

Research Article

The Bass Diffusion Model on Finite Barabasi-Albert Networks

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Using a heterogeneous mean-field network formulation of the Bass innovation diffusion model and recent exact results on the degree correlations of Barabasi-Albert networks, we compute the times of the diffusion peak and compare them with those on scale-free networks which have the same scale-free exponent but different assortativity properties. We compare our results with those obtained for the SIS epidemic model with the spectral method applied to adjacency matrices. It turns out that diffusion times on finite Barabasi-Albert networks are at a minimum. This may be due to a little-known property of these networks: whereas the value of the assortativity coefficient is close to zero, they look disassortative if one considers only a bounded range of degrees, including the smallest ones, and slightly assortative on the range of the higher degrees. We also find that if the trickle-down character of the diffusion process is enhanced by a larger initial stimulus on the hubs (via an inhomogeneous linear term in the Bass model), the relative difference between the diffusion times for BA networks and uncorrelated networks is even larger, reaching, for instance, the 34% in a typical case on a network with 10^4 nodes.

1. Introduction

The study of epidemic diffusion is one of the most important applications of network theory [1–3], the absence of an epidemic threshold on scale-free networks being perhaps the best known result [4]. This result essentially also holds for networks with degree correlations [5], although some exceptions have been pointed out [6, 7]. In [8] the dependence of the epidemic threshold and diffusion time on the network assortativity was investigated, using a degree-preserving rewiring procedure which starts from a Barabasi-Albert network and analysing the spectral properties of the resulting adjacency matrices. In this paper we mainly focus on Barabasi-Albert (BA) networks [9], using the exact results by Fotouhi and Rabbat on the degree correlations [10]. We employ the mean-field method [11] and our network formulation of the Bass diffusion model for the description of the innovation diffusion process [12, 13]. We solve numerically the equations and find the time of the diffusion peak, an important turning point in the life cycle of an innovation, for values of the maximum network degree n of the order of 10^2 , which correspond to medium-size real networks with

$N \approx 10^4$ nodes. Then we compare these diffusion times with those of networks with different kinds of degree correlations.

In Section 2, as a preliminary to the analysis of diffusion, we provide the mathematical expressions of the average nearest neighbor degree function $k_m(k)$ and the Newman assortativity coefficient r [14] of BA networks.

In Section 3 we write the network Bass equations and give the diffusion times found from the numerical solutions. We compare these times with those for uncorrelated scale-free networks with exponent $\gamma = 3$ and with those for assortative networks whose correlation matrices are mathematically constructed and studied in another work. Then we apply a method proposed by Newman in [15] to build disassortative correlation matrices and evaluate the corresponding diffusion times. Our results are in qualitative agreement with those by D'Agostino et al. [8] for the SIS model. The minimum diffusion time is obtained for the BA networks, which have, with $n \approx 10^2$ (corresponding to $N \approx 10^4$ nodes, see Section 4), $r \approx -0.10$. The minimum value of r obtained for the disassortative networks (with $\gamma = 3$) is $r \approx -0.09$, also not far from the values obtained in [8]. For assortative networks, on the contrary, values of r much closer to the maximum $r = 1$ are easily obtained.

Section 4 contains a discussion of the results obtained for the r coefficient and the k_{mn} function of BA networks.

In Section 5 we further reexamine the validity of the mean-field approximation, also by comparison to a version of the Bass model based on the adjacency matrix of BA networks.

In Section 6 we compute diffusion times with heterogeneous p coefficients (“trickle-down” modified Bass model).

In summary, our research objectives in this work have been the following: (a) Analyse the assortativity properties of BA networks, using the correlation functions recently published. These properties are deduced from the Newman coefficient r and from the average nearest neighbor degree function $k_{mn}(k)$. (b) Compute the peak diffusion times of the network Bass model in the mean-field approximation on BA networks, and compare these times with those obtained for other kinds of networks. (c) Briefly discuss the validity of the mean-field approximation, compared to a first-moment closure of the Bass model on BA networks described through the adjacency matrix.

Section 7 contains our conclusions and outlook.

2. The Average Nearest Neighbor Degree Function of Barabasi-Albert (BA) Networks

We emphasize that for the networks considered in this paper a maximal value $n \in \mathbb{N}$ is supposed to exist for the admissible number of links emanating from each node. Notice that, as shown later in the paper (see Section 4.1, 4th paragraph), this is definitely compatible with a high number N of nodes in the network (e.g., with a number N differing from n by various magnitude orders).

Together with the degree distribution $P(k)$ which expresses the probability that a randomly chosen node has k links, other important statistical quantities providing information on the network structure are the degree correlations $P(h | k)$. Each coefficient $P(h | k)$ expresses the conditional probability that a node with k links is connected to one with h links. In particular, an increasing character of the average nearest neighbor degree function

$$k_{mn}(k) = \sum_{h=1}^n hP(h | k) \quad (1)$$

is a hallmark of assortative networks (i.e., of networks in which high degree nodes tend to be linked to other high degree nodes), whereas a decreasing character of this function is to be associated with disassortative networks (networks in which high degree nodes tend to be linked to low degree nodes). We recall that the $P(k)$ and $P(h | k)$ must satisfy, besides the positivity requirements,

$$P(k) \geq 0 \quad (2)$$

$$\text{and } P(h | k) \geq 0,$$

both the normalizations

$$\sum_{k=1}^n P(k) = 1 \quad (3)$$

$$\text{and } \sum_{h=1}^n P(h | k) = 1,$$

and the Network Closure Condition (NCC)

$$hP(k | h)P(h) = kP(h | k)P(k) \quad (4)$$

$$\forall h, k = i = 1, \dots, n.$$

A different tool usually employed to investigate structural properties of networks is the Newman assortativity coefficient r also known as the Pearson correlation coefficient [14]. To define it, we need to introduce the quantities e_{jk} which express the probability that a randomly chosen edge links nodes with excess degree j and k . Here, the excess degree of a node is meant as its total degree minus one, namely, as the number of all edges emanating from the node except the one under consideration. The distribution of the excess degrees is easily found to be given [14] by

$$q_k = \frac{(k+1)}{\sum_{j=1}^n jP(j)} P(k+1). \quad (5)$$

The assortativity coefficient r is then defined as

$$r = \frac{1}{\sigma_q^2} \sum_{k,h=0}^{n-1} kh(e_{kh} - q_k q_h), \quad (6)$$

where σ_q denotes the standard deviation of the distribution $q(k)$, i.e.,

$$\sigma_q^2 = \sum_{k=1}^n k^2 q_k - \left(\sum_{k=1}^n k q_k \right)^2. \quad (7)$$

