# Hopf Bifurcation and Chaos of a Delayed Finance System 

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In this paper, a finance system with delay is considered. By analyzing the corresponding characteristic equations, the local stability of equilibrium is established. The existence of Hopf bifurcations at the equilibrium is also discussed. Furthermore, formulas for determining the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are derived by applying the normal form method and center manifold theorem. Finally, numerical simulation results are presented to validate the theoretical analysis. Numerical simulation results show that delay can lead a stable system into a chaotic state.

## 1. Introduction

Ever since economist Stutzer first revealed the chaotic phenomena in an economic system in 1980, chaotic dynamics which supports an endogenous explanation of the complexity observed in economic series has become a hot topic, and many economic models have been proposed, e.g., Goodwin's nonlinear accelerator model [1, 2], the van der Pol model on business cycle [3-5], the IS-LM model [6, 7], and nonlinear dynamical model on finance system [8-11]. In [8, 9], Ma and Chen proposed a simplified financial model as follows:

$$
\left\{\begin{array}{l}
\dot{x}=(y-a) x+z,  \tag{1}\\
\dot{y}=1-b y-x^{2} \\
\dot{z}=-x-c z
\end{array}\right.
$$

where $x$ is the interest rate, $y$ is the investment demand, $z$ is the price index, $a>0$ denotes saving amount, $b>0$ denotes cost per investment, and $c>0$ denotes elasticity of demand of commercial markets. The variation of $x$ is not only influenced by the surplus between investment and saving but also structurally adjusted by the price. The changing rate of $y$ is proportional to the rate of investment and inversely proportional to the cost of investment and interest rate. The variation of $z$ is influenced by the contradiction between supply
and demand in commercial markets and affected by the inflation rates. The authors studied the focus on bifurcation and topological horseshoe of chaotic financial system (1). Some delay feedback control strategies [12-15] have also been considered for system (1).

It is well known that delays are extensively encountered in many fields such as biology [16-18], chemistry [19, 20], and engineering [21-23]. Also, delay is inevitable in economic activities. For example, changes in the money supply do not cause immediate changes in the economy; there is always a lag period. The production cycle has both long and short phases. Price change always has a delay. Therefore, delay differential equations (DDEs) support a realistic economic mathematical modeling than ordinary differential equations (ODEs) $[6,7]$.

In [24], Wang et al. proposed a delayed fractional order financial system as follows:

$$
\left\{\begin{array}{l}
D_{t}^{\alpha_{1}} x=(y(t-\tau)-a) x+z  \tag{2}\\
D_{t}^{\alpha_{2}} y=1-b y-x^{2}(t-\tau) \\
D_{t}^{\alpha_{3}} z=-x(t-\tau)-c z
\end{array}\right.
$$

where $\tau \geq 0$ is the time delay. The authors studied its dynamic behaviors, such as single-periodic, multiple-periodic, and chaotic motions.

Based on [24], Chen et al. [25] studied the following delayed financial system:

$$
\left\{\begin{array}{l}
\dot{x}=(y(t-\tau)-\mathrm{a}) x+z  \tag{3}\\
\dot{y}=1-b y-x^{2}(t-\tau) \\
\dot{z}=-x(t-\tau)-c z
\end{array}\right.
$$

The authors have studied the asymptotic stability and Hopf bifurcations of the unique equilibrium, and the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions were also considered.

According to the above discussions, we consider a delayed finance system as follows:

$$
\left\{\begin{array}{l}
\dot{x}=(y-a) x+z(t-\tau)  \tag{4}\\
\dot{y}=1-b y-x^{2} \\
\dot{z}=-x-c z
\end{array}\right.
$$

where $\tau$ denotes price change delay, for price change does not immediately affect the interest rate, and it often has a lag period.

The main purpose of this paper is to investigate the stability and Hopf bifurcation for system (4) with delay $\tau$ as the bifurcation parameter.

The structure of this paper is arranged as follows. In Section 2, we study the local stability and the existence of Hopf bifurcation. In Section 3, we give the formula determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. Finally, to support our theoretical predictions, some numerical simulations are given which support the analysis of Sections 2-3.

## 2. Stability and Hopf Bifurcation

2.1. The Existence of Equilibria. In this section, we consider the stability and Hopf bifurcation of the equilibria of system (4). First, we find all possible equilibria of system (4). We make the following hypothesis:
(H1)

$$
\begin{equation*}
c-a b c-b>0 \tag{5}
\end{equation*}
$$

According to system (4), equilibria should satisfy

$$
\left\{\begin{array}{l}
(y-a) x+z=0  \tag{6}\\
1-b y-x^{2}=0 \\
-x-c z=0
\end{array}\right.
$$

Obviously, system (4) has an equilibrium $P_{0}=(0,1 / b, 0)$. For other equilibria, solving for the second and third equations of (6), we have

$$
\begin{align*}
& y=\frac{1-x^{2}}{b}  \tag{7}\\
& z=-\frac{x}{c}
\end{align*}
$$

