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Research Article

Residual Symmetry Reduction and Consistent Riccati Expansion to a Nonlinear Evolution Equation

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The residual symmetry of a $(1+1)$-dimensional nonlinear evolution equation (NLEE) $u_t + u_{xxx} - 6u^2u_x + 6\lambda u_x = 0$ is obtained through Painlevé expansion. By introducing a new dependent variable, the residual symmetry is localized into Lie point symmetry in an enlarged system, and the related symmetry reduction solutions are obtained using the standard Lie symmetry method. Furthermore, the $(1+1)$-dimensional NLEE equation is proved to be integrable in the sense of having a consistent Riccati expansion (CRE), and some new Bäcklund transformations (BTs) are given. In addition, some explicitly expressed solutions including interaction solutions between soliton and cnoidal waves are derived from these BTs.

1. Introduction

In nonlinear science, the study of nonlinear equations plays an important role in analyzing related complex phenomenon, which exists in the fields of fluid dynamics, plasma, optics, and so on [1]. In the past few decades, many effective methods, including Hirota's bilinear method [2], Darbux transformation [3, 4] and Bäcklund transformation, Lie symmetry analysis [5, 6], and inverse scattering transformation [7], have been proposed and developed to investigate abundant properties of nonlinear equations. Among these methods, symmetry analysis plays an important role in simplifying and even completely solving nonlinear equations. For many integrable systems, the standard Lie symmetry method can be used to obtain their Lie symmetry group and symmetry reduction solutions. In addition, a finite transformation group related to a symmetry can also be obtained by using Lie's first theorem.

Traditionally, finite transformation related to a nonlocal symmetry cannot be obtained directly in the same way as those of Lie point symmetries. To concur this difficulty, Cheng et al. [8] proposed localizing a nonlocal symmetry in an enlarged nonlinear system by introducing some new dependent variables to the original system, provided all the Lie point symmetries are closed in the enlarged system. Since then, a lot of works have been done on many important nonlinear systems [9–13]. They prove that this localization method is very efficient in obtaining new Bäcklund transformations (BTs) and also new symmetry reduction solutions related to a nonlocal symmetry in an enlarged system. Especially, for many integrable systems, interaction solutions between a soliton and nonlinear periodic waves could be obtained in this way.

There exist many different ways to obtain nonlocal symmetries, including potential symmetry [14], Lie–Bäcklund symmetries [15], inverse recursion operators [16, 17], the conformal-invariant form [18], Darboux transformation [19], Bäcklund transformation (BT), and Lax pair. It is found in recent years that, for many nonlinear systems, the coefficient of negative first power of a singular manifold in a truncated Painlevé expansion is readily a nonlocal symmetry of the equation [20], which is called residual symmetry. Compared with other methods for obtaining nonlocal symmetries, the method for obtaining a residual symmetry is very simple, and plenty of studies have been carried out by localizing a residual symmetry into a Lie point symmetry [19, 21, 22]. To obtain more abundant interaction solutions, Lou generalized Riccati expansion method and proposed a new concept of integrability in the sense of having consistent Riccati expansion (CRE) [23]. By applying the CRE method, many new BTs and interaction solutions for various nonlinear systems are obtained [12, 24–29].

In this paper, by using residual symmetry localization method and CRE method, we investigate the following $(1+1)$-dimensional nonlinear evolution equation (NLEE):

$$u_t + u_{xxx} - 6u^2 u_x + 6\lambda u_x = 0, \tag{1}$$

which includes the modified KdV equation as a special case by setting $\lambda = 0$. In Reference [30], NLEE (1) is derived with a Bäcklund transformation relating solutions of KdV equation and solutions of equation (1). It reveals that the vacuum state exists in the KdV equation as well as in NLEE equation (1), and the relationship between vacuum parameter of the set of KdV solutions and the vacuum parameter of equation (1) is found [30, 31]. A series of analytical solutions of NLEE (1) are also given in [32].

The paper is organized as follows. In Section 1, the residual symmetry of NLEE (1) is derived from the truncated Painlevé expansion and then localized into a Lie point symmetry by introducing a new dependent variable to enlarge the NLEE. On this basis, the finite transformation related to the residual symmetry is also obtained by applying Lie's first theorem. In Section 2, the general form of Lie point symmetry group as well as symmetry reduction solutions of the enlarged NLEE is obtained by using the standard Lie symmetry method. In Section 3, NLEE (1) is proved to be CRE integrable, and some new BTs are obtained. By applying the CTE method, some concrete explicitly expressed solutions of the NLEE are given, which include the interaction solutions between solitons and background cnoidal waves. The last section contains a summary.

