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Research Article

Convergence Time Calculation for Supertwisting Algorithm and Application for Nonaffine Nonlinear Systems

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In this study, an accurate convergence time of the supertwisting algorithm (STA) is proposed to build up a framework for nonaffine nonlinear systems' finite-time control. The convergence time of the STA is provided by calculating the solution of a differential equation instead of constructing Lyapunov function. Therefore, precise convergence time is presented instead of estimation of the upper bound of the algorithm's reaching time. Regardless of affine or nonaffine nonlinear systems, supertwisting control (STC) provides a general solution based on virtual control law skill ensuring the output of the systems converges to the origin point at exact time. Benchmark tests are simulated to demonstrate the effectiveness and efficiency of the algorithm.

1. Introduction

Sliding mode control (SMC) has become one of the most efficient techniques to control uncertain complex systems and engineering [1–3]. Theoretically, such controllers are able to compensate and match disturbances by confining the systems' trajectories in a properly chosen hypersurface (the so-called sliding manifold) [4–6] and, under the chosen surface, make the origin of the state space an asymptotically stable equilibrium point for the closed-loop systems [7–9]. Both the convergence to the sliding manifold and convergence to the origin are guaranteed in a finite time interval if the control action is large enough to counteract the effect of the uncertain terms and hypersurface is defined suitably [10–12].

STA is a well-known second-order sliding mode (SOSM) algorithm introduced in [13], and it is a possible solution for chattering reduction and widely used for control, observation, and robust exact differentiation. The reaching time estimation and the finite-time control method design are essentially complex problems with the sliding mode control systems' study [14–16]. Finite-time convergence and robustness of the STA have been proved with some methods, such as geometrical methods [17], homogeneity properties of the algorithm [18], and Lyapunov methods [19]. The Lyapunov stability theorem and Lyapunov function provide a means of determining stability without explicit knowledge on system solutions [20–22]. Traditionally, quadratic Lyapunov functions are constructed to analyze and control design of nonlinear dynamic systems [23–25]. It should be noted that there also exist some other formats of Lyapunov functions, such as integral Lyapunov function, barrier Lyapunov function, and vector Lyapunov function. Such attempts have enhanced Lyapunov function applications in control system design. A strict Lyapunov function is provided to ascertain finite-time convergence, and it would provide an estimate of the convergence time, as well as the robustness of the finite-time or ultimate boundedness for the STA [19, 26]. By a detailed analysis of the Lyapunov function in finite time, robust convergence for the STA is proved, and it is not possible to provide necessary and sufficient conditions to estimate the convergence time from it.

However, the form of the estimate of the convergence time contained arbitrary positive matrixes which are related by the algebraic Lyapunov equation (ALE), making it difficult to operate with it for applications or further developments [24, 27, 28]. For nonlinear systems, numerical techniques (open solutions) have played a significant role in the controller design process [29–31]. As general solutions to the complicated nonlinear dynamic problem, the U model uses linear approaches to design nonlinear control [32, 33]. Specifically, it can be concluded that the applicable systems fall into classes of an increasing order of complexity: strict feedback, pure feedback, affine form, and nonaffine systems [34]. Nonaffine systems are difficult to control because of the complexity of the systems.

Motivated by the above observations, this paper presents accurate convergence time of the STA without and with perturbation and designs the nonaffine STA finite-time control. The main contributions are listed as follows:

(1) Accurate convergence time is proposed for the STA based on the analytical solution of the differential equation. The main advantages are that the output of the systems converges to the origin point at exact time and the exact time is determined by the designer before the controller is implemented.

(2) To overcome the main obstacle, the trajectory of the STA is analyzed by the analytical solution of the differential equation and LaSalle's invariance principle instead of Lyapunov stability theory used commonly in the traditional sliding mode control.

(3) Technically, for nonaffine nonlinear systems, sliding mode control based on the STA and backstepping achieves finite-time stabilization. The closed-loop control systems will achieve stability in finite time without violation of the constraint.

The rest of this study is organized as follows: In Section 2, the problem formulation and preliminaries are presented, which also contains some definitions and lemmas about the STA. They are presented to establish a basis for designing and analyzing the STA. In Section 3, accurate convergence time for finite-time convergence of the STA is developed by the parametric equation in different initial conditions. In Section 4, accurate convergence time is developed by the parametric equation in different initial conditions with perturbation. In Section 5, STA control is designed for nonaffine nonlinear systems based on the backstepping skill. The closed-loop systems' trajectory is analyzed, and the output can be obtained effectively by the finite-time algorithm. In Section 6, three simulated case studies are conducted to initially demonstrate the efficiency and effectiveness of the procedure. In Section 7, the conclusions are given to summarize the study.

2. System Description and Preliminaries

The STA can be written as

$$\begin{aligned} \dot{x}_1 &= -k_1 |x_1|^{1/2} \operatorname{sign}(x_1) + x_2, \\ \dot{x}_2 &= -k_2 \operatorname{sign}(x_1), \end{aligned} \tag{1}$$

where x_1 and x_2 are the scalar state variables: $x_1(0) = a$ and $x_2(0) = b$, and k_1 and k_2 are gains to be designed.

If $k_1 > 0$ and $k_2 > 0$, the states x_1 and x_2 of the system reach the origin in finite time T; therefore, the system undergoes finite-time convergence [19], using $\{x_1, x_2 \mid t, k_1, k_2, a, b\}$ to indicate STA (1).

Lemma 1 (see [35]). *For differential equation,*

$$yy'_x - y = Ax + B, \quad A \neq 0. \tag{2}$$

The solution is in the parametric form:

$$\begin{aligned} x &= C \exp\left(-\int \frac{\tau d\tau}{\tau^2 - \tau - A}\right) - \frac{B}{A}, \\ y &= C\tau \exp\left(-\int \frac{\tau d\tau}{\tau^2 - \tau - A}\right). \end{aligned} \tag{3}$$

More details about the solution of the parametric equation (3) are presented in Appendix A.

3. Finite-Time Convergence of STA

As mentioned, when $k_1 > 0$ and $k_2 > 0$, the accurate convergence time is considered. To show the trajectory of the STA, the analytic method is used to describe the system.

If the initial condition $x_1(0) = a$, then x_1 will reach the origin point in finite time at $x_1(t_{a0}) = 0$, where t_{a0} indicates the time of state when x_1 reaches the origin, and let $T_{a0} = \{t \mid 0 < t < t_{a0}\}$; if the initial condition $x_1(0) = 0$, then x_1 will reach the origin point in finite time at $x_1(t_{00}) = 0$, where t_{00} indicates the time of state when x_1 reaches the origin, and let $T_{00} = \{t \mid 0 < t < t_{00}\}$.

The main contribution are the following three facts:

Fact 1. For the STS (1), when $k_1 > 0, k_2 > 0$, and $A \geq -(1/4)$ hold, the initial conditions are $x_1(0) = a$ and $x_2(0) = b$; if $a \neq 0$ holds, the system state x_1 arrives at the zero point in finite time.

As shown in Figure 1, $x_1(0) = a, x_2(0) = b$, and $x_1(t_{a0}) = 0$ hold.

Fact 2. This is the most important fact; for the STS (1), when $k_1 > 0, k_2 > 0$, and $A \geq -(1/4)$ hold, the initial conditions are $x_1(0) = a$ and $x_2(0) = b$; if $a = 0$ holds, the system states x_1 and x_2 arrive at the zero point simultaneously.

As shown in Figure 2, the initial conditions $x_1(0) = 0$ and $x_2(0) = b$ hold, in period T_{00}; then at time t_{00}, $x_1(t_{00}) = 0$ and $x_2(t_{00}) = 0$ hold.

Fact 3. For the STS (1), when $k_1 > 0, k_2 > 0$, and $A \geq -1/4$ hold, the initial conditions are $x_1(0) = a$ and $x_2(0) = b$, and the system states x_1 and x_2 arrive at the zero point simultaneously.

As shown in Figure 3, the initial conditions $x_1(0) = a$ and $x_2(0) = b$ hold, and then system states change as in Figure 1 in period T_{a0}, where state x_1 converges to zero at time t_{a0}; after that, as in Figure 2, states x_1 and x_2 converge to zero in period T_{00} at time $t_{a0} + t_{00}$, where $x_1(t_{a0} + t_{00}) = 0$ and $x_2(t_{a0} + t_{00}) = 0$ hold.

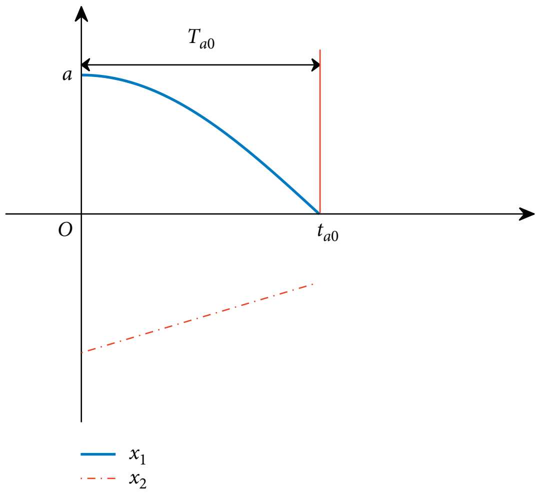

FIGURE 1: Trajectory of the STS with $a \neq 0$.

