# Pattern Formation in a Reaction-Diffusion Predator-Prey Model with Weak Allee Effect and Delay 

<br>${ }^{1}$ School of Mathematics and Computer Science, Northwest Minzu University, Lanzhou 730000, China<br>${ }^{2}$ Experimental Center, Northwest Minzu University, Lanzhou 730000, China

Correspondence should be addressed to Hua Liu; 7783360@qq.com
Received 31 August 2019; Revised 6 November 2019; Accepted 13 November 2019; Published 30 November 2019
Academic Editor: Peter Giesl
Copyright © 2019 Hua Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.


#### Abstract

In this paper, we establish a reaction-diffusion predator-prey model with weak Allee effect and delay and analyze the conditions of Turing instability. The effects of Allee effect and delay on pattern formation are discussed by numerical simulation. The results show that pattern formations change with the addition of weak Allee effect and delay. More specifically, as Allee effect constant and delay increases, coexistence of spotted and stripe patterns, stripe patterns, and mixture patterns emerge successively. From an ecological point of view, we find that Allee effect and delay play an important role in spatial invasion of populations.


## 1. Introduction

Since the Allee effect was proposed by Allee [1] in 1931, the predator-prey model with Allee effect has been studied extensively [2-27]. From the ordinary differential equation predator-prey model with Allee effect to the partial differential equation model, many researchers have achieved rich results [4, 7, 9-16, 28]. Cai et al. [6] established a Leslie-Gower predator-prey model with additive Allee effect on prey, and they found Allee effect can increase the risk of ecological extinction. Sen et al. [5] established a twoprey one-predator model with Allee effect, and the effects of Allee effect on the dynamics of predator population are discussed. Of course, the research on reaction-diffusion predator-prey model with Allee effect is also very rich. For example, Wang et al. [7] established a reaction-diffusion predator-prey model and found the model dynamics exhibits both Allee effect and diffusion controlled pattern formation growth to holes. They also studied Allee effect induced instability in a reaction-diffusion predator-prey model [4]. Petrovskii et al. found that the deterministic system with Allee effect can induce patch invasion [23]. Sun et al. found that predator mortality plays an important role in the pattern formation of populations [13]. It is now believed that the spatial composition of population
interactions has been identified as an important factor in how ecological communities operate and form. Pattern formation in the predator-prey model is an appropriate tool for understanding the basic mechanism of spatiotemporal population dynamics. We find that there are few studies on delays in reaction-diffusion predator-prey model with Allee effect. So next we discuss the effects of Allee effect and delay on pattern formation. First, we consider a predator-prey model with hyperbolic mortality established by Zhang et al. [10], the model is obtained as follows:

$$
\begin{cases}\frac{\partial U}{\partial T}-d_{1} \Delta U=a U\left(1-\frac{U}{K}\right)-\frac{b U V}{c+U}, & x \in \Omega, T>0,  \tag{1}\\ \frac{\partial V}{\partial T}-d_{2} \Delta V=\frac{m U V}{c+U}-h(V), & x \in \Omega, T>0, \\ \frac{\partial U}{\partial n}=\frac{\partial V}{\partial n}=0, & x \in \partial \Omega, T>0, \\ U(x, 0)=U_{0}(x) \geq 0, V(x, 0)=V_{0}(x) \geq 0, & x \in \Omega,\end{cases}
$$

where $U$ and $V$ are the population densities of prey and predator, respectively; $a$ is the birth rate, $K$ is the carrying capacity and $b$ is the maximum uptake rate of the prey; $c$ is the prey density at which the predator has the maximum kill rate; $m$ is the birth rate of predator; function $h(V)$ reflects the predator death rate; the habitat $\Omega \subset R^{n}$ is a bounded domain with smooth boundary $\partial \Omega ; n$ is the outward unit normal vector on $\partial \Omega ; d_{1}$ and $d_{2}$ are the diffusion coefficients, respectively; and $\Delta$ is the Laplacian operator. The homogeneous Neumann boundary condition implies that the system above is self-contained and there is no host across the boundary. After nondimensionalization,

$$
\begin{equation*}
U \longrightarrow K u, V \longrightarrow \frac{a c}{b} v, \frac{K}{c} \longrightarrow \beta, \frac{a}{m} \longrightarrow \alpha, T \longrightarrow \frac{t}{m} . \tag{2}
\end{equation*}
$$

Then, considering that the predator-prey model with Allee effect is more realistic, people begin to introduce delay into the predator-prey model and discuss the effects of Allee effect and delay on the dynamics of the model [2, 17-22]. We try to introduce weak Allee effect and searching delay into model (1), and then we get

$$
\begin{cases}\frac{\partial u}{\partial t}-d_{1} \Delta u=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u(t-\tau) v(t-\tau)}{1+\beta u(t-\tau)}, & x \in \Omega, t>0,  \tag{3}\\ \frac{\partial v}{\partial t}-d_{2} \Delta v=\frac{\beta u v}{1+\beta u}-h(v), & x \in \Omega, t>0, \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x \in \partial \Omega, t>0, \\ u(x, \theta)=\varphi(x, \theta) \geq 0, v(x, \theta)=\psi(x, \theta) \geq 0,(x, \theta) \in \Omega \times(-\tau, 0),\end{cases}
$$

where $h(v)=\gamma v^{2} / e+\eta v$. For hyperbolic mortality, $\gamma$ is the death rate of the predator, $e$ and $\eta$ are coefficients of light attenuation by water and self-shading in the context of plankton mortality, and $\tau$ is the searching delay. The weak Allee effect term is $u / u+A$, where $A>0$ is described as a weak Allee effect constant.

## 2. Turing Instability

First, we consider the model with $\tau=0$ :

$$
\begin{cases}\frac{\partial u}{\partial t}-d_{1} \Delta u=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u v}{1+\beta u}, & x \in \Omega, t>0,  \tag{4}\\ \frac{\partial v}{\partial t}-d_{2} \Delta v=\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v}, & x \in \Omega, t>0, \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x \in \partial \Omega, t>0, \\ u(x, 0)=u_{0}(x) \geq 0, v(x, 0)=v_{0}(x) \geq 0, & x \in \Omega .\end{cases}
$$

Obviously, if $d_{1}=d_{2}=0$, without diffusion in model (4), then we can obtain the following ordinary differential equations:

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} u}{\mathrm{~d} t}=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u v}{1+\beta u}=f(u, v),  \tag{5}\\
\frac{\mathrm{d} v}{\mathrm{~d} t}=\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v}=g(u, v) .
\end{array}\right.
$$

We mainly focus on the stability of the positive equilibrium of model (4). Clearly, the positive equilibrium $E_{*}=$ ( $u_{*}, v_{*}$ ) of the ordinary differential equation (ODE) or the partial differential equation (PDE) model (4) satisfies $f\left(u_{*}, v_{*}\right)=0$ and $g\left(u_{*}, v_{*}\right)=0:$

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} u}{\mathrm{~d} t}=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u v}{1+\beta u}=0  \tag{6}\\
\frac{\mathrm{~d} v}{\mathrm{~d} t}=\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v}=0
\end{array}\right.
$$