2. Localization of Residual Symmetry

By balancing the dispersion term and nonlinear term, the truncated Painlevé expansion of equation (1) is

$$u = \frac{u_0}{\phi} + u_1, \tag{2}$$

where ϕ is the singular manifold and u_0 and u_1 are functions of x and t to be determined later. Substituting (2) into (1) and making the coefficients of different powers of ϕ to zero, we have

$$u_0 = \phi_x, \tag{3}$$

$$u_1 = -\frac{\phi_{xx}}{2\phi_x}, \tag{4}$$

where ϕ satisfies the Schwartzian form of (1)

$$C + K + 6\lambda = 0, \tag{5}$$

with $C = (\phi_{xxx}/\phi_x) - (3/2)(\phi_{xx}^2/\phi_x^2)$ and $K = \phi_t/\phi_x$. By substituting (3) and (4) into (2), the following theorem is readily obtained.

Theorem 1. *If ϕ is a solution of the Schwartzian equation (5), then*

$$u = \frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}, \tag{6}$$

is a solution of equation (1).

As we know, any Schwartzian equation like (5) is form-invariant under Möbious transformation

$$\phi \longrightarrow \frac{a_1\phi + b_1}{a_2\phi + b_2}, \quad a_1 a_2 \neq b_1 b_2, \tag{7}$$

which means equation (5) have three symmetries:

$$\begin{aligned} \sigma_\phi &= d_1, \\ \sigma_\phi &= d_2\phi, \\ \sigma_\phi &= d_3\phi^2, \end{aligned} \tag{8}$$

with arbitrary constants d_1, d_2, and d_3. It is interesting that the residue u_0 in expansion (2) is just a symmetry of the NLEE, which can be verified by substituting it with equations (4) and (5) into the linearized equation of (1). Obviously, the residual symmetry $\sigma_u = u_0$ is linked to the symmetry of (8) by the linearized equation of (4).

To get the finite transformation corresponding to the residual symmetry $\sigma_u = \phi_x$, we have to localize it into a Lie point symmetry first. To this end, we introduce a new dependent variable as

$$g \equiv \phi_x \tag{9}$$

to enlarge the original equation (1).

The linearized equations of the enlarged system (1), (5), and (9) are

$$\sigma_{u,t} + \sigma_{u,xxx} - 12uu_x\sigma_u - 6u^2\sigma_{u,x} + 6\lambda\sigma_{u,x,x} = 0, \tag{10a}$$

$$\begin{aligned} &\phi_{xxx}\sigma_{\phi,x} + \sigma_{\phi,xxx}\phi_x - 3\phi_{xx}\sigma_{\phi,xx} + \sigma_{\phi,x}\phi_t \\ &\quad + \left(12\sigma_{\phi,x}\lambda + \sigma_{\phi,t}\right)\phi_x = 0, \end{aligned} \tag{10b}$$

$$\sigma_{\phi,x} - \sigma_g = 0. \tag{10c}$$

When we fix $\sigma_\phi = -\phi^2$, the equations of (10a)–(10c) have a simple solution

$$\sigma_u = g, \tag{11a}$$

$$\sigma_g = -2g\phi, \tag{11b}$$

$$\sigma_\phi = -\phi^2, \tag{11c}$$

which means the residual symmetry is localized in the enlarged system.

By applying Lie's first theorem to the initial value problem of the symmetry (11a) and (11b), i.e.,

$$\frac{d\widehat{u}(\varepsilon)}{d\varepsilon} = \widehat{g}(\varepsilon), \quad \widehat{u}(0) = u,$$
$$\frac{d\widehat{g}(\varepsilon)}{d\varepsilon} = -2\widehat{\phi}(\varepsilon)\widehat{g}(\varepsilon), \quad \widehat{g}(0) = g, \tag{12}$$
$$\frac{d\widehat{\phi}(\varepsilon)}{d\varepsilon} = -\widehat{\phi}(\varepsilon)^2, \quad \widehat{\phi}(0) = \phi,$$

we get the following Bäcklund transformation.