Substitute (7) into the first equation of (6), we obtain

$$
\begin{equation*}
x= \pm \sqrt{\frac{c-a b c-b}{c}} \tag{8}
\end{equation*}
$$

So, we have following results.
Lemma 1. If (H1) holds, then system (4) has two other equilibria $P_{1}$ and $P_{2}$, where

$$
\begin{align*}
& P_{1}=\left(\sqrt{\frac{c-a b c-b}{c}}, \frac{a c+1}{c},-\frac{1}{c} \sqrt{\frac{c-a b c-b}{c}}\right),  \tag{9}\\
& P_{2}=\left(-\sqrt{\frac{c-a b c-b}{c}}, \frac{a c+1}{c}, \frac{1}{c} \sqrt{\frac{c-a b c-b}{c}}\right) .
\end{align*}
$$

In the following, we consider the stability of the equilibria of system (4) by analyzing the corresponding characteristic equations. Assume that $P^{*}=\left(x^{*}, y^{*}, z^{*}\right)$ denotes an arbitrary equilibrium of system (4), then let $\bar{x}=x-x^{*}$, $\bar{y}=y-y^{*}$, and $\bar{z}=z-z^{*}$ and drop the bars for the simplicity of notations. Then by linearizing system (4) around $P^{*}$, we have

$$
\left\{\begin{array}{l}
\dot{x}=\left(y^{*}-a\right) x+x^{*} y+z(t-\tau)  \tag{10}\\
\dot{y}=-2 x^{*} x-b y \\
\dot{z}=-x-c z
\end{array}\right.
$$

The characteristic equation associated with system (10) is

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+\left(\lambda+b_{4}\right) e^{-\lambda \tau}=0 \tag{11}
\end{equation*}
$$

where

$$
\begin{align*}
& b_{1}=b+c+a-y^{*} \\
& b_{2}=\left(a-y^{*}\right)(b+c)+b c+2 x^{2}  \tag{12}\\
& b_{3}=\left(a-y^{*}\right) b c+2 x^{2} c \\
& b_{4}=b
\end{align*}
$$

2.2. Stability and Hopf Bifurcation of Equilibrium $P_{0}$. Obviously, the characteristic equation of system (4) at the equilibrium $P_{0}=(0,1 / b, 0)$ has the following form:

$$
\begin{equation*}
(\lambda+b)\left(\lambda^{2}+\left(a+c-\frac{1}{b}\right) \lambda+\left(a-\frac{1}{b}\right) c+e^{-\lambda \tau}\right)=0 \tag{13}
\end{equation*}
$$

Clearly, $\lambda=-b$ is negative; we only need to consider the following equation:

$$
\begin{equation*}
\lambda^{2}+\left(a+c-\frac{1}{b}\right) \lambda+\left(a-\frac{1}{b}\right) c+e^{-\lambda \tau}=0 \tag{14}
\end{equation*}
$$

For further discussion, we make following hypotheses:
(H2)

$$
\begin{equation*}
1-a b-b c<0 \tag{15}
\end{equation*}
$$

(H3)

$$
\begin{equation*}
a b c-c+b>0 \tag{16}
\end{equation*}
$$

As $\tau=0$, (14) is equivalent to the following equation:

$$
\begin{equation*}
\lambda^{2}+\left(a+c-\frac{1}{b}\right) \lambda+\left(a-\frac{1}{b}\right) c+1=0 \tag{17}
\end{equation*}
$$

Obviously, $\lambda=0$ is not a root of (17).
Lemma 2. If (H2) and (H3) hold, then equilibrium $P_{0}$ of system (4) is locally asymptotically stable with $\tau=0$.

Proof. Let $\lambda_{1}$ and $\lambda_{2}$ be two roots of (17). Clearly, if (H2) and (H3) hold, then we have

$$
\begin{align*}
\lambda_{1}+\lambda_{2} & =-\left(a-\frac{1}{b}+c\right)<0 \\
\lambda_{1} \lambda_{2} & =\left(a-\frac{1}{b}\right) c+1>0 \tag{18}
\end{align*}
$$

It means that all the roots of (17) have negative real parts. So, equilibrium $P_{0}$ of system (4) with $\tau=0$ is locally asymptotically stable.

Now we discuss the effect of delay $\tau$ on the stability of the equilibrium $P_{0}$ of system (4). Assume that $i \omega(\omega>0)$ is a root of (11). Then $\omega$ should satisfy the following equation:

$$
\begin{array}{r}
-\omega^{2}+i \omega\left(a-\frac{1}{b}+c\right)+\left(a-\frac{1}{b}\right) c  \tag{19}\\
+\cos (\omega \tau)-i \sin (\omega \tau)=0
\end{array}
$$

which implies that

$$
\left\{\begin{array}{l}
-\omega^{2}+\left(a-\frac{1}{b}\right) c=-\cos (\omega \tau)  \tag{20}\\
\omega\left(a-\frac{1}{b}+c\right)=\sin (\omega \tau)
\end{array}\right.
$$

From (20), adding the squared terms for both equations yields
$\omega^{4}+\left(\left(a-\frac{1}{b}+c\right)^{2}-2\left(a-\frac{1}{b}\right) c\right) \omega^{2}+\left(a-\frac{1}{b}\right)^{2} c^{2}-1=0$.