For simplicity of discussion, in this paper, we shall concentrate the case of $\eta=\gamma$ and $e=1$. We easily see that model (4) exhibits a positive equilibrium point $E_{*}=\left(u_{*}, v_{*}\right)$ when $\beta>\gamma / 1-\gamma, 0<\gamma<1$, and $A<\gamma / \beta$. When $\beta>\gamma / 1-\gamma$, $0<\gamma<1$, and $\gamma / \beta<A<\left((\beta \gamma-\gamma-\beta)^{2}+4 \beta \gamma^{2}\right) / 4 \beta^{2} \gamma$, model (4) exhibits two positive equilibrium points $E_{1 *}=\left(u_{1 *}, v_{1 *}\right)$ and $E_{2 *}=\left(u_{2 *}, v_{2 *}\right)$. In this work, we mainly focus on a positive equilibrium point, where

$$
\begin{equation*}
u_{*}=\frac{\sqrt{(\beta \gamma-\gamma-\beta)^{2}+(\gamma-\beta A) \beta \gamma}}{2 \beta \gamma}+\frac{\beta \gamma-\gamma-\beta}{2 \beta \gamma}, v_{*}=\frac{\beta}{\gamma} u_{*} \tag{7}
\end{equation*}
$$

We calculate the Jacobian matrix of model (5) at $E_{*}$, which is given by $J_{*}=\left[\begin{array}{ll}a_{10} & a_{01} \\ b_{10} & b_{01}\end{array}\right]$, where

$$
\begin{align*}
& a_{10}=\frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}-\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}, \\
& a_{01}=-\frac{\alpha u_{*}}{1+\beta u_{*}}, \\
& b_{10}=\frac{\beta^{2} u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}  \tag{8}\\
& b_{01}=\frac{-\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}
\end{align*}
$$

We can easily know that the characteristic polynomial is

$$
\begin{equation*}
H(\lambda)=\lambda^{2}-T \lambda+D \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
T= & \frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}-\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}} \\
& -\frac{\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}, \\
D= & \left(\frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}-\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}\right) \\
& \cdot \frac{-\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}+\frac{\alpha \beta^{2} u_{*}^{2}}{\gamma\left(1+\beta u_{*}\right)^{3}} . \tag{10}
\end{align*}
$$

Thus, we have the following conclusions:
(a) If $T<0$ and $D>0$, then the positive equilibrium is locally asymptotically stable
(b) If $T>0$, then the positive equilibrium is unstable

Next, let us consider the PDE model (4); we choose the perturbation function consisting of the following two-dimensional Fourier modes:

$$
\begin{align*}
\binom{u}{v} & =e^{\lambda_{k} t+i k_{x} x}, \\
J & =\left(\begin{array}{cc}
a_{10}-k^{2} d_{1} & a_{01} \\
b_{10} & b_{01}-k^{2} d_{2}
\end{array}\right), \tag{11}
\end{align*}
$$

so, we can find

$$
\begin{align*}
& T_{k}=a_{10}+b_{01}-k^{2}\left(d_{1}+d_{2}\right) \\
& D_{k}=\left(a_{10}-k^{2} d_{1}\right)\left(b_{01}-k^{2} d_{2}\right)-a_{01} b_{10} \tag{12}
\end{align*}
$$

We know, when $E_{*}$ is stable,

$$
\begin{align*}
T= & \frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}  \tag{13}\\
& -\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}-\frac{\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}<0
\end{align*}
$$

We can easily find that $T_{k}=a_{11}+a_{22}-k^{2}\left(d_{1}+d_{2}\right)<0$. So, if model (3) changes from stable to unstable, it needs to be

$$
\begin{equation*}
D_{k}=\left(a_{10}-k^{2} d_{1}\right)\left(b_{01}-k^{2} d_{2}\right)-a_{01} b_{10}<0 \tag{14}
\end{equation*}
$$

that is,

$$
\begin{equation*}
2 \sqrt{D \frac{d_{1}}{d_{2}}}<a_{10}+b_{01} \frac{d_{1}}{d_{2}}, d_{1}<d_{2} \tag{15}
\end{equation*}
$$

## 3. Delay-Induced Instability

Finally, we consider the PDE model (4) with delay (searching delay), and we get model (3). Considering $\tau$ and spatial diffusion, if $\tau$ is small enough, the following changes are made [29]:

$$
\begin{align*}
u(x, y, t-\tau) & =u(x, y, t)-\tau \frac{\partial u}{\partial t}, v(x, y, t-\tau) \\
& =v(x, y, t)-\tau \frac{\partial v}{\partial t} \tag{16}
\end{align*}
$$

we substitute (16) into model (3) to get

$$
\begin{cases}\frac{\partial u}{\partial t}-d_{1} \Delta u=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha(u(x, y, t)-\tau \partial u / \partial t)(v(x, y, t)-\tau \partial v / \partial t)}{1+\beta(u(x, y, t)-\tau \partial u / \partial t)}, & x \in \Omega, t>0  \tag{17}\\ \frac{\partial v}{\partial t}-d_{2} \Delta v=\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v}, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x \in \partial \Omega, t>0 \\ u(x, \theta)=\varphi(x, \theta) \geq 0, v(x, \theta)=\psi(x, \theta) \geq 0, & (x, \theta) \in \Omega \times(-\tau, 0)\end{cases}
$$

Expanding in Taylor series and neglecting the higherorder nonlinearities, we find

$$
\begin{cases}\frac{\partial u}{\partial t}-d_{1} \Delta u=\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u v}{1+\beta u}  \tag{18}\\ -\tau f_{u(t-\tau)}(u(t-\tau), v(t-\tau)) \frac{\partial u}{\partial t}, & \\ -\tau f_{v(t-\tau)}(u(t-\tau), v(t-\tau)) \frac{\partial v}{\partial t}, & x \in \Omega, t>0 \\ \frac{\partial v}{\partial t}-d_{2} \Delta v=\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v}, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x \in \partial \Omega, t>0 \\ u(x, \theta)=\varphi(x, \theta) \geq 0, v(x, \theta)=\psi(x, \theta) \geq 0 \\ \quad(x, \theta) \in \Omega \times(-\tau, 0),\end{cases}
$$

where

$$
\begin{align*}
f(u(t-\tau), v(t-\tau))= & \alpha u(1-u)\left(\frac{u}{u+A}\right) \\
& -\frac{\alpha u(t-\tau) v(t-\tau)}{1+\beta u(t-\tau)} . \tag{19}
\end{align*}
$$

We can see that if $f\left(u_{*}, v_{*}\right)=0$ and $g\left(u_{*}, v_{*}\right)=0$ are satisfied at equilibrium point $E_{*}=\left(u_{*}, v_{*}\right)$, then we can get the model:

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}-d_{1} \Delta u=f_{u}(u, v)\left(u-u_{*}\right)+f_{v}(u, v)\left(v-v_{*}\right) \\
-\tau f_{u(t-\tau)}(u(t-\tau), v(t-\tau)) \frac{\partial u}{\partial t}, \\
-\tau f_{v(t-\tau)}(u(t-\tau), v(t-\tau)) \frac{\partial v}{\partial t}, \quad x \in \Omega, t>0, \\
\frac{\partial v}{\frac{\partial t}{\partial t}-d_{2} \Delta v=g_{u}(u, v)\left(u-u_{*}\right)+g_{v}(u, v)\left(v-v_{*}\right),} \\
\\
\begin{array}{l}
\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, \\
u(x, \theta)=\varphi(x, \theta) \geq 0, v(x, \theta)=\psi(x, \theta) \geq 0, \\
\quad(x, \theta) \in \Omega \times(-\tau, 0),
\end{array}
\end{array}\right.
$$

where

$$
\begin{align*}
f(u, v) & =\alpha u(1-u)\left(\frac{u}{u+A}\right)-\frac{\alpha u v}{1+\beta u}, g(u, v) \\
& =\frac{\beta u v}{1+\beta u}-\frac{\gamma v^{2}}{e+\eta v} . \tag{21}
\end{align*}
$$

