,x)}{\partial t} &= \int_\Omega \theta(x,y) v(t,y)\,dy - \gamma(x) v(t,x) + p(x) I(t,x) - c(x) v(t,x), \quad x \in \Omega
\end{aligned}
\tag{2}
$$

with the initial condition

$$
\begin{aligned}
T(0,x) &= T_0(x) > 0,\\
I(0,x) &= I_0(x) \geq 0,\\
v(0,x) &= v_0(x) \geq 0 \quad \text{for } x \in \Omega.
\end{aligned}
\tag{3}
$$

Here $\lambda(x)$ is the created rate of uninfected cells at x; $\mu(x)$, $\delta(x)$, and $c(x)$ are, respectively, the death rates of uninfected cells, infected cells, and virions at x; $\int_\Omega f(x,y,T(t,x),v(t,y))dy$ denotes the rate of new infections at time t and location x; $\int_\Omega \theta(x,y)v(t,y)dy$ represents the transfer rate of virions from positions in $\Omega \setminus \{x\}$ to position x with a kernel function $\theta(x,y)$ while $-\gamma(x)v(t,x)$ is the total transfer rate of virions from location x to all the other locations. Note that if we take the kernel function $\theta(x,y) = J(x-y)$, then system (2) becomes a classical within-host model with the nonlocal diffusion.

The main contribution of this paper has three aspects. Firstly, it is easy to see that the equation for I has an integral term and thus the solution of system (2) lacks strong regularity. This implies that the semiflow generated by system (2) is not compact. To overcome this difficulty, we use Arzelà-Ascoli Theorem [25] to establish the asymptotic smoothness of the semiflow, which enables us to pass the dissipativity of system (2) from $L^1(\Omega;\mathbb{R}_+)$ to $C(\Omega;\mathbb{R}_+)$. Secondly, the nonlocal diffusion further weakens the regularity of the solution and this requires nonroutine methods to deal with compactness and nonsupporting of the next generation operator. Finally, the nonlocal diffusion term enhances the difficulty in proving the global stability of the steady states. Inspired by the work of Thieme [26, Section 11], in order to construct a suitable Lyapunov functional, a nonnegative Borel measurable function should be picked to balance the nonlocal term. We explicitly identify such a Borel measurable function as $v^*(x)$ instead of in an abstract form.

The organization of this paper is as follows. Section 2 gives the existence, uniqueness, and nonnegativity of solutions to system (2). Section 3 shows that the solution semiflow is asymptotically smooth by applying Arzelà-Ascoli Theorem. Section 4 focuses on the basic reproduction number $\mathscr{R}_0$ of system (2) defined as the spectral radius of the next-generation operator $\mathscr{R}(x)$ and the relationship between $\mathscr{R}_0$ and the spectral bound of a linear operator. Section 5 is devoted to the existence of endemic steady states by employing Shauder fixed theorem. We discuss the stability of the virus-free steady state in Section 6. The global behavior of system (2) including uniform persistence and global stability of endemic steady states is established in Sections 7 and 8. In Section 9, we carry out numerical experiments to validate the theoretical results. Section 10 concludes the paper with a succinct discussion.

2. Preliminaries

Let $X = C(\Omega;\mathbb{R})$ equip with the supremum norm $\|\cdot\|_X$

$$\|\phi\|_X = \sup_{x\in\Omega} |\phi(x)|, \quad \phi \in X. \tag{4}$$

Moreover, for $\phi \in X$, we denote $\overline{\phi} = \sup_{x\in\Omega}\phi(x)$ and $\underline{\phi} = \inf_{x\in\Omega}\phi(x)$. Let $Y = X^3$ and define

$$\|\psi\|_Y = \sqrt{\|\psi_1\|_X^2 + \|\psi_2\|_X^2 + \|\psi_3\|_X^2} \quad \text{for } \psi = (\psi_1,\psi_2,\psi_3) \in Y. \tag{5}$$

Then Y is also a Banach space equipped with the norm $\|\cdot\|_Y$. Clearly, $X_+ = C(\Omega;\mathbb{R}_+)$ and $Y_+ = X_+^3$ are positive cones of X and Y, respectively.

To study the asymptotic dynamics of system (2), we make the following assumptions.

Assumption 1.

(i) $\lambda(\cdot)$, $\mu(\cdot)$, $\delta(\cdot)$, $p(\cdot)$, and $c(\cdot) \in X$ are all strictly positive.

(ii) $f \in C(\Omega^2 \times \mathbb{R}_+^2, \mathbb{R})$ is continuously differentiable with respect to the third and fourth variables and also satisfies the following:

(ii-1) $f(x, y, 0, v) = f(x, y, T, 0) = 0$ for all $(x, y, T, v) \in \Omega^2 \times \mathbb{R}_+^2$;

(ii-2) for all $(x, y) \in \Omega^2$, $f(x, y, T, v)$ is increasing in both T and v;

(ii-3) for all $(x, y, T) \in \Omega^2 \times \mathbb{R}_+$, the function $f(x, y, T, v)/v$ is decreasing associated with the variable v on $(0, \infty)$;

(ii-4) for any $c > 0$ there exists some $L_c > 0$ such that

$$\left|f(x, y, T, v) - f\left(x, y, \overline{T}, \overline{v}\right)\right| \le L_c\left(\left|T - \overline{T}\right| + |v - \overline{v}|\right) \quad \text{for } T, \overline{T}, v, \overline{v} \in [0, c]. \tag{6}$$

(iii) $\theta \in C(\Omega^2, \mathbb{R}_+)$ and satisfies $\theta(x, x) > 0$. Moreover, it is irreducible and symmetrical for all $x, y \in \Omega$.

(iv) $\gamma \in X_+$ and θ satisfy the balance condition

$$\int_\Omega \theta(x, y)\, dy = \gamma(x) \quad \text{for all } x \in \Omega. \tag{7}$$

To continue the discussion, we define a linear operator A and a nonlinear operator F on Y by

$$(A\psi)(x) = \Big(-\mu(x)\psi_1(x), -\delta(x)\psi_2(x), \int_\Omega \theta(x, y)\psi_3(y)\, dy - (c(x) + \gamma(x))\psi_3(x)\Big) \tag{8}$$

and

$$F(\psi)(x) = \Big(\lambda(x) - \int_\Omega f(x, y, \psi_1(x), \psi_3(y))\, dy, \int_\Omega f(x, y, \psi_1(x), \psi_3(y))\, dy, p(x)\psi_2(x)\Big) \tag{9}$$

for $\psi \in Y$ and $x \in \Omega$, respectively. Here f is extended to $\Omega^2 \times \mathbb{R}^2$ through continuity.

Lemma 2. *The operator A defined by (8) generates a uniformly continuous semigroup $\{e^{tA}\}_{t\ge0}$ on Y. Furthermore, $e^{At}Y_+ \subset Y_+$ for all $t \in \mathbb{R}_+$.*

Proof. We decompose the operator A as $A = A_1 + A_2$, where

$$(A_1\psi)(x) = (-\mu(x)\psi_1(x), -\delta(x)\psi_2(x), -(c(x) + \gamma(x))\psi_3(x)) \tag{10}$$

and

$$(A_2\psi)(x) = \Big(0, 0, \int_\Omega \theta(x, y)\psi_3(y)\, dy\Big) \tag{11}$$

for $\psi \in Y$ and $x \in \Omega$. We readily see that A_1 generates a strongly continuous and positive semigroup $\{e^{A_1 t}\}_{t\ge0}$ with

$$\left(e^{A_1 t}\psi\right)(x) = \left(e^{-\mu(x)t}\psi_1(x), e^{-\delta(x)t}\psi_2(x), e^{-(c(x)+\gamma(x))t}\psi_3(x)\right) \tag{12}$$

for $\psi \in Y$ and $x \in \Omega$. Moreover, it follows from Assumption 1 that the operator A_2 is bounded. Then [27, Corollary VI 1.11] and [28, Theorem 1.2] combined ensure that the operator A generates a positively continuous semigourp $\{e^{At}\}_{t\ge0}$. This completes the proof. □

The following proposition gives the existence and uniqueness of solutions of system (2).

Proposition 3. *For all $u_0 \in Y$, system (2) admits a unique classical solution $u(\cdot, u_0) \in C([0, T_{max}), Y) \cap C^1((0, T_{max}), Y)$ with $T_{max} > 0$. Moreover, the solution has the following properties:*

(i) If $T_{max} < \infty$ then $\lim_{t\to T_{max}^-}\|u(t, u_0)\|_Y = \infty$.

(ii) Let $T \in (0, T_{max})$. If $\{u_n\} \subset Y$ with $u_n \longrightarrow u_0$ in Y as $n \longrightarrow \infty$, then

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u(t, u_n) - u(t, u_0)\|_Y = 0. \tag{13}$$

(iii) $T(t, x) > 0$ for $t \in (0, T_{\max})$ and $x \in \Omega$.

(iv) If $u_0 \in Y_+$, then $u(t, u_0) \in Y_+$ for all $t \in (0, T_{max})$.

Proof. [29, Lemma 3.1], together with Assumption 1, ensures that the operator F defined by (9) is continuously Fréchet differentiable on Y. Therefore,

$$u(t, u_0)(x) = e^{At}u_0(x) + \int_0^t e^{A(t-s)}F(u(s, x))\, ds \quad \text{for } x \in \Omega. \tag{14}$$

Then from [30, Proposition 4.16] and [28, Theorems 1.2–1.5 in Chapter 6] assertions (i) and (ii) follow immediately.

We use by way of contradiction to prove (iii). We claim that there exists $t_0 \in (0, T_{\max})$ and $x_0 \in \Omega$ such that $T(t_0, x_0) = 0$. Notice that $T(t, x_0) > 0$ for all small enough t. Hence, by the continuous dependence of the solution on the initial values, define

$$\hat{t} = \inf\{t \in (0, T_{\max}) \mid T(t, x_0) = 0\}. \tag{15}$$

Thus $\hat{t} \in (0, T_{\max})$ and $T(\hat{t}, x_0) = 0$ with $\partial T(\hat{t}, x_0)/\partial t \le 0$. But then

$$\frac{\partial T(\hat{t}, x_0)}{\partial t} = \lambda(x_0) > 0 \tag{16}$$

is a contradiction. This proves assertion (iii).

In order to establish (iv), note that for each positive r there exists a sufficiently large κ_r such that

$$F(u(t,\cdot)) + \kappa_r u(t,\cdot) > 0 \quad \text{for } t \in \mathbb{R}_+ \text{ and } u \in C(\mathbb{R}_+, Y_+) \cap \mathbb{B}(0,r), \tag{17}$$

where $\mathbb{B}(0,r)$ is an open ball in Y with the center at $(0,0,0)$ and radius r. We rewrite system (2) in the form of

$$\frac{du(t,\cdot)}{dt} = A_{\kappa_r} u(t,\cdot) + F_{\kappa_r}(u(t,\cdot)), \tag{18}$$

$A_{\kappa_r} = A - \kappa Id$ where Id is an indentity operator on X and $F_{\kappa_r}(u) = F(u) + \kappa_r u$. Applying the method of variation of constant, we have

$$u(t,x) = \left(e^{A_{\kappa_r} t} u_0\right)(x) + \int_0^t F_{\kappa_r}(u(s,x))\, e^{A_{\kappa_r}(t-s)} ds. \tag{19}$$

Hence, for $(t,u_0) \in (0,T_{\max}) \times Y_+$, $u(t,u_0) \in Y_+$. □

In fact, as the following result shows, solutions exist globally with initial values in Y_+.

