

Research Article

A Generalization of the Cauchy-Schwarz Inequality and Its Application to Stability Analysis of Nonlinear Impulsive Control Systems

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In this paper, we first present a generalization of the Cauchy-Schwarz inequality. As an application of our result, we obtain a new sufficient condition for the stability of a class of nonlinear impulsive control systems. We end up this note with a numerical example which shows the effectiveness of our method.

1. Introduction

In this paper, the Euclidean norm of $\mathbf{x} \in R^n$ is defined as $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$. We use $\lambda_{\max}(H)$ and $\lambda_{\min}(H)$ to denote the largest and the smallest eigenvalues of a real square matrix H with real eigenvalues, respectively. Let

$$H = Q^T \text{diag}(\lambda_1, \dots, \lambda_n) Q \quad (1)$$

be a spectral decomposition with Q is orthogonal. Then the functional calculus for H is defined as

$$f(H) = Q^T \text{diag}(f(\lambda_1), \dots, f(\lambda_n)) Q \quad (2)$$

where $f(t)$ is a continuous real-valued function defined on a real interval Ω and H is a real symmetrical matrix with eigenvalues in Ω [1].

During the last three decades, many people have studied impulsive control method because it is an efficient way in dealing with the stability of complex systems [2–4]. For example, impulsive control method can be used in the synchronization and stabilization of chaos systems [5–11] and neural network systems [12–23].

In this paper, we consider a class of nonlinear impulsive control systems as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \phi(\mathbf{x}), \\ \mathbf{y} &= C\mathbf{x}, \quad t \neq \tau_k, \\ \Delta\mathbf{x} &= B\mathbf{y}, \quad t = \tau_k, \quad k = 1, 2, \dots, \end{aligned} \quad (3)$$

where $\mathbf{x} \in R^n$ is the state variable and $\mathbf{y} \in R^m$ is output, and $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{m \times n}$ are constant matrices. The nonlinear part $\phi : R^n \rightarrow R^n$ is a continuous function which satisfies $\phi(t, 0) = 0$ and $\|\phi(\mathbf{x})\| \leq L\|\mathbf{x}\|$. If $t = \tau_k$, then there will be a jump in the system and $\Delta\mathbf{x}(\tau_k) = \mathbf{x}(\tau_k^+) - \mathbf{x}(\tau_k^-) = \mathbf{x}(\tau_k^+) - \mathbf{x}(\tau_k)$, with $\mathbf{x}(\tau_k^+) = \lim_{t \rightarrow \tau_k^+} \mathbf{x}(t)$ and $\mathbf{x}(\tau_k^-) = \lim_{t \rightarrow \tau_k^-} \mathbf{x}(t)$. Without loss of generality, we assume that

$$\begin{aligned} t_0 &< \tau_1 < \tau_2 < \dots, \\ \lim_{k \rightarrow \infty} \tau_k &= \infty. \end{aligned} \quad (4)$$

For simplicity, we can rewrite this last system as

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \phi(\mathbf{x}), \quad t \neq \tau_k, \\ \Delta\mathbf{x} &= BC\mathbf{x}, \quad t = \tau_k, \quad k = 1, 2, \dots, \\ \mathbf{x}(t_0) &= \mathbf{x}_0. \end{aligned} \quad (5)$$

The stability problems of nonlinear impulsive control system (5) have been investigated extensively in the literature in the past several decades. For example, a number of sufficient conditions for the stability of nonlinear impulsive control system (5) are derived in [24–27]. Inequalities play an important role in their research, for instance, by using the Cauchy-Schwarz inequality [1] and comparison principle [27], and Yang showed a sufficient condition for the stability of nonlinear impulsive control system (5). For more results on applications of the Cauchy-Schwarz inequality to impulsive control theory, the reader is referred to [4] and the references therein.

In this paper, we first present a generalization of the Cauchy-Schwarz inequality by using some results of matrix analysis and techniques of inequalities. As an application of our result, we obtain a new sufficient condition for the stability of nonlinear impulsive control system (5). We end up this note with a numerical example which will show the effectiveness of our result.

2. A Generalization of the Cauchy-Schwarz Inequality

In this section, we will give a generalized Cauchy-Schwarz inequality.