We consider that the stable equilibrium point $E_{*}=\left(u_{*}\right.$, $v_{*}$ ) is subjected to a small perturbation of $\bar{u}_{*}$ and $\bar{v}_{*}$. Let $u=u_{*}+\bar{u}_{*}$ and $v=v_{*}+\bar{v}_{*}$, so we get

$$
\left\{\begin{array}{lc}
\frac{\partial \bar{u}_{*}}{\partial t}-d_{1} \Delta \bar{u}_{*}=A_{10} \bar{u}_{*}+A_{01} \bar{v}_{*}+\tau \frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}} \frac{\partial \bar{u}_{*}}{\partial t} \\
+\tau \frac{\alpha u_{*}}{1+\beta u_{*}} \frac{\partial \bar{v}_{*}}{\partial t}, & x \in \Omega, t>0, \\
\frac{\partial \bar{v}_{*}}{\partial t}-d_{2} \Delta \bar{v}_{*}=B_{10} \bar{u}_{*}+B_{01} \bar{v}_{*}, & x \in \Omega, t>0,  \tag{22}\\
\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, & x \in \partial \Omega, t>0, \\
u(x, \theta)=\varphi(x, \theta) \geq 0, v(x, \theta)=\psi(x, \theta) \geq 0 \\
\quad(x, \theta) \in \Omega \times(-\tau, 0) .
\end{array}\right.
$$

Assuming that the solution of the system has the following form,

$$
\begin{equation*}
\bar{u}_{*}(x, t)=\bar{u}_{*}^{0} e^{\lambda t} \cos k_{x} x, \bar{v}_{*}(x, t)=\bar{v}_{*}^{0} e^{\lambda t} \cos k_{x} x . \tag{23}
\end{equation*}
$$

So, we get

$$
J_{k}^{\tau}=\left[\begin{array}{cc}
\frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M} & \frac{a_{01}+\tau b_{10} N-\tau d_{2} N k^{2}}{1-M}  \tag{24}\\
b_{10} & b_{01}-k^{2} d_{2}
\end{array}\right]
$$

where

$$
\begin{align*}
M & =\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}  \tag{25}\\
N & =\frac{\alpha u_{*}}{1+\beta u_{*}}
\end{align*}
$$

We can easily find

$$
\begin{align*}
T_{k}^{\tau}= & \frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M}+b_{01}-k^{2} d_{2} \\
D_{k}^{\tau}= & \left(\frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M}\right)\left(b_{01}-k^{2} d_{2}\right)  \tag{26}\\
& -b_{10}\left(\frac{a_{01}+\tau b_{01} N-\tau d_{2} N k^{2}}{1-M}\right)
\end{align*}
$$

When $k=0$, model (3) undergoes Hopf bifurcation at $T_{k}^{\tau}=0$, so the critical value for undergoing Hopf bifurcation can be obtained:

$$
\begin{equation*}
\tau_{H}=\frac{-\left(a_{10}+b_{01}\right)+b_{01} M}{b_{10} N} \tag{27}
\end{equation*}
$$

We know, when $E_{*}$ is stable,

$$
\begin{align*}
a_{10} b_{01}-a_{01} b_{10} & >0 \\
D_{k}^{\tau}= & \left(\frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M}\right)\left(b_{01}-k^{2} d_{2}\right) \\
& -b_{10}\left(\frac{a_{01}+\tau b_{01} N-\tau d_{2} N k^{2}}{1-M}\right)>0 \tag{28}
\end{align*}
$$

So, we just need to judge

$$
\begin{equation*}
T_{k}^{\tau}=\frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M}+b_{01}-k^{2} d_{2}>0 \tag{29}
\end{equation*}
$$

It is easy to know when

$$
\begin{gather*}
\tau>\frac{-a_{10}+k^{2} d_{1}+\left(-b_{01}+k^{2} d_{2}\right)(1-M)}{b_{10} N},  \tag{30}\\
T_{k}^{\tau}=\frac{a_{10}+\tau b_{10} N-k^{2} d_{1}}{1-M}+b_{01}-k^{2} d_{2}>0 .
\end{gather*}
$$

So, the instability condition caused by delay is as follows:

$$
\left\{\begin{align*}
T & =\frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}-\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}-\frac{\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}<0 \\
D & =\left(\frac{-2 \alpha u_{*}^{3}-3 \alpha A u_{*}^{2}+\alpha u_{*}^{2}+2 \alpha A u_{*}}{\left(u_{*}+A\right)^{2}}-\frac{\alpha \beta u_{*}}{\gamma\left(1+\beta u_{*}\right)^{2}}\right)  \tag{31}\\
& \frac{-\beta u_{*}}{\left(1+\beta u_{*}\right)^{2}}+\frac{\alpha \beta^{2} u_{*}^{2}}{\gamma\left(1+\beta u_{*}\right)^{3}}>0 \\
\tau> & \frac{-a_{10}+k^{2} d_{1}+\left(-b_{01}+k^{2} d_{2}\right)(1-M)}{b_{10} N}
\end{align*}\right.
$$

## 4. Amplitude Equations and Pattern Selection

We rewrite the transformed form of system (3) at the positive spatially homogeneous steady state $E_{*}=\left(u_{*}, v_{*}\right)$ as follows and denote by $(U, V)^{T}$ the perturbation solution $\left(U-u_{*}, V-v_{*}\right)^{T}$ of the system:

$$
\begin{equation*}
\frac{\partial X}{\partial t}=L X+H \tag{32}
\end{equation*}
$$

where $X=(U, V)^{T}$.
Then, let the linear operator $L$ be defined as follows:

$$
\begin{align*}
L= & J^{\tau}+D_{a} \Delta=\left(\begin{array}{cc}
\frac{a_{10}+\tau a_{01} N}{1-M} & \frac{b_{10}+\tau b_{01} N}{1-M} \\
b_{10} & b_{01}
\end{array}\right) \\
& +\left(\begin{array}{cc}
\frac{-k^{2} d_{1}}{1-M} & \frac{-\tau d_{2} N k^{2}}{1-M} \\
0 & -k^{2} d_{2}
\end{array}\right), \tag{33}
\end{align*}
$$