The coefficient r takes values in $[-1, 1]$ and it owes its name to the fact that if $r < 0$, the network is disassortative, if $r = 0$, the network is neutral, and if $r > 0$, the network is assortative. A formula expressing the e_{kh} in terms of known quantities is necessary if one wants to calculate r . This can be obtained as discussed next. Let us move from the elements E_{kh} expressing the number of edges which link nodes with degree k and h , with the only exception that edges linking nodes with the same degree have to be counted twice [16, 17]. Define now $\tilde{e}_{kh} = E_{kh} / (\sum_{k,h} E_{kh})$. Each \tilde{e}_{kh} corresponds then to the fraction of edges linking nodes with degree k and h (with the mentioned interpretation of the \tilde{e}_{kk}). We also observe that $e_{k,h} = \tilde{e}_{k+1,h+1}$ holds true. The degree correlations can be related to the \tilde{e}_{kh} through the formula

$$P(h | k) = \frac{\tilde{e}_{kh}}{\sum_{j=1}^n \tilde{e}_{kj}} \quad \forall h, k = i = 1, \dots, n. \quad (8)$$

What is of interest for us here is the following “inverse” formula:

$$\tilde{e}_{hk} = \frac{P(h | k)kP(k)}{\sum_{j=1}^n jP(j)}. \quad (9)$$

In the rest of this section we discuss the quantities which allow us to calculate the average nearest neighbor degree function $k_m(k)$ and the coefficient r for finite Barabasi-Albert (BA) networks.

The degree distribution of the Barabasi-Albert networks is known to be given by

$$P(k) = \frac{2\beta(\beta+1)}{k(k+1)(k+2)}, \quad (10)$$

where $\beta \geq 1$ is the parameter in the preferential attachment procedure characterizing them [9, 17]. In particular, (10) yields $P(k) \sim c/k^3$ with a suitable constant c for large k .

An explicit expression for the degree correlations $P(h|k)$ was given by Fotouhi and Rabbat in [10]. They showed that, for a growing network in the asymptotic limit as $t \rightarrow \infty$,

$$P(h|k) = \frac{\beta}{kh} \left(\frac{k+2}{h+1} - B_{\beta+1}^{2\beta+2} \frac{B_{h-\beta}^{k+h-2\beta}}{B_h^{k+h+2}} \right), \quad (11)$$

with B_j^m denoting the binomial coefficient

$$B_j^m = \frac{m!}{j!(m-j)!}. \quad (12)$$

Since the networks we consider have a maximal number of links n [18], we must normalize the matrix with elements $P(h|k)$. We calculate $C_k = \sum_{h=1}^n P(h|k)$ and take as a new degree correlation matrix the matrix whose (h, k) -element is

$$P_n(h|k) = \frac{P(h|k)}{C_k}, \quad (13)$$

with $P(h|k)$ as in (11).

The average nearest neighbor degree function $k_m(k)$ and the coefficient r can be now easily calculated with software like *Mathematica*, by using (1), (6), (10), and (13). Results are reported and discussed in Section 4.

3. The Bass Diffusion Equation on Complex Networks

In [12, 13] we have reformulated the well-known Bass equation of innovation diffusion

$$\frac{dF(t)}{dt} = [1 - F(t)] [p + qF(t)] \quad (14)$$

(where $F(t)$ is the cumulative adopter fraction at the time t and p and q are the innovation and the imitation coefficient, respectively), providing versions suitable for the case in which the innovation diffusion process occurs on a network. The model can be expressed in such a case by a system of n ordinary differential equations:

$$\frac{dG_i(t)}{dt} = [1 - G_i(t)] \left[p + iq \sum_{h=1}^n P(h|i) G_h(t) \right] \quad (15)$$

$i = 1, \dots, n.$

The quantity $G_i(t)$ in (15) represents for any $i = 1, \dots, n$ the fraction of potential adopters with i links that at the time t have adopted the innovation. More precisely, denoting by $F_i(t)$ the fraction of the total population composed by individuals with i links, who at the time t have adopted the innovation, we set $G_i(t) = F_i(t)/P(i)$. Further heterogeneity can be introduced allowing also the innovation coefficient (sometimes called publicity coefficient) to be dependent on i . In this case, the equations take the form

$$\frac{dG_i(t)}{dt} = [1 - G_i(t)] \left[p_i + iq \sum_{h=1}^n P(h|i) G_h(t) \right] \quad (16)$$

$i = 1, \dots, n,$

and, for example, the p_i can be chosen to be inversely proportional to $P(i)$ or to have a linear dependence, decreasing in i ; in the first case more publicity is delivered to the hubs, with ensuing “trickle-down” diffusion, while in the second case a “trickle-up” diffusion from the periphery of the network can be simulated.

The function $f_i(t) = \dot{F}_i(t)$ gives the fraction of new adoptions per unit time in the “link class i ” (or “degree class i ”) i.e., in the subset of individuals having i links. The left panel in Figure 1 shows an example of a numerical solution with plots of all the f_i 's, in a case where for graphical reasons we have taken n small ($n = 15$). The underlying network is a BA with $\beta = 1$. As is clear from the plot, the largest fraction of new adopters belongs at all times to the link class with $i = 1$, which reaches its adoption peak later than the others. In general, the more connected individuals are, the earlier they adopt. This phenomenon is quite intuitive and has been evidenced in previous works on mean-field epidemic models; see, for instance, [19]. As discussed in our paper [13], in applications of the Bass model to marketing this may allow to estimate the q coefficient, when it is not known in advance, by monitoring the early occurrence of adoption in the most connected classes. The right panel in Figure 1 shows the total adoption rate $f(t)$ corresponding to the same case as in the left panel. The simple Bass curve (homogeneous model, without underlying network) is also shown for comparison.