Lemma 1. Let P be positive definite and suppose that λ_2, λ_1 are the largest and the smallest eigenvalues of P , respectively. If $\mathbf{x}, \mathbf{y} \in R^n$ satisfy

$$|\mathbf{x}^T \mathbf{y}|^2 \leq \sigma(\mathbf{x}^T \mathbf{x})(\mathbf{y}^T \mathbf{y}) \quad (6)$$

for a certain $\sigma \in [0, 1]$, then

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}), \quad (7)$$

where

$$g(\sigma) = \frac{1 - \sigma}{1 + \sigma}. \quad (8)$$

Proof. First we assume that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Let

$$X = [\mathbf{x}, \mathbf{y}], \quad (9)$$

and then, we have

$$X^T X = \begin{bmatrix} 1 & \mathbf{x}^T \mathbf{y} \\ \mathbf{y}^T \mathbf{x} & 1 \end{bmatrix} \quad (10)$$

and

$$X^T P X = \begin{bmatrix} \mathbf{x}^T P \mathbf{x} & \mathbf{x}^T P \mathbf{y} \\ \mathbf{y}^T P \mathbf{x} & \mathbf{y}^T P \mathbf{y} \end{bmatrix}. \quad (11)$$

Small calculations show that $1 + |\mathbf{x}^T \mathbf{y}|$ and $1 - |\mathbf{x}^T \mathbf{y}|$ are the eigenvalues of $X^T X$. Suppose that μ_2, μ_1 are the largest and the smallest eigenvalues of $X^T P X$, respectively. Then we have

$$\begin{aligned} \mu_2 &= \lambda_{\max}(X^T P X) \leq \lambda_2 \lambda_{\max}(X^T X) \\ &= (1 + |\mathbf{x}^T \mathbf{y}|) \lambda_2 \end{aligned} \quad (12)$$

and

$$\begin{aligned} \mu_1 &= \lambda_{\min}(X^T P X) \geq \lambda_1 \lambda_{\min}(X^T X) \\ &= (1 - |\mathbf{x}^T \mathbf{y}|) \lambda_1. \end{aligned} \quad (13)$$

It follows from (12) and (13) that

$$\frac{(1 - |\mathbf{x}^T \mathbf{y}|) \lambda_1}{(1 + |\mathbf{x}^T \mathbf{y}|) \lambda_2} \leq \frac{\mu_1}{\mu_2}. \quad (14)$$

It can easily be seen that the function

$$g(t) = \frac{1 - t}{1 + t}, \quad 0 \leq t \leq 1, \quad (15)$$

is decreasing and so

$$g\left(\frac{\mu_1}{\mu_2}\right) \leq g\left(\frac{(1 - q) \lambda_1}{(1 + q) \lambda_2}\right), \quad (16)$$

which is equivalent to

$$\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \leq \frac{\lambda_2 - g(q) \lambda_1}{\lambda_2 + g(q) \lambda_1}, \quad (17)$$

where

$$q = |\mathbf{x}^T \mathbf{y}| \in [0, 1]. \quad (18)$$

Note that

$$(\mu_2 + \mu_1)^2 = (\mathbf{x}^T P \mathbf{x} + \mathbf{y}^T P \mathbf{y})^2 \quad (19)$$

and

$$\begin{aligned} (\mu_2 - \mu_1)^2 &= (\mu_2 + \mu_1)^2 - 4\mu_1\mu_2 \\ &= (\mathbf{x}^T P \mathbf{x} + \mathbf{y}^T P \mathbf{y})^2 - 4(\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}) \\ &\quad + 4(\mathbf{x}^T P \mathbf{y})^2. \end{aligned} \quad (20)$$

It follows that

$$\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \right)^2 = 1 - 4 \frac{(\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}) - (\mathbf{x}^T P \mathbf{y})^2}{(\mathbf{x}^T P \mathbf{x} + \mathbf{y}^T P \mathbf{y})^2}. \quad (21)$$

Meanwhile, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (\mathbf{x}^T P \mathbf{y})^2 &= \left((P^{1/2} \mathbf{x})^T (P^{1/2} \mathbf{y}) \right)^2 \\ &\leq (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \end{aligned} \quad (22)$$

On the other hand, the arithmetic-geometric mean inequality for scalars implies that

$$(\mathbf{x}^T P \mathbf{x} + \mathbf{y}^T P \mathbf{y})^2 \geq 4(\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \quad (23)$$

It follows from (21), (22), and (23) that

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \quad (24)$$

By using inequalities (17) and (24), we obtain

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\lambda_2 - g(q) \lambda_1}{\lambda_2 + g(q) \lambda_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \quad (25)$$