Theorem 2. *If $\{u, g, \phi\}$ is a solution of the prolonged system (1), (5), and (9), then so is $\{\widehat{u}, \widehat{g}, \widehat{\phi}\}$ with*

$$\widehat{u} = u + \frac{\varepsilon g}{\varepsilon\phi + 1}, \tag{13a}$$

$$\widehat{g} = \frac{g}{(\varepsilon\phi + 1)^2}, \tag{13b}$$

$$\widehat{\phi} = \frac{\phi}{\varepsilon\phi + 1}, \tag{13c}$$

with an arbitrary group parameter ε.

3. Residual Symmetry Reduction Solutions of Equation (1)

The general form of a Lie point symmetry of the enlarged NLEE system (1), (5), and (9) can be written in the form as follows:

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + G\frac{\partial}{\partial g} + \Phi\frac{\partial}{\partial \phi}, \tag{14}$$

which means the system is invariant under the following transformation:

$$\{x, t, u, g, \phi\} \longrightarrow \{x + \varepsilon X, t + \varepsilon T, u + \varepsilon U, g + \varepsilon G, \phi + \varepsilon\Phi\}, \tag{15}$$

with the infinitesimal parameter ε. Equivalently, the symmetry in the form (13a)-(13b) can be written as a function form as follows:

$$\sigma_u = Xu_x + Tu_t - U, \tag{16a}$$

$$\sigma_g = Xg_x + Tg_t - G, \tag{16b}$$

$$\sigma_\phi = X\phi_x + T\phi_t - \Phi. \tag{16c}$$

Substituting equation (16a)–(16c) with the enlarged NLEE system into equation (10a)–(10c) and vanishing all the derivatives of dependent variables $u, g,$ and ϕ, over-determined linear equations for the infinitesimals $X, T, U, G,$ and Φ are obtained. After calculation by computer using the software *Maple*, we get the result

$$\begin{aligned} X &= 4c_1\lambda t + \frac{c_1}{3}x + c_4, \\ T &= c_1 t + c_2, \\ U &= -\frac{c_1}{3}u + c_3 g, \\ G &= -\frac{1}{3}g\left(6c_3\phi + c_1 - 3c_5\right), \\ \Phi &= -c_3\phi^2 + c_5\phi + c_6, \end{aligned} \tag{17}$$

with arbitrary constants c_i $(i = 1, \ldots, 6)$. It is obvious that the symmetry for $u, g,$ and ϕ in equation (17) contains the symmetry of (11a)–(11c) as a special case.

In consideration of equation (17), the symmetries in (16a)–(16c) can be written as

$$\begin{aligned} \sigma_u &= \left(4c_1\lambda t + \frac{c_1}{3}x + c_4\right)u_x + (c_1 t + c_2)u_t + \frac{c_1}{3}u - c_3 g, \\ \sigma_g &= \left(4c_1\lambda t + \frac{c_1}{3}x + c_4\right)g_x + (c_1 t + c_2)g_t \\ &\quad + \frac{1}{3}g\left(6c_3\phi + c_1 - 3c_5\right), \\ \sigma_\phi &= \left(4c_1\lambda t + \frac{c_1}{3}x + c_4\right)\phi_x + (c_1 t + c_2)\phi_t + c_3\phi^2 - c_5\phi - c_6. \end{aligned} \tag{18}$$

The group invariant solutions of the enlarged NLEE system can be obtained by applying the symmetry constraints $\sigma_u = \sigma_g = \sigma_\phi = 0$ in equation (19), which is equivalent to solving the characteristic equation

$$\begin{aligned} \frac{dx}{4c_1\lambda t + (c_1/3)x + c_4} &= \frac{dt}{c_1 t + c_2} = \frac{du}{-(c_1/3)u + c_3 g} \\ &= \frac{dg}{-(1/3)g\left(6c_3\phi + c_1 - 3c_5\right)} \\ &= \frac{d\phi}{-c_3\phi^2 + c_5\phi + c_6}. \end{aligned} \tag{19}$$

Without loss of generality, we consider symmetry reduction solutions of the enlarged NLEE system in the following two cases.

Case 1 ($c_i \neq 0\,(i = 1, \ldots, 6)$).