Proposition 4. *Let $u_0 \in Y_+$. Then the solution of (2) with the initial value u_0 exists on $\mathbb{R}_+$.*

Proof. Let $J = [0,T_{\max})$ be the maximal existence interval of the solution through u_0. Define $Z(t,x) = T(t,x) + I(t,x)$, for $(t,x) \in (J \times \Omega)$. Then, for any $t \in J$, first, we add the first two equations of (2) to obtain

$$\begin{aligned} \frac{\partial Z(t,x)}{\partial t} &= \lambda(x) - \mu(x) T(t,x) - \delta(x) I(t,x) \\ &\le \lambda(x) - \nu(x) Z(t,x), \end{aligned} \tag{20}$$

where $\nu(x) = \min\{\mu(x), \delta(x)\}$. It follows that

$$\begin{aligned} Z(t,x) &\le e^{-t\nu(x)} Z(0,x) + \frac{\lambda(x)}{\nu(x)}\left(1 - e^{-t\nu(x)}\right) \\ &\le M_1(x), \end{aligned} \tag{21}$$

where $M_1(x) = Z(0,x) + \lambda(x)/\nu(x)$. Next we integrate the third equation of system (2) on the set Ω to obtain

$$\begin{aligned} \frac{d\int_\Omega v(t,x)\,dx}{dt} &= \int_\Omega \left(\int_\Omega \theta(x,y) v(t,y)\,dy\right) dx \\ &- \int_\Omega \gamma(x) v(t,x)\,dx + \int_\Omega p(x) I(t,x)\,dx \\ &- \int_\Omega c(x) v(t,x)\,dx \\ &= \int_\Omega \left(\int_\Omega \theta(x,y) v(t,y)\,dx\right) dy \ \ (\text{interchange order of integration}) \\ &- \int_\Omega \gamma(x) v(t,x)\,dx + \int_\Omega p(x) I(t,x)\,dx \\ &- \int_\Omega c(x) v(t,x)\,dx \\ &= \int_\Omega \left(\int_\Omega \theta(y,x) v(t,y)\,dx\right) dy - \int_\Omega \gamma(x) \\ &\cdot v(t,x)\,dx \ \ (\text{by Assumption 1}) + \int_\Omega p(x) \\ &\cdot I(t,x)\,dx - \int_\Omega c(x) v(t,x)\,dx = \int_\Omega \gamma(y) \\ &\cdot v(t,y)\,dy - \int_\Omega \gamma(x) v(t,x)\,dx + \int_\Omega p(x) \\ &\cdot I(t,x)\,dx - \int_\Omega c(x) v(t,x)\,dx = \int_\Omega p(x) \\ &\cdot I(t,x)\,dx - \int_\Omega c(x) v(t,x)\,dx \le \int_\Omega p(x) \\ &\cdot M_1(x)\,dx - \underline{c}\int_\Omega v(t,x)\,dx. \end{aligned} \tag{22}$$

Solving this differential inequality, we get

$$\begin{aligned} \int_\Omega v(t,x)\,dx &\le e^{-\underline{c}t} \int_\Omega v(0,x)\,dx \\ &+ \frac{\int_\Omega p(x) M_1(x)\,dx}{\underline{c}}\left(1 - e^{-\underline{c}t}\right) \\ &\le \int_\Omega v(0,x)\,dx + \frac{\int_\Omega p(x) M_1(x)\,dx}{\underline{c}} \\ &:= M_2 \end{aligned} \tag{23}$$

for $t \in J$. Lastly, it also follows from the last equation of (2) that

$$\frac{\partial v(t,\cdot)}{\partial t} \le \overline{\theta} M_2 + \overline{p}\overline{M_1} - \left(\underline{\gamma} + \underline{c}\right) v(t,\cdot), \tag{24}$$

which gives

$$v(t,\cdot) \le v(0,\cdot)\, e^{-(\underline{\gamma}+\underline{c})t} + \frac{\overline{\theta} M_2 + \overline{p}\overline{M_1}}{\underline{\gamma} + \underline{c}}\left(1 - e^{-(\underline{\gamma}+\underline{c})t}\right) \quad \text{for } t \in J. \tag{25}$$

In summary, we have obtained that

$$\|u(t,\cdot)\|_Y \le \sqrt{2\overline{M_1}^2 + \left(\|v(0,\cdot)\|_X + \frac{\overline{\theta} M_2 + \overline{p}\overline{M_1}}{\underline{\gamma} + \underline{c}}\right)^2} \quad \text{for } t \in J. \tag{26}$$

By (i) of Proposition 3, we get $T_{\max} = \infty$ and this completes the proof. □

By Propositions 3 and 4, a solution semiflow $\Phi : \mathbb{R}_+ \times Y_+ \longrightarrow Y_+$ is defined by

$$\Phi(t,\psi) = (T(t,\cdot), I(t,\cdot), v(t,\cdot)) \tag{27}$$

for $(t,\psi) \in \mathbb{R}_+ \times Y_+$. Moreover, the existence of the solution of (2) is indeed global and the semiflow Φ is bounded and dissipative. Define

$$\Gamma = \left\{ \psi \in Y_+ \;\middle|\; \begin{array}{c} 0 < \psi_1(x) + \psi_2(x) \le \dfrac{\lambda(x)}{\min\{\mu(x), \delta(x)\}} \quad \text{for } x \in \Omega \\ \text{and } \int_\Omega \psi_3(x)\,dx \le \dfrac{\int_\Omega p(x)\left(\lambda(x)/\min\{\mu(x),\delta(x)\}\right)dx}{\underline{c}} \end{array} \right\}. \tag{28}$$

Then it follows easily from the proof of Proposition 4 (actually using the same differential inequalities and the resulting inequalities in the proof) that Γ is positively invariant and attracts all the bounded subsets of (2) in Y_+. Therefore, to study the asymptotic behavior of (2), we only need to focus on solutions with initial values in Γ.

3. The Asymptotic Smoothness

A semiflow Φ is *asymptotically smooth* if there exists a nonempty compact set attracting each forward invariant bounded closed set (see [31, Definition 2.25]), in other words, if the semiflow is asymptotically compact on every forward invariant bounded closed set.

Theorem 5. *The semiflow $\{\Phi\}_{t\ge0}$ defined by (27) is asymptotically compact and hence it is asymptotically smooth.*

Proof. The proof is inspired by the ideal in [26, Section 4]. Assume that B is any forward invariant bounded subset of Y_+. We show that Φ is asymptotically compact on B [32, pp. 28]. This needs to show that a sequence of solutions $u_n = (S_n, I_n, v_n)$ is equi-bounded. That is, there exists a sequence $\{t_n\}$ with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $\{u_n(t_n)\}$ has a convergent subsequence in Y_+. Let us consider the translated solutions $\overline{u}_n = (\overline{T}_n, \overline{I}_n, \overline{v}_n) = u_n(t_n + \cdot)$. Then

$$\frac{\partial \overline{T}_n(t,x)}{\partial t} = \lambda(x) - \int_\Omega f\left(x,y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)dy - \mu(x)\overline{T}_n(t,x), \tag{29}$$

$$\frac{\partial \overline{I}_n(t,x)}{\partial t} = \int_\Omega f\left(x,y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)dy - \delta(x)\overline{I}_n(t,x), \tag{30}$$

$$\frac{\partial \overline{v}_n(t,x)}{\partial t} = \int_\Omega \theta(x,y)\overline{v}_n(t,y)\,dy - (\gamma(x)+c(x))\overline{v}_n(t,x) + p(x)\overline{I}_n(t,x). \tag{31}$$

By the Arzelà-Ascoli theorem, it suffices to show that $\{\overline{T}_n(0,\cdot)\}$, $\{\overline{I}_n(0,\cdot)\}$, and $\{\overline{v}_n(0,\cdot)\}$ are equi-continuous. Then, for each $x, \tilde{x} \in \Omega$, we obtain

$$\begin{aligned}
\frac{\partial\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)^2}{\partial t} &= 2\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right) \\
&\quad\cdot\Big\{\lambda(x)-\lambda(\tilde{x})-\mu(x)\overline{T}_n(t,x)+\mu(\tilde{x})\overline{T}_n(t,\tilde{x}) \\
&\quad-\int_\Omega\left[f\left(x,y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x},y,\overline{T}_n(t,\tilde{x}),\overline{v}_n(t,y)\right)\right]dy\Big\} \\
&= 2\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)(\lambda(x)-\lambda(\tilde{x}))-2\mu(x) \\
&\quad\cdot\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)^2-2\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right) \\
&\quad\cdot(\mu(x)-\mu(\tilde{x}))\overline{T}_n(t,\tilde{x})-2\int_\Omega\left(\overline{T}_n(t,x)\right. \\
&\quad\left.-\overline{T}_n(t,\tilde{x})\right)\left[f\left(x,y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x},y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right]dy \\
&\quad-2\int_\Omega\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right) \\
&\quad\cdot\left[f\left(\tilde{x},y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x},y,\overline{T}_n(t,\tilde{x}),\overline{v}_n(t,y)\right)\right]dy \\
&\le \frac{(\lambda(\tilde{x})-\lambda(x))^2}{\zeta^2}+3\zeta^2\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)^2 \\
&\quad-2\mu(x)\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)^2 \\
&\quad+\frac{(\mu(x)-\mu(\tilde{x}))^2\overline{T}_n(t,\tilde{x})^2}{\zeta^2}+\frac{1}{\zeta^2} \\
&\quad\cdot\int_\Omega\left[f\left(\tilde{x},y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x},y,\overline{T}_n(t,\tilde{x}),\overline{v}_n(t,y)\right)\right]^2dy.
\end{aligned} \tag{32}$$

Here we have used the fact that f is increasing with respect to T and $ab \le [(\zeta a)^2 + (b/\zeta)^2]/2$, where $\zeta > 0$ is small enough and satisfies

$$3\zeta^2 - 2\mu(x) < -\zeta \quad \text{for } x \in \Omega. \tag{33}$$

Integrating the inequality (32) from $-t_n$ to $t \geq -t_n$, we obtain

$$\begin{aligned}
&\left(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x})\right)^2 \leq \left(T_n(0,x)-T_n(0,\tilde{x})\right)^2 \\
&\quad\cdot e^{-\zeta(t+t_n)}+\frac{(\lambda(x)-\lambda(\tilde{x}))^2}{\zeta^3}+\frac{(\mu(x)-\mu(\tilde{x}))^2}{\zeta^3} \\
&\quad\cdot \sup_{-t_n\leq l\leq t}\overline{T}_n^2(l,x)+\frac{1}{\zeta^3} \\
&\quad\cdot \sup_{-t_n\leq l\leq t}\left(\int_\Omega\left[\left(f\left(\tilde{x},y,\overline{T}_n(l,x),\overline{v}_n(l,y)\right)\right.\right.\right. \\
&\quad\left.\left.\left.-f\left(\tilde{x},y,\overline{T}_n(l,\tilde{x}),\overline{v}_n(l,y)\right)\right]^2dy\right).
\end{aligned} \tag{34}$$

By Assumption 1, for each bounded set $\Omega^2\times[0,c]^2 (c>0)$, f is uniformly continuous. The boundedness of f, together with $e^{-\zeta t_n}\longrightarrow 0$ as $n\longrightarrow\infty$, implies that $(\overline{T}_n(t,x)-\overline{T}_n(t,\tilde{x}))^2\longrightarrow 0$ as $\tilde{x}\longrightarrow x$. This limitation is uniform for both $n\in\mathbb{N}$ and t in compact subsets of $\mathbb{R}$. The equi-continuity of $\{\overline{T}_n(t,\cdot): n\in\mathbb{N}, t\in[-T,T]\}$ holds immediately for any $T>0$.

Similarly, for $t\geq -t_n$, we can get

$$\begin{aligned}
&\left(\overline{I}_n(t,x)-\overline{I}_n(t,\tilde{x})\right)^2 \leq \left(I_n(0,x)-I_n(0,\tilde{x})\right)^2 e^{-\zeta(t+t_n)} \\
&\quad+\frac{(\delta(x)-\delta(\tilde{x}))^2}{\zeta^3}\sup_{-t_n\leq r\leq t}\overline{I}_n^2(r,x)+\frac{1}{\zeta^2} \\
&\quad\cdot\int_{-t_n}^t e^{-\delta(x)(t-s)}\int_\Omega\left[f\left(x,y,\overline{T}_n(s,x),\overline{v}_n(s,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x},y,\overline{T}_n(s,\tilde{x}),\overline{v}_n(s,y)\right)\right]^2dyds
\end{aligned} \tag{35}$$

Arguing directly, we claim that $\{\overline{I}_n(\hat{t},\cdot)\}$ is not equi-continuous for some $\hat{t}\geq 0$. Then we can pick up a sequence $\{\tilde{x}_n\}_n$ such that $\tilde{x}_n\longrightarrow x$ as $n\longrightarrow\infty$. Moreover, we choose a subsequence of $\{\overline{I}_n\}$ satisfying

$$\limsup_{n\longrightarrow\infty}\left(\overline{I}_n(\hat{t},x)-\overline{I}_n(\hat{t},\tilde{x}_n)\right)^2>0. \tag{36}$$

Since, for each t, $\{\overline{T}_n(t,\cdot)\}$ is equi-continuous, $T_n(t,\tilde{x}_n)\longrightarrow T_n(t,x)$ as $n\longrightarrow\infty$. Applying Fatou's lemma and Assumption 1, we get from (35) that

$$\begin{aligned}
&\limsup_{n\longrightarrow\infty}\left(\overline{I}_n(\hat{t},x)-\overline{I}_n(\hat{t},\tilde{x}_n)\right)^2\leq\frac{1}{\zeta^2}\int_{-\infty}^{\hat{t}}e^{\zeta t}dt \\
&\quad\cdot\limsup_{n\longrightarrow\infty}\int_\Omega\left[f\left(x,y,\overline{T}_n(t,x),\overline{v}_n(t,y)\right)\right. \\
&\quad\left.-f\left(\tilde{x}_n,y,\overline{T}_n(t,\tilde{x}_n),\overline{v}_n(t,y)\right)\right]^2dyds
\end{aligned} \tag{37}$$

is a contradiction. This proves that $\{\overline{I}_n(t,\cdot)\}_n : n\in\mathbb{N}, t\in[-T,T]\}$ is also equi-continuous for any $T>0$.