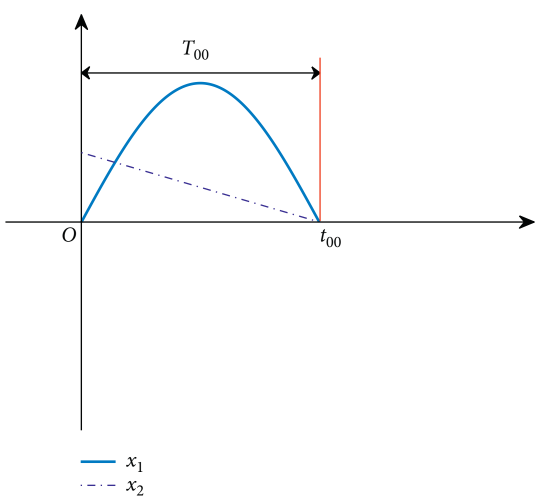

FIGURE 2: Trajectory of the STS with $a = 0$.

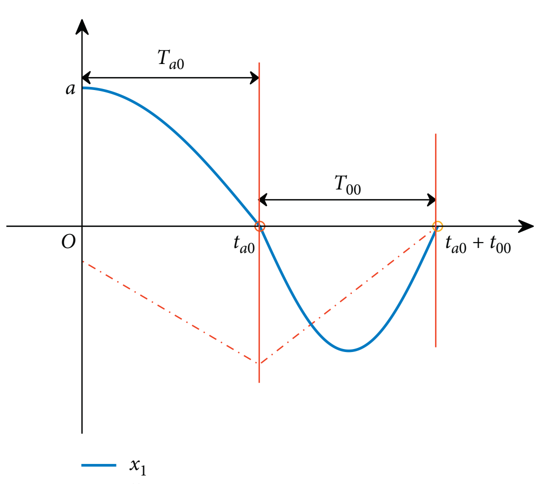

FIGURE 3: Trajectory of STS in the convergence procedure.

To prove the facts above, in $t \in T_{a0}$, let

$$y = -\frac{2}{k_1}|x_1|^{1/2}. \tag{4}$$

Its derivative on the trajectories of (B.2) and (B.6) can be obtained as follows:

$$y\dot{y} - y = At + B, \tag{5}$$

where

$$\begin{aligned} A &= -\frac{2k_2}{k_1^2}, \\ B &= \frac{2b}{k_1^2}\operatorname{sign}(2\operatorname{sign}(a) + \operatorname{sign}(b)). \end{aligned} \tag{6}$$

The details about the parameters of equation (6) are presented in Appendix B.

Then, based on the parametric solution in Appendix A and general form of (5), define the solution in different situations by the parameter A and define the trajectory in different parameters and different initial conditions.

Case I. For the supertwisting system, when $(A = (-1/4))$ and $a \neq 0$, the trajectory is

$$\begin{cases} t(s) = Cs\exp(s) - \dfrac{B}{A}, \\ y(s) = \dfrac{C}{2}(s+1)\exp(s), \end{cases} \tag{7}$$

where the parameter C is determined by initial conditions $s = s_0$, $t = 0$, and $y = a_1 = -(2/k_1)|a|^{1/2}$ as follows:

$$\begin{cases} 0 = Cs_0\exp(s_0) - \dfrac{B}{A}, \\ a_1 = \dfrac{C}{2}(s_0+1)\exp(s_0). \end{cases} \tag{8}$$

Therefore, the initial parameter can be obtained as

$$s_0 = \frac{B}{2a_1A - B}, \tag{9}$$

and then the parameter C is obtained as

$$C = \frac{B}{As_0\exp(s_0)}. \tag{10}$$

Then, the whole trajectory can be obtained for the parametric equation, when $s_1 = -1$, $s_2 = -\infty$, and $y = 0$:

$$\begin{aligned} t(s_1) &= -C\exp(-1) - \frac{B}{A}, \\ t(s_2) &= -\frac{B}{A}. \end{aligned} \tag{11}$$

The state x_1 will arrive at the origin point at t_{a0} where the following conditions are met:

(a1) When $s_2 < s_1 < s_0$, $t_{a0} = t(s_1)$.

(a2) When $s_2 < s_0 < s_1$, if $(\partial t/\partial s) > 0$, $t_{a0} = t(s_1)$.

(a3) When $s_2 < s_0 < s_1$, if $(\partial t/\partial s) < 0$, $t_{a0} = t(s_2)$.

Therefore, for the STS at time t_{a0},

$$\begin{cases} x_1(t_{a0}) = 0, \\ x_2(t_{a0}) = -k_2 \operatorname{sign}(a) t_{a0} + b. \end{cases} \tag{12}$$

Remark 1. If $t_{a0} = b/(k_2 \operatorname{sign}(a))$ holds, $x_2(t_{a0}) = 0$ in (12), and the system converges to the origin point at time t_{a0}; if not, the system changes into another STA with initial conditions $a = 0$ and $b \neq 0$ because of the time invariance of the STA.

Case II. For the supertwisting system, when $A = -(1/4)$ and $a = 0$,

$$\begin{cases} t(s) = Cs\exp(s) - \dfrac{B}{A}, \\ y(s) = \dfrac{C}{2}(s+1)\exp(s). \end{cases} \tag{13}$$

For the initial conditions, (13) becomes

$$\begin{cases} 0 = Cs_0\exp(s_0) - \dfrac{B}{A}, \\ 0 = \dfrac{C}{2}(s_0+1)\exp(s_0). \end{cases} \tag{14}$$

Therefore, the initial parameter can be obtained as

$$s_0 = -1, \tag{15}$$

and then the parameter C can be obtained as

$$C = -\frac{B}{A\exp(-1)}. \tag{16}$$

Then, for the parametric equation, when $s_1 = -\infty$ and $y = 0$,

$$t_{00} = -\frac{B}{A}. \tag{17}$$

According to system (7) and convergence time (17), the states of the system reach the origin point at the same time:

$$\begin{cases} x_1(t_{00}) = 0, \\ x_2(t_{00}) = 0. \end{cases} \tag{18}$$

Therefore, the system reaches finite-time stability.

Remark 2. If the initial condition $a = 0$, based on the convergence time t_{00}, both the states reach the origin point at the same time.

The supertwisting system is the time invariance system, and states reach finite-time stability when $k_1, k_2 > 0$; therefore, if the initial condition $a \neq 0$, based on the convergence time t_{a0}, by calculating both the states at t_{a0}, a new supertwisting system with the initial condition $a = 0$ can be obtained. Combining process initial conditions $a = 0$ and $a \neq 0$, the supertwisting system convergence time can be obtained.

Theorem 1. *The supertwisting system (1) reaches finite-time stability, if gains $k_1 > 0$ and $k_2 > 0$ satisfy $k_1^2 = 8k_2$ and system's initial condition $x_1(0) = a$ and $x_2(0) = b$; the finite time T satisfies $T = t_{a0} + t_{00}$, where t_{a0} is the time when state x_1 reaches the origin point in first time $x_2 = b_1 \neq 0$, and then time $t_{00} = -(B_1/A)$ is elapsed, when both states reach the origin point at the same time; therefore, the STS reaches the origin point at time T where the following conditions are met:*

(b1) When $s_2 < s_1 < s_0$, $t_{a0} = -C\exp(-1) - (B/A)$.

(b2) When $s_2 < s_0 < s_1$, if $C < 0$, $t_{a0} = -C\exp(-1) - (B/A)$.

(b3) When $s_2 < s_0 < s_1$, if $C > 0$, $t_{a0} = -(B/A)$.

Here,

$$\begin{aligned} A &= -\frac{2k_2}{k_1^2}, \\ B &= \frac{2b}{k_1^2}\operatorname{sign}(2\operatorname{sign}(a) + \operatorname{sign}(b)), \\ B_1 &= \frac{2b_1}{k_1^2}\operatorname{sign}(b_1), \\ b_1 &= -k_2\operatorname{sign}(a)t_{a0} + b, \\ s_0 &= \frac{-B}{(4/k_1)|a|^{1/2}A + B}, \\ s_1 &= -1, \\ s_2 &= -\infty, \\ C &= \frac{B}{As_0\exp(s_0)}. \end{aligned} \tag{19}$$

Proof of Theorem 1. Based on the supertwisting system and gains k_1 and k_2, if $A = -(1/4)$, the parametric solution can be obtained in the form of (7); based on the initial condition, parameters s_0 and C can be obtained by (9) and (10), and then t_{a0} can be obtained in following different situations: (b1)–(b3).