Make the following assumptions:

$$
\begin{equation*}
a b c-c-b>0 \tag{H4}
\end{equation*}
$$

(H5)

$$
\begin{equation*}
a b c-c-b<0 \tag{23}
\end{equation*}
$$

Theorem 1. If (H2) and (H4) hold, then the equilibrium $P_{0}$ of system (4) is locally asymptotically stable for all $\tau \geq 0$.

Proof. Clearly, if (H4) holds, then we have

$$
\begin{array}{r}
\left(a-\frac{1}{b}+c\right)^{2}-2\left(a-\frac{1}{b}\right) c=\left(a-\frac{1}{b}\right)^{2}+c^{2}>0 \\
\left(a-\frac{1}{b}\right)^{2} c^{2}-1=\left(\left(a-\frac{1}{b}\right) c+1\right)\left(\left(a-\frac{1}{b}\right) c-1\right)>0 \tag{24}
\end{array}
$$

which means that (21) has no positive roots. That is to say, all roots of (14) have negative real parts. Combining with Lemma 2, it thus follows from the Routh-Hurwitz criterion that the equilibrium $P_{0}$ of system (4) is locally asymptotically stable for all $\tau \geq 0$.

Lemma 3. If (H5) holds, then (21) has a unique positive root.
Proof. (H5) holds, so we have

$$
\begin{align*}
& \left(a-\frac{1}{b}+c\right)^{2}-2\left(a-\frac{1}{b}\right) c=\left(a-\frac{1}{b}\right)^{2}+c^{2}>0  \tag{25}\\
& \left(a-\frac{1}{b}\right)^{2} c^{2}-1=\left(a-\frac{1}{b}+1\right)\left(a-\frac{1}{b}-1\right)<0
\end{align*}
$$

Hence, (21) has a unique positive root as follows:

$$
\begin{equation*}
\omega_{0}=\sqrt{\frac{-(a-1 / b)^{2}-c^{2}+\sqrt{\left((a-1 / b)^{2}-c^{2}\right)^{2}+4}}{2}} . \tag{26}
\end{equation*}
$$

According to Lemma 3, (21) has a unique positive root $\omega_{0}$. By (20), we have

$$
\begin{equation*}
\cos \left(\omega_{0} \tau\right)=\omega_{0}^{2}-\left(a-\frac{1}{b}\right) c \tag{27}
\end{equation*}
$$

Thus, if we denote
$\tau_{0}^{j}=\frac{1}{\omega_{0}}\left(\arccos \left(\omega_{0}^{2}-\left(a-\frac{1}{b}\right) c\right)+2 j \pi\right), \quad j=0,1,2, \ldots$,
then $\pm i \omega_{0}$ is a pair of purely imaginary roots of (14) with $\tau=\tau_{0}^{j}$. Clearly, sequence $\left\{\tau_{0}^{j}\right\}_{j=0}^{\infty}$ is increasing and

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \tau_{0}^{j}=+\infty \tag{29}
\end{equation*}
$$

Thus, we can define

$$
\begin{equation*}
\tau_{0}=\tau_{0}^{0}=\min \left\{\tau_{0}^{j}\right\} \tag{30}
\end{equation*}
$$

Lemma 4. Let $\lambda(\tau)=\alpha(\tau) \pm i \omega(\tau)$ be the root of (14) near $\tau=\tau_{0}^{j}$ satisfying $\alpha\left(\tau_{0}^{j}\right)=0$ for $\omega\left(\tau_{0}^{j}\right)=\omega_{0}$. Then the following transversal condition holds:

$$
\begin{equation*}
\left.\left(\alpha^{\prime}(\tau)\right)^{-1}\right|_{\tau=\tau_{0}^{j}}>0, \quad j=0,1,2, \ldots \tag{31}
\end{equation*}
$$

Proof. Differentiating the two sides of (14) with respect to $\tau$ yields

$$
\begin{equation*}
\frac{d \lambda}{d \tau}\left(2 \lambda+a-\frac{1}{b}+c-\tau e^{-\lambda \tau}\right)=\lambda e^{-\lambda \tau} \tag{32}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left(\frac{d \lambda}{d \tau}\right)^{-1}=\frac{2 \lambda+a-1 / b+c-\tau e^{-\lambda \tau}}{\lambda e^{-\lambda \tau}}=2 e^{\lambda \tau}+\frac{a-1 / b+c}{\lambda} e^{\lambda \tau}-\frac{\tau}{\lambda} . \tag{33}
\end{equation*}
$$

Substituting $\tau_{0}^{j}$ into the above equation, we obtain

$$
\begin{align*}
\left.\left(\alpha^{\prime}(\tau)\right)^{-1}\right|_{\tau=\tau_{0}^{j}}= & \operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)_{\tau=\tau_{0}^{j}}^{-1}=2 \cos \left(\omega_{0} \tau_{0}^{j}\right) \\
& +\frac{(a-1 / b+c) \sin \left(\omega \tau_{0}^{j}\right)}{\omega_{0}} \tag{34}
\end{align*}
$$