Finally, again similarly as before, we can obtain

$$\begin{aligned}
&\left(\overline{v}_n(0,x)-\overline{v}_n(0,\tilde{x})\right)^2 \\
&\quad\leq\left(v_n(0,x)-v_n(0,\tilde{x})\right)^2e^{-\zeta(t+t_n)} \\
&\quad+\frac{(p(x)-p(\tilde{x}))^2}{\zeta^3}\sup_{-t_n\leq l\leq t}\overline{I}_n^2(l,x) \\
&\quad+\frac{(\gamma(x)+c(x)-\gamma(\tilde{x})-c(\tilde{x}))^2}{\zeta^3}\sup_{-t_n\leq l\leq t}\overline{v}_n^2(l,x) \\
&\quad+\int_\Omega\left|\theta(x,y)-\theta(\tilde{x},y)\right|dy\sup_{-t_n\leq l\leq t}\overline{v}_n(l,x).
\end{aligned} \tag{38}$$

By Assumption 1 and $\sup_{\tilde{x}\in\Omega}v_n(0,\tilde{x})<\infty$, we have $(\overline{v}_n(0,x)-\overline{v}_n(0,\tilde{x}))^2\longrightarrow 0$ as $\tilde{x}\longrightarrow x$ uniformly for $n\in\mathbb{N}$. This proves the equi-continuity of $\{\overline{v}_n(0,\cdot)\}$ and the hence the proof is complete. □

4. The Basic Reproduction Number

This section is conducted for estimation of the basic reproduction number, which is defined as the expected numbers of secondary cases created by a typical infected individual among a completely susceptible population. Clearly, system (2) has a virus-free steady state $E_0=(T^0(x),0,0)$, where $T^0(x)=\lambda(x)/\mu(x)$ for $x\in\Omega$. Linearize (2) around E_0 in the disease invasion phase to obtain

$$\begin{aligned}
\frac{\partial I(t,x)}{\partial t}&=\int_\Omega f_v\left(x,y,T^0(x),0\right)v(t,y)dy \\
&\quad-\delta(x)I(t,x), \\
\frac{\partial v(t,x)}{\partial t}&=\int_\Omega\theta(x,y)v(t,y)dy+p(x)I(t,x) \\
&\quad-(c(x)+\gamma(x))v(t,x).
\end{aligned} \tag{39}$$

Solving them, we have

$$\begin{aligned}
&I(t,x) \\
&\quad=I_0(x)e^{-\delta(x)t} \\
&\quad+\int_0^t e^{-\delta(x)(t-s)}\int_\Omega f_v\left(x,y,T^0(x),0\right)v(s,y)dyds
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
v(t,x)&=v_0(x)e^{-(c(x)+\gamma(x))t} \\
&\quad+\int_0^t e^{-(c(x)+\gamma(x))(t-s)}\left(\int_\Omega\theta(x,y)v(s,y)dy\right. \\
&\quad\left.+p(x)I(s,x)\right)ds,
\end{aligned} \tag{41}$$

respectively. Plugging (40) into (41) yields

$$\begin{aligned} v(t,x) &= v_0(x)e^{-(c(x)+\gamma(x))t} \\ &+ \int_0^t e^{-(c(x)+\gamma(x))(t-s)} \int_\Omega \theta(x,y)\, v(s,y)\, dy ds \\ &+ p(x)\int_0^t e^{-(c(x)+\gamma(x))(t-s)} I_0(x)\, e^{-\delta(x)s} ds + p(x) \\ &\cdot \int_0^t e^{-(c(x)+\gamma(x))(t-s)} \int_0^s e^{-\delta(x)(s-\tau)} \\ &\cdot \int_\Omega f_v\left(x,y,T^0(x),0\right) v(\tau,y)\, dy d\tau ds. \end{aligned} \tag{42}$$

Using change of variables gives us

$$\begin{aligned} v(t,x) &= v_0(x)e^{-(c(x)+\gamma(x))t} + p(x) \\ &\cdot \int_0^t e^{-(c(x)+\gamma(x))(t-s)} I_0(x)\, e^{-\delta(x)s} ds \\ &+ \int_0^t e^{-(c(x)+\gamma(x))s} \int_\Omega \theta(x,y)\, v(t-s,y)\, dy ds \\ &+ p(x)\int_0^t e^{-(c(x)+\gamma(x))s} \int_\tau^t e^{-\delta(x)(\tau-s)} \\ &\cdot \int_\Omega f_v\left(x,y,T^0(x),0\right) v(t-\tau,y)\, dy d\tau ds. \end{aligned} \tag{43}$$

Therefore, following the approach of Diekmann et al. [33], we define the *next-generation operator* $\mathscr{R} : X \longrightarrow X$ by

$$\begin{aligned} (\mathscr{R}\phi)(x) &= \int_\Omega \int_0^\infty e^{-(c(x)+\gamma(x))\tau} \theta(x,y)\, \phi(y)\, d\tau dy \\ &+ \int_\Omega \int_0^\infty \int_\tau^\infty p(x) \\ &\cdot e^{-(c(x)+\gamma(x))\tau} e^{-\delta(x)(s-\tau)} f_v\left(x,y,T^0(x),0\right) \\ &\cdot \phi(y)\, ds d\tau dy = \frac{\int_\Omega \theta(x,y)\, \phi(y)\, dy}{c(x)+\gamma(x)} \\ &+ \frac{p(x)}{\delta(x)(\gamma(x)+c(x))} \int_\Omega f_v\left(x,y,T^0(x),0\right) \\ &\cdot \phi(y)\, dy \end{aligned} \tag{44}$$

for $\phi \in X$ and $x \in \Omega$. Based on Assumption 1, the operator $\mathscr{R}$ is well defined, continuous, and positive. In the following, we show that $\mathscr{R}$ is compact and nonsupporting. Nonsupporting means that, for any $v \in X_+ \backslash \{0\}$ and $\xi \in (X^*)_+ \backslash \{0\}$, there exists positive $N_0 = N_0(v,\xi)$ such that $\int_\Omega (\mathscr{R}^{N_0} v)(y)\xi(y)dy > 0$.

Proposition 6. *Suppose that Assumption 1 holds. Then the next-generation operator $\mathscr{R}$ defined by (44) is compact and nonsupporting.*

Proof. Let $B \subset X_+$ be a bounded set. It follows from Assumption 1 that $\mathscr{R}(B)$ is bounded. To show that $\mathscr{R}$ is compact, it suffices to show that $\mathscr{R}(B)$ is equi-continuous by Arzelà-Ascoli theorem. For $x, \tilde{x} \in \Omega$, and $\phi \in B$,

$$\begin{aligned} (\mathscr{R}\phi)(x) - (\mathscr{R}\phi)(\tilde{x}) &= \left[\frac{p(x)}{\delta(x)(c(x)+\gamma(x))}\right. \\ &\left. - \frac{p(\tilde{x})}{\delta(\tilde{x})(c(\tilde{x})+\gamma(\tilde{x}))}\right] \int_\Omega f_v\left(x,y,T^0(x),0\right) \\ &\cdot \phi(y)\, dy + \frac{p(\tilde{x})}{\delta(\tilde{x})(c(\tilde{x})+\gamma(\tilde{x}))} \\ &\cdot \int_\Omega \left[f_v\left(x,y,T^0(x),0\right) - f_v\left(\tilde{x},y,T^0(x),0\right)\right] \\ &\cdot \phi(y)\, dy + \frac{p(\tilde{x})}{\delta(\tilde{x})(c(\tilde{x})+\gamma(\tilde{x}))} \\ &\cdot \int_\Omega \left[f_v\left(\tilde{x},y,T^0(x),0\right) - f_v\left(\tilde{x},y,T^0(\tilde{x}),0\right)\right] \\ &\cdot \phi(y)\, dy + \frac{\int_\Omega [\theta(x,y) - \theta(\tilde{x},y)]\, \phi(y)\, dy}{\gamma(x)+c(x)} \\ &+ \left[\frac{1}{\gamma(x)+c(x)} - \frac{1}{\gamma(\tilde{x})+c(\tilde{x})}\right] \int_\Omega \theta(\tilde{x},y) \\ &\cdot \phi(y)\, dy. \end{aligned} \tag{45}$$

Note that the uniform continuity by Assumption 1 and $\|\phi\|_X \le M$, where M is a positive constant. We can easily see that $\mathscr{R}(B)$ is equi-continuous.

Now, we show that $\mathscr{R}$ is nonsupporting. For any $\phi \in X_+ \backslash \{0\}$, by Proposition 3 and the monotonicity property of f, we have

$$(\mathscr{R}\phi)(x) \ge \frac{\int_\Omega \theta(x,y)\, \phi(y)\, dy}{\overline{c}+\overline{\gamma}}, \quad x \in \Omega. \tag{46}$$

From the assumption on θ, it follows that $(\mathscr{R}\phi)(x) > 0$ for $x \in \Omega$. Then, for any $\xi \in (X^*)_+ \backslash \{0,\}$, we have $\int_\Omega (\mathscr{R}\phi)(x)\xi(x)dx > 0$ (with $N_0 = N_0(\phi,\xi) = 1$). This proves that $\mathscr{R}$ is nonsupporting. □

From the definition of the basic reproduction number by Diekmann [33], such value is defined by

$$\mathscr{R}_0 := r(\mathscr{R}) \tag{47}$$

where $r(\cdot)$ represents the spectral radius of an operator. Hence, $\mathscr{R}(\cdot)$ can be considered as a next-generation operator in [33–35].

Now, in order to clarify the relationship between the next-generation operator and a linear operator, we define $\mathscr{L} : X \longrightarrow X$:

$$\begin{aligned} \mathscr{L}[\phi](x) &= \frac{p(x)}{\delta(x)} \int_\Omega f_v\left(x,y,T^0(x),0\right) \phi(y)\, dy \\ &+ \int_\Omega \theta(x,y)\, \phi(y)\, dy \\ &- (c(x)+\gamma(x))\, \phi(x) \end{aligned} \tag{48}$$

for $\phi \in X$ and $x \in \Omega$. From definitions of operators $\mathscr{R}$ and $\mathscr{L}$, we immediately see that $\mathscr{L} = (c+\gamma)(\mathscr{R} - Id)$ or $\mathscr{R} = Id + (1/(c+\gamma))\mathscr{L}$, where Id is the identity operator on X.

Theorem 7. *$r(\mathscr{R}) > 1$, $r(\mathscr{R}) < 1$, and $r(\mathscr{R}) = 1$ if and only if $s(\mathscr{L}) > 0$, $s(\mathscr{L}) < 0$, and $s(\mathscr{L}) = 0$, respectively, where $s(\mathscr{L}) = \sup\{\operatorname{Re}\xi : \xi \in \sigma(\mathscr{L})\}$ denotes the spectral bound of $\mathscr{L}$.*

Proof. First, suppose that $r(\mathscr{R}) = 1$. It follows from [36, Proposition 4.4] that there exists a positive function $\phi \in X$ such that $\mathscr{R}\phi = \phi$ or $(\mathscr{R} - Id)\phi = 0$. This implies that $(c+\gamma)(\mathscr{R} - Id)\phi = \mathscr{L}\phi = 0$. Therefore, $s(\mathscr{L}) = 0$. On the other hand, suppose that $s(\mathscr{L}) = 0$. Then, with the help of the irreducibility of θ and [37, Theorem 2.2], we conclude that there exists an eigenfunction $\phi > 0$ with respect to $s(\mathscr{L}) = 0$; namely, $\mathscr{L}\phi = 0$. It follows that $\mathscr{R}\phi = \phi + (1/(c+\gamma))\mathscr{L}\phi = \phi$, which gives $r(\mathscr{R}) = 1$. This proves theta $r(\mathscr{R}) = 1$ if and only if $s(\mathscr{L}) = 0$.