When $t = t_{a0}$, the states of the system change as in (12), and then the system will elapse the period t_{00}, given by (17), which the parameter B satisfies:

$$B = \frac{2b_1}{k_1^2}\operatorname{sign}(b_1), \tag{20}$$

where

$$b_1 = -k_2\operatorname{sign}(a)t_{a0} + b. \tag{21}$$

Then, at $t_{00} = -(B/A)$, the states of the supertwisting system reach the origin point:

$$\begin{cases} x_1(t_{00}) = 0, \\ x_2(t_{00}) = 0. \end{cases} \quad (22)$$

This completes the proof of Theorem 1. □

Case III. For the supertwisting system, when $A > -(1/4)$ and $a \neq 0$, the trajectory is

$$\begin{cases} t(s) = Cs(1+ps)^q - \dfrac{B}{A}, \\ y(s) = C\left(1 + \dfrac{1}{2}s + \dfrac{p}{2}s\right)(1+ps)^q, \end{cases} \quad (23)$$

where $p = \sqrt{4A+1}$ $(0 < p < 1)$ and $q = (1/2p) - (1/2)$ $(q > 0)$.

$$\begin{cases} 0 = Cs_0(1+ps_0)^q - \dfrac{B}{A}, \\ a_1 = C\left(1 + \dfrac{1}{2}s_0 + \dfrac{p}{2}s_0\right)(1+ps_0)^q. \end{cases} \quad (24)$$

Therefore,

$$s_0 = \frac{1}{(a_1 A/B) - (1/2) - (p/2)}, \quad (25)$$

$$C = \frac{B}{As_0(1+ps_0)^q}. \quad (26)$$

Then, the whole trajectory can be obtained for the parametric equation, when $s_1 = -(2/1+p), s_2 = -(1/p)$, $y = 0$, and $s_2 < s_1$:

$$t(s_1) = C\frac{((p/2)-(1/2))^q}{(-(p/2)-(1/2))^{q+1}} - \frac{B}{A}, \quad (27)$$

$$t(s_2) = -\frac{B}{A}. \quad (28)$$

The state x_1 will reach the origin point at t_{a0}, where the following conditions are met:

(c1) When $s_2 < s_1 < s_0$, $t_{a0} = t_1$.

(c2) When $s_0 < s_2 < s_1$, $t_{a0} = t_1$.

(c3) When $s_2 < s_0 < s_1$, if $\partial t/\partial s|_{s=s_0} > 0$, $t_{a0} = t(s_1)$.

(c4) When $s_2 < s_0 < s_1$, if $\partial t/\partial s|_{s=s_0} > 0$, $t_{a0} = t(s_2)$.

Then, the states of the supertwisting system at time t_{a0} become

$$\begin{cases} x_1(t_{a0}) = 0, \\ x_2(t_{a0}) = -k_2 \operatorname{sign}(a) t_{a0} + b. \end{cases} \quad (29)$$

Case IV. For the supertwisting system, when $A > -(1/4)$ and $a = 0$, the trajectory is

$$\begin{cases} t(s) = Cs(1+ps)^q - \dfrac{B}{A}, \\ y(s) = C\left(1 + \dfrac{1}{2}s + \dfrac{p}{2}s\right)(1+ps)^q. \end{cases} \quad (30)$$

For the initial conditions $t = 0$ and $y = 0$,

$$\begin{cases} 0 = Cs_0(1+ps_0)^q - \dfrac{B}{A}, \\ 0 = C\left(1 + \dfrac{1}{2}s_0 + \dfrac{p}{2}s_0\right)(1+ps_0)^q. \end{cases} \quad (31)$$

Therefore,

$$s_0 = -\frac{2}{1+p}, \quad (32)$$

$$C = \frac{B}{As_0(1+ps_0)^q}, \quad (33)$$

for the parametric equation

$$s_2 = -\frac{1}{p}. \quad (34)$$

Therefore, the convergence time is

$$t_{00} = -\frac{B}{A}, \quad (35)$$

$$\begin{cases} x_1(t_{00}) = 0, \\ x_2(t_{00}) = 0. \end{cases} \quad (36)$$

Therefore, the system reaches finite-time stability.

Theorem 2. *The supertwisting system (1), if gains $k_1 > 0$ and $k_2 > 0$ satisfy $k_1^2 > 8k_2$ and system initial conditions $x_1(0) = a$ and $x_2(0) = b$, reaches finite-time stability; the finite time T satisfies $T = t_{a0} + t_{00}$, where t_{a0} is the time when state x_1 reaches the origin point in first time $x_2 = b_1 \neq 0$, and then time $t_{00} = -(B_1/A)$ is elapsed, when both states reach the origin point at the same time.*

(d1) When $s_2 < s_1 < s_0$, $t_{a0} = t(s_1)$.

(d2) When $s_0 < s_2 < s_1$, $t_{a0} = t(s_1)$.

(d3) When $s_2 < s_0 < s_1$, if $\partial t/\partial s|_{s=s_0} > 0$, $t_{a0} = t(s_1)$.

(d4) When $s_2 < s_0 < s_1$, if $\partial t/\partial s|_{s=s_0} < 0$, $t_{a0} = t(s_2)$.

Here,

$$\begin{aligned} A &= -\frac{2k_2}{k_1^2}, \\ B &= \frac{2b}{k_1^2} \operatorname{sign}(2\operatorname{sign}(a) + \operatorname{sign}(b)), \\ B_1 &= \frac{2b_1}{k_1^2} \operatorname{sign}(b_1), \\ b_1 &= -k_2 \operatorname{sign}(a) t_{a0} + b. \end{aligned} \quad (37)$$

Proof of Theorem 2. Based on the supertwisting system and gains k_1 and k_2, if $A > -(1/4)$, the parametric solution can be obtained in the form of (23); based on the initial condition, parameters s_0 and C can be obtained by (25) and (26), and then t_{a0} can be obtained in following different situations: (d1)–(d4).

When $t = t_{a0}$, the states of the system change as in (29), and then the system will elapse period t_{00}, given by (35), which the parameter B satisfies:

$$B = \frac{2b_1}{k_1^2}\operatorname{sign}(b_1), \tag{38}$$

where

$$b_1 = -k_2 \operatorname{sign}(a) t_{a0} + b. \tag{39}$$

Then, $t_{00} = -(B/A)$, and the states of the supertwisting system reach the origin point at time T:

$$\begin{cases} x_1(T) = 0, \\ x_2(T) = 0. \end{cases} \tag{40}$$

This completes the proof of Theorem 2. □

Remark 3. From Theorem 1 and Theorem 2, the convergence time is $T = t_{a0} + t_{00}$ and then k_1 and k_2 exist in both t_{a0} and t_{00} in the form of B/A; therefore, the gain parameter k_1 is not influenced by the convergence time.

4. Finite-Time Convergence of STA with Perturbation

The STA with perturbation can be written as

$$\begin{aligned} \dot z_1 &= -h_1 |z_1|^{1/2}\operatorname{sign}(z_1) + z_2 + \rho_1(t), \\ \dot z_2 &= -h_2 \operatorname{sign}(z_1) + \rho_2(t), \end{aligned} \tag{41}$$

where z_1 and z_2 are the scalar state variables, with initial conditions $z_1(0) = a$ and $z_2(0) = b$; $h_1, h_2 > 0$ are gains to be designed; and ρ_1 and ρ_2 are the perturbation terms. It is well known that the STA is robustly stable to perturbations globally bounded, and the equilibrium point is $(0 \;\; -\rho_1)$; therefore, suppose the perturbation terms of the system.

Assume that $|\rho_1| \le d_1$, $|\rho_2| \le d_2$, $|\dot\rho_1| \le d_3$, and $|\dot\rho_2| \le d_4$.

Let

$$\begin{aligned} x_1 &= z_1, \\ x_2 &= z_2 + \rho_1, \end{aligned} \tag{42}$$

then system (41) can be changed as

$$\begin{aligned} \dot x_1 &= -h_1 |x_1|^{1/2}\operatorname{sign}(x_1) + x_2, \\ \dot x_2 &= -h_2 \operatorname{sign}(x_1) + \rho_2 + \dot\rho_1. \end{aligned} \tag{43}$$

Next, let

$$\begin{aligned} k_1 &= h_1, \\ k_2 &= h_2 - (\rho_2 + \dot\rho_1)\operatorname{sign}(x_1), \end{aligned} \tag{44}$$

then the system can be written as

$$\begin{aligned} \dot x_1 &= -k_1 |x_1|^{1/2}\operatorname{sign}(x_1) + x_2, \\ \dot x_2 &= -k_2 \operatorname{sign}(x_1). \end{aligned} \tag{45}$$

The initial conditions are $x_1(0) = a$ and $\underline{b} < x_2(0) < \overline{b}$ and parameters are $k_1 = h_1$ and $\underline{k}_2 < k_2 < \overline{k}_2$, where $\underline{b} = b_z - d_1$, $\overline{b} = b_z + d_1$, $\underline{k}_2 = h_2 - d_2 - d_3$, and $\overline{k}_2 = h_2 + d_2 + d_3$.