Since $\cos \left(\omega_{0} \tau_{0}^{j}\right)=\omega_{0}^{2}-(a-1 / b) c$ and $\sin \left(\omega_{0} \tau_{0}^{j}\right)=(a-$ $1 / b+c) \omega_{0}$, then we have

$$
\begin{aligned}
\left.\left(\alpha^{\prime}(\tau)\right)^{-1}\right|_{\tau=\tau_{0}^{j}}= & \operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)_{\tau=\tau_{0}^{j}}^{-1}=2 \omega_{0}^{2}-2\left(a-\frac{1}{b}\right) c \\
& +\left(a-\frac{1}{b}+c\right)^{2}
\end{aligned}
$$

By (26), we have
$\left(\alpha_{n}{ }^{\prime}(\tau)\right)^{-1}=\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)_{\tau=\tau_{0}^{j}}^{-1}=\sqrt{\left(\left(a-\frac{1}{b}\right)^{2}-c^{2}\right)^{2}+4}>0$.

On the basis of Lemmas 2-4, we have the following result:
Theorem 2. If (H2), (H3), and (H5) hold, then the following statements are true:
(i) When $\tau \in\left[0, \tau_{0}\right)$, the equilibrium $P_{0}$ of (4) is asymptotically stable
(ii) The Hopf bifurcation occurs at $\tau=\tau_{0}$. That is, system (4) has a branch of periodic solutions bifurcating from $P_{0}$ near $\tau=\tau_{0}$
2.3. Stability and Hopf Bifurcation of Equilibrium $P_{1}$ and $P_{2}$. In this section, we consider stability and Hopf bifurcation of equilibria $P_{1}$ and $P_{2}$. At the equilibria $P_{1}$ and $P_{2}$, the characteristic (11) takes the following form:

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}+\left(\lambda+b_{4}\right) e^{-\lambda \tau}=0 \tag{37}
\end{equation*}
$$

where

$$
\begin{aligned}
& b_{1}=b+c-\frac{1}{c} \\
& b_{2}=b c+2\left(1-b\left(a+\frac{1}{c}\right)\right)-\frac{b}{c}-1 \\
& b_{3}=2\left(1-b\left(a+\frac{1}{c}\right)\right) c-b \\
& b_{4}=b
\end{aligned}
$$

As $\tau=0$, (37) becomes

$$
\begin{equation*}
\lambda^{3}+b_{1} \lambda^{2}+\left(b_{2}+1\right) \lambda+b_{3}+b_{4}=0 \tag{39}
\end{equation*}
$$

Make the following assumptions:
(H6)

$$
\begin{equation*}
\frac{1}{c}<\min \{b, c\} \tag{40}
\end{equation*}
$$

$$
\begin{equation*}
c-a b c-2 b<0 \tag{H7}
\end{equation*}
$$

Lemma 5. Based on Lemma 1, if (H6) holds, then equilibria $P_{1}$ and $P_{2}$ are both locally asymptotically stable with $\tau=0$.

Proof. As (H1) and (H6) hold, we have

$$
\begin{align*}
b_{1}= & b+c-\frac{1}{c}>0, \\
b_{3} & =2 c-2 a b c-2 b>0, \\
b_{1}\left(b_{2}+1\right)-b_{3}= & b\left(b-\frac{1}{c}\right)\left(c-\frac{1}{c}\right)+2 x^{*^{2}}+b c\left(c-\frac{1}{c}\right)>0 . \tag{42}
\end{align*}
$$

By the Routh-Hurwitz criteria, all the roots of (39) have negative real parts. Therefore, $P_{1}$ and $P_{2}$ are both locally asymptotically stable with $\tau=0$.

Now we discuss the effect of delay $\tau$ on the stability of the equilibria $P_{1}$ and $P_{2}$ of system (4). Assume that $i \omega(\omega>0)$ is a root of (37). Then $\omega$ should satisfy the following equation:
$-i \omega^{3}-b_{1} \omega^{2}+i b_{2} \omega+b_{3}+\left(i \omega+b_{4}\right)(\cos (\omega \tau)-i \sin (\omega \tau))=0$,
which implies that

$$
\left\{\begin{array}{l}
-\omega^{3}+b_{2} \omega=b_{4} \sin (\omega \tau)-\omega \cos (\omega \tau)  \tag{44}\\
-b_{1} \omega^{2}+b_{3}=-\omega \sin (\omega \tau)-b_{4} \cos (\omega \tau)
\end{array}\right.
$$

From (44), adding up the squares of both equations, we have

$$
\begin{equation*}
\omega^{6}+\left(b_{1}^{2}-2 b_{2}\right) \omega^{4}+\left(b_{2}^{2}-2 b_{1} b_{3}-1\right) \omega^{2}+b_{3}^{2}-b_{4}^{2}=0 \tag{45}
\end{equation*}
$$