Now we consider the general situation. For arbitrary $\mathbf{x}, \mathbf{y} \in R^n$, we have

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| = \left\| \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| = 1 \quad (26)$$

By inequality (25), we have

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\lambda_2 - g(q) \lambda_1}{\lambda_2 + g(q) \lambda_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}) \quad (27)$$

where

$$g(q) = \frac{1-q}{1+q}, \quad q = \frac{|\mathbf{x}^T \mathbf{y}|}{\|\mathbf{x}\| \|\mathbf{y}\|}. \quad (28)$$

Inequality (6) implies that $q \leq \sqrt{\sigma}$ and so

$$g(\sqrt{\sigma}) \leq g(q). \quad (29)$$

Small calculations show that the function

$$f(t) = \frac{a-t}{a+t}, \quad a > 1, \quad 0 \leq t \leq 1, \quad (30)$$

is decreasing and so

$$\frac{\lambda_2/\lambda_1 - g(q)}{\lambda_2/\lambda_1 + g(q)} \leq \frac{\lambda_2/\lambda_1 - g(\sqrt{\sigma})}{\lambda_2/\lambda_1 + g(\sqrt{\sigma})}. \quad (31)$$

It follows from (27) and (31) that

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \quad (32)$$

This completes the proof of our result. \square

Remark 2. By the Cauchy-Schwarz inequality, we know that condition (6) holds for any $\mathbf{x}, \mathbf{y} \in R^n$ if we choose $\sigma = 1$. And so Lemma 1 is a generalization of the Cauchy-Schwarz inequality:

$$\begin{aligned} (\mathbf{x}^T P \mathbf{y})^2 &= \left((P^{1/2} \mathbf{x})^T (P^{1/2} \mathbf{y}) \right)^2 \\ &\leq (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \end{aligned} \quad (33)$$

Remark 3. If $\mathbf{x}, \mathbf{y} \in R^n$ is orthogonal, then we can choose $\sigma = 0$ and Lemma 1 is the well-known Wielandt inequality:

$$(\mathbf{x}^T P \mathbf{y})^2 \leq \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} \right)^2 (\mathbf{x}^T P \mathbf{x})(\mathbf{y}^T P \mathbf{y}). \quad (34)$$

3. An Application of Lemma 1

Let us recall the definition of the angle between two vectors $\mathbf{y}, \mathbf{z} \in R^n$:

$$\theta = \arccos \frac{\mathbf{y}^T \mathbf{z}}{\|\mathbf{y}\| \|\mathbf{z}\|}, \quad \theta \in [0, \pi]. \quad (35)$$

In the course of experiment, we note that for some systems the state variable \mathbf{x} and nonlinear part $\phi(\mathbf{x})$ have special relationships. For instance, Lü et al. [28] presented the following chaotic system:

$$\begin{aligned} \dot{x} &= (25\alpha + 10)(y - x), \\ \dot{y} &= (28 - 35\alpha)x + (29\alpha - 1)y - xz, \\ \dot{z} &= -\frac{\alpha + 8}{3}z + xy, \end{aligned} \quad (36)$$

where $\alpha \in [0, 1]$. Note that $\mathbf{x} = [x, y, z]^T$, $\phi(\mathbf{x}) = [0, -xz, xy]^T$ and so $\mathbf{x}^T \phi(\mathbf{x}) = 0$. That is, they are orthogonal. So we want to know whether the angle between \mathbf{x} and $\phi(\mathbf{x})$ has an effect on the stability of systems. And the results showed in [24–27] do not take into account this factor. This is the motivation for the present paper.

In this section, as an application of Lemma 1, we present a new sufficient condition for the stability of nonlinear impulsive control system (5). Compared with Theorem 3 in [27] (see also Theorem 3.1.5 in [4]), if we consider the angle factor, then we will get a larger stable region for some systems.

Lemma 4 (see [1]). *Suppose that H is a real symmetrical matrix and let λ_2, λ_1 be the largest and smallest eigenvalues of H , respectively. Then*

$$\lambda_1 \mathbf{y}^T \mathbf{y} \leq \mathbf{y}^T H \mathbf{y} \leq \lambda_2 \mathbf{y}^T \mathbf{y}, \quad (37)$$

for any $\mathbf{y} \in R^n$.