After solving equation (19), we get the symmetry reduction solutions of the enlarged NLEE system (1), (5), and (9) as

$$\phi = \frac{\tanh(\Delta_1(c_1\Phi + \ln(\Delta))/2c_1)\Delta_1 + c_5}{2c_3},$$
$$\Delta = c_1 t + c_2, \quad (20)$$
$$\Delta_1 = \sqrt{4c_3c_6 + c_5^2},$$

$$g = -\frac{2G}{\Delta^{1/3}[\cosh(\Delta_1(c_1\Phi + \ln(\Delta))/c_1) + 1]}, \quad (21)$$

$$u = \frac{U}{\Delta^{1/3}} + \frac{4c_3G}{\Delta_1[\exp(\Delta_1\Phi)\Delta^{\Delta_1/c_1} + 1]\Delta^{1/3}}, \quad (22)$$

where U, G, and Φ are group invariant functions of a group invariant variable $\xi = -6c_1\lambda t - c_1x + 18c_2\lambda - 3c_4/(c_1 t + c_2)^{1/3}c_1$.

The symmetry reduction equations for U, G, and Φ can be obtained by substituting equations (20)–(22) into the enlarged NLEE system (1), (5), and (9). The results are

$$G = -\frac{\Delta_1^2\Phi_\xi}{4c_3}, \quad (23)$$

$$U = \frac{\Delta_1\Phi_\xi^2 - \Phi_{\xi\xi}}{2\Phi_\xi}, \quad (24)$$

where Φ satisfies the following symmetry reduction equation:

$$-3\Delta_1^2\Phi_\xi^4 - 2c_1\xi\Phi_\xi^2 + 6(1 + \Phi_{\xi\xi\xi})\Phi_\xi - 9\Phi_{\xi\xi}^2 = 0. \quad (25)$$

It is obvious that once any solution of equation (25) is given, the solution of the NLEE can be obtained by substituting it with equations (23) and (24) into equation (22). To give a concrete example, we take a simple solution for (25) under the condition $\Delta_1 = c_1/3$ as

$$\Phi = \frac{d_2c_1 + 3\ln(d_1\xi)}{c_1}, \quad (26)$$

which leads to a simple solution for NLEE (1)

$$u = \frac{\exp(1/3d_2c_1)d_1}{\exp(1/3d_2c_1)d_1\xi\Delta^{1/3} + 1}, \quad (27)$$

with arbitrary constants d_1 and d_2.

Case 2 ($c_1 = 0$ and $c_i \neq 0\,(i = 2, \ldots, 6)$).

In this case, similar to case 1, the symmetry reduction solutions of the enlarged NLEE system (1), (5), and (9) are

$$\phi = \frac{\tanh(\Delta_1(\phi' + t)/2c_2)\Delta_1 + c_5}{2c_3},$$
$$g = -\frac{2g'}{\cosh(\Delta_1(\phi' + t)/c_2) + 1}, \quad (28)$$
$$u = u' - \frac{2c_3g'\tanh(\Delta_1(\phi' + t)/2c_2)}{\Delta_1},$$

with u', g', and ϕ' being group invariant functions of group invariant variable $x' = x - c_4/c_2t$.

The corresponding symmetry reduction equations for u', g', and ϕ' are

$$g' = -\frac{\Delta_1^2\phi'_{x'}}{4c_2c_3}, \quad (29)$$

$$u' = -\frac{\phi'_{x'x'}}{2\phi'_{x'}}, \quad (30)$$

$$\Delta_1^2\phi_{x'}^{\prime 4} - 2c_2(6c_2\lambda - c_4)\phi_{x'}^{\prime 2} - 2c_2^2(\phi'_{x'x'x'} + 1)\phi'_{x'} + 3c_2^2\phi_{x'x'}^{\prime 2} = 0. \quad (31)$$

4. CRE Solvability and Interaction Solutions

4.1. CRE Integrable. By leading order analysis, the Riccati expansion of NLEE (1) is

$$u = v_0 + v_1R(w),$$
$$w = w(x, y, t), \quad (32)$$

where v_0 and v_1 are functions of (x, t) and $R(w)$ is a solution of the Riccati equation

$$R_w = a_0 + a_1R + a_2R^2. \quad (33)$$

Substituting equations (32) with (33) into equation (1) and vanishing all the coefficients of different powers of $R(w)$, we get

$$v_0 = \frac{a_1w_x^2 + w_{xx}}{2w_x},$$
$$v_1 = a_2w_x, \quad (34)$$

and three different equations of w. Fortunately, these three equations are consistent with each other, one of which is

$$\delta w_x^2 + 2(K' + C' + 6\lambda) = 0, \quad (35)$$

with $C' = w_{xxx}/w_x - (3/2)(w_{xx}^2/w_x^2)$, $K' = w_t/w_x$, and $\delta = 4a_0a_2 - a_1^2$.