Now, we only show that $r(\mathscr{R}) > 1$ if and only if $s(\mathscr{R}) > 0$ as the proof of $r(\mathscr{R}) < 1$ if and only if $s(\mathscr{L}) < 0$ is similar. On one hand, let $r(\mathscr{R}) > 1$ hold. Then Proposition 6 implies that there exists a positive eigenfunction associated with $r(\mathscr{R}) > 1$; that is, $\mathscr{R}\phi = \phi + (1/(c+\gamma))\mathscr{L}\phi = r(\mathscr{R})\phi$. It follows that $\mathscr{L}\phi = (c+\gamma)(r(\mathscr{R}) - 1)\phi$. Based on Assumption 1, we know that $(c+\gamma)(r(\mathscr{R}) - 1) > 0$. It follows that $s(\mathscr{L}) > 0$. On the other hand, let $s(\mathscr{L}) > 0$. Following the above approach, we obtain the existence of a positive eigenfunction ϕ with respect to the eigenvalue $s(\mathscr{L}) > 0$; that is, $\mathscr{L}\phi = (c+\gamma)(\mathscr{R} - Id)\phi = s(\mathscr{L})\phi$. Then $\mathscr{R}\phi = ((1/(c+\gamma))s(\mathscr{L}) + 1)\phi > \phi$, which implies that $r(\mathscr{R}) > 1$. This completes the proof. □

5. Existence of Endemic Steady States

This section is conducted for the existence of endemic steady states of (2). Let $E^* = (T^*(\cdot), I^*(\cdot), v^*(\cdot)) \in Y_+$ be a feasible steady state and then it satisfies

$$\begin{aligned}
&\lambda(x) - \int_\Omega f(x, y, T^*(x), v^*(y))\,dy \\
&\quad - \mu(x) T^*(x) = 0, \\
&\int_\Omega f(x, y, T^*(x), v^*(y))\,dy - \delta(x) I^*(x) = 0, \\
&\int_\Omega \theta(x, y) v^*(y)\,dy - (c(x) + \gamma(x)) v^*(x) \\
&\quad + p(x) I^*(x) = 0.
\end{aligned} \tag{49}$$

We apply Shauder fixed point theorem to find solutions of (49). To overcome the difficulty caused by the nonlinear function f, we make the following modifications on f. For $n \in \mathbb{N}$, define

$$f_n(x, y, T, v) = f(x, y, T, v \wedge n) \quad \text{for } (x, y, T, v) \in \Omega^2 \times \mathbb{R}_+^2, \tag{50}$$

where $v \wedge n = \min\{v, n\}$. We focus on the following perturbation systems:

$$\begin{aligned}
&\lambda(x) - \int_\Omega f_n(x, y, T^*(x), v^*(y))\,dy \\
&\quad - \mu(x) T^*(x) = 0, \\
&\int_\Omega f_n(x, y, T^*(x), v^*(y))\,dy - \delta(x) I^*(x) = 0, \\
&\int_\Omega \theta(x, y) v^*(y)\,dy - (c(x) + \gamma(x))(v^*(x) - \epsilon_n) \\
&\quad + p(x) I^*(x) = 0,
\end{aligned} \tag{51}$$

where $\{\epsilon_n\}$ is a bounded and decreasing sequence in $\mathbb{R}_+$ with $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Define

$$\begin{aligned}
g_n(T, v) = {}& \lambda(x) - \int_\Omega f_n(x, y, T(x), v(y))\,dy \\
& - \mu(x) T(x).
\end{aligned} \tag{52}$$

Since f is, and so is f_n, increasing in T, for each $x \in \Omega$ and $v \in X_+$ there exists a unique $G_n(v)(x) \in [0, T^0(x)]$ such that

$$\begin{aligned}
&\lambda(x) - \int_\Omega f_n(x, y, G_n(v)(x), v(y))\,dy \\
&\quad - \mu(x) G_n(v)(x) = 0.
\end{aligned} \tag{53}$$

Lemma 8 ([26, Lemma 7.3]). *For fixed $n \in \mathbb{N}$, $G_n(v)$ is continuous with respect to x, uniformly for v in bounded subsets of X_+.*

Theorem 9. *Suppose that $\mathscr{R}_0 = r(\mathscr{R}) > 1$ holds. Then system (2) admits at least one feasible endemic steady state $E^* = (T^*, I^*, v^*) \in Y_+$ with $v^* \in X_+ \setminus \{0\}$.*

Proof. Solving the second equation of (51) yields $I^*(x) = (\int_\Omega f_n(x, y, T^*(x), v^*(y))dy)/\delta(x)$. Then substitute this expression of $I^*(x)$ into the third equation of (51) to get

$$\begin{aligned}
&\int_\Omega \theta(x, y) v^*(y)\,dy - (c(x) + \gamma(x))(v^*(x) - \epsilon_n) \\
&\quad + \frac{p(x)\int_\Omega f_n(x, y, T^*(x), v^*(y))\,dy}{\delta(x)} = 0.
\end{aligned} \tag{54}$$

Therefore, we can define a map $\mathscr{F}_n (n \in \mathbb{N}) : X_+ \longrightarrow X_+$ by

$$\begin{aligned}
&\mathscr{F}_n(v)(x) \\
&\quad = \frac{p(x)\int_\Omega f(x, y, G_n(v)(x), \min\{v(y), n\})\,dy}{\delta(x)(c(x) + \gamma(x))} \\
&\qquad + \frac{\int_\Omega \theta(x, y)\min\{v(y), n\}\,dy}{c(x) + \gamma(x)} + \epsilon_n
\end{aligned} \tag{55}$$

for $v \in X_+$ and $x \in \Omega$. Clearly, $\mathcal{F}_n$ is continuous and nonincreasing by Assumption 1 and Lemma 8. Furthermore, for all $v \in X_+$,

$$\mathcal{F}_n(v)(x) \le \frac{n\int_\Omega \theta(x,y)\,dy}{c(x)+\gamma(x)} + \frac{p(x)\int_\Omega f(x,y,G_n(v)(x),n)\,dy}{\delta(x)(c(x)+\gamma(x))} + \epsilon_1 \quad \text{for } v \in X_+ \text{ and } x \in \Omega. \tag{56}$$

Let $B_n = \{v \in X_+ : \overline{v} \le r_n\}$ be a closed nonnegative ball with the radius $r_n \gg 1$ and the center at 0. If we pick up large enough r_n, then $\mathcal{F}_n$ maps B_n into itself. Since B_n is closed and convex, applying Shauder fixed theorem gives the existence of some $v \in B_n$ such that $\mathcal{F}_n(v) = v$. Note that $v \ne 0$. Then $T = G_n(v)$, $I = (\int_\Omega f(x,y,G_n(v)(x),v(y))dy)/\delta(x)$, and v satisfy

$$\begin{aligned}
&\lambda(x) - \int_\Omega f(x,y,T(x),v(y))\,dy - \mu(x)T(x) = 0,\\
&\int_\Omega f(x,y,T(x),v(y))\,dy - \delta(x)I(x) = 0,\\
&\int_\Omega \theta(x,y)\min\{v(y),n\}\,dy\\
&\quad - (c(x)+\gamma(x))(v(x)-\epsilon_n) + p(x)I(x) = 0.
\end{aligned} \tag{57}$$

Adding the first two equations of (57) gives

$$Z(x) \le \frac{\lambda(x)}{\nu(x)}, \tag{58}$$

where $Z(x) = T(x) + I(x)$ and $\nu = \min\{\mu(x),\delta(x)\}$. Integrating the last equation of (57) on Ω with respect to x, together with $\int_\Omega \theta(x,y)\min\{v(y),n\}dy \le \int_\Omega \theta(x,y)v(y)dy$, we have

$$\int_\Omega c(x)v(x)\,dx \le \int_\Omega p(x)I(x)\,dx + \epsilon_n\int_\Omega (c(x)+\gamma(x))\,dx. \tag{59}$$

Consequently, $\{\int_\Omega v(x)dx\}$ is bounded. Thus $\{v\}$ is uniformly bounded. Furthermore, we can apply the similar argument as that in the proof of Proposition 6 to show that $\{v\}$ is equicontinuous. Thus it is precompact in X_+. Then we can choose a subsequence, say itself, such that $(T,I,v) \longrightarrow (T^*,I^*,v^*)$ uniformly in $x \in \Omega$ as $n \longrightarrow \infty$. Letting $n \longrightarrow \infty$ in (57), we see that (T^*,I^*,v^*) is a steady state of system (2).

Now, we show that (T^*,I^*,v^*) is an endemic steady state by showing that $v^* \ne 0$. By way of contradiction, assume that $v^* \equiv 0$, which implies that $I(x) \longrightarrow 0$, and $T(x) \longrightarrow T^0(x)$ uniformly for $x \in \Omega$ as $n \longrightarrow \infty$ and $v \longrightarrow 0$. For large enough n, we know that $v(x) \le n$ for all $x \in \Omega$. Define

$$\mathcal{G}(T,v)(x) = \frac{\int_\Omega \theta(x,y)v(y)\,dy}{c(x)+\gamma(x)} + \frac{p(x)\int_\Omega f(x,y,T(x),v(y))\,dy}{\delta(x)(c(x)+\gamma(x))} \quad \text{for } x \in \Omega. \tag{60}$$

Then $v = \mathcal{F}(v) = \mathcal{G}(T,v) + \epsilon_n$. It follows that

$$\frac{v}{\|v\|_X} = \frac{\mathcal{G}(T,v) - \mathcal{R}(v)}{\|v\|_X} + \mathcal{R}\left(\frac{v}{\|v\|_X}\right) + \frac{\epsilon_n}{\|v\|_X}. \tag{61}$$

Note that

$$\begin{aligned}
\frac{\|\mathcal{G}(T,v) - \mathcal{R}(v)\|_X}{\|v\|_X} &= \sup_{x\in\Omega}\frac{p(x)}{\delta(x)(c(x)+\gamma(x))}\\
&\quad\cdot \int_\Omega \left|\frac{f(x,y,T(x),v(y))}{v(y)} - f_v(x,y,T^0(x),0)\right|\\
&\quad\cdot \frac{v(y)}{\|v\|_X}dy \le \sup_{x\in\Omega}\frac{p(x)}{\delta(x)(c(x)+\gamma(x))}\\
&\quad\cdot \int_\Omega \left|\frac{f(x,y,T(x),v(y))}{v(y)}\right.\\
&\quad\left. - f_v(x,y,T(x),0)\right| dy\\
&\quad + \sup_{x\in\Omega}\frac{p(x)}{\delta(x)(c(x)+\gamma(x))}\int_\Omega \left|f_v(x,y,T(x),0)\right.\\
&\quad\left. - f_v(x,y,T^0(x),0)\right| dy.
\end{aligned} \tag{62}$$

Lemma 8, together with Assumption 1, implies that the above inequality converges to zero as $n \longrightarrow \infty$. From the compactness of $\mathcal{R}$ and boundedness of $\epsilon_n/\|v\|_X$, we can pick up a subsequence, again say itself, such that

$$\mathcal{R}\left(\frac{\epsilon_n}{\|v\|_X}\right) \longrightarrow L \quad \text{and } \frac{\epsilon_n}{\|v\|_X} \longrightarrow \xi \quad \text{as } n \longrightarrow \infty \tag{63}$$

for some $\xi \ge 0$ and $L \in X_+$. Set $u_n = v/\|v\|_X$. Then $u_n \longrightarrow u$ as $n \longrightarrow \infty$ in X. Proposition 6 ensures that $u \in X_+$. By Krein-Rutman theorem, there exists some $u^* \in X^* \setminus \{0\}$ and $u^* \ge 0$ such that $\mathcal{R}^*u^* = \mathcal{R}_0u^*$. Hence

$$\begin{aligned}
\int_\Omega u(x)\,du^*(x) &= \int_\Omega \mathcal{R}(u)(x)\,du^*(x) + \xi u^*(\Omega)\\
&\ge \int_\Omega u(x)\,d\mathcal{R}^*u^*(x)\\
&= \mathcal{R}_0\int_\Omega u(x)\,du^*(x).
\end{aligned} \tag{64}$$

This leads to a contradiction with $\mathcal{R}_0 > 1$. □

6. Stability of the Virus-Free Steady State E_0

The objective of this section is to establish the stability of the virus-free steady state $E_0 = (T^0, 0, 0)$.

Lemma 10. *If $T_0(x) \le T^0(x)$, then $T(t,x) \le T^0(x)$ for all $(t,x) \in \mathbb{R}_+ \times \Omega$.*

Proof. Define $w(t) = T(t,x) - T^0(x)$. Noting that $\lambda(x) = \mu(x)T^0(x)$, we get

$$\frac{\partial w(t,x)}{\partial t} = -\left(\mu(x) + \int_\Omega f(x,y,T(t,x),v(t,y))\,dy\right) \cdot w(t,x) - T^0(x)\int_\Omega f(x,y,T(t,x),v(t,y))\,dy. \quad (65)$$

Solving this equation, we have

$$w(t,x) = w_0(x)\,e^{-\int_0^t(\mu(x)+\int_\Omega f(x,y,T(s,x),v(s,y))dy)ds} - T^0(x)\int_0^t e^{-\int_s^t(\mu+\int_\Omega f(x,y,T(\xi,x),v(\xi,y))dy)d\xi} \cdot \int_\Omega f(x,y,T(s,x),v(s,y))\,dyds. \quad (66)$$

Because f is nonnegative on $\Omega^2 \times \mathbb{R}_+^2$ by Assumption 1(ii) and $w_0(x) = T_0(x) - T^0(x) \le 0$, we conclude that, for all $(t,x) \in (\mathbb{R}_+ \times \Omega)$, $w(t,x) = T(t,x) - T^0(x) \le 0$. □

Define $\widehat{\Phi} : D(\widehat{\Phi}) \longrightarrow X_+^2$ by

$$\widehat{\Phi}\begin{pmatrix}\psi_I\\ \psi_v\end{pmatrix}(x) = \begin{pmatrix}\int_\Omega f_v(x,y,T^0(x),0)\,\psi_v(y)\,dy - \delta(x)\,\psi_I(x)\\ \int_\Omega \theta(x,y)\,\psi_v(y)\,dy - (c(x)+\gamma(x))\,\psi_v(x) + p(x)\,\psi_I(x)\end{pmatrix}, \quad (67)$$

where $(\psi_I, \psi_v) \in D(\widehat{\Phi})$ and $x \in \Omega$. We can separate $\widehat{\Phi}$ into two operators $\widehat{F}$ and $\widehat{A}$ defined by

$$\widehat{F}\begin{pmatrix}\psi_I\\ \psi_v\end{pmatrix}(x) = \begin{pmatrix}0\\ \int_\Omega \theta(x,y)\,\psi_v(y)\,dy + p(x)\,\psi_I(x)\end{pmatrix} \quad (68)$$

and

$$\widehat{A}\begin{pmatrix}\psi_I\\ \psi_v\end{pmatrix}(x) = \begin{pmatrix}\int_\Omega f_v(x,y,T^0(x),0)\,\psi_v(y)\,dy - \delta(x)\,\psi_I(x)\\ -(c(x)+\gamma(x))\,\psi_v(x)\end{pmatrix}, \quad (69)$$

respectively. Denote $\widehat{A} = (\widehat{A}_I, \widehat{A}_v)^T$. Observe that $\widehat{A}_v$ actually acts on ψ_v by

$$\widehat{A}_v\psi_v = -(c(x)+\gamma(x))\,\psi_v, \quad \psi_v \in X,\ x \in \Omega. \quad (70)$$