Theorem 5. *Let P be positive definite and suppose that λ_2, λ_1 are the largest and smallest eigenvalues of P , respectively. Let λ_3 be the largest eigenvalue of $P^{-1}Q$ with $Q = PA + A^T P$. Suppose that λ_4 is the largest eigenvalue of $P^{-1}(I + BC)^T P(I + BC)$. If*

$$|\mathbf{x}^T \phi(\mathbf{x})|^2 \leq \sigma (\mathbf{x}^T \mathbf{x}) (\phi(\mathbf{x})^T \phi(\mathbf{x})) \quad (38)$$

for a certain $\sigma \in [0, 1]$ and

$$\lambda_3 + 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1}} \geq 0, \quad (39)$$

$$\begin{aligned} &\left(\lambda_3 + 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1}} \right) (\tau_{k+1} - \tau_k) \\ &\leq -\ln(\gamma \lambda_4), \end{aligned} \quad (40)$$

where

$$\begin{aligned} g(\sigma) &= \frac{1-\sigma}{1+\sigma}, \\ \gamma &> 1, \end{aligned} \quad (41)$$

then the origin of nonlinear impulsive control system (5) is asymptotically stable.

Proof. Let

$$V(\mathbf{x}(t)) = \mathbf{x}^T P \mathbf{x}. \quad (42)$$

For $t \neq \tau_k$, we have

$$D^+ V(\mathbf{x}(t)) = \mathbf{x}^T (PA + A^T P) \mathbf{x} + 2\mathbf{x}^T P\phi(\mathbf{x}). \quad (43)$$

By Lemma 4 and noting that the matrices $P^{-1/2}(PA + A^T P)P^{-1/2}$ and $P^{-1}(PA + A^T P)$ have the same eigenvalues, we obtain

$$\begin{aligned} \mathbf{x}^T (PA + A^T P) \mathbf{x} &= (\mathbf{x}^T P^{1/2}) \\ &\times (P^{-1/2} (PA + A^T P) P^{-1/2}) \\ &\times (P^{1/2} \mathbf{x}) \\ &\leq \lambda_3 (\mathbf{x}^T P^{1/2}) (P^{1/2} \mathbf{x}) = \lambda_3 V(\mathbf{x}). \end{aligned} \quad (44)$$

By Lemmas 1 and 4 and $\|\phi(\mathbf{x})\| \leq L\|\mathbf{x}\|$, we have

$$\begin{aligned} &2\mathbf{x}^T P\phi(\mathbf{x}) \\ &\leq 2 \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{(\mathbf{x}^T P \mathbf{x})(\phi(\mathbf{x})^T P \phi(\mathbf{x}))} \\ &\leq 2 \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\lambda_2 (\mathbf{x}^T P \mathbf{x})(\phi(\mathbf{x})^T \phi(\mathbf{x}))} \\ &\leq 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\lambda_2 (\mathbf{x}^T P \mathbf{x})(\mathbf{x}^T \mathbf{x})} \\ &= 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \\ &\quad \times \sqrt{\lambda_2 (\mathbf{x}^T P \mathbf{x})(\mathbf{x}^T P^{1/2} P^{-1} P^{1/2} \mathbf{x})} \\ &\leq 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1} (\mathbf{x}^T P \mathbf{x})(\mathbf{x}^T P^{1/2} P^{1/2} \mathbf{x})} \\ &= 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1} V(\mathbf{x})}. \end{aligned} \quad (45)$$

It follows from (43), (44), and (45) that

$$\begin{aligned} D^+ V(\mathbf{x}(t)) &\leq \left(\lambda_3 + 2L \frac{\lambda_2 - g(\sqrt{\sigma}) \lambda_1}{\lambda_2 + g(\sqrt{\sigma}) \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1}} \right) V(\mathbf{x}). \end{aligned} \quad (46)$$

For $t = \tau_k$, by using Lemma 4 again and noting that the matrices $P^{-1}(I + BC)^T P(I + BC)$ and $P^{-1/2}(I + BC)^T P(I + BC)P^{-1/2}$ have the same eigenvalues, we obtain

$$\begin{aligned} V(\mathbf{x} + BC\mathbf{x})|_{t=\tau_k} &= (\mathbf{x} + BC\mathbf{x})^T P(\mathbf{x} + BC\mathbf{x})|_{t=\tau_k} \\ &= \mathbf{x}^T (I + BC)^T P(I + BC) \mathbf{x}|_{t=\tau_k} \\ &= (\mathbf{x}^T P^{1/2}) P^{-1/2} \\ &\quad \times (I + BC)^T P(I + BC) \\ &\quad \times P^{-1/2} (P^{1/2} \mathbf{x})|_{t=\tau_k} \\ &\leq \lambda_4 (\mathbf{x}^T P^{1/2}) (P^{1/2} \mathbf{x})|_{t=\tau_k} \\ &= \lambda_4 V(\mathbf{x})|_{t=\tau_k}. \end{aligned} \quad (47)$$