The consistency of different equations of w means that NLEE (1) is integrable under the meaning of having a consistent Riccati expansion. In summary, we have the following theorem.

Theorem 3. *If w is a solution of*

$$\delta w_x^2 + 2(K' + C' + 6\lambda) = 0, \quad (36)$$

then

$$u = a_2w_xR(w) + \frac{a_1w_x^2 + w_{xx}}{2w_x}, \quad (37)$$

is a solution of NLEE (1), with $R = R(w)$ being a solution of the Riccati equation (33).

4.2. Consistent tanh-Function Expansion. When we take a special solution of the Riccati equation (33) as $R(w) = \tanh(w)$, the consistent Riccati expansion (31) reduces to

$$u = u_1' \tanh(w) + u_0', \tag{38}$$

which is called consistent tanh expansion (CTE).

Following the same logic as in the CRE case, we get the following nonauto BT.

Theorem 4. *If w satisfies the following equation:*

$$2w_x^2 - K - C - 6\lambda = 0, \tag{39}$$

then

$$u = w_x \tanh(w) - \frac{w_{xx}}{2w_x} \tag{40}$$

is a solution of NLEE (1).

To give some concrete exact solutions of NLEE (1), we first take the form of w in (39) as

$$w = k_0 x + \omega_0 t + g, \tag{41}$$

where g is a arbitrary function of x and t, k_0, and ω_0 are arbitrary constants. Three special cases of (41) are listed as follows.

Case 3. We take a simple solution of equation (39) as

$$w = 2k^3 t - 6k\lambda t + kx + d, \tag{42}$$

with k and d being arbitrary constants. By using Theorem 4, we obtain a kink soliton solution for NLEE (1):

$$u = k \tanh\left(2k^3 t - 6k\lambda t + kx + d\right). \tag{43}$$

Case 4. We further constrain the form of (41) as

$$w = k_0 x + \omega_0 t + W(X), \quad X = k_1 x + \omega_1 t, \tag{44}$$

where k_0, ω_0, k_1, and ω_1 are arbitrary constants. By substituting (44) into (39), it is interesting to find that $W_1(X) \equiv W(X)_X$ satisfies the following elliptic function equation:

$$W_{1X}^2 = C_0 + C_1 W_1 + C_2 W_1^2 + C_3 W_1^3 + C_4 W_1^4, \tag{45}$$

with

$$\begin{aligned} C_0 &= \frac{\left(C_1 k_1^3 - C_2 k_0 k_1^2 + C_3 k_0^2 k_1 - 4k_0^3\right) k_0}{k_1^4}, \\ C_4 &= 4, \\ \omega_0 &= C_1 k_1^3 - \frac{3}{2} C_2 k_0 k_1^2 + \frac{3}{2} C_3 k_0^2 k_1 - 4k_0^3 - 6k_0 \lambda, \\ \omega_1 &= \frac{k_1\left(C_2 k_1^2 - 3C_3 k_0 k_1 + 24k_0^2 - 12\lambda\right)}{2}, \end{aligned} \tag{46}$$

with arbitrary constants C_1, C_2, and C_3. By Theorem 4, equation (44) leads to a solution of NLEE (1) as

$$u = (k_0 + W_1 k_1) \tanh(k_0 x + \omega_0 t + W) - \frac{k_1^2}{2(k_0 + W_1 k_1)} W_{1,X}, \tag{47}$$

which describes an interaction mode between soliton and elliptic waves. To illustrate this point more clearly, we take a special Jacobi elliptic solution of equation (45) as