Under the conditions on STA without perturbation in Theorem 2, the robust finite time is presented for STA with perturbation. Assume $A = -(2k_2/k_1^2) > -(1/4)$, then $h_1^2 > 8(h_2 - d_2 - d_3) > 0$. The finite time is presented by $T_P = t_{Pa0} + t_{P00}$, where t_{Pa0} and t_{P00} can be obtained by Theorem 2 in different initial conditions and gains.

In Theorem 2, t_{Pa0} is dependent on k_1, k_2, a, and b and is then calculated in different situations to get the maximum. To compare the influence of perturbation, three STAs are built:

$$\begin{cases} \text{STA 1: } \{x_1, x_2 | t, k_1, k_2, a, b\}, \\ \text{STA 2: } \{\overline{x}_1, \overline{x}_2 | t, k_1, \overline{k}_2, a, \overline{b}\}, \\ \text{STA 3: } \{\underline{x}_1, \underline{x}_2 | t, k_1, \underline{k}_2, a, \underline{b}\}. \end{cases} \tag{46}$$

Without loss of generality, assume that $a > 0$, in time phasing $t \in [0, t_{Pa0}]$.

For the perturbation of the STS at different gains k_2 and different initial conditions b, assume states x_2, $\underline{x}_2$, and $\overline{x}_2$ as $x_2 = b - k_2 t$, $\underline{x}_2 = \underline{b} - \overline{k}_2 t$, and $\overline{x}_2 = \overline{b} - \underline{k}_2 t$; therefore, $\underline{x}_2 \le x_2 \le \overline{x}_2$ and $\underline{x}_1 \le x_1 \le \overline{x}_1$ in $t \in [0, t_{Pa0}]$.

The convergence time satisfies $t_{a0}(k_1, \overline{k}_2, a, \underline{b}) \le t_{Pa0} \le t_{a0}(k_1, \underline{k}_2, a, \overline{b})$.

Assume that $a = 0$ and $b > 0$; to compare the influence of perturbation, three STAs are built:

$$\begin{cases} \text{STA 1: } \{x_1, x_2 | t, k_1, k_2, a, b\}, \\ \text{STA 2: } \{\overline{x}_1, \overline{x}_2 | t, k_1, \overline{k}_2, a, b\}, \\ \text{STA 3: } \{\underline{x}_1, \underline{x}_2 | t, k_1, \underline{k}_2, a, b\}. \end{cases} \tag{47}$$

Choose a sliding mode surface as $s = c_1 x_1 + c_2 x_2$, $\overline{s} = c_1 \overline{x}_1 + c_2 \overline{x}_2$, and $\underline{s} = c_1 \underline{x}_1 + c_2 \underline{x}_2$, where $c_1 > 0$ and $c_2 > 0$; therefore,

$$\begin{aligned} \dot s &= c_1 \dot x_1 + c_2 \dot x_2 \\ &= -c_1 k_1 |x_1|^{1/2}\operatorname{sign}(x_1) + c_1 x_2 - c_2 k_2 \operatorname{sign}(x_1) \\ &= -c_1 k_1 |x_1|^{1/2} + c_1 b - c_1 k_2 t - c_2 k_2, \end{aligned} \tag{48}$$

because of

$$-c_1 k_1 |x_1|^{1/2} + c_1 b - c_1 \overline{k}_2 t - c_2 \overline{k}_2 \le \dot s \le -c_1 k_1 |x_1|^{1/2} + c_1 b - c_1 \underline{k}_2 t - c_2 \underline{k}_2. \tag{49}$$

Therefore,

$$\dot{\bar{s}} \le \dot{s} \le \dot{\underline{s}}. \tag{50}$$

Because trajectories $\overline{x}_1, \overline{x}_2, \underline{x}_1, \underline{x}_2, \overline{s}$, and $\underline{s}$ converge to origin in finite time, the trajectory s reaches origin in finite time. Parameters c_1 and c_2 are arbitrary positive constants; therefore, $s = 0$ could be equivalent to $x_1 = 0$ and $x_2 = 0$. The convergence time satisfies

$$t_{a0}\left(k_1, \overline{k}_2, a, b\right) \le t_{P00} \le t_{a0}\left(k_1, \underline{k}_2, a, b\right). \tag{51}$$

Theorem 3. *For the supertwisting algorithm with perturbation (41), if $h_1, h_2 > 0$ and $h_1^2 > 8(h_2 - d_2 - d_3) > 0$ hold, the system converges to the origin point in finite time, and the maximum convergence time is $T_P = t_{Pa0} + t_{P00}$, where $t_{Pa0} = t_{a0}(k_1, \underline{k}_2, a, \overline{b})$ and $t_{P00} = \max\left\{|\overline{b}|, |\underline{b} - \overline{k}_2 t_{Pa0}|\right\}/\underline{k}_2$.*

Proof of Theorem 3. For STA with perturbation, when the initial condition $a \ne 0$, in the first stage, the state x_1 would arrive at the origin point in finite time $t_{Pa0} = t_{a0}(k_1, \underline{k}_2, a, \overline{b})$, and then when the system is in the second stage, both states x_1 and x_2 arrive at the origin point in finite time:

$$t_{P00} = \frac{\max\left\{|\overline{b}|, \left|\underline{b} - \overline{k}_2 t_{Pa0}\right|\right\}}{\underline{k}_2}. \tag{52}$$

Then, the maximum finite time is presented as $T_P = t_{Pa0} + t_{P00}$. □

5. STC for Nonaffine Systems

To illustrate the STA, nonlinear system STC is proposed for the nonaffine system:

$$\dot{y} = f(y, u). \tag{53}$$

Using the coordinate transform,

$$x_1 = y - y_d. \tag{54}$$

The virtual control law is

$$\alpha = -k_1 |x_1|^{1/2} \operatorname{sign}(z_1) + u - f(y, u) + \dot{y}_d. \tag{55}$$

Then,

$$\dot{x}_1 = -k_1 |x_1|^{1/2} \operatorname{sign}(x_1) + x_2, \tag{56}$$

where

$$x_2 = u - \alpha. \tag{57}$$

Choose the ideal control law

$$v = -k_2 \operatorname{sign}(x_1) + \dot{\alpha}, \tag{58}$$

then

$$\dot{x}_2 = -k_2 \operatorname{sign}(x_1). \tag{59}$$

Therefore, the system satisfies Theorem 1 and Theorem 2 by choosing the gain parameters k_1 and k_2.

To overcome the nonaffinity, the virtual control and backstepping skill are used. They are extremely important to ensure the successful design of the proposed STC.

6. Simulation

Several simulated examples were selected to conduct bench tests of the STA and STC. Example 1–Example 6 were testing simple STA, and the purpose of testing these examples was to investigate whether the convergence time is solved from the parametric equation. Examples 7–10 were designed for the test of STC for nonaffine nonlinear systems and demonstrated the superiority of STC and convergence time of the STA.

Example 1. Consider the STS given by (1), with $k_1 = 1$, $k_2 = 1/8$, $a = 2$, and $b = -1$ in Case I; based on (9) and (10), we can get $s_0 = -0.5858$ and $C = -24.5332$, respectively, and then we can get $t_{a0} = 1.0253$; therefore, the state x_1 reaches the zero point at time 1.0253.

Example 2. Consider the STS given by (1), with $k_1 = 1$, $k_2 = 1/8$, $a = 0$, and $b = -1.1282$ in Case II; based on (16) and (17), we can get $s_0 = -1$ and $C = -(B/A \exp(-1))$, respectively, and then we can get $t_{00} = 9.0256$; therefore, the states x_1 and x_2 reach the zero point at time 9.0256.

Example 3. Consider the STS given by (1), with $k_1 = 3$, $k_2 = 1$, $a = 3$, and $b = -3$ in Case III; based on (25) and (26), we can get $s_0 = -0.9510$ and $C = -4.6188$, respectively, and then we can get $t_{a0} = 0.4641$; therefore, the state x_1 reaches the zero point at time 0.4641.

Example 4. Consider the STS given by (1), with $k_1 = 3$, $k_2 = 1$, $a = 0$, and $b = -3.4641$ in Case IV; based on (32) and (33), we can get $s_0 = -(2/1 + p)$ and $C = (B/As_0(1 + ps_0)^q)$, respectively, and then we can get $t_{00} = 3.4641$; therefore, the states x_1 and x_2 reach the origin point at time 3.4641.

Example 5. Consider the supertwisting system given by

$$\begin{aligned} \dot{x}_1 &= -|x_1|^{1/2} \operatorname{sign}(x_1) + x_2, \\ \dot{x}_2 &= -\frac{1}{8} \operatorname{sign}(x_1), \end{aligned} \tag{60}$$

with $k_1 = 1, k_2 = 1/8$, and $A = -(1/4)$ and initial conditions $x_1(0) = 2$ and $x_2(0) = -1$.