To avoid repetition, we omit the following proof because it is same as that of Theorem 3 in [27]. This completes the proof of our result. \square

Remark 6. If we choose $\sigma = 1$, then by the Cauchy-Schwarz inequality we know that inequality (38) holds for any $\mathbf{x}, \phi(\mathbf{x})$ and condition of (40) becomes

$$\left(\lambda_3 + 2L \sqrt{\frac{\lambda_2}{\lambda_1}} \right) (\tau_{k+1} - \tau_k) \leq -\ln(\gamma \lambda_4), \quad \gamma > 1, \quad (48)$$

which is the condition of Theorem 3 in [27] (see also [4]). So, our result is a generalization of Theorem 3 in [27].

Remark 7. If $P = I$, condition of (40) will be replaced by

$$(\lambda_3 + 2\sqrt{\sigma}L)(\tau_{k+1} - \tau_k) \leq -\ln(\gamma \lambda_4), \quad \gamma > 1. \quad (49)$$

Remark 8. Let us discuss Lü's [28] chaotic system again. Noting that $\mathbf{x}^T \phi(\mathbf{x}) = 0$ and taking into consideration that we can choose $\sigma = 0$, then inequality (38) holds and condition of (40) becomes

$$\begin{aligned} &\left(\lambda_3 + 2L \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} \sqrt{\frac{\lambda_2}{\lambda_1}} \right) (\tau_{k+1} - \tau_k) \leq -\ln(\gamma \lambda_4), \\ &\gamma > 1. \end{aligned} \quad (50)$$

Furthermore, if we choose $P = I$, then this last condition can be simplified as

$$\lambda_3 (\tau_{k+1} - \tau_k) \leq -\ln(\gamma \lambda_4), \quad \gamma > 1, \quad (51)$$

which contains the condition of Theorem 3.2.1 in [4] (see also [10]).

Remark 9. Lemma 1 has some other applications in impulsive control theory; for example, by using Lemma 1 and comparison lemmas on the sufficient condition for the stability of nonlinear impulsive differential systems shown in [24–26], some results presented in [24–26] can be generalized.

4. A Numerical Example

We end up this paper with a numerical example which shows the effectiveness of our method.

In 2005, Qi and Chen et al. [29] produced a new system which is described by

$$\dot{\mathbf{x}} = A\mathbf{x} + \phi(\mathbf{x}), \quad (52)$$

where

$$A = \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad (53)$$

$$\phi(\mathbf{x}) = \begin{bmatrix} yz \\ -xz \\ xy \end{bmatrix}.$$

This system is chaotic when

$$a = 35,$$

$$b = \frac{8}{3}, \quad (54)$$

$$c = 25.$$

By definition of the Euclidean norm, we have

$$\begin{aligned} \|\phi(\mathbf{x})\| &= \sqrt{y^2z^2 + x^2z^2 + x^2y^2} \\ &\leq \max\{|x|, |y|, |z|\} \sqrt{x^2 + y^2 + z^2} \\ &= \max\{|x|, |y|, |z|\} \|\mathbf{x}\| \end{aligned} \quad (55)$$

By Figure 1, we know that $\max\{|x|, |y|, |z|\} \leq 45$, so we can choose $L = 45$. By the arithmetic-geometric mean inequality for scalars we know that

$$\begin{aligned} |\mathbf{x}^T \phi(\mathbf{x})|^2 &= x^2 y^2 z^2 = \sqrt[3]{x^2 y^2 z^2} \times \sqrt[3]{x^2 y^2 z^2 x^2 z^2} \\ &\leq \frac{1}{9} (x^2 + y^2 + z^2) (x^2 y^2 + y^2 z^2 + x^2 z^2) \quad (56) \\ &= \frac{1}{9} (\mathbf{x}^T \mathbf{x}) (\phi(\mathbf{x})^T \phi(\mathbf{x})). \end{aligned}$$