In the Bass model, unlike in other epidemic models where infected individuals can return to the susceptible state, the diffusion process always reaches all the population. The function $f(t) = \sum_{i=1}^n \dot{F}_i(t)$, which represents the total number of new adoptions per unit time, usually has a peak, as we have seen in the previous example. We choose the time of this peak as a measure of the diffusion time; it is computed for each numerical solution of the diffusion equations by sampling the function $f(t)$. For fixed coefficients of publicity p and imitation q , the time depends on the features of the network. In this paper we consider only scale-free networks with $\gamma = 3$, for comparison with BA networks.

Figure 2 shows the peak times obtained for different networks with maximum degree $n = 100$, as a function of the imitation parameter q . This value of n has been chosen because it corresponds to a number N of nodes of the order of 10^4 ; this allows a comparison with the results of D'Agostino et al. (see below) and displays finite-size effects, as discussed

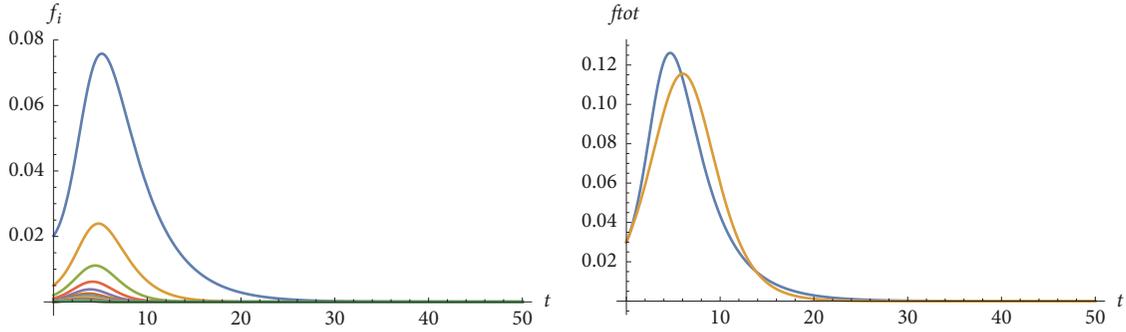


FIGURE 1: *Left panel*: fraction f_i of new adoptions per unit time in the “link class i ” (the subset of all individuals having i links), as a function of time, in the Bass innovation diffusion model on a BA network. The parameter β of the network (number of child nodes in the preferential attachment scheme) is $\beta = 1$. The maximum number of links in this example is $n = 15$, while in the comprehensive numerical solutions, whose results are summarized in Figure 2, it is $n = 100$. The adoption peak occurs later for the least connected (and most populated) class with $i = 1$. *Right panel*: cumulative new adoptions $f_{tot} = \sum_{i=1}^n f_i$ per unit time, as a function of time, compared with the same quantity for the homogeneous Bass model without an underlying network. The peak of the homogeneous Bass model is slightly lower and shifted to the right. For the values of the model parameters p and q and the measuring unit of time, see Figure 2.

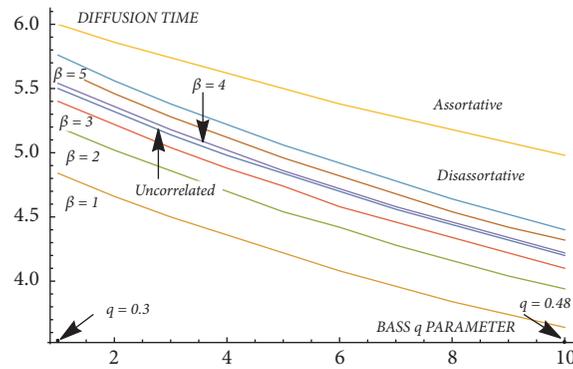


FIGURE 2: Time of the diffusion peak (in years) for the Bass model on different kinds of scale-free networks with exponent $\gamma = 3$, as a function of the imitation coefficient q . All networks have maximum degree $n = 100$ ($N \approx 10^4$). The q coefficient varies in the range $0.3 - 0.48$, corresponding to a typical set of realistic values in innovation diffusion theory [20]. The publicity coefficient is set to the value $p = 0.03$ (see Figure 9 for results with an inhomogeneous p_k depending on the link class $k = 1, \dots, n$). The lines with $\beta = 1, 2, 3, 4, 5$ correspond to BA networks with those values of β . Their assortativity coefficients are, respectively, $r = -0.104, -0.089, -0.078, -0.071, -0.065$. The disassortative network is built with the Newman recipe (Section 3.4) with $d = 4$ and has $r = -0.084$. The assortative network is built with our recipe (Section 3.3), with $\alpha = 1/2$, and has $r = 0.863$.

in Section 4 (note, however, that such effects are still present with $N \approx 10^6$ and larger).

It is clear from Figure 2 that diffusion is faster on the BA networks with $\beta = 1, 2, 3$ than on an uncorrelated network. For $\beta = 4$ the diffusion time is almost the same, and for $\beta = 5$ (and $\beta > 5$, not shown) diffusion on the BA network is slower than on the uncorrelated network. On purely disassortative networks diffusion is slightly slower than on an uncorrelated network and much slower than on assortative networks.

3.1. Comparison with the SIS Model. In [8] D’Agostino et al. have studied the dependence of the epidemic threshold and diffusion time for the SIS epidemic model on the assortative or disassortative character of the underlying network. Both the epidemic threshold and the diffusion time were evaluated from the eigenvalues of the adjacency matrix. The networks employed had a number of nodes $N = 10^4$, a scale-free degree

distribution with exponent $\gamma = 3$, and were obtained from a BA seed network through a Monte Carlo rewiring procedure which preserves the degree distribution but changes the degree correlations. The Monte Carlo algorithm employs an “Hamiltonian” function which is related to the Newman correlation coefficient r . The values of r explored in this way lie in the range from -0.15 to 0.5 and are therefore comparable with those obtained for our assortative and disassortative matrices.