Let $z=\omega^{2}$, then (45) can be rewritten into the following form:

$$
\begin{equation*}
z^{3}+\left(b_{1}^{2}-2 b_{2}\right) z^{2}+\left(b_{2}^{2}-2 b_{1} b_{3}-1\right) z+b_{3}^{2}-b_{4}^{2}=0 \tag{46}
\end{equation*}
$$

## Denote

$$
\begin{equation*}
h(z)=z^{3}+R_{0} z^{2}+Q_{0} z+V_{0} \tag{47}
\end{equation*}
$$

Lemma 6. If (H7) holds, then (46) has at least a root.
Proof. Obviously,
$b_{3}^{2}-b_{4}^{2}=\left(b_{3}-b_{4}\right)\left(b_{3}+b_{4}\right)=2(c-a b c-2 b)(c-a b c-b)<0$.

Therefore, (46) has at least a positive root.
According to Lemma 6, (46) has a positive root, denoted by $z_{0}$, and thus, (45) has a positive root $\omega_{0}=\sqrt{z_{0}}$. By (44), we have

$$
\begin{equation*}
\cos \left(\omega_{0} \tau\right)=\frac{\left(\omega_{0}^{3}-b_{2} \omega_{0}\right) \omega_{0}-\left(-\omega_{0}^{2} b_{1}+b_{3}\right) b_{4}}{\omega_{0}^{2}+b_{4}^{2}} \tag{49}
\end{equation*}
$$

Thus, if we denote

$$
\begin{align*}
\tau_{0}^{j}= & \frac{1}{\omega_{0}}\left(\arccos \left(\frac{\left(\omega_{0}^{3}-b_{2} \omega_{0}\right) \omega_{0}-\left(-\omega_{0}^{2} b_{1}+b_{3}\right) b_{4}}{\omega_{0}^{2}+b_{4}^{2}}\right)\right.  \tag{50}\\
& +2 j \pi), \quad j=0,1,2, \ldots
\end{align*}
$$

then $\pm i \omega_{0}$ is a pair of purely imaginary roots of (21) with $\tau=\tau_{0}^{j}$. Clearly, sequence $\left\{\tau_{0}^{j}\right\}_{j=0}^{\infty}$ is increasing and

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \tau_{0}^{j}=+\infty \tag{51}
\end{equation*}
$$

Thus, we can define

$$
\begin{equation*}
\tau_{0}=\tau_{0}^{0}=\min \left\{\tau_{0}^{j}\right\} \tag{52}
\end{equation*}
$$

Lemma 7. Let $\lambda(\tau)=\alpha(\tau) \pm i \omega(\tau)$ be the root of (37) near $\tau=\tau_{0}^{j}$ satisfying $\alpha\left(\tau_{0}^{j}\right)=0, \omega\left(\tau_{0}^{j}\right)=\omega_{0}$. Suppose that $h^{\prime}\left(z_{0}\right) \neq$ 0 , where $h(z)$ is defined by (47). Then the following transversal condition holds:

$$
\begin{equation*}
\left.\frac{d(\operatorname{Re} \lambda(\tau))}{d \tau}\right|_{\tau=\tau_{0}^{j}} \neq 0 \tag{53}
\end{equation*}
$$

and the sign of $d(\operatorname{Re\lambda }(\tau)) /\left.d \tau\right|_{\tau=\tau_{0}^{j}}$ is consistent with that of $h^{\prime}\left(z_{0}\right)$.

Proof. Denote

$$
\begin{align*}
& R(\lambda)=\lambda^{3}+b_{1} \lambda^{2}+b_{2} \lambda+b_{3}  \tag{54}\\
& Q(\lambda)=\lambda+b_{4}
\end{align*}
$$

Then (37) can be written as

$$
\begin{equation*}
R(\lambda)+Q(\lambda) e^{-\lambda \tau}=0 \tag{55}
\end{equation*}
$$

and (45) can be transformed into the following form:

$$
\begin{equation*}
R(i \omega) \bar{R}(i \omega)-Q(i \omega) \bar{Q}(i \omega)=0 \tag{56}
\end{equation*}
$$

Thus, together with (46) and (47), we have

$$
\begin{equation*}
h\left(\omega^{2}\right)=R(i \omega) \bar{R}(i \omega)-Q(i \omega) \bar{Q}(i \omega) \tag{57}
\end{equation*}
$$

Differentiating both sides of (57) with respect to $\omega$, we obtain

$$
\begin{align*}
2 \omega h^{\prime}\left(\omega^{2}\right)= & i\left[R^{\prime}(i \omega) \bar{R}(i \omega)+R(i \omega) \bar{R}^{\prime}(i \omega)-Q^{\prime}(i \omega) \bar{Q}(i \omega)\right. \\
& \left.+Q(i \omega) \bar{Q}^{\prime}(i \omega)\right] \tag{58}
\end{align*}
$$