and $H$ be given by

$$
\begin{equation*}
H=\binom{A_{20} U^{2}+A_{11} U V+A_{02} V^{2}+A_{30} U^{3}+A_{21} U^{2} V+A_{12} U V^{2}+A_{03} V^{3}+o\left(\varepsilon^{3}\right)}{B_{20} U^{2}+b_{11} U V+B_{02} V^{2}+B_{30} U^{3}+B_{21} U^{2} V+B_{12} U V^{2}+B_{03} V^{3}+o\left(\varepsilon^{3}\right)} \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
& A_{20}=\frac{1}{1-M}\left(\frac{-\alpha u_{*}^{3}-3 \alpha A u_{*}^{3}+3 \alpha A u_{*}^{2}-\alpha A u_{*}+\alpha A^{2}}{\left(u_{*}+A\right)^{3}}+\frac{\alpha \beta^{2} u_{*}}{\gamma\left(1+\beta u_{*}\right)^{3}}\right)-\frac{2 \tau N \beta^{2} v_{*}}{(1-M)\left(1+\beta u_{*}\right)^{3}}, \\
& A_{11}=-\frac{\alpha}{2\left(1+\beta u_{*}\right)(1-M)}+\frac{\tau N \beta}{\left(1+\beta u_{*}\right)^{2}(1-M)}, \\
& A_{02}=0, \\
& B_{20}=-\frac{2 \beta^{2} v_{*}}{\left(1+\beta u_{*}\right)^{3}}, \\
& B_{11}=\frac{\beta}{\left(1+\beta u_{*}\right)^{2}}, \\
& B_{02}=-\frac{2 \gamma}{\left(1+\gamma \nu_{*}\right)^{2}}, \\
& A_{30}=\frac{1}{1-M}\left(\frac{-9 \alpha A^{2} u_{*}^{2}-6 \alpha A u_{*}^{2}+6 \alpha A^{2} u_{*}+2 \alpha A u_{*}-4 \alpha A^{2}}{3\left(u_{*}+A\right)^{4}}-\frac{\alpha \beta^{3} u_{*}}{\gamma\left(1+\beta u_{*}\right)^{4}}\right)+\frac{6 \tau N \beta^{3} v_{*}}{(1-M)\left(1+\beta u_{*}\right)^{4}},  \tag{35}\\
& A_{21}=-\frac{2 \tau N \beta^{2}}{(1-M)\left(1+\beta u_{*}\right)^{3}}, \\
& A_{12}=0, \\
& A_{03}=0 \text {, } \\
& B_{30}=\frac{6 \beta^{3} v_{*}}{\left(1+\beta u_{*}\right)^{4}}, \\
& B_{21}=-\frac{2 \beta^{2}}{\left(1+\beta u_{*}\right)^{3}}, \\
& B_{12}=0, \\
& B_{03}=-\frac{2 \gamma}{\left(1+\gamma v_{*}\right)^{3}} .
\end{align*}
$$

Next, near the Turing bifurcation threshold, we expand the control parameter $\tau$ as

$$
\begin{equation*}
\tau_{T}-\tau=\varepsilon \tau_{1}+\varepsilon^{2} \tau_{2}+\varepsilon^{3} \tau_{3}+o\left(\varepsilon^{3}\right) \tag{36}
\end{equation*}
$$

where $|\varepsilon| \ll 1$. Similarly, expand the solution $X$, linear operator $L$, and the nonlinear term $H$ into Taylor series at $\varepsilon=0$ :

$$
\begin{align*}
X & =\varepsilon\binom{U_{1}}{V_{1}}+\varepsilon^{2}\binom{U_{2}}{V_{2}}+\varepsilon^{3}\binom{U_{3}}{V_{3}}+o\left(\varepsilon^{3}\right),  \tag{37}\\
H & =\varepsilon^{2} h_{2}+\varepsilon^{3} h_{3}+o\left(\varepsilon^{3}\right)  \tag{38}\\
L & =L_{T}+\left(\tau_{T}-\tau\right) M \tag{39}
\end{align*}
$$

where

$$
\begin{aligned}
h_{2}= & \binom{h_{2}^{1}}{h_{2}^{2}}=\binom{A_{20}^{T} U_{1}^{2}+A_{11}^{T} U_{1} V_{1}+A_{02}^{T} V_{1}^{2}}{B_{20}^{T} U_{1}^{2}+B_{11}^{T} U_{1} V_{1}+B_{02}^{T} V_{1}^{2}}, \\
h_{3}= & \binom{h_{3}^{1}}{h_{3}^{2}}=\binom{A_{30}^{T} U^{3}+A_{21}^{T} U^{2} V+A_{12}^{T} U V^{2}+A_{03}^{T} V^{3}+2\left(A_{20}^{T} U_{1} U_{2}+A_{02}^{T} V_{1} V_{2}\right)+A_{11}^{T}\left(U_{1} V_{2}+V_{1} U_{2}\right)}{B_{30}^{T} U^{3}+B_{21}^{T} U^{2} V+B_{12}^{T} U V^{2}+B_{03}^{T} V^{3}+2\left(B_{20}^{T} U_{1} U_{2}+B_{02}^{T} V_{1} V_{2}\right)+B_{11}^{T}\left(U_{1} V_{2}+V_{1} U_{2}\right)} \\
& -\binom{\alpha_{1}\left(A_{20}^{\prime} U_{1}^{2}+A_{11}^{\prime} U_{1} V_{1}+A_{02}^{\prime} V_{1}^{2}\right)}{\alpha_{1}\left(B_{20}^{\prime} U_{1}^{2}+B_{11}^{\prime} U_{1} V_{1}+B_{02}^{\prime} V_{1}^{2}\right)} .
\end{aligned}
$$

are terms corresponding to the second and third orders in the expansion of the nonlinear term and for the linear operator

$$
\begin{equation*}
L=L_{T}+\left(\tau_{T}-\tau\right) M \tag{41}
\end{equation*}
$$

We have,

$$
\begin{gather*}
L_{T}=\left(\begin{array}{cc}
\frac{a_{10}+\tau a_{01} N-k^{2} d_{1}}{1-M} & \frac{b_{10}+\tau b_{01} N-\tau d_{2} N k^{2}}{1-M} \\
b_{10} & b_{01}-k^{2} d_{2}
\end{array}\right)_{\tau=\tau_{T}},  \tag{42}\\
M=\left(\begin{array}{ll}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{array}\right) \tag{43}
\end{gather*}
$$

where $m_{11}=-a_{21} N / 1-M, m_{12}=-a_{22} N+d_{2} N k^{2} / 1-M$, $m_{21}=0$, and $m_{22}=0$ at $U=u_{*}$.

Finally, we introduce multiple time scales:

$$
\begin{equation*}
\frac{\partial}{\partial t}=\varepsilon \frac{\partial}{\partial T_{1}}+\varepsilon^{2} \frac{\partial}{\partial T_{2}}+o\left(\varepsilon^{2}\right) \tag{44}
\end{equation*}
$$

Then, substituting equations (33)-(44) into equation (32) and expanding it with respect to different orders of $\varepsilon^{i},(i=1,2,3)$,

$$
\begin{align*}
& \varepsilon: L_{T}\binom{u_{1}}{v_{1}}=0 \\
& \varepsilon^{2}: L_{T}\binom{u_{2}}{v_{2}}=\frac{\partial}{\partial T_{1}}\binom{u_{1}}{v_{1}}-\alpha_{1} M\binom{u_{1}}{v_{1}}-h_{2}, \\
& \varepsilon^{3}: L_{T}\binom{u_{3}}{v_{3}}=\frac{\partial}{\partial T_{1}}\binom{u_{2}}{v_{2}}+\frac{\partial}{\partial T_{2}}\binom{u_{1}}{v_{1}}-\alpha_{1} M\binom{u_{2}}{v_{2}} \\
& \quad-\alpha_{2} M\binom{u_{1}}{v_{1}}-h_{3} . \tag{45}
\end{align*}
$$

In what follows, we seek the amplitude equations by solving system (45). Since $L_{T}$ has an eigenvector associated with the zero eigenvalue,

$$
\begin{equation*}
(f, 1)^{T}, f=\frac{-a_{01}-\tau b_{01} N+\tau d_{2} N k^{2}}{a_{10}+\tau b_{10} N-k^{2} d_{1}} \tag{46}
\end{equation*}
$$