Lemma 11. *If the problem*

$$\widehat{A}\psi = \eta\widehat{F}\psi, \quad \psi \in X^2 \quad (71)$$

has a positive eigenvalue η^0 with a positive eigenfunction ψ, then

$$\mathcal{R}_0 = r\left(-\widehat{F}\widehat{A}^{-1}\right) = r\left(-\widehat{A}^{-1}\widehat{F}\right) = \frac{1}{\eta^0}. \quad (72)$$

Proof. Let $\eta \in \rho(\widehat{A})$, where $\rho(\widehat{A})$ represents the resolvent set of the operator $\widehat{A}$. For any $\phi \in X^2$, let $\psi \in D(\widehat{A})$ such that

$$\psi = \left(\eta Id - \widehat{A}\right)^{-1}(\phi). \quad (73)$$

Then

$$\psi_I = \frac{1}{\eta+\delta}\left(\phi_I + \int_\Omega f_v(x,y,T^0(x),0)\,\psi_v(y)\,dy\right) \quad (74)$$

and

$$\psi_v = \frac{\phi_v}{\eta+c+\gamma}. \quad (75)$$

Let Z be a positive semigroup generated by the operator $\widehat{A}$. Then

$$\left(\eta Id - \widehat{A}\right)^{-1}\phi = \int_0^\infty e^{-\eta t}Z(t,\phi)\,dt, \quad \phi \in X^2. \quad (76)$$

Without loss of the generality, letting $\eta = 0$ and $\phi_I = 0$ gives $-\widehat{A}^{-1}\phi = \int_0^\infty Z(t,\phi)dt$ for each $\phi \in X^2$. By the definition of $\widehat{F}$, we have

$$\widehat{F}\left(-\widehat{A}\right)^{-1}\phi(x) = \begin{pmatrix}0\\ \dfrac{p(x)\int_\Omega f_v(x,y,T^0(x),0)\,\phi_v(t,y)\,dy}{\delta(x)(c(x)+\gamma(x))} + \dfrac{\int_\Omega \theta(x,y)\,\phi_v(y)\,dy}{c(x)+\gamma(x)}\end{pmatrix} = \begin{pmatrix}0\\ \mathcal{R}[\phi_v](x)\end{pmatrix}. \quad (77)$$

Let $\widetilde{\psi} := -\widehat{A}^{-1}\phi$. Then the above equality can be rewritten:

$$\widehat{F}[\widetilde{\psi}](x) = \begin{pmatrix} 0 \\ -\mathscr{R}\widehat{A}_v[\widetilde{\psi}_v](x) \end{pmatrix}. \quad (78)$$

This means that

$$\begin{pmatrix} 0 \\ -\widehat{A}_v[\widetilde{\psi}_v](x) \end{pmatrix} = (\mathscr{R})^{-1}\widehat{F}[\widetilde{\psi}](x). \quad (79)$$

Following the approach in [38], we derive that the eigenvalue problem

$$\widehat{A}(x)\psi = \eta\widehat{F}(x)\psi, \quad x \in \Omega \quad (80)$$

has a positive eigenvalue η^0 with a positive eigenvector $\psi(x)$ for $x \in \Omega$. Since $\widetilde{\psi}$ is positive, it follows that $\eta^0 = 1/\mathscr{R}_0$. □

Theorem 12. *Suppose that $\mathscr{R}_0 < 1$. Then the virus-free steady state $E_0 = (T^0(x), 0, 0)$ is locally asymptotically stable.*

Proof. Linearizing system (2) around the virus-free steady state E_0, we have

$$\begin{aligned}
\frac{\partial T(t,x)}{\partial t} &= -\mu(x)T(t,x) \\
&\quad - \int_\Omega f_v\left(x,y,T^0(x),0\right)v(t,y)\,dy, \\
\frac{\partial I(t,x)}{\partial t} &= \int_\Omega f_v\left(x,y,T^0(x),0\right)v(t,y)\,dy \\
&\quad - \delta(x)I(t,x), \\
\frac{\partial v(t,x)}{\partial t} &= \int_\Omega \theta(x,y)v(t,y)\,dy \\
&\quad - \left(c(x)+\gamma(x)\right)v(t,x) \\
&\quad + p(x)I(t,x).
\end{aligned} \quad (81)$$

Let $T(t,x) = \psi_T(x)e^{\eta t}$, $I(t,x) = \psi_I(x)e^{\eta t}$, and $v(t,x) = \psi_v(x)e^{\eta t}$ be a solution of system (81). After substitution, we arrive at

$$\begin{aligned}
\eta\psi_T(x) &= -\mu(x)\psi_T(x) \\
&\quad - \int_\Omega f_v\left(x,y,T^0(x),0\right)\psi_v(y)\,dy, \\
\eta\psi_I(x) &= \int_\Omega f_v\left(x,y,T^0(x),0\right)\psi_v(y)\,dy \\
&\quad - \delta(x)\psi_I(x), \\
\eta\psi_v(x) &= \int_\Omega \theta(x,y)\psi_v(y)\,dy \\
&\quad - \left(c(x)+\gamma(x)\right)\psi_v(x) + p(x)\psi_I(x).
\end{aligned} \quad (82)$$

Observe that the first equation of system (82) decouples with the other two equations. Furthermore, the last two equations can be rewritten in the form

$$\left(\eta Id - \widehat{A}\right)[\psi](x) = \widehat{F}[\psi](x) \quad (83)$$

where $\widehat{F}$ and $\widehat{A}$ are defined by (68) and (69), respectively. By [Theorem 2.2, [39]], the eigenvalue problem (83) has a principal eigenvalue η^0 with a unique positive eigenfunction. Lemma 11 implies that $\eta^0 < 0$ if $\mathscr{R}_0 < 1$. Otherwise, $(\psi_I, \psi_v) = (0,0)$ and then it follows from the first equation of (82) that $\eta = 0$. Therefore, local stability of the virus-free steady state E_0 follows immediately. □

Next, we show the global attractivity of the virus-free steady state E_0. To establish this result, we need the following Volterra-type:

$$g(z) = z - 1 - \ln z, \quad z > 0. \quad (84)$$

g has the property as follows: for all $z > 0$ $g(z) \geq 0$ and the quality holds if and only if $z = 1$. This function has been used very often to construct Lyapunov functionals (see, e.g., Yang et al. [40], Kuniya and Wang [29], and McCluskey [41] and the references therein).

Theorem 13. *Suppose that $\mathscr{R}_0 < 1$ and $u \in \Gamma$. The virus-free steady state E_0 is globally asymptotically stable.*

Proof. For $(t,x) \in \mathbb{R}_+ \times \Omega$, we consider the following Lyapunov functional:

$$\widehat{V}[u](t,x) = \Delta(x)V_T(t,x) + V_I(t,x) + V_v(t,x), \quad (85)$$

where $V_I(t,x) = I(t,x)$, $V_v(t,x) = v(t,x)$, $V_T(,x) = g(T(t,x)/T^0(x))$, and $\Delta(x)$ is a weighted positive function to be determined later. Based on Lemma 10, $\widehat{V}$ is well defined.

Taking the derivative of $V_T(t,x)$ along solutions of (2) with respect to t, one arrives at

$$\begin{aligned}
\left.\frac{\partial V_T(t,x)}{\partial t}\right|_{(2)} &= \frac{1}{T^0(x)}\left(1 - \frac{T_0(x)}{T(t,x)}\right)\frac{\partial T(t,x)}{\partial t} \\
&= \frac{1}{T^0(x)}\left(1 - \frac{T_0(x)}{T(t,x)}\right)\Big(\lambda(x) \\
&\quad - \int_\Omega f(x,y,T(t,x),v(t,y)\,dy - \mu T(t,x)\Big) \\
&= \frac{1}{T^0(x)}\left(1 - \frac{T_0(x)}{T(t,x)}\right)\Big(\mu\left(T^0(x) - T(t,x)\right) \\
&\quad - \int_\Omega f(x,y,T(t,x),v(t,y)\,dy\Big) \\
&= \frac{1}{T^0(x)}\Bigg[-\frac{\mu\left(T(t,x) - T^0(x)\right)^2}{T(t,x)} \\
&\quad - \int_\Omega f\left(x,y,T(t,x),v(t,y)\right)dy \\
&\quad + \frac{T^0(x)}{T(t,x)}\int_\Omega f\left(x,y,T(t,x),v(t,y)\right)dy\Bigg].
\end{aligned} \quad (86)$$

Similarly, we get

$$\left.\frac{\partial V_I(t,x)}{\partial t}\right|_{(2)} = \int_\Omega f(x,y,T(t,x),v(t,y))\,dy - \delta(x) I(t,x) \tag{87}$$

and

$$\left.\frac{\partial V_v(t,x)}{\partial t}\right|_{(2)} = \int_\Omega \theta(x,y) v(t,y)\,dy + p(x) I(t,x) - (c(x)+\gamma(x)) v(t,x). \tag{88}$$

Therefore, we can pick up $\Delta(x) = T^0(x)$ and take a Lyapunov functional in the form of

$$V[u](t) = \int_\Omega \frac{p(x)\psi_v(x)}{\delta(x)}\left[T^0(x) V_T(t,x) + V_I(t,x) + V_v(t,x)\right] dx, \tag{89}$$

where ψ_v defined in (82) is a positive eigenvector function associated with eigenvalue η^0. Then

$$\begin{aligned}\left.\frac{dV[u](t)}{dt}\right|_{(2)} &= \int_\Omega \left[\frac{p(x)\psi_v(x)}{\delta(x)}\left(T^0(x)\frac{\partial V_T(t,x)}{\partial t} + \frac{\partial V_I(t,x)}{\partial t}\right) + \frac{\partial V_v(t,x)}{\partial t}\right] dx \le \int_\Omega \psi_v(x) \\ &\cdot \left[\frac{p(x)}{\delta(x)}\int_\Omega f(x,y,T^0(x),0) v(t,y)\,dy + \int_\Omega \theta(x,y) v(t,y)\,dy - (\gamma(x)+c(x)) \cdot v(t,x)\right] dx = \int_\Omega \mathscr{L}[\psi_v](x) v(t,x)\,dx \\ &= \eta^0 \int_\Omega \psi_v(x) v(t,x)\,dx,\end{aligned} \tag{90}$$

where $\mathscr{L}$ is defined by (48). It follows from Theorem 7 that $dV[u](t)/dt \le 0$ provided that $\mathscr{R}_0 < 1$. The equality holds if and only if $v(t,x) = 0$ and $T(t,x) = T^0(x)$ for each $x \in \Omega$. Therefore, the largest invariant set in $M = \{u \in \Gamma \mid \dot V(t) = 0\}$ is $\{E_0\}$. We employ the LaSalle invariant principle to conclude that E_0 is globally asymptotically stable. □

7. Uniform Persistence

Denote

$$\Gamma_0 = \{\psi \in \Gamma \mid \psi_3 \not\equiv 0\}, \quad \partial\Gamma_0 = \{\psi \in \Gamma \mid \psi_3 = 0\}. \tag{91}$$

Let

$$M_\partial = \{\psi \in \partial\Gamma_0 \mid \Phi(t,\psi) \in \partial\Gamma_0 \text{ for } t \in \mathbb{R}_+\}, \tag{92}$$

And let $O_+(\phi)$ be the positive orbit $\{\Phi_t\phi\}_{t\ge0}$. From Theorem 13, we readily have the following result.

Lemma 14. *For every $u_0 \in M_\partial$, $\omega(\Phi(t,u_0)) = \{E_0\}$, where $\omega(\Phi(t,u_0))$ represents the omega limit set of $O_+(\phi)$.*

The semiflow $\{\Phi_t\}_{t\ge0}$ is said to be persistent associated with $(\Gamma_0, \partial\Gamma_0)$ if there exists an ϵ such that

$$\liminf_{t\to\infty} \underline{T(t,\cdot)} \ge \epsilon, \quad \liminf_{t\to\infty} \underline{I(t,\cdot)} \ge \epsilon, \quad \liminf_{t\to\infty} \underline{v(t,\cdot)} \ge \epsilon \tag{93}$$

for any solution $u(t) = (T(t,\cdot), I(t,\cdot), v(t,\cdot))$ with $u(0) \in \Gamma_0$.