Although the epidemic model considered and the definition of the diffusion time adopted in [8] are different from ours, there is a qualitative agreement in the conclusions: the diffusion time increases with the assortativity of the network and is at a minimum for values of r approximately equal to -0.10 . This value of r corresponds to those of finite BA networks with $\beta = 1$ and $N \approx 10^4$ and is slightly smaller than the minimum value of r obtained for disassortative networks with a matrix e_{jk} built according to Newman’s

recipe (Section 3.4). Note that for those networks the function $k_{nn}(k)$ is decreasing for any k , while for BA networks it is decreasing at small values of k and increasing at large k (Section 4); nevertheless, their r coefficient never becomes significantly less than ≈ -0.1 , even with other choices in the recipe, as long as $\gamma = 3$. We also recall that the clustering coefficient of the BA networks is exactly zero for $\beta = 1$; for an analysis of the role of the clustering coefficient in epidemic spreading on networks, see, for instance, [21].

A rewiring procedure comparable to that of [8] (without evaluation of the epidemic threshold and diffusion time) has been mathematically described by Van Mieghem et al. [22].

In the next subsections we provide information on the degree correlation matrices and other features of the networks used above for comparison with the BA networks. To compare diffusion peak times on networks which, although characterized by different assortativity/disassortativity properties, share some similarity, we consider networks whose degree distribution $P(k)$ obeys a power-law with exponent three, i.e., is of the form

$$P(k) = \frac{c}{k^3}, \quad (17)$$

where c is the normalization constant.

3.2. Uncorrelated Networks. Let us start with uncorrelated networks. As their name suggests, in these networks the degree correlations $P(h | k)$ do not depend on k . They can be easily seen to be given by

$$P(h | k) = \frac{hP(h)}{\langle h \rangle}, \quad (18)$$

with $\langle h \rangle = \sum_{h=1}^n hP(h)$, and hence their average nearest neighbor degree function (1) is

$$k_{nn}(k) = \frac{\langle h^2 \rangle}{\langle h \rangle}, \quad (19)$$

a constant. The coefficient r is trivially found to be equal to zero.

3.3. A Family of Assortative Networks. We consider now a family of assortative networks we have introduced in [12]. To give here the expressions of their degree correlations $P(h | k)$, we need to recall their construction. We start defining the elements of a $n \times n$ matrix P_0 as

$$P_0(h | k) = |h - k|^{-\alpha} \quad \text{if } h < k \\ \text{and } P_0(h | k) = 1 \quad \text{if } h = k, \quad (20)$$

for some parameter $\alpha > 0$, and we define the elements $P_0(h | k)$ with $h > k$ in such a way that formula (4) is satisfied by the $P_0(h | k)$. Hence, since the normalization $\sum_{h=1}^n P_0(h | k) = 1$ has to hold true, we compute for any $k = 1, \dots, n$ the sum $C_k = \sum_{h=1}^n P_0(h | k)$ and call $C_{max} = \max_{k=1, \dots, n} C_k$. Then, we introduce a new matrix P_1 , requiring that its diagonal elements be given by $P_1(k | k) = C_{max} - C_k$ for any $k =$

$1, \dots, n$, whereas the nondiagonal elements are the same as those of the matrix P_0 : $P_1(h | k) = P_0(h | k)$ for $h \neq k$. For any $k = 1, \dots, n$ the column sum $\sum_{h=1}^n P_1(h | k)$ is then equal to $C_k - 1 + C_{max} - C_k = C_{max} - 1$. Finally, we normalize the entire matrix by setting

$$P(h | k) = \frac{1}{(C_{max} - 1)} P_1(h | k) \quad \text{for } h, k = 1, \dots, n. \quad (21)$$

Again, the average nearest neighbor degree function $k_{nn}(k)$ and the coefficient r can be calculated with a software. The increasing character of $k_{nn}(k)$ for a network of the family in this subsection with $\alpha = 1/2$ and $n = 101$ is shown for example in Figure 3.

3.4. A Family of Disassortative Networks. Different models of disassortative networks can be constructed based on a suggestion by Newman contained in [15]. According to it, one can set $e_{kh} = q_k x_h + x_k q_h - x_k x_h$ for $h, k = 0, \dots, n - 1$, where q_k is the distribution of the excess degrees and x_k is any distribution satisfying $\sum_{k=0}^{n-1} x_k = 1$, and with x_k decaying faster than q_k . Choosing, to fix ideas, $x_k = (k + 1)^{-\gamma} / \sum_{j=0}^{n-1} (j + 1)^{-\gamma}$ with a parameter $\gamma > 2$, we denote $S = \sum_{j=0}^{n-1} (j + 1)^{-\gamma}$ and $T = \sum_{j=0}^{n-1} (j + 1)^{-2}$ and then set for all $h, k = 0, \dots, n - 1$,

$$e_{kh} = \left(\frac{1}{ST} \left((k + 1)^{-2} (h + 1)^{-\gamma} + (h + 1)^{-2} (k + 1)^{-\gamma} \right) - \frac{1}{S^2} (k + 1)^{-\gamma} (h + 1)^{-\gamma} \right). \quad (22)$$

We show in Appendix that the inequalities $0 \leq e_{kh} \leq 1$ hold true for any $h, k = 0, \dots, n - 1$. In view of (8), the degree correlations are then obtained as

$$P(h | k) = \frac{e_{k-1, h-1}}{\sum_{j=1}^n e_{k-1, j-1}}, \quad \forall h, k = 1, \dots, n, \quad (23)$$

with the e_{kh} as in (22). It is immediate to check that the coefficient r is negative, see, e.g., [15]. As for the average nearest neighbor degree function $k_{nn}(k)$, it can be calculated with a software. The decreasing character of $k_{nn}(k)$ for a network of the family in this subsection with $\gamma = 4$ and $n = 101$ is shown for example in Figure 4.