If $i \omega_{0}$ is not simple, then $\omega_{0}$ must satisfy

$$
\begin{equation*}
\left.\frac{d}{d \lambda}\left[R(\lambda)+Q(\lambda) e^{-\lambda \tau_{0}}\right]\right|_{\lambda=i \omega_{0}}=0 \tag{59}
\end{equation*}
$$

that is, $\omega_{0}$ must satisfy

$$
\begin{equation*}
R^{\prime}(i \omega)+Q^{\prime}(i \omega) e^{-i \omega_{0} \tau_{0}}-\tau_{0} Q\left(i \omega_{0}\right) e^{-i \omega_{0} \tau_{0}}=0 \tag{60}
\end{equation*}
$$

With (55), we have

$$
\begin{equation*}
\tau_{0}=\frac{Q^{\prime}\left(i \omega_{0}\right)}{Q\left(i \omega_{0}\right)}-\frac{R^{\prime}\left(i \omega_{0}\right)}{R\left(i \omega_{0}\right)} \tag{61}
\end{equation*}
$$

Thus, by (56) and (57), we obtain

$$
\begin{align*}
\operatorname{Im}\left(\tau_{0}\right) & =\operatorname{Im}\left(\frac{Q^{\prime}\left(i \omega_{0}\right)}{Q\left(i \omega_{0}\right)}-\frac{R^{\prime}\left(i \omega_{0}\right)}{R\left(i \omega_{0}\right)}\right)=\operatorname{Im}\left(\frac{Q^{\prime}\left(i \omega_{0}\right) \bar{Q}\left(i \omega_{0}\right)}{Q\left(i \omega_{0}\right) \bar{Q}\left(i \omega_{0}\right)}-\frac{R^{\prime}\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)}{R\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)}\right)=\operatorname{Im}\left(\frac{Q^{\prime}\left(i \omega_{0}\right) \bar{Q}\left(i \omega_{0}\right)-R^{\prime}\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)}{R\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)}\right) \\
& =\frac{-i\left[Q^{\prime}\left(i \omega_{0}\right) \bar{Q}\left(i \omega_{0}\right)-R^{\prime}\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)+\bar{Q}^{\prime}\left(i \omega_{0}\right) Q\left(i \omega_{0}\right)+\bar{R}^{\prime}\left(i \omega_{0}\right) R\left(i \omega_{0}\right)\right]}{2 R\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)}=\frac{\omega_{0} h^{\prime}\left(\omega_{0}^{2}\right)}{\left|R\left(i \omega_{0}\right)\right|^{2}} . \tag{62}
\end{align*}
$$

Since $\tau_{0}$ is real, i.e., $\operatorname{Im}\left(\tau_{0}\right)=0$, we have $h^{\prime}\left(\omega_{0}^{2}\right)=0$. We get a contradiction to the condition $h^{\prime}\left(\omega_{0}^{2}\right) \neq 0$. This proves the first conclusion. Differentiating both sides of (55) with respect to $\tau$, we obtain

$$
\begin{equation*}
\left[R^{\prime}(\lambda)+Q^{\prime}(\lambda) e^{-\lambda \tau}-\tau Q(\lambda) e^{-\lambda \tau}\right] \frac{d \lambda}{d \tau}-\lambda Q(\lambda) e^{-\lambda \tau}=0 \tag{63}
\end{equation*}
$$

which implies

$$
\begin{align*}
\frac{d \lambda}{d \tau} & =\frac{\lambda Q(\lambda)}{R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)} \\
& =\frac{\lambda Q(\lambda)\left[\overline{R^{\prime}}(\lambda) e^{\lambda \tau}+\overline{Q^{\prime}}(\lambda)-\tau \bar{Q}(\lambda)\right]}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}}  \tag{64}\\
& =\frac{\lambda\left[-R(\lambda) \bar{R}^{\prime}(\lambda) e^{\lambda \tau}+Q(\lambda) \overline{Q^{\prime}}(\lambda)-\tau|Q(\lambda)|^{2}\right]}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}} .
\end{align*}
$$

It follows together with (58) that

$$
\begin{align*}
\left.\frac{d(\operatorname{Re} e \lambda(\tau))}{d \tau}\right|_{\tau=\tau_{0}, \lambda=i \omega_{0}} & =\frac{\operatorname{Re}\left\{\lambda\left[-R(\lambda) \bar{R}^{\prime}(\lambda) e^{\lambda \tau}+Q(\lambda) \bar{Q}^{\prime}(\lambda)-\tau|Q(\lambda)|^{2}\right]\right\}_{\tau=\tau_{0}, \lambda=i \omega_{0}}}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}} \\
& =\frac{i \omega_{0}\left[-R\left(i \omega_{0}\right) \bar{R}^{\prime}\left(i \omega_{0}\right)+Q\left(i \omega_{0}\right) \overline{Q^{\prime}}\left(i \omega_{0}\right)+R^{\prime}\left(i \omega_{0}\right) \bar{R}\left(i \omega_{0}\right)-Q^{\prime}\left(i \omega_{0}\right) \bar{Q}\left(i \omega_{0}\right)\right]}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}}  \tag{65}\\
& =\frac{\omega_{0}^{2} h^{\prime}\left(\omega_{0}^{2}\right)}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}}=\frac{\omega_{0}^{2} h^{\prime}\left(z_{0}\right)}{\left|R^{\prime}(\lambda) e^{\lambda \tau}+Q^{\prime}(\lambda)-\tau Q(\lambda)\right|^{2}} \neq 0 .
\end{align*}
$$

Clearly, the sign of $d(\operatorname{Re} \lambda(\tau)) /\left.d \tau\right|_{\tau=\tau_{0}}$ is determined by that of $h^{\prime}\left(z_{0}\right)$.