Based on Theorem 1, the system (60) reaches finite-time stability and calculates the time from parametric equation (7) step by step. Firstly, the initial parameter $s_0 = -0.5858$ is calculated from (9) and $C = -24.5332$ from (10) because $s_2 < s_1 < s_0$ holds; therefore, $t_{a0} = t(s_1)$ and $t(s_1)$ can be obtained from (11) as $t_{a0} = 1.0253$ and $x_2(t_{a0}) = -1.1282$ in Figure 4. Secondly, $t_{00} = 9.0256$ is calculated from (17) in Figure 5. Finally, the reaching time $t = t_{a0} + t_{00} = 10.0509$ in Figure 6.

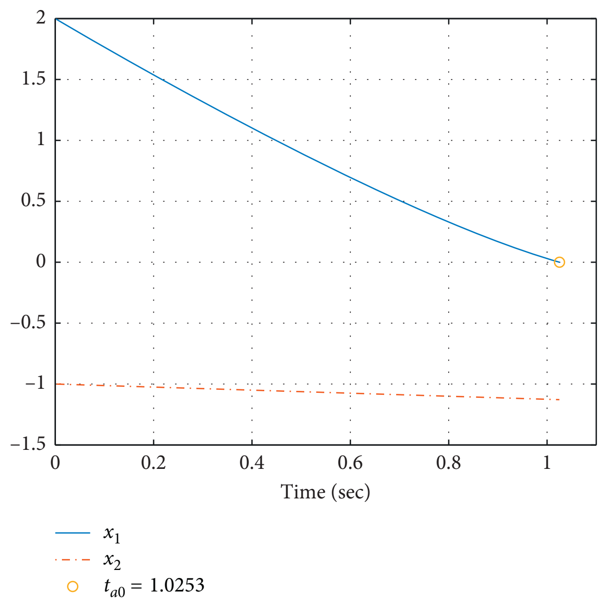

Figure 4: Finite-time convergence to the origin point with the nonzero initial condition of the state ($A = (-1/4)$).

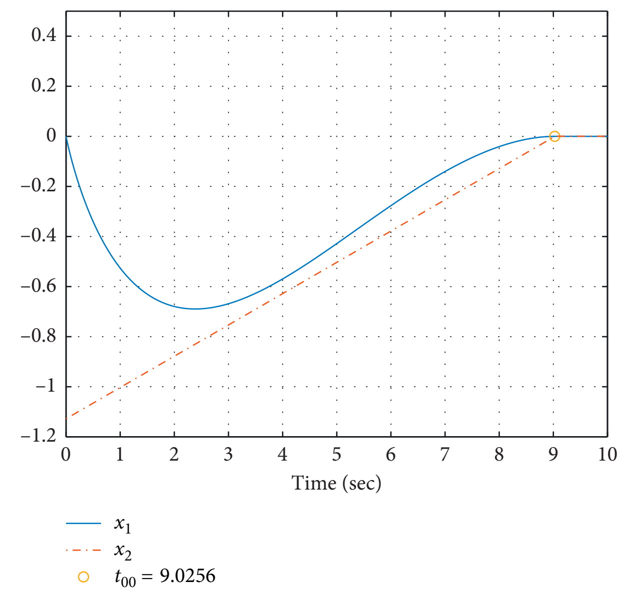

Figure 5: Finite-time convergence with the zero initial condition of the state ($A = (-1/4)$).

Example 6. Consider the supertwisting system given by

$$\begin{aligned}\dot{x}_1 &= -3\left|x_1\right|^{1/2}\operatorname{sign}(x_1) + x_2,\\ \dot{x}_2 &= -\operatorname{sign}(x_1),\end{aligned} \tag{61}$$

with $k_1 = 3, k_2 = 1$, and $A = -(2/9)$ and initial conditions $x_1(0) = 3$ and $x_2(0) = -3$.

Firstly, the initial parameter $s_0 = -0.9510$ is calculated from (25) and $C = -4.6188$ from (26) because $s_1 = -1.5$, $s_2 = -3$, and $s_2 < s_1 < s_0$ hold; therefore, $t_{a0} = t(s_1)$ and $t(s_1)$ can be obtained from (27) as $t_{a0} = 0.4641$ and $x_2(t_{a0}) = -3.4641$ in Figure 7. Secondly, $t_{00} = 3.4641$ is calculated from (35) in Figure 8. Finally, the reaching time $t = t_{a0} + t_{00} = 3.9282$ in Figure 9.

Example 7. Consider the STA with perturbation as given by (45), as $k_1 = 4, 0.5 \le k_2 \le 1.5, a = 3$, and $-5 \le b \le -1$:

$$\begin{aligned}
&\text{STA 1: } \begin{cases} \dot{x}_1 = -4\left|x_1\right|^{1/2}\operatorname{sign}(x_1) + x_2,\\ \dot{x}_2 = -\operatorname{sign}(x_1) + 0.3\cos(t) + 0.2\cos(0.1t),\\ x_1(0) = 3,\\ x_2(0) = -3,\end{cases}\\
&\text{STA 2: } \begin{cases} \dot{\overline{x}}_1 = -4\left|\overline{x}_1\right|^{1/2}\operatorname{sign}(\overline{x}_1) + \overline{x}_2,\\ \dot{\overline{x}}_2 = -0.5\operatorname{sign}(\overline{x}_1),\\ \overline{x}_1(0) = 3,\\ \overline{x}_2(0) = -1,\end{cases}\\
&\text{STA 3: } \begin{cases} \dot{\underline{x}}_1 = -4\left|\underline{x}_1\right|^{1/2}\operatorname{sign}(\underline{x}_1) + \underline{x}_2,\\ \dot{\underline{x}}_2 = -1.5\operatorname{sign}(\underline{x}_1),\\ \underline{x}_1(0) = 3,\\ \underline{x}_2(0) = -5.\end{cases}
\end{aligned} \tag{62}$$

In Figure 10, states of three STAs are presented to indicate stability in finite time, and convergence time is $t_{a0} = 0.4099$, $\overline{t}_{a0} = 0.5816$, and $\underline{t}_{a0} = 0.3135$, respectively. The trajectory indicates states x_1, $\overline{x}_1$, and $\underline{x}_1$ with the same initial condition reach zero at different time because of different x_2, $\overline{x}_2$, and $\underline{x}_2$. The trajectories x_1 and x_2 are in between $\underline{x}_1$ and $\overline{x}_1$ and between $\underline{x}_2$ and $\overline{x}_2$, respectively. State $\overline{x}_1$ is the last state that reaches zero; therefore, the maximum convergence time $\overline{t}_{a0} = 0.5816$. In addition, the maximum of the state $|x_2|$ is $|\underline{x}_2(\overline{t}_{a0})| = 5.8723$ and absolutely necessary to estimate finite time t_{00} in the next step.

Example 8. Consider the STA with perturbation as given by (45), as $k_1 = 4, 0.5 \le k_2 \le 1.5, a = 0$, and $b = -5.8723$:

$$\begin{aligned}
&\text{STA 1: } \begin{cases} \dot{x}_1 = -4\left|x_1\right|^{1/2}\operatorname{sign}(x_1) + x_2,\\ \dot{x}_2 = -\operatorname{sign}(x_1) + 0.3\cos(t) + 0.2\cos(0.1t),\\ x_1(0) = 0,\\ x_2(0) = -5.8723,\end{cases}\\
&\text{STA 2: } \begin{cases} \dot{\overline{x}}_1 = -4\left|\overline{x}_1\right|^{1/2}\operatorname{sign}(\overline{x}_1) + \overline{x}_2,\\ \dot{\overline{x}}_2 = -0.5\operatorname{sign}(\overline{x}_1),\\ \overline{x}_1(0) = 0,\\ \overline{x}_2(0) = -5.8723,\end{cases}\\
&\text{STA 3: } \begin{cases} \dot{\underline{x}}_1 = -4\left|\underline{x}_1\right|^{1/2}\operatorname{sign}(\underline{x}_1) + \underline{x}_2,\\ \dot{\underline{x}}_2 = -1.5\operatorname{sign}(\underline{x}_1),\\ \underline{x}_1(0) = 0,\\ \underline{x}_2(0) = -5.8723.\end{cases}
\end{aligned} \tag{63}$$

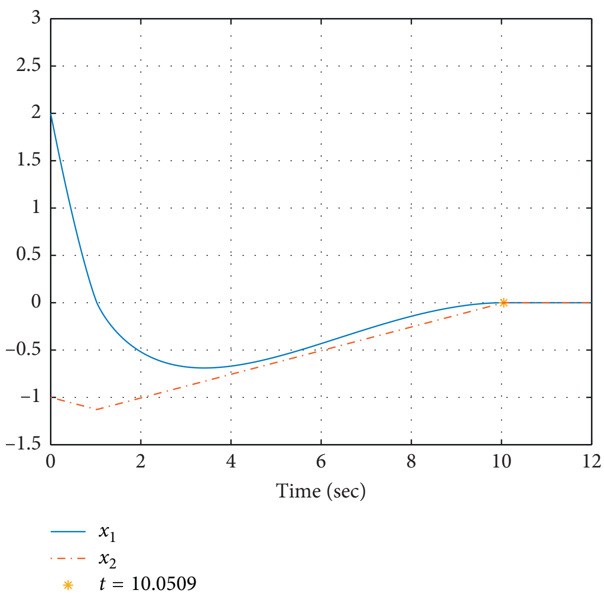

Figure 6: Finite-time convergence ($A = -(1/4)$).