4. Discussion

4.1. The Newman Assortativity Coefficient for BA Networks. In [14] Newman reported in a brief table the values of r for some real networks. More data are given in his book [23], Table 8.1. Focusing on disassortative scale-free networks for which the scale-free exponent γ is available, one realizes that their negative r coefficient is generally small in absolute value, especially when γ is equal or close to 3, for instance,

- (i) www.nd.edu: γ from 2.1 to 2.4, $r = -0.067$
- (ii) Internet: $\gamma = 2.5$, $r = -0.189$
- (iii) Electronic circuits: $\gamma = 3.0$, $r = -0.154$

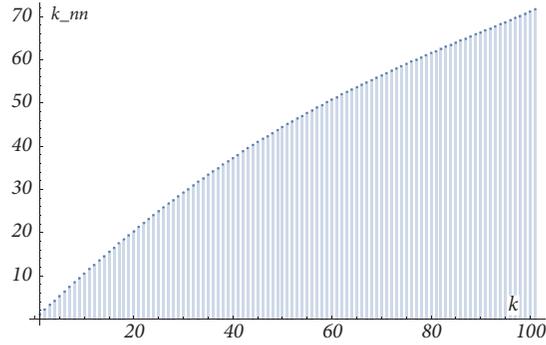


FIGURE 3: Function k_m for an assortative network as in Section 3.3 with $\alpha = 1/2$, $n = 101$ ($N \approx 10^4$).

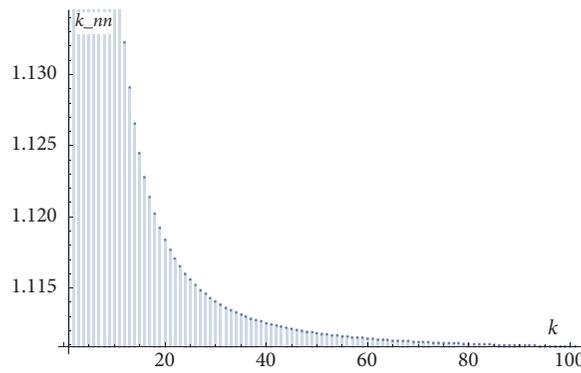


FIGURE 4: Function k_m for a disassortative network as in Section 3.4 with $\gamma = 4$, $n = 101$ ($N \approx 10^4$).

The size of these networks varies, being of the magnitude order of $N \approx 10^4$ to $N \approx 10^6$. The data mentioned above have been probably updated and extended, but in general it seems that scale-free disassortative networks with these values of γ tend to have an r coefficient that is much closer to 0 than to -1 . As we have seen, this also happens with disassortative scale-free networks mathematically defined, whose degree correlations are given by a procedure also introduced by Newman.

For ideal BA networks, Newman gave in [14] an asymptotic estimate of the r coefficient based on the correlations computed in [24] and concluded that r is negative but vanishes as $(\log N)^2/N$ in the large- N limit. Other authors share the view that BA networks are essentially uncorrelated [19, 25]. The smallness of their r coefficient is also confirmed by numerical simulations, in which the network is grown according to the preferential attachment scheme, even though the results of such simulations are affected by sizable statistical errors when N varies between $\approx 10^4$ and $\approx 10^6$.

In the recent paper [26] Fotouhi and Rabbat use their own expressions for the $p(l, k)$ correlations (expressing the fraction of links whose incident nodes have degrees l and k) to compute the asymptotic behavior of r according to an alternative expression given by Dorogovtsev and Mendes [27]. They conclude that the estimate $r \approx (\log N)^2/N$ given by Newman is not correct. They find $|r| \approx (\log n)^2/n$. In order to relate the network size N to the largest degree n ,

they use the relation $n \approx \sqrt{N}$ based on the continuum-like criterium $\int_n^\infty P(k)dk = N^{-1}$ [28]. So in the end $|r| \approx (\log N)^2/\sqrt{N}$. They check that this is indeed the correct behavior by performing new large-scale simulations.

We have computed r using the original definition by Newman [14], based on the matrix e_{jk} , and the exact $P(h | k)$ coefficients, with n up to 15000, which corresponds to $N \approx 225000000$ (Figure 5). We found a good agreement with the estimate of Fotouhi and Rabbat. Note, however, that although the “thermodynamic” limit of r for infinite N is zero, for finite N the value of r cannot be regarded as vanishingly small, in consideration of what we have seen above for disassortative scale-free networks with $\gamma = 3$.

4.2. The Function $k_m(k)$. An early estimate of the function $k_m(k)$ valid also for BA networks has been given by Vespignani and Pastor-Satorras in an Appendix of their book on the structure of the Internet [29]. Their formula is based in turn on estimates of the behavior of the conditional probability $P(k' | k)$ which hold for a growing scale-free network with a degree distribution $P(k) \propto k^{-\gamma}$. This formula reads

$$P(k' | k) \propto k'^{-(\gamma-1)} k^{-(3-\gamma)} \quad (24)$$

and holds under the condition $1 \ll k' \ll k$. Using the exact $P(k' | k)$ coefficients of Fotouhi and Rabbat it is possible to check the approximate validity of this formula.

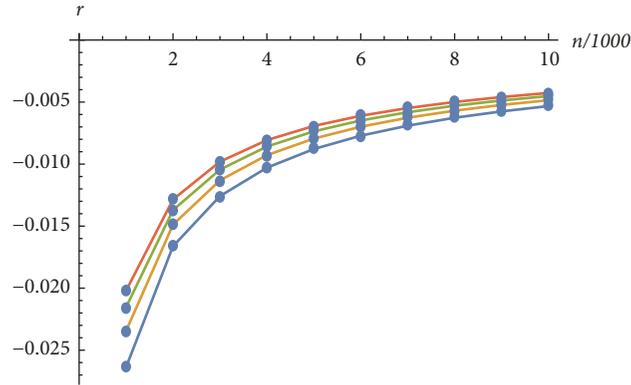


FIGURE 5: Newman assortativity coefficient r for BA networks with $\beta = 1, 2, 3, 4$ and largest degree $n = 1000, 2000, \dots, 10000$. The upper curve is for $\beta = 1$, and the curves for the values $\beta = 2, 3, 4$ follow, from the top to the bottom.

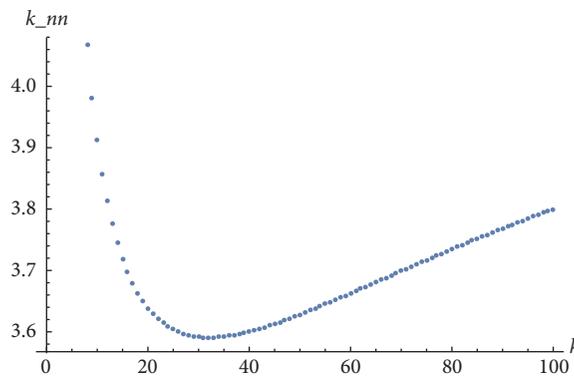


FIGURE 6: Function k_m for a BA network with $\beta = 1$, $n = 100$ (largest degree; the corresponding number of nodes is $N \approx 10^4$).