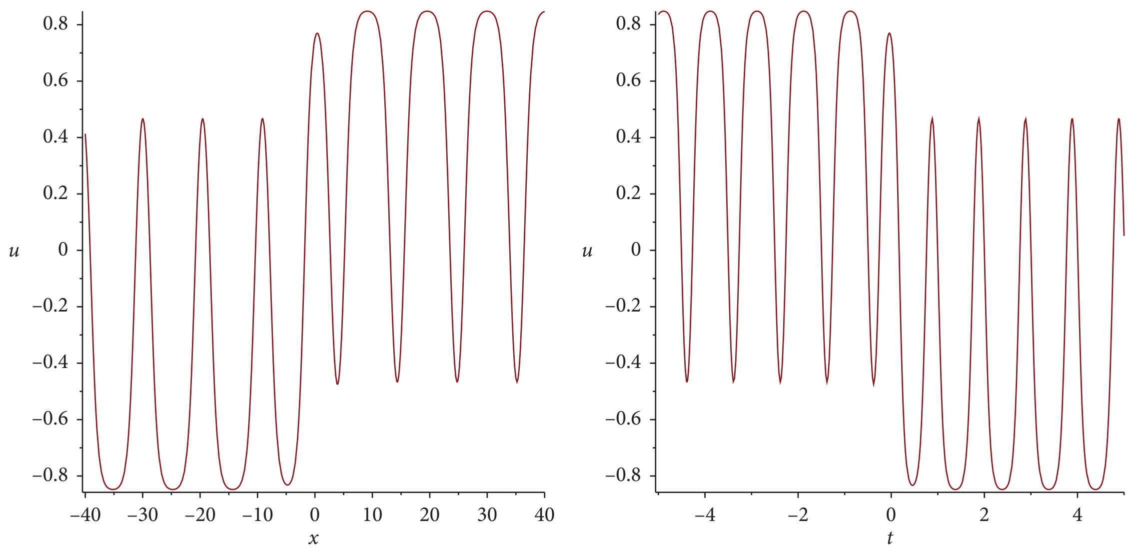
$$W_1 = \mu_0 + \mu_1 \operatorname{sn}(b_1 X, p), \tag{48}$$

with arbitrary constants μ_0 and μ_1. By substituting equation (48) into equation (45) and vanishing coefficients of different powers of sn, we get

$$\begin{aligned} C_1 &= \frac{C_3}{128}\left(16C_2 - C_3^2\right), \\ C_3 &= 8m + 16\frac{k_0}{k_1}, \\ \mu_0 &= -\frac{C_3}{16}, \\ \mu_1 &= \frac{\sqrt{5C_3^2 k_1^2 - 64C_2 k_1^2 + 32C_3 k_0 k_1 - 256k_0^2}}{16k_1}, \\ b_1 &= -\frac{C_3 k_1 - 16k_0}{8k_1}, \\ p &= \frac{\sqrt{5C_3^2 k_1^2 - 64C_2 k_1^2 + 32C_3 k_0 k_1 - 256k_0^2}}{C_3 k_1 - 16k_0}. \end{aligned} \tag{49}$$

Figure 1 displays the interaction solution of (47) with equations (46), (48), and (49) and the parameters are fixed by

$$\begin{aligned} C_0 &= \frac{11607}{102400}, \\ C_1 &= -\frac{8669}{51200}, \\ C_2 &= -\frac{8269}{6400}, \\ C_3 &= 1, \\ b_1 &= \frac{7}{8}, \\ \mu_0 &= -\frac{1}{16}, \\ \mu_1 &= -\frac{63}{160}, \\ k_0 &= \frac{1}{2}, \\ k_1 &= 1, \\ \lambda &= 2, \\ \omega_0 &= -\frac{54531}{10240}, \\ \omega_1 &= -\frac{133069}{12800}, \\ p &= \frac{9}{10}. \end{aligned} \tag{50}$$



(a)

(b)

Figure 1: The interaction solution (46) with parameters being fixed by (49): (a) two-dimensional plot with $t = 0$; (b) two-dimensional plot with $x = 0$.

Figures 1(a) and 1(b) describe the interaction structure between a kink soliton and background cnoidal waves in one-dimensional variables of x and t, respectively, which have similar structures.

Case 5. We take the form of equation (41) as

$$w = k_1 x + l_1 t + a_1 E_\pi\left(\operatorname{sn}\left(k_2 x + l_2 t, p\right), q, p\right), \tag{51}$$

where E_π is the third type of incomplete elliptic integral and $k_1, l_1, a_1, k_2, l_2, p$, and q are all arbitrary constants. Substituting equation (51) into equation (39) and vanishing coefficients of different powers of sn function, we get several types of solutions, two of which are

$$\begin{aligned} a_1 &= \sqrt{-\frac{p^2 q - p^2 - q^2 + q}{q}}, \\ k_1 &= 0, \\ l_1 &= -\frac{4\sqrt{-\left(p^2 q - p^2 - q^2 + q/q\right) k_2^3 p^2}}{q}, \\ l_2 &= -\frac{2\left(k_2^2 p^2 q - 3k_2^2 p^2 + k_2^2 q + 3\lambda q\right) k_2}{q}, \end{aligned} \tag{52}$$

$$\begin{aligned} l_1 &= 2k_1^3 + 6a_1 k_1^2 k_2 + \left(6a_1^2 k_2^2 - 6\lambda\right) k_1 \\ &\quad + a_1\left(2a_1^2 k_2^3 - 6k_2\lambda - l_2\right), \\ q &= 0. \end{aligned} \tag{53}$$