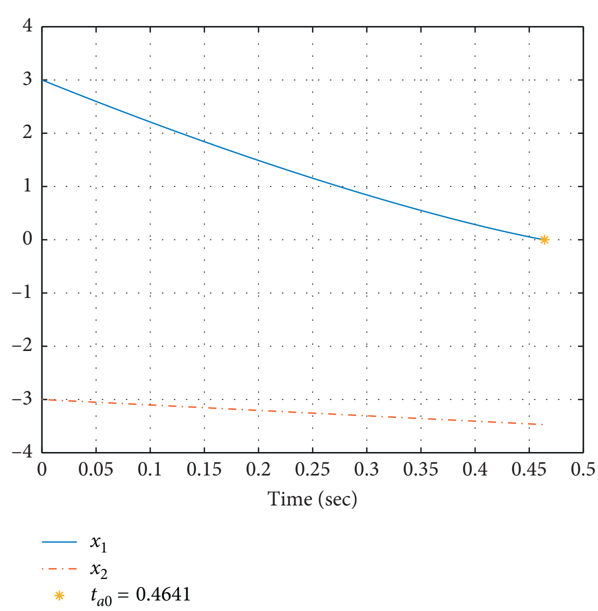

Figure 7: Finite-time convergence to the origin point with the nonzero initial condition of the state ($A = (-2/9)$).

In Figure 11, states of three STAs are presented to indicate stability in finite time, and convergence time is $t_{00} = 5.6589$, $\overline{t}_{00} = 11.7447$, and $\underline{t}_{00} = 3.9149$, respectively. The trajectories x_1, x_2, and s are in between $\underline{x}_1$ and $\overline{x}_1, \underline{x}_2$ and $\overline{x}_2$, and $\underline{s}$ and $\overline{s}$, respectively. State $\overline{x}_1$ is the last state that reaches zero; therefore, the maximum convergence time $\overline{t}_{00} = 11.7447$. In addition, states of the sliding mode surface indicate the state s is in between $\underline{s}$ and $\overline{s}$, providing evidence that $\overline{t}_{00}$ is the maximum convergence time.

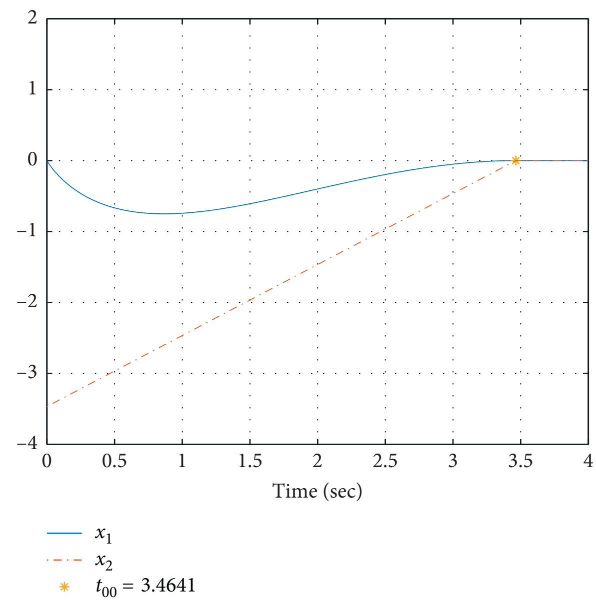

Figure 8: Finite-time convergence with the zero initial condition of the state ($A = (-2/9)$).

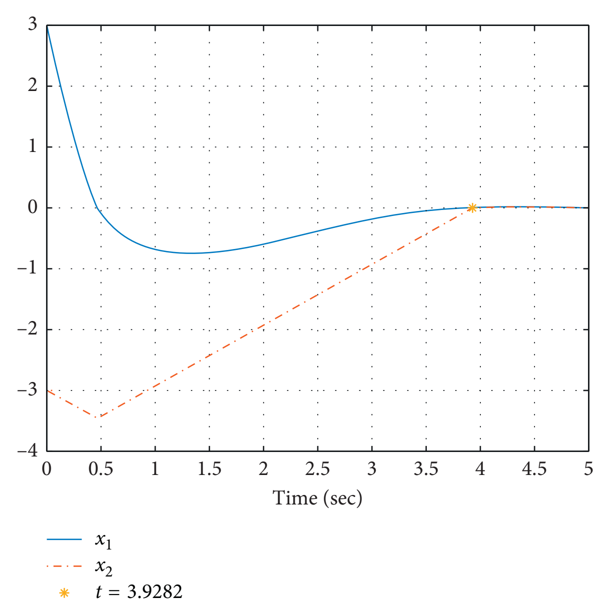

Figure 9: Finite-time convergence ($A = -(2/9)$).

Example 9. Consider the supertwisting system with perturbation given by

$$\begin{aligned}\dot{z}_1 &= -h_1 |z_1|^{1/2} \operatorname{sign}(z_1) + z_2 + \rho_1(t),\\ \dot{z}_2 &= -h_2 \operatorname{sign}(z_1) + \rho_2(t),\end{aligned} \tag{64}$$

where z_1 and z_2 are the scalar state variables, with initial conditions $z_1(0) = 3$ and $z_2(0) = -3$; $h_1 = 4$ and $h_2 = 1$ are gains; and $\rho_1(t) = 2\sin(0.1t)$ and $\rho_2(t) = 0.3\cos(t)$ are perturbation terms. Therefore, the system (64) changes as

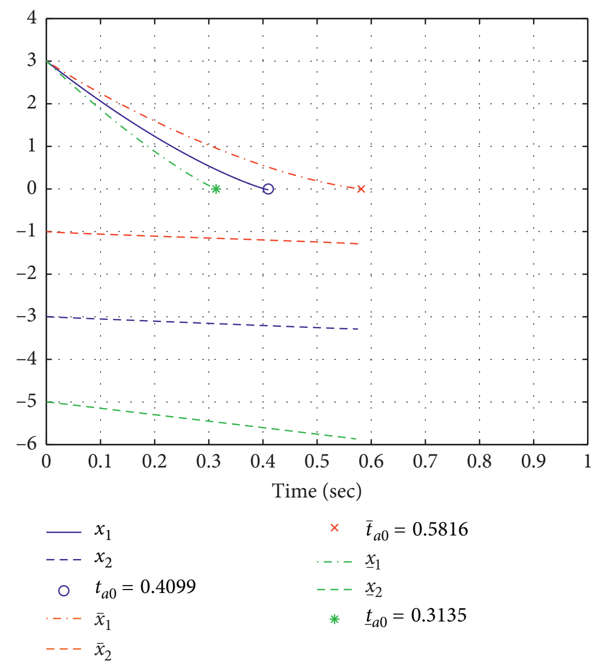

Figure 10: Finite-time convergence of the output with perturbation.