In [29] the expression (24) is then used to estimate $k_m(k)$. Since, for $\gamma = 3$, $P(k' | k)$ does not depend on k , the conclusion is that k_m is also independent from k . It is not entirely clear, however, how the condition $1 \ll k' \ll k$ can be fulfilled when the sum over k' is performed, and how the diverging factor $\sum_{k'=1}^n (1/k') \approx \ln(n) \approx (1/2) \ln(N)$ should be treated.

In their recent work [26], Fotouhi and Rabbat use their own results for the $P(k' | k)$ coefficients in order to estimate $k_m(k)$ in the limit of large k . They find $k_m(k) \approx \beta \ln(n)$ and cite a previous work [30] which gives the same result, in the form $k_m(k) \approx (1/2)\beta \ln(N)$ (we recall that $n \approx \sqrt{N}$). In this estimate we can observe, as compared to [29], the explicit presence of the parameter β and the diverging factor $\ln(n)$.

Concerning the Newman assortativity coefficient r , Fotouhi and Rabbat also make clear that even though $r \rightarrow 0$ when $N \rightarrow \infty$, this does not imply that the BA networks are uncorrelated, and in fact the relation $k_m(k) = \langle k^2 \rangle / \langle k \rangle$, valid for uncorrelated networks, does not apply to BA networks.

A direct numerical evaluation for finite n of the function $k_m(k)$ based on the exact $P(k' | k)$ shows further interesting features. As can be seen in Figures 6 and 7, the function is decreasing at small k and slightly and almost linearly increasing at large k . Note that this happens for networks of medium size ($n = 100$, $N \approx 10^4$) like those employed in

our numerical solution of the Bass diffusion equations and employed in [8], but also for larger networks (for instance, with $n = 1000$, $N \approx 10^6$, compare Figure 7). It seems that the “periphery” of the network, made of the least connected nodes, has a markedly disassortative character, while the hubs are slightly assortative. (On general grounds one would instead predict for finite scale-free networks a small structural disassortativity at large k [17].)

Since evidence on epidemic diffusion obtained so far indicates that generally the assortative character of a network lowers the epidemic threshold and the disassortative character tends to make diffusion faster once it has started, this “mixed” character of the finite BA networks appears to facilitate spreading phenomena and is consistent with our data on diffusion time (Section 3). Note in this connection that some real networks also turn out to be both assortative and disassortative, in different ranges of the degree k ; compare the examples in [17], Ch. 7.

Finally, we would like to relate our numerical findings for the function k_m to the general property (compare, for instance, [5])

$$\langle k^2 \rangle = \sum_{k=1}^n k P(k) K(k, n), \quad (25)$$

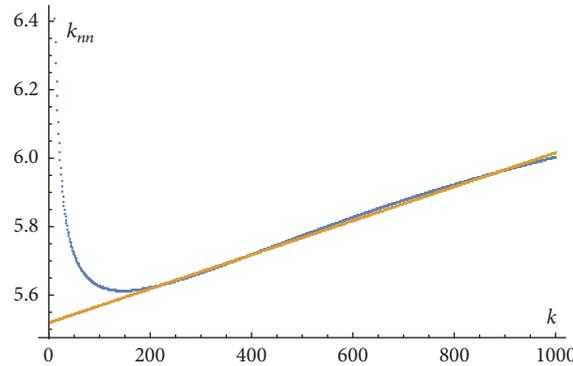


FIGURE 7: Function k_m for a BA network with $\beta = 1$, $n = 1000$ ($N \approx 10^6$). A linear fit for large k is also shown. The slope is approximately equal to $5 \cdot 10^{-4}$; a comparison with Figure 6 shows that the slope decreases with increasing n , but is still not negligible.

where for simplicity K denotes the function k_m and the dependence of this function on the maximum degree n is explicitly shown. For BA networks with $\beta = 1$ we have at large n on the l.h.s. of (25), from the definition of $\langle k^2 \rangle$,

$$\langle k^2 \rangle = \sum_{k=1}^n k^2 \frac{4}{k(k+1)(k+2)} = 4 \ln(n) + o(n), \quad (26)$$

where the symbol $o(n)$ denotes terms which are constant or do not diverge in n .

For the expression on the r.h.s. of (25) we obtain

$$\sum_{k=1}^n k P(k) K(k, n) = \sum_{k=1}^n \frac{4}{(k+1)(k+2)} K(k, n) \quad (27)$$

Equations (26) and (27) are compatible, in the sense that their diverging part in n is the same, in two cases: (1) if for large k we have $K(k, n) \approx a \ln(n)$, where a is a constant; this is true because in that case the sum on k is convergent; (2) if more generally $K(k, n) \approx a \ln(n) + b(n)k$; this is still true because also for the term $b(n)k$ the sum in k leads to a result proportional to $\ln(n)$. Case (2) appears to be what happens, according to Figures 6 and 7.

5. Validity of the Mean-Field Approximation

The validity of the heterogeneous mean-field approximation in epidemic diffusion models on networks has been discussed in [11, 31] and references. As far as the form of the equation is concerned, the Bass model is very similar to other epidemic models. A network which is completely characterized by the functions $P(k)$, $P(h | k)$, is called in [31] a “Markovian network”. Mean-field statistical mechanics on these networks has been treated in [32]. Recent applications can be found, for instance, in [33]. It has, in a certain sense, an axiomatic value and it is well defined in the statistical limit of infinite networks. Also the BA networks studied by Fotouhi and Rabbat are defined in this limit. In our work we are mainly interested into the properties of the networks, therefore when we speak of their properties with respect to diffusion, we make reference to this kind of idealized diffusion. Note that the mean-field approximation employed has a quite

rich structure (whence the denomination “heterogeneous”), since the population can be divided into a large number of connectivity classes, and the functions $P(k)$, $P(h | k)$ contain a lot of networking information about these classes.