The general solution of the first system of (45) can be written as

$$
\begin{equation*}
\binom{U_{1}}{V_{1}}=\binom{f}{1}\left(\sum_{j=1}^{3} W_{j} e^{i k_{j} \cdot r}+\text { c.c. }\right) \tag{47}
\end{equation*}
$$

where $W_{j}$ is the amplitude of the mode $e^{i k_{j} \cdot r}$. Notice that the second system of (45) is nonhomogeneous, and $L_{T}^{*}$, the adjoint operator of $L_{T}$, has zero eigenvectors in the form of

$$
\begin{equation*}
\binom{1}{g} e^{i k_{j} \cdot r}+\text { c.c. }, \quad j=1,2,3 \tag{48}
\end{equation*}
$$

with $g=-b_{01}(1-M) / a_{10}+\tau b_{10} N-k^{2} d_{1}$. Let

$$
\begin{align*}
\binom{F_{U}}{F_{V}}= & \frac{\partial}{\partial T_{1}}\binom{U_{1}}{V_{1}}-\beta_{1}\binom{m_{11} U_{1}+m_{12} V_{1}}{m_{21} U_{1}+m_{22} V_{1}}  \tag{49}\\
& -\alpha_{2} M\binom{h_{2}^{1}}{h_{2}^{2}}
\end{align*}
$$

Then, in view of the Fredholm solvability conditions,

$$
\begin{equation*}
(1, g)\binom{F_{U}^{j}}{F_{V}^{j}}=0 \tag{50}
\end{equation*}
$$

where $F_{U}^{j}$ and $F_{V}^{j}$ are the coefficients of $e^{i k_{j} \cdot r}$ in $F_{U}$ and $F_{V}$, respectively. It follows after some routine calculation that, for $j_{l}=1,2,3$ and $j_{l} \neq l_{m}$, if $l \neq m$,

$$
\begin{equation*}
(f+g) \frac{\partial W_{j 1}}{\partial T_{1}}=\alpha_{1} h_{3} W_{j 1}-2\left(h_{1}+g h_{2}\right) \bar{W}_{j 2} \bar{W}_{j 3}, \tag{51}
\end{equation*}
$$

where

$$
\begin{align*}
& h_{1}=-\left(f^{2} A_{20}^{T}+f A_{11}^{T}+A_{02}^{T}\right) \\
& h_{2}=-\left(f^{2} B_{20}^{T}+f B_{11}^{T}+B_{02}^{T}\right)  \tag{52}\\
& h_{3}=f m_{11}+m_{12}+g\left(f m_{21}+m_{22}\right) .
\end{align*}
$$

Notice the forms of $U_{1}$ and $V_{1}$ given by (47). We have a particular solution for the second system of (45) as follows:

$$
\begin{align*}
\binom{U_{2}}{V_{2}}= & \binom{\bar{U}_{0}}{\bar{V}_{0}}+\sum_{j=1}^{3}\binom{\bar{U}_{j}}{\bar{V}_{j}} e^{i k_{j} \cdot r}+\sum_{j=1}^{3}\binom{\bar{U}_{j j}}{\bar{V}_{j j}} e^{i 2 k_{j} \cdot r} \\
& +\binom{\bar{U}_{12}}{\bar{V}_{12}} e^{i\left(k_{1}-k_{2}\right) \cdot r}+\binom{\bar{U}_{23}}{\bar{V}_{23}} e^{i\left(k_{2}-k_{3}\right) \cdot r} \\
& +\binom{\bar{U}_{31}}{\bar{V}_{31}} e^{i\left(k_{3}-k_{1}\right) \cdot r}+\text { c.c. } \tag{53}
\end{align*}
$$

with the coefficients being given below at $\alpha_{T}=\alpha$ :

$$
\left.\begin{array}{rl}
\binom{\bar{U}_{0}}{\bar{V}_{0}}= & \binom{\frac{2\left(B_{01} h_{1}-A_{01} h_{2}\right)}{\Delta_{0}}}{\frac{2\left(A_{10} h_{2}-B_{10} h_{1}\right)}{\Delta_{0}}} \sum_{j=1}^{3}\left|W_{j}\right|^{2}, \\
\equiv & \binom{z_{U 0}}{z_{V 0}} \sum_{j=1}^{3}\left|W_{j}\right|^{2}, \bar{U}_{j}=f \bar{V}_{j,}\binom{X_{j j}}{Y_{j j}} \\
= & \binom{z_{U 1}}{z_{V 1}} W_{j}^{2}, \\
\left(A_{10}-4 d_{1} k_{c}^{2}\right)\left(B_{01}-4 d_{2} k_{c}^{2}\right)-A_{01} B_{10}  \tag{54}\\
Y_{j k}
\end{array}\right)=\binom{\left(B_{01}-4 d_{2} k_{c}^{2}\right) h_{1}-A_{01} h_{2}}{\left(A_{10}-4 d_{1} k_{c}^{2}\right) h_{2}-B_{10} h_{1}} W_{j}^{2}, W_{j 2} \bar{W}_{k}, \quad \begin{aligned}
& z_{U 2} \\
& z_{j k} \\
& =
\end{aligned}
$$

Again, apply the Fredholm solvability condition to the third system of (45). We have, for $j=1$,

$$
\begin{align*}
(f+g)\left(\frac{\partial V_{j}}{\partial T_{1}}+\frac{\partial W_{j}}{\partial T_{2}}\right)= & h\left(\alpha_{1} V_{j}+\alpha_{2} W_{j}\right)+h_{4} \bar{W}_{l} \bar{W}_{m} \\
& +H\left(\bar{V}_{l} \bar{W}_{m}+\bar{V}_{m} \bar{W}_{l}\right)-\left(G_{1}\left|W_{1}\right|^{2}\right. \\
& \left.+G_{2}\left(\left|W_{2}\right|^{2}+\left|W_{3}\right|^{2}\right)\right) W_{j} \tag{55}
\end{align*}
$$

with

$$
\begin{align*}
h_{4}= & -2 \alpha_{1}\left(A_{20}^{\prime} f^{2}+A_{11}^{\prime} f+A_{02}^{\prime}+g\left(B_{20}^{\prime} f^{2}+A_{11}^{\prime} f+B_{02}^{\prime}\right)\right) \\
H= & -2\left(h_{1}+g h_{2}\right), \\
G_{1}= & -\left(3 A_{30} f^{3}+2 A_{11} f z_{V 0}+A_{11} f z_{V 1}+4 A_{20} f z_{U 0}\right. \\
& +2 A_{20} f z_{U 1}+3 A_{21} f^{2}+4 A_{02} z_{V 0}+2 A_{02} z_{V 1} \\
& \left.+2 A_{11} z_{U 0}+A_{11} z_{U 1}+3 A_{12} f+3 A_{03}\right)-g\left(B_{30} f^{3}\right. \\
& +2 B_{11} f z_{V 0}+B_{11} f z_{V 1}+4 B_{20} f z_{U 0}+2 B_{20} f z_{U 1} \\
& +3 B_{21} f^{2}+4 B_{02} z_{V 0}+2 B_{02} z_{V 1}+2 B_{11} z_{U 0}+B_{11} z_{U 1} \\
& \left.+3 B_{12} f+3 B_{03}\right), \\
G_{2}= & -\left(6 A_{30} f^{3}+2 A_{11} f z_{V 0}+A_{11} f z_{V 2}+4 A_{20} f z_{U 0}\right. \\
& +2 A_{20} f z_{U 2}+6 A_{21} f^{2}+4 A_{02} z_{V 0}+2 A_{02} z_{V 2}+2 A_{11} z_{U 0} \\
& \left.+A_{11} z_{U 2}+6 A_{12} f+6 A_{03}\right)-g\left(6 B_{30} f^{3}+2 B_{11} f z_{V 0}\right. \\
& +B_{11} f z_{V 2}+4 B_{20} f z_{U 0}+2 B_{20} f z_{U 2}+6 B_{21} f^{2}+4 B_{02} z_{V 0} \\
& \left.+2 B_{02} z_{V 2}+2 B_{11} z_{U 0}+B_{11} z_{U 2}+6 B_{12} f+6 B_{03}\right) . \tag{56}
\end{align*}
$$