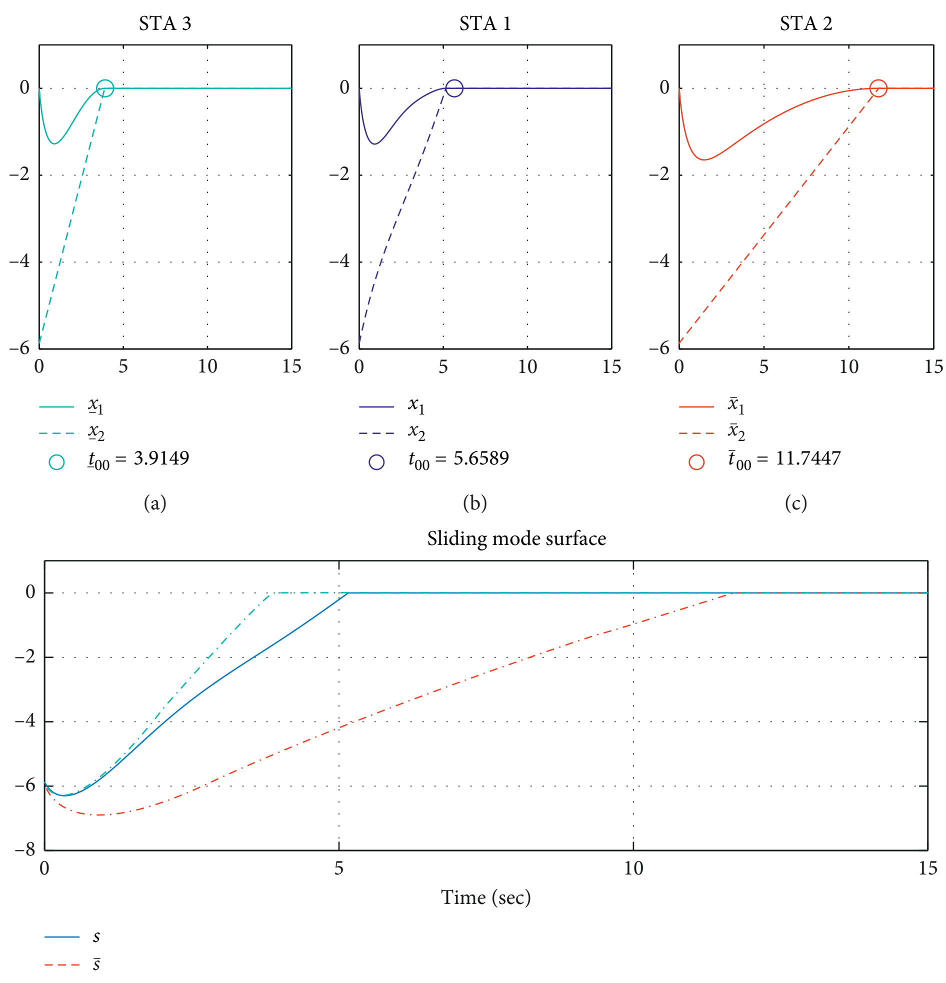

Figure 11: Finite-time convergence of the output and sliding mode surface.

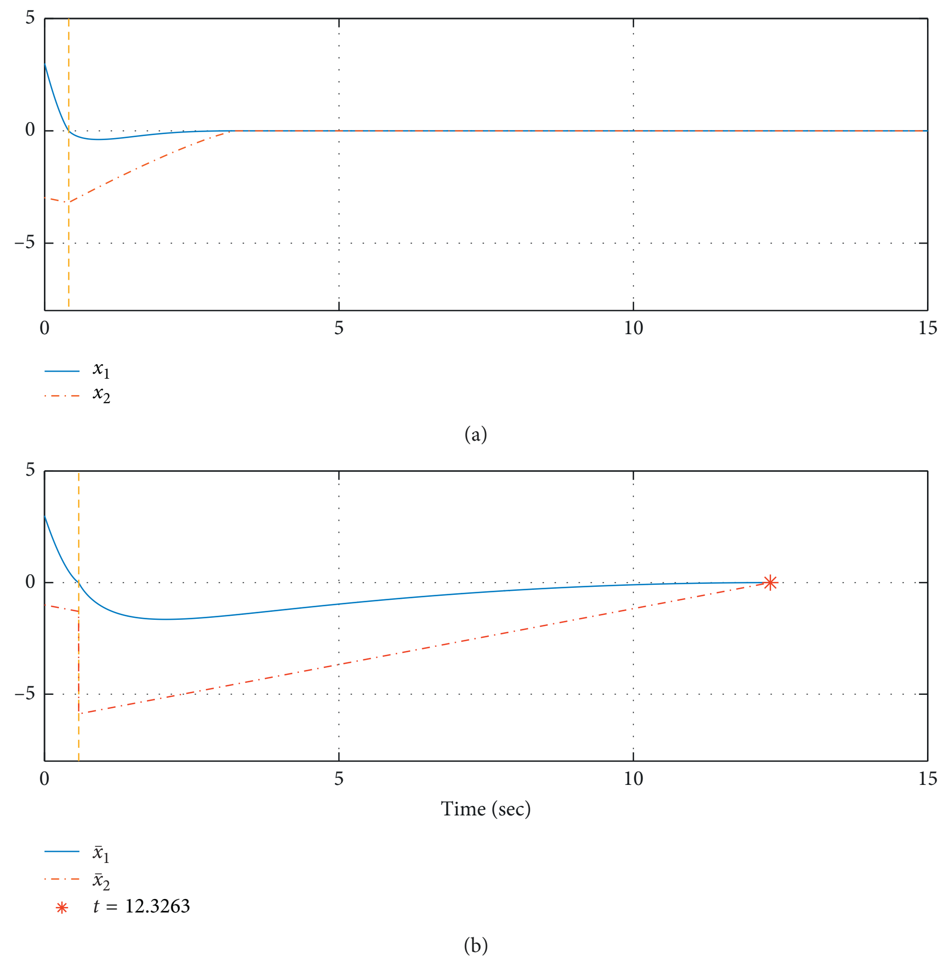

Figure 12: Finite-time convergence of the output and maximum stage.

$$\begin{aligned}\dot{x}_1 &= -h_1|x_1|^{1/2}\operatorname{sign}(x_1) + x_2,\\ \dot{x}_2 &= -h_2\operatorname{sign}(x_1) + \rho_2 + \dot{\rho}_1,\end{aligned} \quad (65)$$

where $x_1 = z_1$ and $x_2 = z_2 + \rho_1$; therefore, parameters are $k_1 = 4$ and $0.5 \leq k_2 \leq 1.5$, and initial conditions are $a = 3$ and $-5 \leq b \leq -1$; based on Theorem 3, the convergence time can be obtained.

The whole convergence process can be split into three stages. Firstly, the state x_1 from initial condition $x_1(0)$ reaches zero in finite time t_{pa0}, and at this time, $x_2(t_{pa0}) \neq 0$. Secondly, both states x_1 and x_2 reach zero at the same time and elapse time t_{p00} in this stage. Finally, after last two stages, the states x_1 and x_2 would stay at zero.

From Figure 12, it can be seen that, in the first stage, state x_1 from the initial condition reaches zero in finite time as in Figure 10 for Example 7, and the maximum convergence time $t_{a0} = 0.5816$ is calculated, and in this stage, the maximum of state $|x_2|$ is $|x_2(t_{a0})| = 5.8723$. Therefore, in the next stage, the initial condition is $x_2(0) = -5.8723$, such as in Figure 11 for Example 8, and then the maximum convergence time $t_{00} = 11.7447$ is calculated. Finally, the total finite convergence time $t = 12.3263$; therefore, the system converges to zero after 12.3263.

Example 10. Consider a nonaffine nonlinear system

$$\dot{y} = y + y^3 + 2u + \sin(u), \quad (66)$$

where y and u are the output and input, respectively, and the reference output is taken as y_d. The control objective is to design the STC for the system such that the output follows the given reference signal y_d in finite time.

Choose states $x_1 = y - y_d$ and $x_2 = u - \alpha$ with virtual control where

$$\begin{aligned}\alpha &= -3|x_1|^{1/2}\operatorname{sign}(x_1) + y_d - y - y^3 - u - \sin(u),\\ v &= -\operatorname{sign}(x_1) + \dot{\alpha}.\end{aligned} \quad (67)$$

The states satisfy the STA condition $A = -(2/9)$ and initial conditions $x_1(0) = 3$ and $x_2(0) = -3$. Therefore, the states x_1 and x_2 satisfy the condition in Figure 9 and output and reference output in Figure 10.

It can be observed that, for the nonaffine system, the closed-loop system has been controlled in finite time effectively in Figures 13–15, and time can be calculated and designed by control gain and initial condition.

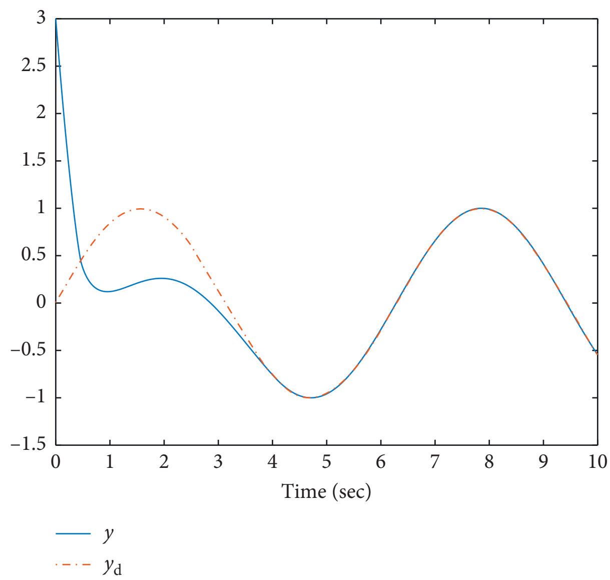

FIGURE 13: Finite-time convergence of the output and reference signal in control.

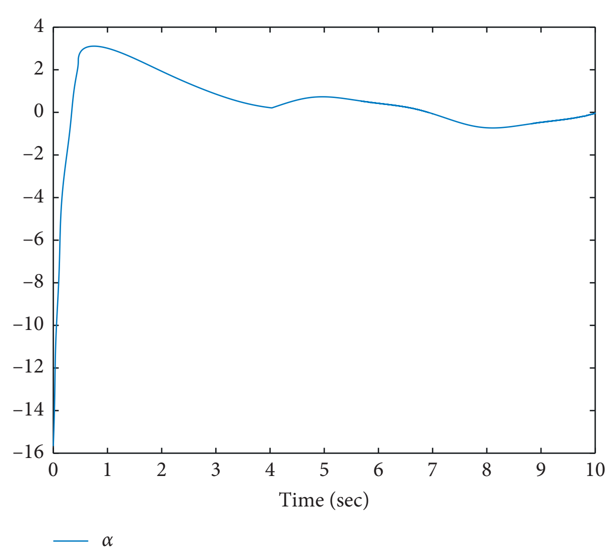

FIGURE 14: State of virtual control.