It may therefore be of interest to check at least numerically if the mean-field approximation is a good approximation of the full Bass model in this case. For this purpose one may consider a network formulation of the model which employs the adjacency matrix $\{a_{ij}\}$. The coupled differential equations for a network with N nodes are in this formulation

$$\frac{dF_i(t)}{dt} = [1 - F_i(t)] \left(p + q \sum_{j=1}^N a_{ij} F_j(t) \right), \quad (28)$$

$$i = 1, \dots, N.$$

The solution $F_i(t)$ defines, for each node, an adoption level which grows as a function of time from $F_i = 0$ at $t = 0$, to $F_i = 1$ when t goes to infinity. The “bare” q parameter is renormalized with the average connectivity $\langle k \rangle$. This model can be seen as a first-moment closure of a stochastic model [23] and is actually more suitable to describe innovation diffusion among firms or organizations, under the assumption that inside each organization innovations diffuse in a gradual way. In fact, in forthcoming work we apply the model to real data concerning networks of enterprises. It is also useful, however, in order to illustrate the difference between a mean-field approach and a completely realistic approach. For the case of a BA network it is possible to generate the adjacency matrix with a random preferential attachment algorithm. This matrix is then fed to a code which solves the differential equations (28) and computes for each node the characteristic diffusion times. The results clearly show that in general different nodes with the same degree adopt at different rates (unlike in the mean-field approximation). It happens, for instance, as intuitively expected, that a node with degree 1 which is connected to a big central hub adopts more quickly than a node with the same degree that has been attached to the network periphery in the final phase of the growth. However, these fluctuations can be estimated numerically, and it turns out that mean-field approximations for the diffusion times agree with the average times within

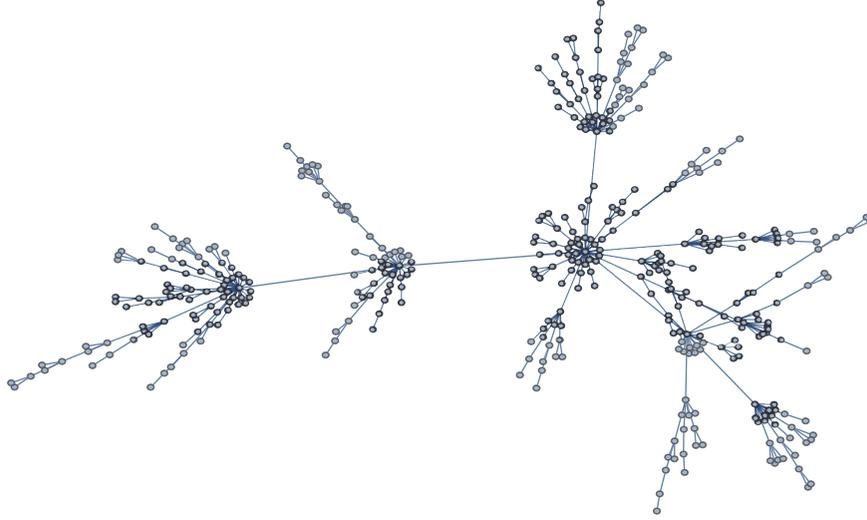


FIGURE 8: An example of a BA network with 400 nodes and $\beta = 1$, to which (28) has been applied for comparison to the corresponding mean-field approximation.

standard errors. For instance, in a realization of a BA network with $\beta = 1$, $N = 400$ (Figure 8), we obtain (considering the most numerous nodes, for which statistics is reliable) the following:

Degree 1: 266 nodes, with average diffusion peak time 5.2, $\sigma = 1.3$

Degree 2: 67 nodes, with average diffusion peak time 4.8, $\sigma = 1.1$

Degree 3: 33 nodes, with average diffusion peak time 4.3, $\sigma = 0.9$

With the mean-field model, with the same bare $p = 0.03$ and $q = 0.35$, we obtain the following:

Degree-1 nodes: diffusion peak time 5.2

Degree-2 nodes: diffusion peak time 4.9

Degree-3 nodes: diffusion peak time 4.6

The agreement is reasonable, also in consideration of the relatively small value of N . Concerning this, a final remark is in order: while random generation of BA networks mainly yields networks with maximum degree close to the most probable value $n = \sqrt{N}$, sizable fluctuations may occur as well. The network in Figure 8, for instance, has $N = 400$ and a hub with degree 35. We intentionally included this network in our checks, in order to make sure that the mean-field approximation can handle also these cases. This turns out to be true, provided the *real* value of the maximum degree is used (the mean-field peak times given above are obtained with $n = 35$). In fact, the full numerical solutions of (28) routinely show, as expected, that for fixed values of N the presence of large hubs speeds up the overall adoption. For an “average” network, the mean-field approximation works well just with $n = \sqrt{N}$.

6. Diffusion Times with Heterogeneous p Coefficients

In the Bass model the publicity coefficient p gives the adoption probability per unit time of an individual who has not yet adopted the innovation, independently from the fraction of current adopters. Therefore, it is not due to the word-of-mouth effect, but rather to an external stimulus (advertising) which is received in the same way by all individuals. The intensity of this stimulus is in turn proportional to the advertising expenses of the producer or seller of the innovation. One can therefore imagine the following alternative to the uniform dissemination of ads to all the population: the producer or seller invests for advertising to each individual in inverse proportion to the population of the individual’s link class, so as to speed up adoption in the most connected classes and keep the total expenses unchanged. In terms of the p_i coefficients in (16) this implies $p_i \propto 1/P(i)$, with normalization $\sum_{i=1}^n p_i P(i) = p$ [12].

As can be seen also from Figure 9, this has the effect of making the total diffusion faster or slower, depending on the kind of network. The differences observed in the presence of a homogeneous p coefficient (Figure 2) are now amplified. We observe that diffusion on BA networks is now always faster than on uncorrelated networks, independently from β . It is also remarkable how slow diffusion becomes on assortative networks in this case. A possible interpretation is the following: due to the low epidemic threshold of assortative networks, the targeted advertising on the hubs (which are well connected to each other) causes them to adopt very quickly, but this is not followed by easy diffusion on the whole network. The BA networks appear instead to take advantage, in the presence of advertising targeted on the hubs, both from their low threshold and from a stronger linking to the periphery.