Lemma 15. *Let $u(t) = (T(t,\cdot), I(t,\cdot), v(t,\cdot))$ be any solution of system (2) with the initial value $u_0 \in \Gamma_0$. Then $u(t,\cdot) > 0$ for all $t > 0$.*

Proof. Proposition 3(iii) has already asserted that $T(t,\cdot) > 0$ for $t > 0$. Now we show that $I(t,\cdot) > 0$ and $v(t,\cdot) > 0$ for all $t > 0$. Moreover, since $u_0 = (\psi_1,\psi_2,\psi_3) \in \Gamma_0$, we have $\psi_3 \not\equiv 0$. Then a similar argument to that for the proof of Proposition 3(iii) can also give $v(t,\cdot) > 0$ for $t > 0$. Now, from the second equation of (2), it follows that, for $t > 0$,

$$I(t,\cdot) \ge e^{-\delta(x)t}\int_0^t \int_\Omega f(\cdot,y,T(s,\cdot),v(s,y))\,dy\,ds > 0. \tag{94}$$

□

Lemma 15 implies that Γ_0 is a positive invariant of (2). Let $\Phi|_{\Gamma_0}$ be the restricted semiflow of Φ on $\mathbb{R}_+ \times \Gamma_0$.

Lemma 16. *Suppose that $\mathscr{R}_0 > 1$. Then system (2) is uniformly weakly persistent. That is to say that there exists a positive value ϵ such that, for all $\psi \in \Gamma_0$,*

$$\limsup_{t\to\infty} \|\Phi(t,\psi) - E_0\|_\infty \ge \epsilon. \tag{95}$$

Proof. For any $\epsilon > 0$, define $\widehat{F}_\epsilon$ by

$$\widehat{F}_\epsilon \begin{pmatrix}\psi_I \\ \psi_v\end{pmatrix}(x) = \begin{pmatrix} \displaystyle\int_\Omega \frac{f(x,y,T^0(x)-\epsilon,\epsilon)}{\epsilon}\psi_v(y)\,dy \\ 0 \end{pmatrix}. \tag{96}$$

Clearly, $\widehat{F}_\epsilon \longrightarrow \widehat{F}$ as $\epsilon \longrightarrow 0^+$. Note that $s(\widehat{\Phi}) = r(\widehat{F} - \widehat{A}) > 0$ since $\mathscr{R}_0 > 1$. Then, for small enough ϵ_0, $\eta\psi = (\widehat{F}_{\epsilon_0} - \widehat{A})\psi$ has a positive principle eigenvalue $\eta_{\epsilon_0} > 0$ with an associated eigenvector function $\psi_{\epsilon_0} > 0$.

By way of contradiction, assume that the conclusion fails. Then, for ϵ_0 chosen above, there exists $\psi^{\varepsilon_0} \in \Gamma_0$ (will be denoted by ψ for simplicity of notation) such that

$$\limsup_{t\to\infty} \|\Phi(t,\psi) - E_0\|_Y < \frac{\epsilon_0}{2}. \tag{97}$$

Then, with a possible shift in time t and by Lemma 15, we can assume that, for $t \in \mathbb{R}_+$,

$$0 < v(t,x) \le \epsilon_0 \quad \text{and } T^0(x) - \epsilon_0 \le T(t,x) \le T^0(x) + \epsilon_0. \tag{98}$$

Then, by the monotonicity of f in T and v, we get

$$\frac{\partial I(t,x)}{\partial t} \ge \int_\Omega f\left(x, y, T^0(x) - \epsilon_0, v(t,y)\right) dy - \delta(x) I(t,x). \tag{99}$$

From Assumption 1 (ii-3), we obtain

$$\frac{\partial I(t,x)}{\partial t} \ge \int_\Omega \frac{f\left(x, y, T^0(x) - \epsilon_0, \epsilon_0\right)}{\epsilon_0} v(t,y)\, dy - \delta(x) I(t,x). \tag{100}$$

Let $\widetilde{u} = (\widetilde{I}, \widetilde{v})$ be the solution of the following auxiliary system:

$$\begin{aligned}
\frac{\partial \widetilde{I}(t,x)}{\partial t} &= \int_\Omega \frac{f\left(x, y, T^0(x) - \epsilon_0, \epsilon_0\right)}{\epsilon_0} \widetilde{v}(t,y)\, dy - \delta(x) \widetilde{I}(t,x), \quad x \in \Omega, \\
\frac{\partial \widetilde{v}(t,x)}{\partial t} &= \int_\Omega \theta(x,y) \widetilde{v}(t,y)\, dy - (\gamma(x) + c(x)) \widetilde{v}(t,x) + p(x) \widetilde{I}(t,x), \quad x \in \Omega
\end{aligned} \tag{101}$$

with $\widetilde{\psi} = (\widetilde{\psi}(0,\cdot), \widetilde{v}(0,\cdot)) = (\psi_2, \psi_3)$. Then we have $u(t,\cdot) \ge \widetilde{u}(t,\cdot)$ for $t \in \mathbb{R}_+$. Let $\xi > 0$ such that $\psi_2 \ge \xi \psi_{I\epsilon_0}$ and $\psi_3 \ge \xi \psi_{v\epsilon_0}$. By comparison principle, we get

$$u(t,\cdot) \ge \widetilde{u}(t,\cdot) \ge \xi e^{\eta_{\epsilon_0} t} \widetilde{\psi}(\cdot) \tag{102}$$

for $t \in \mathbb{R}_+$. This contradicts with the fact that $(I(t,\cdot), v(t,\cdot))$ is bounded and hence the proof is complete. □

With the help of Lemmas 15 and 16, we can establish the uniform persistence.

Theorem 17. *If $\mathscr{R}_0 > 1$ and $u_0 \in \Gamma_0$, then system (2) is uniformly strongly persistent and hence the deduced semiflow $\Phi|_{\Gamma_0}$ has a global attractor $\mathscr{A}$ in Γ_0.*

Proof. Define $\rho : Y_+ \longrightarrow \mathbb{R}_+$ by

$$\rho(\psi) = \underline{\psi_3} \quad \text{for } \psi = (\psi_1, \psi_2, \psi_3) \in Y_+. \tag{103}$$

Clearly, $\rho^{-1}(\mathbb{R}_+) \subset \Gamma_0$. It follows from Lemma 15 that ρ has the property that $\rho(\Phi(t,\psi)) > 0$ for $t > 0$ if $\rho(\psi) > 0$ or $\psi \in \Gamma_0$ with $\rho(\psi) = 0$. Lemma 14 ensures that, for any $\psi \in M_\partial$, we have $\lim_{t \longrightarrow \infty} \Phi(t,\psi) = E_0$ and hence there is no cycle in $\partial\Gamma_0$ from E_0 to itself. Moreover, Lemma 16 tells us that $W^S(E_0) \cap \Gamma_0 = \emptyset$, where $W^S(E_0)$ is the stable manifold of E_0. Applying [42, Theorem 3] gives a positive number ϵ_v such that

$$\min_{\psi \in \omega(\phi)} \rho(\psi) > \epsilon_v \quad \text{for all } \phi \in \Gamma_0, \tag{104}$$

that is

$$\liminf_{t \longrightarrow \infty} \underline{v(t,\cdot)} > \widetilde{\epsilon}_v. \tag{105}$$

Since f is continuously differentiable and system (2) is dissipative, there exists a positive constant M such that

$$\frac{\partial T(t,\cdot)}{\partial t} \ge \lambda(\cdot) - (\mu(\cdot) + M) T(t,\cdot). \tag{106}$$

Using comparison principle again, one admits

$$\liminf_{t \longrightarrow \infty} \underline{T(t,\cdot)} \ge \frac{\underline{\lambda}}{\overline{\mu} + M} := \epsilon_T. \tag{107}$$

Then, it follows from the second equation of system (2) that

$$\frac{\partial I(t,x)}{\partial t} \ge \int_\Omega f(x, y, \epsilon_T, \epsilon_v)\, dy - \delta(x) I(t,x). \tag{108}$$

Therefore,

$$\liminf_{t \longrightarrow \infty} \underline{I(t,\cdot)} \ge \frac{\sup_{x \in \Omega} \int_\Omega f(x, y, \epsilon_T, \epsilon_v)\, dy}{\overline{\delta}} := \epsilon_I. \tag{109}$$

Letting $\epsilon = \{\epsilon_T, \epsilon_I, \widetilde{\epsilon}_v\}$ completes the proof. □

8. Global Attractivity of Endemic Steady State

This section is devoted to the uniqueness and the global stability of endemic steady states under $\mathscr{R}_0 > 1$ and some additional condition on f. The approach is Lyapunov functional method. Recall that the semiflow Φ on Y_+ is bounded. Then the following result, which follows directly from this and Theorem 17, is helpful to construct the Lyapunov functional.

Lemma 18. *Suppose that $u(t,x) = (T(t,x), I(t,x), v(t,x)) \in \Gamma_0$ is a solution and $E^* = (T^*, I^*, v^*)$ is an endemic steady state of system (2). Then there exist two positive numbers δ_1 and δ_2 such that*

$$\delta_1 \le \frac{T(t,x)}{T^*(x)}, \frac{I(t,x)}{I^*(x)}, \frac{v(t,x)}{v^*(x)} \le \delta_2 \quad \text{for } t > 0 \text{ and } x \in \Omega. \tag{110}$$

In order to establish the global attractivity of an endemic steady state, we impose the following hypothesis.

Assumption 19. Suppose that an endemic steady state $E^* = (T^*, I^*, v^*)$ of system (2) satisfies

$$\left(\frac{T^*(x)}{T} - \frac{f(x, y, T^*(x), v^*(y))}{f(x, y, T, v)}\right) \cdot \left(\frac{T^*(x)}{T} - \frac{f(x, y, T^*(x), v^*(y))\, v}{f(x, y, T, v)\, v^*(y)}\right) \le 0 \tag{111}$$

for all $x, y \in \Omega$ and $T, I, v > 0$.

Assumption 19 is obviously satisfied if

$$f(x,y,T(t,x),v(t,x)) = \beta(x)T(t,x)v^{p}(t,x) \tag{112}$$

or

$$f(x,y,T(t,x),v(t,x)) = \frac{\beta(x)T(t,x)v^{p}(t,x)}{1+v^{p}(t,x)}, \tag{113}$$

where $p \in (0,1]$.

Theorem 20. *Suppose that $\mathscr{R}_0 > 1$ and $u_0 \in \Gamma_0$. Under Assumption 19 and $f(x,y,T(x),v(y)) = f(T(x),v(x))$, the endemic steady state E^* of system (2) is globally attractive.*

Proof. Consider the Lyapunov functional defined by

$$V[u](t) = \int_{\Omega}\mathscr{G}(x)U(t,x)\,dx, \tag{114}$$

where $\mathscr{G}(x)$ is a strictly positive function to be defined later and

$$U(t,x) = V_T(t,x) + V_I(t,x) + \frac{\delta(x)}{p(x)}V_v(t,x) \tag{115}$$

with

$$\begin{aligned} V_T(t,x) &= T^*(x)\,g\left(\frac{T(t,x)}{T^*(x)}\right),\\ V_I(t,x) &= I^*(x)\,g\left(\frac{I(t,x)}{I^*(x)}\right),\\ V_v(t,x) &= v^*(x)\,g\left(\frac{v(t,x)}{v^*(x)}\right), \end{aligned} \tag{116}$$

and g is defined as in the proof of Theorem 13. Lemmas 18 and 15 ensure that $V_j(t,x)$ $(j = T, I, v)$ are well defined for $(t,x) \in (\mathbb{R}_+ \times \Omega)$.

Differentiating $V_T(t,x)$ along trajectories of system (2) yields

$$\begin{aligned} \left.\frac{\partial V_T(t,x)}{\partial t}\right|_{(2)} &= \left(1-\frac{T^*(x)}{T(t,x)}\right)\frac{\partial T(t,x)}{\partial t} = \Bigg(1\\ &\quad -\frac{T^*(x)}{T(t,x)}\Bigg)\Bigg(\lambda(x)-\mu(x)T(t,x)\\ &\quad -\int_{\Omega}f(x,y,T(t,x),v(t,y))\,dy\Bigg) = \mu(x)T^*(x)\\ &\quad \cdot\left(2-\frac{T^*(x)}{T(t,x)}-\frac{T(t,x)}{T^*(x)}\right)+\left(1-\frac{T^*(x)}{T(t,x)}\right)\\ &\quad \cdot\Bigg(\int_{\Omega}f(x,y,T^*(x),v^*(y))\,dy\\ &\quad -\int_{\Omega}f(x,y,T(t,x),v(t,y))\,dy\Bigg)\\ &\le \int_{\Omega}f(x,y,T^*(x),v^*(y))\Bigg[1-\frac{T^*(x)}{T(t,x)}\\ &\quad -\frac{f(x,y,T(t,x),v(t,y))}{f(x,y,T^*(x),v^*(y))}\\ &\quad +\frac{T^*(x)f(x,y,T(t,x),v(t,y))}{T(t,x)f(x,y,T^*(x),v^*(y))}\Bigg]dy. \end{aligned} \tag{117}$$