The combination of equations (51) and (55) gives the amplitude equation (57) for the amplitude

$$
\begin{equation*}
\tau_{0} \frac{\partial A_{j}}{\partial t}=\mu A_{j}+h \bar{A}_{l} \bar{A}_{m}-\left(g_{1}\left|A_{1}\right|^{2}+g_{2}\left(\left|A_{2}\right|^{2}+\left|A_{3}\right|^{2}\right)\right) A_{j} \tag{57}
\end{equation*}
$$

where

$$
\begin{align*}
\tau_{0} & =\frac{f+g}{\tau_{T}\left[f m_{11}+m_{12}+g\left(f m_{21}+m_{22}\right)\right]}, \\
\mu & =\frac{\tau_{T}-\tau}{\tau_{T}}, \\
h & =\frac{H}{\tau_{T}\left[f m_{11}+m_{12}+g\left(f m_{21}+m_{22}\right)\right]},  \tag{58}\\
g_{i} & =\frac{G_{i}}{\tau_{T}\left[f m_{11}+m_{12}+g\left(f m_{21}+m_{22}\right)\right]}
\end{align*}
$$

Please notice that system (57) is in complex form. Following to reference [30], for the purpose of convenience of discussion, we convert it into the real form by $A_{j}=\rho_{j} \exp \left(i \varphi_{j}\right)$ with $\rho_{j}$ as the real amplitudes and $\varphi_{j}$ as phase angles:

$$
\left\{\begin{array}{l}
\tau_{0} \frac{\partial \varphi}{\partial t}=-h \frac{\rho_{1}^{2} \rho_{2}^{2}+\rho_{1}^{2} \rho_{3}^{2}+\rho_{2}^{2} \rho_{3}^{2}}{\rho_{1} \rho_{2} \rho_{3}} \sin \varphi  \tag{59}\\
\tau_{0} \frac{\partial \rho_{1}}{\partial t}=-\mu \rho_{1}+h \rho_{2} \rho_{3} \cos \varphi-g_{1} \rho_{1}^{3}-g_{2}\left(\rho_{3}^{2}+\rho_{2}^{2}\right) \rho_{1} \\
\tau_{0} \frac{\partial \rho_{2}}{\partial t}=-\mu \rho_{2}+h \rho_{1} \rho_{3} \cos \varphi-g_{1} \rho_{2}^{3}-g_{2}\left(\rho_{3}^{2}+\rho_{1}^{2}\right) \rho_{2} \\
\tau_{0} \frac{\partial \rho_{3}}{\partial t}=-\mu \rho_{3}+h \rho_{2} \rho_{1} \cos \varphi-g_{1} \rho_{3}^{3}-g_{2}\left(\rho_{1}^{2}+\rho_{2}^{2}\right) \rho_{3}
\end{array}\right.
$$

where $\varphi=\varphi_{1}+\varphi_{2}+\varphi_{3}$. Since we are only interested in the stable steady states and notice the fact that $h \rho_{i} \neq 0$, from the first equation of (59), we have $\varphi=0$ or $\pi$. Also, noticing the fact that $\tau_{0}>0$, it implies that when $h>0$, the state corresponding to $\varphi=0$ is stable, but the one corresponding to $\varphi=\pi$ when $h<0$. Then, system of amplitude equation (59) becomes

$$
\left\{\begin{array}{l}
\tau_{0} \frac{\partial \rho_{1}}{\partial t}=\mu \rho_{1}+|h| \rho_{2} \rho_{3}-g_{1} \rho_{1}^{3}-g_{2}\left(\rho_{3}^{2}+\rho_{2}^{2}\right) \rho_{1}  \tag{60}\\
\tau_{0} \frac{\partial \rho_{2}}{\partial t}=\mu \rho_{2}+|h| \rho_{1} \rho_{3}-g_{1} \rho_{2}^{3}-g_{2}\left(\rho_{3}^{2}+\rho_{1}^{2}\right) \rho_{2} \\
\tau_{0} \frac{\partial \rho_{3}}{\partial t}=\mu \rho_{3}+|h| \rho_{2} \rho_{1}-g_{1} \rho_{3}^{3}-g_{2}\left(\rho_{1}^{2}+\rho_{2}^{2}\right) \rho_{3}
\end{array}\right.
$$

Please notice that generally the amplitude equations are valid only when the control parameter is in the Turing space. It is easy to see that the above system of ordinary differential equation (60) has five equilibria, which corresponds five kinds of steady states [10, 30, 31]. Noticing the symmetry of the system, we have the following:
(1) System (60) always has an equilibrium $E_{0}=(0,0,0)$, which is stable for $\mu<\mu_{2}=0$ and unstable for $\mu>\mu_{2}$
(2) System (60) has an equilibrium $E_{s}=\left(\sqrt{\mu / g_{1}}, 0,0\right)$ corresponding to stripe patterns, which is stable for $\mu>\mu_{3}=h^{2} g_{1} /\left(g_{2}-g_{1}\right)^{2}$ and unstable for $\mu>\mu_{3}$
(3) System (60) has an equilibrium $E_{h}=\left(\rho_{1}, \rho_{2}, \rho_{3}\right)$ corresponding to hexagon patterns, with $\varphi=0$ or $\varphi=\pi$, and $\rho_{1}^{+}=|h|+\sqrt{h^{2}+4\left(g_{1}+2 g_{2}\right) \mu} / 2\left(g_{1}+\right.$ $\left.2 g_{2}\right)$ is stable for $\mu<\mu_{4}=h^{2}\left(2 g_{1}+g_{2}\right) /\left(g_{2}-g_{1}\right)^{2}$ and $\rho_{1}^{-}=|h|-\sqrt{h^{2}+4\left(g_{1}+2 g_{2}\right) \mu} / 2\left(g_{1}+2 g_{2}\right)$ is unstable, where $\rho_{1}=\rho_{2}=\rho_{3}=|h| \pm \sqrt{h^{2}+4\left(g_{1}+\right.}$ $\left.2 g_{2}\right) \mu / 2\left(g_{1}+2 g_{2}\right)$
(4) System (60) has an equilibrium $E_{m}=\left(\rho_{1}, \rho_{2}, \rho_{3}\right)$ corresponding to mixed patterns, with $g_{1}>g_{2}$, $\mu>g_{1} \rho_{1}^{2}$ which is unstable, where $\rho_{1}=|h| / g_{2}-g_{1}$,