So, we can choose $\sigma = 1/9$. In this example, we choose the matrices B, P, C as follows:

$$B = \begin{bmatrix} -0.60 & -0.01 & 0.01 \\ -0.01 & -0.60 & 0 \\ 0.01 & 0 & -0.60 \end{bmatrix}, \quad (57)$$

$$P = C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

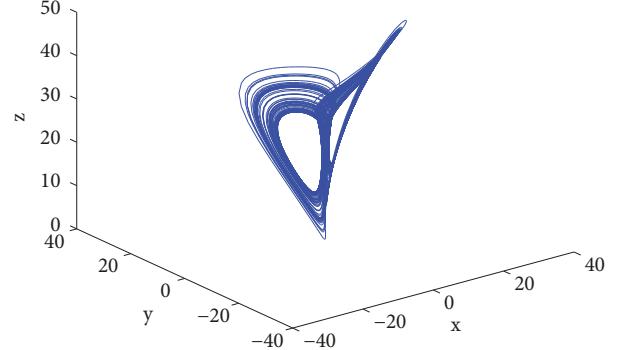


FIGURE 1: The state trajectory of the uncontrolled chaotic system with the initial condition $\mathbf{x}(0) = (3, 4, 5)^T$.

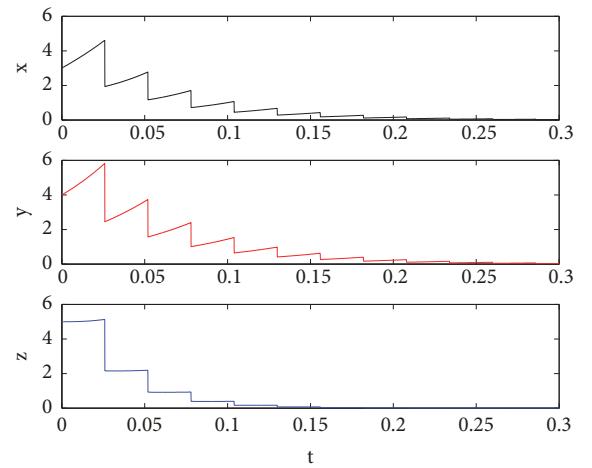


FIGURE 2: The state trajectory of the controlled chaotic system with the initial condition $\mathbf{x}(0) = (3, 4, 5)^T$.

Simple calculations show that

$$\lambda_3 = 32.9638, \quad (58)$$

$$\lambda_4 = 0.1715,$$

and so we have

$$\tau_{k+1} - \tau_k \leq -\frac{\ln(0.1715\gamma)}{62.9638}. \quad (59)$$

We choose $\gamma = 1.1$, and then

$$\tau_{k+1} - \tau_k \leq 0.0265. \quad (60)$$

Putting

$$\tau_{k+1} - \tau_k = 0.0260, \quad (61)$$

the simulation results are shown in Figure 2.

On the other hand, by Yang's [27] result we know that if

$$\tau_{k+1} - \tau_k \leq -\frac{\ln(0.1715\gamma)}{122.9638}, \quad (62)$$

then the origin of Qi's system [29] is asymptotically stable. Figure 3 shows the stable region for different γ 's.

From Figure 3 we know that if we consider the angle factor, then we get a larger stable region for Qi's system.

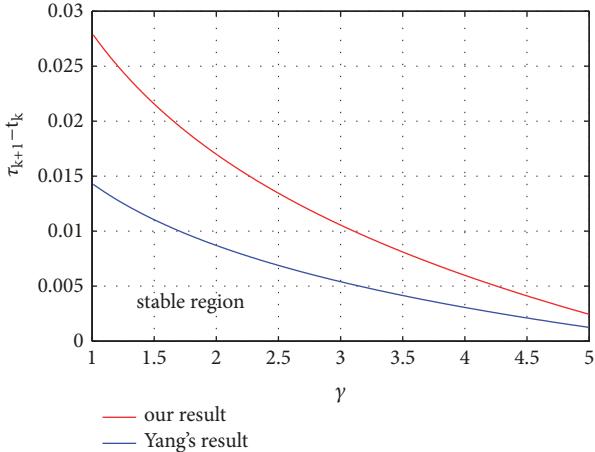


FIGURE 3: The estimation of boundaries of stable region with different γ 's.

5. Conclusion

In this paper, a generalization of the Cauchy-Schwarz inequality is presented. Then we use this inequality to analyze asymptotic stability for a class of nonlinear impulsive control systems. We think that Lemma 1 may have other applications in related fields of control theory.

Data Availability

The Matlab code data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final version of this paper.

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