Similarly, we get

$$\begin{aligned} \left.\frac{\partial V_I(t,x)}{\partial t}\right|_{(2)} &= \left(1-\frac{I^*(x)}{I(t,x)}\right)\frac{\partial I(t,x)}{\partial t} = \Bigg(1\\ &\quad -\frac{I^*(x)}{I(t,x)}\Bigg)\Bigg(\int_{\Omega}f(x,y,T(t,x),v(t,y))\,dy\\ &\quad -\delta(x)I(t,x)\Bigg) = \int_{\Omega}f(x,y,T^*(x),v^*(y))\Bigg[1\\ &\quad +\frac{f(x,y,T(t,x),v(t,y))}{f(x,y,T^*(x),v^*(y))}-\frac{I(t,x)}{I^*(x)}\\ &\quad -\frac{I^*(x)}{I(t,x)}\frac{f(x,y,T(t,x),v(t,y))}{f(x,y,T^*(x),v^*(y))}\Bigg]dy \end{aligned} \tag{118}$$

and

$$\begin{aligned} \left.\frac{\partial V_v(t,x)}{\partial t}\right|_{(2)} &= \left(1-\frac{v^*(x)}{v(t,x)}\right)\frac{\partial v(t,x)}{\partial t} = \Bigg(1\\ &\quad -\frac{v^*(x)}{v(t,x)}\Bigg)\Bigg(\int_{\Omega}\theta(x,y)v(t,y)\,dy+p(x)I(t,x)\\ &\quad -(c(x)+\gamma(x))v(t,x)\Bigg) = p(x)I^*(x)\Bigg(1\\ &\quad -\frac{v(t,x)}{v^*(x)}+\frac{I(t,x)}{I^*(x)}-\frac{I(t,x)}{I^*(x)}\frac{v^*(x)}{v(t,x)}\Bigg)\\ &\quad +\int_{\Omega}\theta(x,y)v^*(y)\\ &\quad \cdot\left[1+\frac{v(t,y)}{v^*(y)}-\frac{v^*(x)v(t,y)}{v(t,x)v^*(y)}-\frac{v(t,x)}{v^*(x)}\right]dy. \end{aligned} \tag{119}$$

Remember that $I^*(x) = (\int_{\Omega}f(x,y,T^*(x),v^*(y))dy)/\delta(x)$. Then we substitute (117)–(119) into $U(t,x)$ to obtain

$$\begin{aligned} \left.\frac{\partial U(t,x)}{\partial t}\right|_{(2)} &\le \int_{\Omega}f(x,y,T^*(x),v^*(y))\Bigg[3\\ &\quad -\frac{T^*(x)}{T(t,x)}+\frac{T^*(x)f(x,y,T(t,x),v(t,y))}{T(t,x)f(x,y,T^*(x),v^*(y))}\\ &\quad -\frac{I^*(x)f(x,y,T(t,x),v(t,y))}{I(t,x)f(x,y,T^*(x),v^*(y))}-\frac{v(t,x)}{v^*(x)}\\ &\quad -\frac{I(t,x)}{I^*(x)}\frac{v^*(x)}{v(t,x)}\Bigg]dy+\frac{\delta(x)}{p(x)}\int_{\Omega}\theta(x,y) \end{aligned}$$

$$
\cdot v^{*}(y)\left[1+\frac{v(t,y)}{v^{*}(y)}-\frac{v^{*}(x)\,v(t,y)}{v(t,x)\,v^{*}(y)}-\frac{v(t,x)}{v^{*}(x)}\right]dy. \tag{120}
$$

We will rearrange the terms in (120) as follows.

First note that

$$
\begin{aligned}
&3-\frac{T^{*}(x)}{T(t,x)}+\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{I^{*}(x)\,f(x,y,T(t,x),v(t,y))}{I(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{v(t,x)}{v^{*}(x)}\\
&-\frac{v^{*}(x)\,v(t,y)}{v(t,x)\,v^{*}(y)}=-\left[g\left(\frac{T^{*}(x)}{T(t,x)}\right)+g\left(\frac{I^{*}(x)\,f(x,y,T(t,x),v(t,y))}{I(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}\right)+g\left(\frac{I(t,x)\,v^{*}(x)}{I^{*}(x)\,v(t,x)}\right)\right]\\
&+\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{v(t,x)}{v^{*}(x)}\\
&+\ln\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,x)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(x)}.
\end{aligned} \tag{121}
$$

Now the last term of (121) can be rewritten as

$$
\begin{aligned}
&\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{v(t,x)}{v^{*}(x)}+\ln\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,x)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(x)}\\
&=-g\left(\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(y)}\right)+\ln\frac{v(t,x)}{v^{*}(x)}-\frac{v(t,x)}{v^{*}(x)}-\ln\frac{v(t,y)}{v^{*}(y)}+\frac{v(t,y)}{v^{*}(y)}\\
&+\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(y)}-1\\
&+\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{v(t,y)}{v^{*}(y)}.
\end{aligned} \tag{122}
$$

Furthermore, the last term of equation (122) can be rearranged to be

$$
\begin{aligned}
&\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(y)}-1+\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-\frac{v(t,y)}{v^{*}(y)}\\
&=\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-1+\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(y)}\\
&\quad\times\left(1-\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}\right)\\
&=\left(\frac{T^{*}(x)\,f(x,y,T(t,x),v(t,y))}{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}-1\right)\cdot\left(1-\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y))\,v^{*}(y)}\right)\\
&=\frac{f(x,y,T(t,x),v(t,y)}{f(x,y,T^{*}(x),v^{*}(y))}\cdot\frac{T(t,x)}{T^{*}(x)}\left(\frac{T^{*}(x)}{T(t,x)}-\frac{f(x,y,T^{*}(x),v^{*}(y))}{f(x,y,T(t,x),v(t,y))}\right)\\
&\quad\times\left(\frac{T^{*}(x)}{T(t,x)}-\frac{f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{f(x,y,T(t,x),v(t,y))\,v^{*}(y)}\right).
\end{aligned} \tag{123}
$$

Next we can simplify the factor in the last term of (120) as

$$
\begin{aligned}
&\int_{\Omega}\theta(x,y)\,v^{*}(y)\cdot\left[1+\frac{v(t,y)}{v^{*}(y)}-\frac{v^{*}(x)\,v(t,y)}{v(t,x)\,v^{*}(y)}-\frac{v(t,x)}{v^{*}(x)}\right]dy\\
&=-\int_{\Omega}\theta(x,y)\,v^{*}(y)\,g\left(\frac{v^{*}(x)\,v(t,y)}{v(t,x)\,v^{*}(y)}\right)dy\\
&+\int_{\Omega}\theta(x,y)\,v^{*}(y)\left[\frac{v(t,y)}{v^{*}(y)}-\ln\frac{v(t,y)}{v^{*}(y)}-\frac{v(t,x)}{v^{*}(x)}+\ln\frac{v(t,x)}{v^{*}(x)}\right]dy
\end{aligned} \tag{124}
$$

In summary, we have found

$$
\begin{aligned}
&\left.\frac{\partial W(t,x)}{\partial t}\right|_{(2)}\\
&\leq-\int_{\Omega}f(x,y,T^{*}(x),v^{*}(y))\left[g\left(\frac{T^{*}(x)}{T(t,x)}\right)+g\left(\frac{I^{*}(x)\,f(x,y,T(t,x),v(t,y))}{I(t,x)\,f(x,y,T^{*}(x),v^{*}(y))}\right)+g\left(\frac{I(t,x)\,v^{*}(x)}{I^{*}(x)\,v(t,x)}\right)+g\left(\frac{T(t,x)\,f(x,y,T^{*}(x),v^{*}(y))\,v(t,y)}{T^{*}(x)\,f(x,y,T(t,x),v(t,y)\,v^{*}(y))}\right)\right]dy\\
&+\int_{\Omega}f(x,y,T(t,x),v(t,y))
\end{aligned}
$$

$$\begin{aligned}
&\cdot \frac{T(t,x)}{T^*(x)} \left(\frac{T^*(x)}{T(t,x)} - \frac{f(x,y,T^*(x),v^*(y))}{f(x,y,T(t,x),v(t,y))} \right) \times \left(\frac{T^*(x)}{T(t,x)} \right. \\
&\left. - \frac{f(x,y,T^*(x),v^*(y))\,v(t,y)}{f(x,y,T(t,x),v(t,y))\,v^*(y)} \right) dy \\
&+ \int_\Omega \left(f(x,y,T^*(x),v^*(y)) \right. \\
&\left. + \frac{\delta(x)}{p(x)} \theta(x,y) \right) \left[\frac{v(t,y)}{v^*(y)} - \ln \frac{v(t,y)}{v^*(y)} - \frac{v(t,x)}{v^*(x)} + \ln \frac{v(t,x)}{v^*(x)} \right] dy \\
&- \frac{\delta(x)}{p(x)} \int_\Omega \theta(x,y)\, v^*(y)\, g\left(\frac{v^*(x)\, v(t,y)}{v(t,x)\, v^*(y)} \right) dy.
\end{aligned} \tag{125}$$

If $f(x,y,T(x),v(y)) = f(T(x),v(x))$, then the last but two term of (125) is equal to zero. Following [26, Proposition 11.1], we catch $v^*(x)$ as a positive Borel function on Ω such that

$$\begin{aligned}
&\int_\Omega \int_\Omega \theta(x,y)\, v^*(x)\, v^*(y) \left[\frac{v(t,y)}{v^*(y)} - \ln \frac{v(t,y)}{v^*(y)} \right. \\
&\quad \left. - \frac{v(t,x)}{v^*(x)} + \ln \frac{v(t,x)}{v^*(x)} \right] dydx \\
&\quad = \int_\Omega \int_\Omega \theta(x,y)\, v^*(x)\, v^*(y) \left[\frac{v(t,y)}{v^*(y)} \right. \\
&\quad \left. - \ln \frac{v(t,y)}{v^*(y)} - \frac{v(t,x)}{v^*(x)} + \ln \frac{v(t,x)}{v^*(x)} \right] dxdy \\
&\quad = \int_\Omega \int_\Omega \theta(y,x)\, v^*(y)\, v^*(x) \left[\frac{v(t,x)}{v^*(x)} \right. \\
&\quad \left. - \ln \frac{v(t,x)}{v^*(x)} - \frac{v(t,y)}{v^*(y)} + \ln \frac{v(t,y)}{v^*(y)} \right] dydx \\
&\quad = \int_\Omega \int_\Omega \theta(x,y)\, v^*(x)\, v^*(y) \left[\frac{v(t,x)}{v^*(x)} \right. \\
&\quad \left. - \ln \frac{v(t,x)}{v^*(x)} - \frac{v(t,y)}{v^*(y)} + \ln \frac{v(t,y)}{v^*(y)} \right] dydx.
\end{aligned} \tag{126}$$

Therefore,

$$\begin{aligned}
&2 \int_\Omega \theta(x,y)\, v^*(x)\, v^*(y) \left[\frac{v(t,y)}{v^*(y)} - \ln \frac{v(t,y)}{v^*(y)} \right. \\
&\quad \left. - \frac{v(t,x)}{v^*(x)} + \ln \frac{v(t,x)}{v^*(x)} \right] dydx = 0.
\end{aligned} \tag{127}$$

Setting $\mathscr{G}(x) = 2(p(x)v^*(x)/\delta(x))$ and plugging (127) into (125), we arrive at

$$\begin{aligned}
&\left. \frac{dV(t)}{dt} \right|_{(2)} \\
&\le -2 \int_\Omega \int_\Omega \frac{p(x)\, f(x,y,T^*(x),v^*(y))\, v^*(x)}{\delta(x)} \left[g\left(\frac{T^*(x)}{T(t,x)} \right) + g\left(\frac{I^*(x)\, f(x,y,T(t,x),v(t,y))}{I(t,x)\, f(x,y,T^*(x),v^*(y))} \right) + g\left(\frac{I(t,x)\, v^*(x)}{I^*(x)\, v(t,x)} \right) + g\left(\frac{T(t,x)\, f(x,y,T^*(x),v^*(y))\, v(t,y)}{T^*(x)\, f(x,y,T(t,x),v(t,y)\, v^*(y)} \right) \right] dydx \\
&+ 2 \int_\Omega \int_\Omega f(x,y,T(t,x),v(t,y)) \\
&\cdot \frac{p(x)\, T(t,x)}{\delta(x)\, T^*(x)} v^*(x) \left(\frac{T^*(x)}{T(t,x)} \right. \\
&\left. - \frac{f(x,y,T^*(x),v^*(y))}{f(x,y,T(t,x),v(t,y)} \right) \times \left(\frac{T^*(x)}{T(t,x)} \right. \\
&\left. - \frac{f(x,y,T^*(x),v^*(y))\, v(t,y)}{f(x,y,T(t,x),v(t,y)\, v^*(y)} \right) dydx \\
&- 2 \int_\Omega \int_\Omega \theta(x,y) \\
&\cdot v^*(x)\, v^*(y)\, g\left(\frac{v^*(x)\, v(t,y)}{v(t,x)\, v^*(y)} \right) dydx.
\end{aligned} \tag{128}$$

By virtue of g and Assumption 19, $(dV(t)/dt)|_{(2)} \le 0$ and the equality holds if and only if

$$T(t,x) = T^*(x),$$
$$\frac{f(x,y,T^*(x),v^*(y))}{f(x,y,T(t,x),v(t,y))} = 1, \quad (129)$$
$$\frac{I(t,x)v^*(x)}{I^*(x)v(t,x)} = 1.$$

Replacing $T(t,x)$ in the first equation of system (2) by $T^*(x)$, we have

$$0 = \lambda(x) - \int_\Omega f(x,y,T^*(x),v(t,y))\,dy - \mu(x)T^*(x). \quad (130)$$

From Assumption 1, it is easy to see that $v(t,x) = v^*(x)$ for all $(t,x) \in \mathbb{R}_+ \times \Omega$. Using the second equation of system (2), we conclude that the largest invariant set $M = \{(T(\cdot), I(\cdot), v(\cdot)) \in \Gamma_0 \mid (dV(t)/dt)|_{(2)} = 0\} = \{E^*\}$. Consequently, it follows from LaSalle Invariance Principle that the endemic steady state E^* is globally attractive. □

Remark 21. The existence and attractivity of endemic steady states (see Theorems 9 and 20) indicate that system (2) has a unique endemic steady state if the conditions of Theorem 20 hold.