$$
\rho_{2}=\rho_{3}=\sqrt{\mu-g_{1} \rho_{1}^{2} / g_{2}+g_{1}}
$$

## 5. Numerical Simulations

In this section, we will further study the dynamic behavior of the coexistence equilibrium of the delayed reaction-diffusion model (3) using numerical simulation in two-dimensional space. In this paper, a two-dimensional delay reaction-diffusion model is treated by the finite difference method in the discrete region of $100 \times 100$. The spatial distance between two lattices is defined as the step size $\Delta x$ and $\Delta y$, using the standard five-point approximation for the 2D Laplacian with the zero-flux boundary conditions, and the time step is expressed as $\Delta t$. Take a fixed time step $\Delta t=0.01$. What needs to be further explained is that the concentrations $\left(S_{i, j}^{n+1}, I_{i, j}^{n+1}\right)$ at the moment $(n+1) \Delta t$ at the mesh position $(i, j)$ are given by

$$
\begin{align*}
& S_{i, j}^{n+1}=S_{i, j}^{n}+\Delta t d_{1} \Delta_{h} S_{i, j}^{n}+\Delta t f\left(S_{i, j}^{n}, I_{i, j}^{n}\right),  \tag{61}\\
& I_{i, j}^{n+1}=I_{i, j}^{n}+\Delta t d_{2} \Delta_{h} I_{i, j}^{n}+\Delta \operatorname{tg}\left(S_{i, j}^{n} I_{i, j}^{n}\right),
\end{align*}
$$

with the diffusion term (Laplacian) are defined by

$$
\begin{align*}
& \Delta_{h} S_{i, j}^{n}=\frac{S_{i+1, j}^{n}+S_{i, j+1}^{n}+S_{i-1, j}^{n}+S_{i, j-1}^{n}-4 S_{i, j}^{n}}{h^{2}},  \tag{62}\\
& \Delta_{h} I_{i, j}^{n}=\frac{I_{i+1, j}^{n}+I_{i, j+1}^{n}+I_{i-1, j}^{n}+I_{i, j-1}^{n}-4 I_{i, j}^{n}}{h^{2}}
\end{align*}
$$

Other parameters are fixed as

$$
\begin{align*}
\alpha & =0.65, \\
\beta & =6 \\
\gamma & =0.5 \\
e & =1  \tag{63}\\
\eta & =0.5 \\
d_{1} & =0.001 \\
d_{2} & =0.1 .
\end{align*}
$$

First, we discuss the effect of weak Allee effect on Turing pattern information. We try to take the Allee effect constant to $A=0, A=0.02$, and $A=0.1$. Here, we first discuss the situation without delay. When $A=0$, Zhang et al. [10] give the condition of Turing instability. Here, we just give the pattern formations. By comparing Figures 1 and $2(A=0$ and $A=0.02$ ), we find that the initial state is the coexistence of stripes and spots, and the stripes are very long. With the increase of Allee parameters, the length of stripes decreases and some stripes even form a circle. Then, we continue to increase the value of the weak Allee parameter like Figure 3 ( $A=0.1$ ), and we find that pattern formations have changed again. As time goes on, we find that pattern formations show a cycle when $t=100$ and when $t=500$ and we find that the cycle diffuses outward (indicating that pattern formations are not stable) to form a butterfly-like shape; and finally, when we increase to $t=2000$, we find that the pattern formations are not stable. It was found that pattern formations became stripes and spots again. After we tried to add more time, we found that the pattern formation did not change again.

Next, let us discuss the pattern formation change of the model with time delay and without Allee effect. We change the delay to $\tau=0.25$. By comparing with Figure 1, we find that the stripes and spots of the original pattern formations change to stripes like Figure 4, and the pattern formations will not change as time goes on.

Finally, we discuss the pattern formations of models with Allee effect and time delay ( $A=0.02$ and $\tau=0.02$ ). We find that pattern formations are spots when $t=100$; as time goes on $t=500$, we find that pattern formations change again, similar to Figure 3, but the final pattern formations change differently. We can see that pattern formations change into strips surrounded by spots; we increase the time again $(t=2000)$ to find that the pattern will spread outward in this form, forming the phenomenon of strip pattern surrounded by spots pattern; we further increase the time $(t=5000)$ to find that the pattern such as in Figure 5 tends to stabilize and does not change again.


Figure 1: Mixture patterns obtained with model (1) for $A=0$ and $\tau=0$. Time: (a) $t=100$, (b) $t=500$, (c) $t=2000$, and (d) $t=5000$.


Figure 2: Continued.


Figure 2: Mixture patterns obtained with model (1) for $A=0.02$ and $\tau=0$. Time: (a) $t=100$, (b) $t=500$, (c) $t=2000$, and (d) $t=5000$.


Figure 3: Mixture patterns obtained with model (1) for $A=0.1$ and $\tau=0$. Time: (a) $t=100$, (b) $t=500$, (c) $t=2000$, and (d) $t=5000$.

## 6. Conclusion

This paper is based on a model that considers a predatorprey model with nonlinear mortality and Holling II functional response. The weak Allee effect is introduced and the effect of the Allee effect on pattern formations is considered. Furthermore, we consider a class of reaction-diffusion predator-prey models with searching delay and weak Allee
effect, considering the effects of delay on pattern formations. We give the stability and Turing instability of the positive equilibrium point $E_{*}$. As a result of diffusion, model (3) and model (4) exhibits stationary Turing pattern. Furthermore, through numerical simulation, comparing Figures 1 and 2, we find that the Allee effect will reduce the length of the strip pattern in Figure 1, and there will be some "cycle" pattern as shown in Figure 2. From an ecological point of view, we


Figure 4: Stripes pattern obtained with model (1) for $A=0$ and $\tau=0.25$. Time: (a) $t=100$, (b) $t=500$, (c) $t=1000$, and (d) $t=2000$.


Figure 5: Continued.


Figure 5: Mixture patterns obtained with model (1) for $A=0.02$ and $\tau=0.02$. Time: (a) $t=100$, (b) $t=500$, (c) $t=2000$, and (d) $t=5000$.
know that the Allee effect increases the risk of population extinction, while the effect of the longer stripe pattern in Figure 1 increases the likelihood of predation. However, the shorter stripes and spots in Figure 2 reduce the likelihood of predation. As the Allee effect parameter continues to increase, we find that the pattern has changed again. The type of the pattern is similar to that of Figure 1, but the density and size of the pattern will change slightly as shown in Figure 3. We believe that in order to avoid predator hunting, predators are concentrated in a certain area rather than scattered throughout the habitat, which further reduces the contact area between predator and prey. Over time, prey needs to migrate to new habitats. The aggregation pattern diffuses slowly, the predator follows the pursuit, and the aggregation point enlarges gradually. It is worth noting when the Allee effect parameter is $A=0.1$, there are two positive equilibrium points in model (4). Next, we consider the effect of delay on pattern formations. By comparing Figure 1 with Figure 4, we find when the delay is $\tau=0.25$, the pattern changes from the state where the starting spots pattern and the strip pattern coexist to the case where only the strip pattern exists. Finally, we try to consider the Allee effect and delay to observe the changes in pattern formations, where $A=0.02$ and $\tau=0.02$. We find that when both are present, the spots pattern is surrounded by strip patterns as shown in Figure 5. This reminds us of animals in the natural world at the lower end of the food chain, often with a means of protection. Juvenile animals are surrounded by adult animals to reduce the probability of their juvenile animals being preyed. This may be an interesting finding or not. So, we find that Allee effect and delay play an important role in spatial invasion of populations.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (31560127), Fundamental Research Funds for the Central Universities (31920180116, 31920180044, and 31920170072), Program for Young Talent of State Ethnic Affairs Commission of China (No. [2014] 121), Gansu Provincial First-Class Discipline Program of Northwest Minzu University (No. 11080305), and Central Universities Fundamental Research Funds for the Graduate Students of Northwest Minzu University (Yxm2019109).