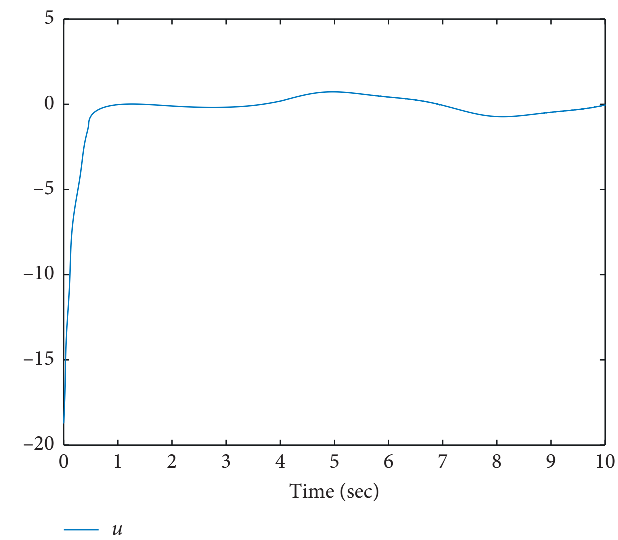

FIGURE 15: State of the controller.

7. Conclusion

A method to calculate the accurate convergence time of the STA has been provided, and an STC method for nonaffine nonlinear systems based on the STA is given. The trajectory of the STA is solved in the form of the parametric equation, and reaching time is calculated accurately. The result indicated the gain parameter k_1 is not influenced by the convergence time. For the application of the STA, STC is designed to solve the nonaffine systems' control problem by backstepping skill.

Furthermore, after a certain prefixed gain and initial conditions, the reaching time of the STA is also useful to design STC. In the future, there still exist some problems to face with, such as $A < -(1/4)$ in the STA and higher order twisting algorithm.

Appendix

A. Parameter Solution

Base on Lemma 1, the following solution can be obtained:

$$\begin{cases} t = Cf(\tau) - \dfrac{B}{A}, \\[2ex] y = C\tau f(\tau), \end{cases} \tag{A.1}$$

where $f(\tau) = \exp(-\int \tau\, d\tau/(\tau^2 - \tau - A))$; calculating the integral $\int \tau\, d\tau/(\tau^2 - \tau - A)$, we get

$$\int \frac{\tau}{\tau^2 - \tau - A} d\tau = \frac{1}{2}\ln\left|\tau^2 - \tau - A\right| + \frac{1}{2}\int \frac{1}{\tau^2 - \tau - A} d\tau, \tag{A.2}$$

and then calculating the integral $\int 1/(\tau^2 - \tau - A)d\tau$, we get

$$\int \frac{1}{\tau^2-\tau-A}dx = \begin{cases} -\dfrac{1}{\tau-(1/2)}+C_3, & \left(A=-\dfrac{1}{4}\right), \\ \dfrac{1}{\sqrt{4A+1}}\ln\left|\dfrac{2\tau-1-\sqrt{4A+1}}{2\tau-1+\sqrt{4A+1}}\right|+C_3, & \left(A>-\dfrac{1}{4}\right), \\ \dfrac{2}{\sqrt{-4A-1}}\arctan\left(\dfrac{2\tau-1}{\sqrt{-4A-1}}\right)+C_3, & \left(A<-\dfrac{1}{4}\right). \end{cases} \tag{A.3}$$

Therefore, the solution of the parametric equation is discussed in different cases.

Case I. $A=-(1/4)$; based on equations (A.2) and (A.3), the solution can be obtained as

$$\begin{cases} t = C\dfrac{\exp(1/2\tau-1)}{2\tau-1}-\dfrac{B}{A}, \\ y = C\tau\dfrac{\exp(1/2\tau-1)}{2\tau-1}. \end{cases} \tag{A.4}$$

To get finite time clearly using $s=1/(2\tau-1)$, equation (A.4) is changed as

$$\begin{cases} t = Cs\exp(s)-\dfrac{B}{A}, \\ y = \dfrac{C}{2}(s+1)\exp(s), \end{cases} \tag{A.5}$$

where s is a new parameter variable.

ase II. $A>-(1/4)$; based on equations (A.2) and (A.3), the solution can be obtained as

$$\begin{cases} t = C\dfrac{|2\tau-1+\sqrt{4A+1}|^{(1/2\sqrt{4A+1})-(1/2)}}{|2\tau-1-\sqrt{4A+1}|^{(1/2\sqrt{4A+1})-(1/2)}}-\dfrac{B}{A}, \\ y = C\dfrac{\tau|2\tau-1+\sqrt{4A+1}|^{(1/2\sqrt{4A+1})-(1/2)}}{|2\tau-1-\sqrt{4A+1}|^{(1/2\sqrt{4A+1})-(1/2)}}. \end{cases} \tag{A.6}$$

To get finite time clearly using $s=1/(\tau-(1/2)-(p/2))$, equation (A.6) is changed as

$$\begin{cases} t = Cs(1+ps)^q-\dfrac{B}{A}, \\ y = C\left(1+\dfrac{1}{2}s+\dfrac{p}{2}s\right)(1+ps)^q, \end{cases} \tag{A.7}$$

where $p=\sqrt{4A+1}$ and $q=(1/2p)-(1/2)$ because of $A=-2k_2/k_1^2$ and $A>-(1/4)$; therefore, $0<p<1$ and $q>0$.

Case III. $A<-(1/4)$; based on equations (A.2) and (A.3), the solution can be obtained as

$$\begin{cases} t = \dfrac{C}{\sqrt{\tau^2-\tau-A}\,\exp\left((1/\sqrt{-4A-1})\arctan(2\tau-1/\sqrt{-4A-1})\right)} \\ -\dfrac{B}{A}, \\ y = \dfrac{C\tau}{\sqrt{\tau^2-\tau-A}\,\exp\left((1/\sqrt{-4A-1})\arctan(2\tau-1/\sqrt{-4A-1})\right)}. \end{cases} \tag{A.8}$$

B. Parameter Equation

To indicate (5) holds, three different situations are considered based on the sign of the initial condition a.

Firstly, assume that $a>0$, then $x_1(t)>0$, for $t\in T_{a0}$. Let us discuss the trajectory when $t\in T_{a0}$, $x_1(0)=a$, $x_2(0)=b$, and $x_1(t_{a0})=0$, then based on the STA in $t\in T_{a0}$

$$x_2 = -k_2t+b. \tag{B.1}$$

Substituting (B.1) into (1) leads to

$$\dot{x}_1 = -k_1x_1^{1/2}-k_2t+b. \tag{B.2}$$

Let

$$y = -\frac{2}{k_1}x_1^{1/2}, \tag{B.3}$$

then its derivative on the trajectory of (B.2) can be obtained as follows:

$$y\dot{y}-y = -\frac{2k_2}{k_1^2}t+\frac{2b}{k_1^2}. \tag{B.4}$$

Secondly, if the initial condition $a<0$, then

$$x_2 = k_2t+b. \tag{B.5}$$

Substituting (B.5) into (1) leads to

$$\dot{x}_1 = k_1\sqrt{(-x_1)}+k_2t+b. \tag{B.6}$$

Let

$$y = -\frac{2}{k_1}(-x_1)^{1/2}, \tag{B.7}$$

then its derivative on the trajectory of (B.6) can be obtained as follows:

$$y\dot{y} - y = -\frac{2k_2}{k_1^2}t - \frac{2b}{k_1^2}. \quad \text{(B.8)}$$

Finally, if $a = 0$ because $x_2(0) = b \neq 0$, $x_1(t) \neq 0$, and $\text{sign}(x_1) = \text{sign}(b)$ when $t \in T_{a0}$, then based on the STA in $t \in T_{a0}$

$$x_2 = -k_2 \text{sign}(b)t + b. \quad \text{(B.9)}$$

Substituting (B.9) into (1) leads to

$$\dot{x}_1 = -k_1 \text{sign}(b)\left|x_1\right|^{1/2} - k_2 \text{sign}(b)t + b. \quad \text{(B.10)}$$

Let

$$y = -\frac{2}{k_1}\left|x_1\right|^{1/2}, \quad \text{(B.11)}$$

then its derivative on the trajectory of (B.10) can be obtained as follows:

$$y\dot{y} - y = -\frac{2k_2}{k_1^2}t + \frac{2|b|}{k_1^2}. \quad \text{(B.12)}$$

Based on (B.4), (B.8), and (B.12), the general form of (5) holds.

Data Availability

The data in this paper are simulated with MATLAB. No other data are present.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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