9. Numerical Simulations

In this section, we perform numerical experiments to illustrate our main theoretical results and compare the effects of diffusion rate and incidence. For this purpose, we take $\Omega = [-1,1] \subset \mathbb{R}$ and use the initial values in [29].

$$T(0,x) = 0.99\cos\left(\frac{\pi}{2}x\right),$$
$$I(0,x) = 0.01\cos\left(\frac{\pi}{2}x\right), \quad (131)$$
$$v(0,x) = 10^{-6}\cos\left(\frac{\pi}{2}x\right).$$

Moreover, we fix the following parameters:

$$\lambda(x) = 2.0,$$
$$\mu(x) = 0.5,$$
$$\delta(x) = 1,$$
$$p(x) = 2, \quad (132)$$
$$\gamma(x) = 10,$$
$$c(x) = 0.5.$$

The incidence rate takes the form

$$f(x,y,T,v) = \frac{\beta(x)T(t,x)v(t,y)}{1+\alpha v(t,x)}, \quad (133)$$

where

$$\beta(x) = \tilde{\beta}(1 + 0.1\cos(5\pi x)), \quad x \in [-1,1] \quad (134)$$

with $\tilde{\beta}$ varying. For the nonlocal effect, we set

$$\theta(x,y) = 1.5d(x-y)^2, \quad (135)$$

where d is the diffusive coefficient. Firstly, we set $\tilde{\beta} = 1.1$, $d = 10^{-5}$, and $\alpha = 0.5$. Then $\mathscr{R}_0 \approx 0.9219 < 1$. Figures 1(a) and 1(b) show that the density of free viruses approaches zero as t goes to infinity.

Now, we enlarge $\tilde{\beta}$ to $\tilde{\beta} = 1.5$ and get $\mathscr{R}_0 \approx 1.257 > 1$. In view of Theorem 20, the endemic steady state converges to a spatially positive endemic steady state (see Figures 1(c) and 1(d)).

Secondly, Figure 2(a) shows that increasing the diffusion rate enhances the infected risk and increases the final infected size. This means that controlling virions diffusion beats the viral replication. Figure 2(b) reflects that viral infection models with bilinear incidence rate have higher infected risk than ones with saturating incidence rates.

10. Discussion

In this paper, we firstly proposed a within-host viral infection model with nonlocal diffusion and nonlocal transmission. The model can be considered as a spatial generalization of that proposed by Nowark and Bangham [43], a continuous spatial model of Funk et al. [9]. We derived the next-generation operator $\mathscr{R}$ and built the relationship between the basic reproduction number $\mathscr{R}_0 = r(\mathscr{R})$ and the spectral bound of the operator $\mathscr{L}$ (see Theorem 7). The asymptotic smoothness of the semiflow Φ was established by Arzelà-Ascoli theorem in Section 3. The threshold dynamics of system (2) has been established by constructing suitable Lyapunov functionals. Biologically, whether or not the viral infection outbreaks is determined by the basic reproduction number $\mathscr{R}_0$.

Compared with other within-host models with diffusion (discrete style [9] and continuous style (Laplace operator) [5]), our model can be considered as a generalization of the model proposed by Zhao and Ruan [18], where for the incidence they took the particular form

$$\int_\Omega f(x,y,T(t,x),v(t,y))\,dy = \beta(x)T(t,x)v(t,x). \quad (136)$$

In [18], they adopted the semigroup method to investigate the asymptotical behavior of system (2). In this paper, we used functional analysis method together with the Lyapunov functional method to study the asymptotical stability of the system. We believe that the method used here can also be applicable to or be generalized to deal with other nonscalar systems with nonlocal diffusions.

Data Availability

The artificial data used to support the findings of this study are included within the article.

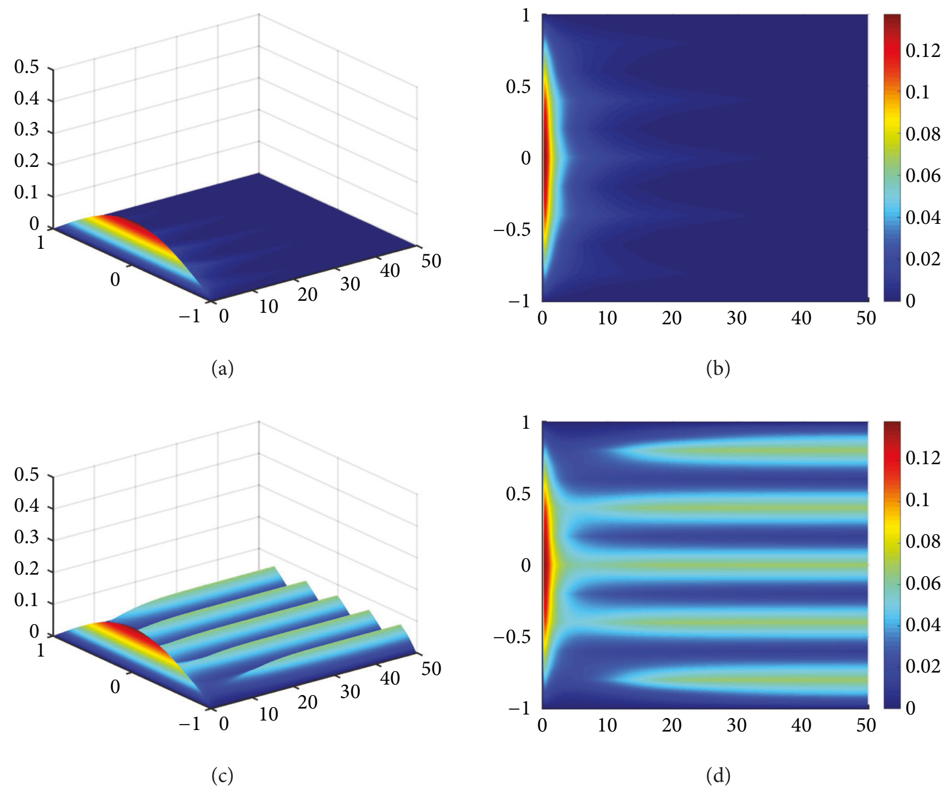

FIGURE 1: The evolution of the density of free viruses with parameters listed in the text. ((a) and (b)) The variation of the free virus density with $\widetilde{\beta} = 1.1$. ((c) and (d)) The variation of the free virus density with $\widetilde{\beta} = 1.5$.

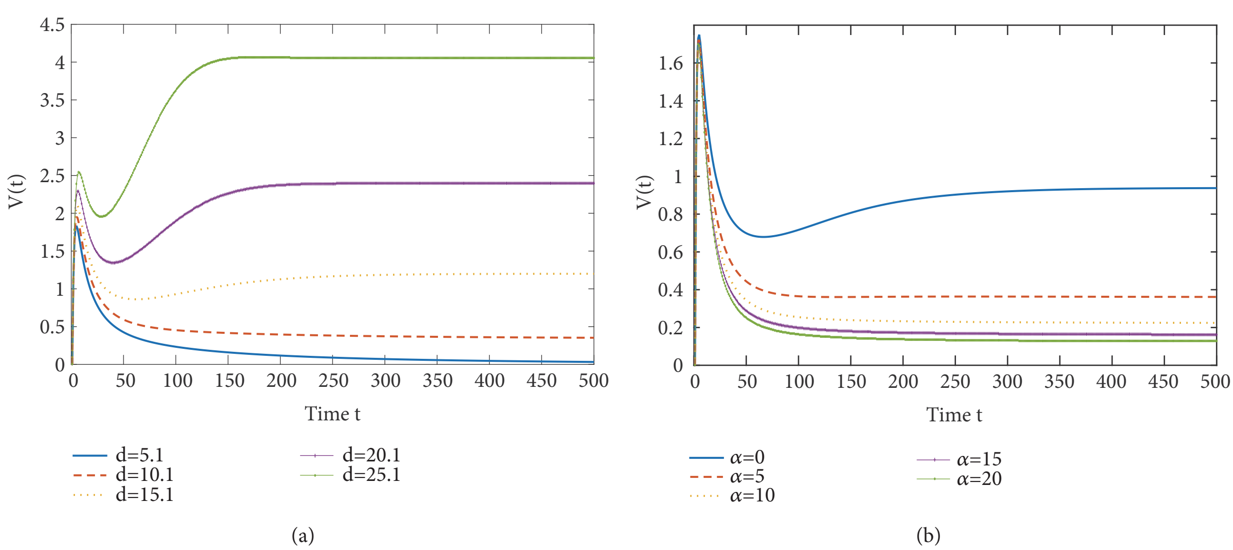

FIGURE 2: The evolution of the cumulative density of free viruses with parameters as those for Figure 1 except d and α. (a) $d \in [5.1 : 5 : 25.1]$ and $\alpha = 0.5$; (b) $\alpha \in [0 : 5 : 20]$ and $d = 10^{-5}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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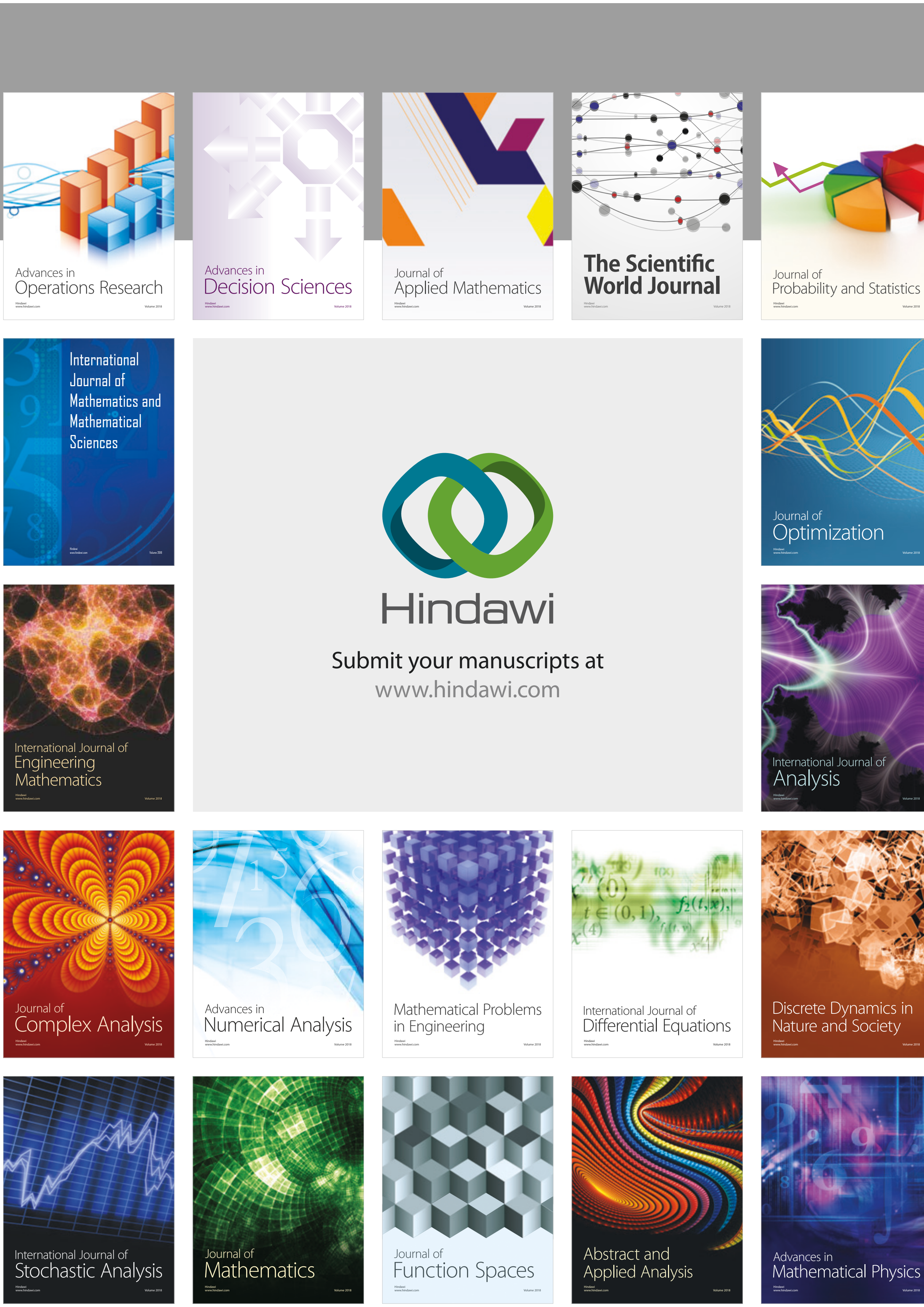
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