## References

[1] F. Courchamp, L. Berec, and J. Gascoigne, Allee Effects in Ecology and Conservation, Oxford University Press, Oxford, UK, 2008.
[2] P. J. Pal, T. Saha, M. Sen, and M. Banerjee, "A delayed predator-prey model with strong Allee effect in prey population growth," Nonlinear Dynamics, vol. 68, no. 1-2, pp. 23-42, 2012.
[3] M. Sen and M. Banerjee, "Rich global dynamics in a preypredator model with Allee effect and density dependent death rate of predator," International Journal of Bifurcation and Chaos, vol. 25, no. 3, Article ID 1530007, 2015.
[4] W. Wang, Y. Cai, Y. Zhu, and Z. Guo, "Allee-effect-induced instability in a reaction-diffusion predator-prey model," Abstract and Applied Analysis, vol. 2013, Article ID 487810, 10 pages, 2013.
[5] M. Sen, M. Banerjee, and Y. Takeuchi, "Influence of Allee effect in prey populations on the dynamics of two-prey-onepredator model," Mathematical Biosciences \& Engineering, vol. 15, no. 4, pp. 883-904, 2018.
[6] Y. Cai, C. Zhao, W. Wang, and J. Wang, "Dynamics of a Leslie-Gower predator-prey model with additive Allee effect," Applied Mathematical Modelling, vol. 39, no. 7, pp. 2092-2106, 2015.
[7] W. Wang, Y.-N. Zhu, Y. Cai, and W. Wang, "Dynamical complexity induced by Allee effect in a predator-prey model," Nonlinear Analysis: Real World Applications, vol. 16, pp. 103-119, 2014.
[8] H. Liu, Y. Ye, Y. Wei, M. Ma, and J. Ye, "Dynamic study of a predator-prey system with weak allee effect and holling type-

III functional response," Dynamic Systems and Applications, vol. 27, no. 4, pp. 943-953, 2018.
[9] B. Zhang, Y. Cai, B. Wang, and W. Wang, "Pattern formation in a reaction-diffusion parasite-host model," Physica A: Statistical Mechanics and its Applications, vol. 525, pp. 732-740, 2019.
[10] T. Zhang, Y. Xing, H. Zang, and M. Han, "Spatio-temporal dynamics of a reaction-diffusion system for a predator-prey model with hyperbolic mortality," Nonlinear Dynamics, vol. 78, no. 1, pp. 265-277, 2014.
[11] J. Shi and R. Shivaji, "Persistence in reaction diffusion models with weak Allee effect," Journal of Mathematical Biology, vol. 52, no. 6, pp. 807-829, 2006.
[12] Y. Peng and T. Zhang, "Turing instability and pattern induced by cross-diffusion in a predator-prey system with Allee effect," Applied Mathematics and Computation, vol. 275, pp. 1-12, 2016.
[13] G.-Q. Sun, L. Li, Z. Jin, Z.-K. Zhang, and T. Zhou, "Pattern dynamics in a spatial predator-prey system with allee effect," Abstract and Applied Analysis, vol. 2013, Article ID 921879, 12 pages, 2013.
[14] Y. Du and J. Shi, "Allee effect and bistability in a spatially heterogeneous predator-prey model," Transactions of the American Mathematical Society, vol. 359, no. 9, pp. 45574594, 2007.
[15] J. Wang, J. Shi, and J. Wei, "Dynamics and pattern formation in a diffusive predator C prey system with strong Allee effect in prey," Journal of Differential Equations, vol. 251, no. 4-5, pp. 1276-1304, 2011.
[16] R. Cui, J. Shi, and B. Wu, "Strong Allee effect in a diffusive predator-prey system with a protection zone," Journal of Differential Equations, vol. 256, no. 1, pp. 108-129, 2014.
[17] C. Çelik, O. H. Merdan, Ö. Duman, and Ö. Akın, "Allee effects on population dynamics with delay," Chaos, Solitons \& Fractals, vol. 37, no. 1, pp. 65-74, 2008.
[18] G.-Q. Sun, "Mathematical modeling of population dynamics with Allee effect," Nonlinear Dynamics, vol. 85, no. 1, pp. 1-12, 2016.
[19] H. Merdan and Ö. A. K. Gümüş, "Stability analysis of a general discrete-time population model involving delay and Allee effects," Applied Mathematics and Computation, vol. 219, no. 4, pp. 1821-1832, 2012.
[20] A. Y. Morozov, M. Banerjee, and S. V. Petrovskii, "Long-term transients and complex dynamics of a stage-structured population with time delay and the Allee effect," Journal of Theoretical Biology, vol. 396, pp. 116-124, 2016.
[21] S. Biswas, S. K. Sasmal, S. Samanta, M. Saifuddin, N. Pal, and J. Chattopadhyay, "Optimal harvesting and complex dynamics in a delayed eco-epidemiological model with weak Allee effects," Nonlinear Dynamics, vol. 87, no. 3, pp. 15531573, 2017.
[22] S. Biswas, M. Saifuddin, S. K. Sasmal et al., "A delayed preypredator system with prey subject to the strong Allee effect and disease," Nonlinear Dynamics, vol. 84, no. 3, pp. 15691594, 2016.
[23] S. V. Petrovskii, A. Y. Morozov, and E. Venturino, "Allee effect makes possible patchy invasion in a predator-prey system," Ecology Letters, vol. 5, no. 3, pp. 345-352, 2010.
[24] H. Liu, Z. Li, M. Gao, H. Dai, and Z. Liu, "Dynamics of a hostparasitoid model with allee effect for the host and parasitoid aggregation," Ecological Complexity, vol. 6, no. 3, pp. 337-345, 2009.
[25] L. Shi, H. Liu, Y. Wei, M. Ma, and J. Ye, "The permanence and periodic solution of a competitive system with infinite delay,
feedback control, and Allee effect," Advances in Difference Equations, vol. 2018, no. 1, 400 pages, 2018.
[26] Y. Ye, H. Liu, Y. Wei, M. Ma, and K. Zhang, "Dynamic study of a predator-prey model with weak Allee effect and delay," Advances in Mathematical Physics, vol. 2019, Article ID 7296461, 15 pages, 2019.
[27] Y. Ye, H. Liu, Y. Wei, K. Zhang, M. Ma, and J. Ye, "Dynamic study of a predator-prey model with Allee effect and Holling type-I functional response," Advances in Difference Equations, vol. 2019, no. 1, 369 pages, 2019.
[28] S. Huang and Q. Tian, "Marcinkiewicz estimates for solution to fractional elliptic Laplacian equation," Computers \& Mathematics with Applications, vol. 78, no. 5, pp. 1732-1738, 2019.
[29] S. Sen, P. Ghosh, S. S. Riaz, and D. S. Ray, "Time-delay-induced instabilities in reaction-diffusion systems," Physical Review E, vol. 80, no. 4, 046212 pages, 2009.
[30] Q. Ouyang, Nonlinear Science and the Pattern Dynamics Introduction, Peking University Press Beijing, Beijing, China, 2010.
[31] S. Yuan, C. Xu, and T. Zhang, "Spatial dynamics in a predatorprey model with herd behavior," Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 23, no. 3, Article ID 033102, 2018.


Advances in
Operations Research
$=$



Decision Sciences
Journal of
Applied Mathematics
$=$


The Scientific World Journal


Journal of
Probability and Statistics