On the basis of Lemma 1 and Lemma 5-Lemma 7, we have the following result.

Theorem 3. If (H1), (H6), and (H7) hold, and $h^{\prime}\left(z_{0}\right)>0$, then the following statements are true:
(i) When $\tau \in\left[0, \tau_{0}\right)$, the equilibria $P_{1}$ and $P_{2}$ of system (4) are both locally asymptotically stable
(ii) The Hopf bifurcation occurs at $\tau=\tau_{0}$, i.e., system (4) has a branch of periodic solutions that bifurcates from $P_{1}$ and $P_{2}$ near $\tau=\tau_{0}$, respectively

## 3. Direction and Stability of Hopf Bifurcation

In the previous section, we have shown that system (4) admits a series of periodic solutions bifurcating from the equilibrium at the critical value $\tau_{0}^{j}\left(j \in N_{0}\right)$. In this section, we derive explicit formulae to determine the properties of the Hopf bifurcation
at the critical value $\tau_{0}^{j}$ by using the normal form theory and center manifold reduction developed by [26].

Denote $\tau_{0}^{j}$ by $\tau^{*}$ and introduce the new parameter $\mu=$ $\tau-\tau^{*}$. Normalizing the delay $\tau$ by the time-scaling $t \rightarrow t / \tau$, (4) can then be rewritten as

$$
\begin{equation*}
\frac{d U(t)}{d t}=L\left(\tau^{*}\right)\left(U_{t}\right)+F\left(U_{t}, \mu\right) \tag{66}
\end{equation*}
$$

where

$$
\begin{align*}
& L(\mu)(\varphi)=\mu\left(\begin{array}{c}
\left(y^{*}-a\right) \varphi_{1}(0)+\varphi_{3}(-1) \\
-2 x^{*} \varphi_{1}(0)-b \varphi_{2}(0) \\
-\varphi_{1}(0)-c \varphi_{3}(0)
\end{array}\right) \\
& F(\varphi, \mu)=L(\mu) \varphi+f(\varphi, \mu) \\
& f(\varphi, \mu)=\left(\tau^{*}+\mu\right)\left(\begin{array}{c}
\varphi_{1}(0) \varphi_{2}(0) \\
\varphi_{1}^{2}(0) \\
0
\end{array}\right)+\text { h.o.t., } \tag{68}
\end{align*}
$$

for $\varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)^{T} \in \mathscr{C}$.
Then the linearized system of (66) at $(0,0,0)$ is

$$
\begin{equation*}
\frac{d U(t)}{d t}=L\left(\tau^{*}\right)\left(U_{t}\right) \tag{69}
\end{equation*}
$$

Based on the discussion in Section 2, we can easily know that for $\tau=\tau^{*}$, the characteristic equation of (11) has a pair of simple purely imaginary eigenvalues $\Lambda_{0}=\left\{i \omega_{0} \tau^{*},-i \omega_{0} \tau^{*}\right\}$.

Let $\mathscr{C}:=C\left([-1,0], \mathbb{R}^{3}\right)$, considering the following FDE on $\mathscr{C}$ :

$$
\begin{equation*}
\dot{z}=L\left(\tau^{*}\right)\left(z_{t}\right) \tag{70}
\end{equation*}
$$

Obviously, $L\left(\tau^{*}\right)$ is a continuous linear function mapping $C\left([-1,0], \mathbb{R}^{3}\right)$ into $\mathbb{R}^{3}$. By the Riesz representation theorem, there exists a $3 \times 3$ matrix function $(\theta, \tau)(-1 \leq \theta \leq 0)$, whose elements are of bounded variation such that

$$
\begin{equation*}
L\left(\tau^{*}\right)(\varphi)=\int_{-1}^{0}\left[d \eta\left(\theta, \tau^{*}\right)\right] \varphi(\theta), \quad \text { for } \varphi \in C \tag{71}
\end{equation*}
$$

In fact, we can choose

$$
\begin{align*}
\eta\left(\theta, \tau^{*}\right)= & \tau^{*}\left(\begin{array}{ccc}
y^{*}-a & x^{*} & 0 \\
-2 x^{*} & -b & 0 \\
-1 & 0 & -c
\end{array}\right) \delta(\theta)  \tag{72}\\
& -\tau^{*}\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \delta(\theta+1)
\end{align*}
$$

where $\delta$ is the Dirac delta function.