Figures 2 and 3 display the solution (40) with equations (51) and (52) in three dimensions and two dimensions, respectively, and the parameters are fixed by

$$\begin{aligned} p &= \frac{9}{10}, \\ q &= \frac{1}{2}, \\ k_2 &= 1, \\ \lambda &= \frac{1}{2}, \\ a_1 &= \frac{\sqrt{31}}{10}, \\ k_1 &= 0, \\ l_1 &= -\frac{81\sqrt{31}}{125}, \\ l_2 &= \frac{31}{10}. \end{aligned} \tag{54}$$

Figure 2 shows that the kink soliton can be seen as being composed by cnoidal waves, which can be seen more clearly by the density plot of Figure 2(b). As for Figures 3(a) and 3(b), they describe this solution one dimensionally with $t = 1/2$ and $x = 0$, respectively.

When $q = 0$, the third-type incomplete elliptic integral in equation (50) becomes the first type. Figure 4 displays the

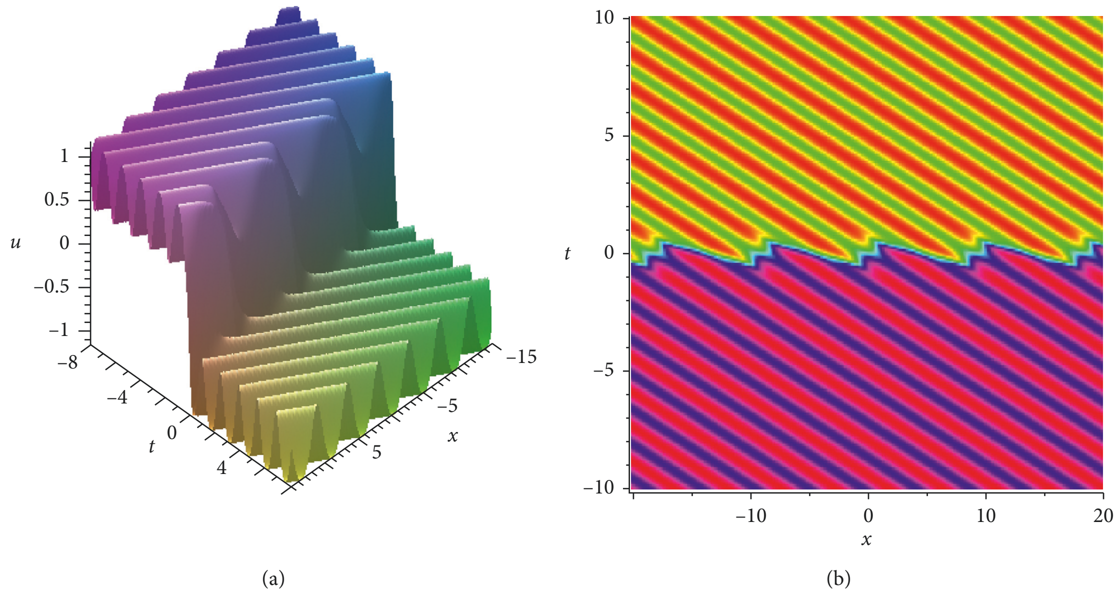


Figure 2: The interaction solution (40) with equations (51) and (52) and the parameters are fixed by (54): (a) the three-dimensional plot; (b) the corresponding density plot.