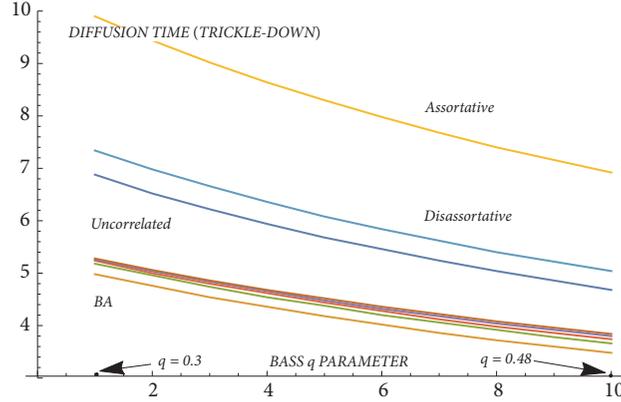


FIGURE 9: Time of the diffusion peak (in years) for the “trickle-down” modified Bass model on different kinds of scale-free networks with exponent $\gamma = 3$, as a function of the imitation coefficient q . All networks have maximum degree $n = 100$. The q coefficient varies in the range $0.3 - 0.48$, corresponding to a typical set of realistic values in innovation diffusion theory [20]. The publicity coefficient p_k of the link class k , with $k = 1, \dots, n$, varies in inverse proportion to $P(k)$ (Section 6). The low-lying group of lines corresponds to BA networks with $\beta = 1, 2, 3, 4, 5$ and the same assortativity coefficients as in Figure 2. The disassortative network is built with the Newman recipe (Section 3.4) with $d = 4$ and has $r = -0.084$. The assortative network is built with our recipe (Section 3.3), with $\alpha = 1/4$, and has $r = 0.827$.

7. Conclusions

In this work we have studied the assortativity properties of BA networks with maximum degree n which is finite, but large (up to $n \approx 10^4$, corresponding to a number of nodes $N \approx 10^8$). These properties were not known in detail until recently; a new decisive input has come from the exact calculation of the conditional probabilities $P(h | k)$ by Fotouhi and Rabbat [10]. We have done an explicit numerical evaluation of the average nearest neighbor degree function $k_{nn}(k)$, whose behavior turns out to be peculiar and unexpected, exhibiting a coexistence of assortative and disassortative correlations, for different intervals of the node degree k . These results have been compared with previous estimates, both concerning the function $k_{nn}(k)$ and the Newman assortativity index r .

The next step has been to write the Bass innovation diffusion model on BA networks, following the mean-field scheme we have recently introduced and tested on generic scale-free networks. This allows computing numerically the dependence of the diffusion peak time (for $n \approx 10^2$) from the model’s parameters and especially from the features of the network. We have thus compared the diffusion times on BA networks with those on uncorrelated, assortative and disassortative networks (the latter built, respectively, with our mathematical recipes (20) and (21) and with Newman’s recipe (22)).

The BA networks with small values of β (β is the number of child nodes in the preferential attachment scheme) turn out to have the shortest diffusion time, probably due to their mixed assortative/disassortative character: diffusion appears to start quite easily among the (slightly assortative) hubs and then to proceed quickly in the (disassortative) periphery of the network. This interpretation is confirmed by the fact that, in a modified “trickle-down” version of the model with enhanced publicity on the hubs, the anticipation effect of BA networks compared to the others is stronger and almost independent from β .

Concerning the dependence of the diffusion time on the values of the r coefficient, we have found a qualitative agreement with previous results by D’Agostino et al. [8]. We stress, however, that the use of two-point correlations in this analysis is entirely new.

In forthcoming work we shall analyse mathematically the construction and the properties of the mentioned families of assortative networks, from which we only have chosen here a few samples for comparison purposes, because the focus in this paper has been on BA networks.

Appendix

We show here that e_{kh} in (22) satisfies $0 \leq e_{kh} \leq 1$ for all $h, k = 0, \dots, n-1$. If $\gamma = 2 + d$ with $d > 0$, $e_{k-1, h-1}$ can be rewritten for any $h, k = 1, \dots, n$ as

$$e_{k-1, h-1} = \frac{(k^d + h^d) \sum_{j=1}^n (1/j^\gamma) - \sum_{j=1}^n (1/j^2)}{k^\gamma h^\gamma \sum_{j=1}^n (1/j^2) \left(\sum_{j=1}^n (1/j^\gamma) \right)^2}. \quad (\text{A.1})$$

The nonnegativity of $e_{k-1, h-1}$ is hence equivalent to that of the numerator of (A.1). This is, for all $h, k = 1, \dots, n$, greater than or equal to

$$2 \sum_{j=1}^n \frac{1}{j^\gamma} - \sum_{j=1}^n \frac{1}{j^2} \geq 2 - \frac{\pi}{6} > 0. \quad (\text{A.2})$$

To show that $e_{k-1, h-1} \leq 1$, we distinguish two cases:

(i) if $k = 1$ and $h = 1$, the expression on the r.h.s. in (A.1) is equal to

$$\begin{aligned}
& \frac{2 \sum_{j=1}^n (1/j^\gamma) - \sum_{j=1}^n (1/j^2)}{\sum_{j=1}^n (1/j^2) \left(\sum_{j=1}^n (1/j^\gamma) \right)^2} \\
& \leq \frac{2 \sum_{j=1}^n (1/j^2) - \sum_{j=1}^n (1/j^2)}{\sum_{j=1}^n (1/j^2) \left(\sum_{j=1}^n (1/j^\gamma) \right)^2} \leq \frac{1}{\left(\sum_{j=1}^n (1/j^\gamma) \right)^2} \quad (\text{A.3}) \\
& \leq 1;
\end{aligned}$$

(ii) otherwise, i.e., if at least one among k and h is greater than 1, the expression on the r.h.s. in (A.1) is no greater than

$$\begin{aligned}
& \frac{(k^d + h^d) \sum_{j=1}^n (1/j^{2+d})}{k^{2+d} h^{2+d} \sum_{j=1}^n (1/j^2) \left(\sum_{j=1}^n (1/j^{2+d}) \right)^2} \\
& \leq \frac{(k^d + h^d)}{k^d h^d} \frac{1}{k^2 h^2} \leq 1. \quad (\text{A.4})
\end{aligned}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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