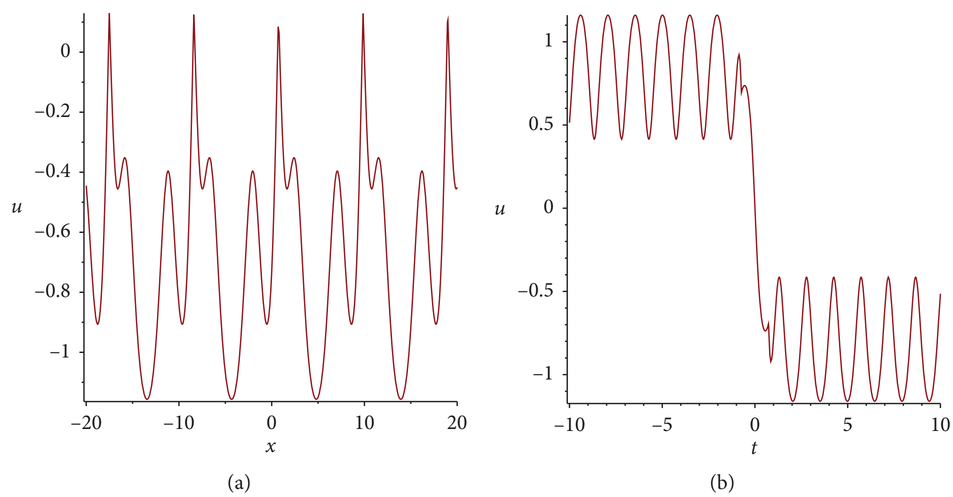


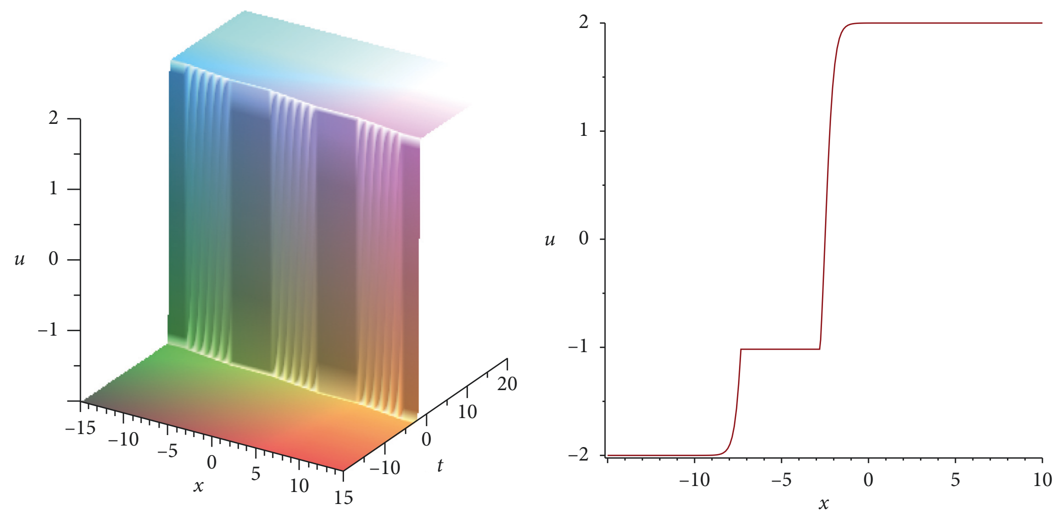
Figure 3: The interaction solution (39) with equations (51) and (52) and the parameters are fixed by (54): (a) the two-dimensional plot with $t = 1/2$; (b) the two-dimensional plot with $x = 0$.

interaction solution of (40) with equations (51) and (53). The parameters are fixed by

$$\begin{aligned} a_1 &= 1, \\ k_1 &= 1, \\ k_2 &= 1, \\ l_1 &= 9, \\ l_2 &= 1, \\ \lambda &= \frac{1}{2}, \\ p &= \frac{9}{10}, \\ q &= 0. \end{aligned} \tag{55}$$

5. Conclusion

In summary, the NLEE is investigated through residual symmetry and the CRE method, respectively. The residual symmetry is obtained through the truncated Painlevé expansion and localized into a Lie point symmetry by enlarging the original NLEE into a new system. By applying the standard symmetry method, the general form of a Lie point symmetry of the enlarged NLEE as well as the corresponding symmetry reduction solutions is obtained, which could be used to describe the interaction mode between soliton and nonlinear waves. The NLEE is proved to be CRE integrable, and some new BTs are derived from this property. As far as we know, the literatures giving interaction solutions between soliton and cnoidal waves are mainly focused on $(2+1)$-dimensional systems. As for $(1+1)$-dimensional case, this



(a) (b)

Figure 4: The interaction solution (39) with equations (50) (in Section 4.2) and the parameters are fixed by (54): (a) the three-dimensional plot; (b) the two-dimensional plot with $t = 1/2$.

kind of soliton-cnoidal wave interaction solutions are hard to give in an explicitly expression form (see, e.g. [10]). Fortunately, in this paper, two kinds of soliton-cnoidal wave solutions for equation (1) are obtained with a detailed analysis.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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