Let $A\left(\tau^{*}\right)$ denote the infinitesimal generator of the semigroup induced by the solutions of (70) and $A^{*}$ be the formal adjoint of $A\left(\tau^{*}\right)$ under the bilinear pairing

$$
\begin{align*}
(\psi, \phi) & =(\psi(0), \phi(0))-\int_{-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi \\
& =(\psi(0), \phi(0))+\tau^{*} \int_{-1}^{0} \psi(\theta+1)\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \phi(\theta) d \theta \tag{73}
\end{align*}
$$

for $\phi \in C$ and $\psi \in C^{*}=C\left([0,1], R^{3}\right)$. Then $A\left(\tau^{*}\right)$ and $A^{*}$ are a pair of adjoint operators. From the discussion in Section 2, we know that $A\left(\tau^{*}\right)$ has a pair of simple purely imaginary eigenvalues $\pm i \omega_{0} \tau^{*}$, and they are also eigenvalues of $A^{*}$ since $A\left(\tau^{*}\right)$ and $A^{*}$ are a pair of adjoint operators. Let $P$ and $P^{*}$ be the center spaces, that is, the generalized eigenspaces of $A\left(\tau^{*}\right)$ and $A^{*}$, respectively, associated with $\Lambda_{0}$. Then $P^{*}$ is the adjoint space of $P$ and $\operatorname{dim} P=\operatorname{dim} P^{*}=2$. Direct computations give the following results.

Lemma 8. Let

$$
\left\{\begin{array}{l}
\alpha=-\frac{-2 x^{*}}{i \omega_{0}+b}  \tag{74}\\
\beta=-\frac{1}{i \omega_{0}+c} \\
\alpha^{*}=\frac{x^{*}}{i \omega_{0}+b} \\
\beta^{*}=\frac{1}{i \omega_{0}+c}
\end{array}\right.
$$

Then,

$$
\begin{align*}
p_{1}(\theta) & =e^{i \omega_{0} \tau^{*} \theta}(1, \alpha, \beta)^{T} \\
p_{2}(\theta) & =\bar{p}_{1}(\theta)  \tag{75}\\
-1 & \leq \theta \leq 0
\end{align*}
$$

is a basis of P associated with $\Lambda_{0}$ and

$$
\begin{align*}
q_{1}(s) & =\left(1, \alpha^{*}, \beta^{*}\right) e^{-i \omega_{0} \tau^{*} s}, \\
q_{2}(s) & =\bar{q}_{1}(s)  \tag{76}\\
0 & \leq s \leq 1
\end{align*}
$$

Let $\Phi=\left(\Phi_{1}, \Phi_{2}\right)$ and $\Psi^{*}=\left(\Psi_{1}^{*}, \Psi_{2}^{*}\right)^{T}$ with

$$
\begin{align*}
& \Phi_{1}(\theta)=\frac{p_{1}(\theta)+p_{2}(\theta)}{2}=\left(\begin{array}{c}
\operatorname{Re}\left\{e^{i \omega_{0} \tau^{*} \theta}\right\} \\
\operatorname{Re}\left\{\alpha e^{i \omega_{0} \tau^{*} \theta}\right\} \\
\operatorname{Re}\left\{\beta e^{i \omega_{0} \tau^{*} \theta}\right\}
\end{array}\right) \\
& =\left(\begin{array}{c}
\cos \omega_{0} \tau^{*} \theta \\
\frac{-2 x^{*}\left(b \cos \left(\omega_{0} \tau^{*} \theta\right)+\omega_{0} \sin \left(\omega_{0} \tau^{*} \theta\right)\right)}{b^{2}+\omega_{0}^{2}} \\
\frac{-\omega_{0} \sin \left(\omega_{0} \tau^{*} \theta\right)-c \cos \left(\omega_{0} \tau^{*} \theta\right)}{c^{2}+\omega_{0}^{2}}
\end{array}\right), \\
& \Phi_{2}(\theta)=\frac{p_{1}(\theta)-p_{2}(\theta)}{2 i}=\left(\begin{array}{c}
\operatorname{Im}\left\{e^{i \omega_{0} \tau^{*} \theta}\right\} \\
\operatorname{Im}\left\{\alpha e^{i \omega_{0} \tau^{*} \theta}\right\} \\
\operatorname{Im}\left\{\beta e^{i \omega_{0} \tau^{*} \theta}\right\}
\end{array}\right)  \tag{77}\\
& =\left(\begin{array}{c}
\sin \omega_{0} \tau^{*} \theta \\
\frac{2 x^{*}\left(\omega_{0} \cos \left(\omega_{0} \tau^{*} \theta\right)-b \sin \left(\omega_{0} \tau^{*} \theta\right)\right)}{b^{2}+\omega_{0}^{2}} \\
\frac{\omega_{0} \cos \left(\omega_{0} \tau^{*} \theta\right)-c \sin \left(\omega_{0} \tau^{*} \theta\right)}{c^{2}+\omega_{0}^{2}}
\end{array}\right),
\end{align*}
$$

for $\theta \in[-1,0]$, and

$$
\left.\begin{array}{rl}
\Psi_{1}^{*}(s) & =\frac{q_{1}(s)+q_{2}(s)}{2}=\left(\begin{array}{c}
\operatorname{Re}\left\{e^{-i \omega_{0} \tau^{*} s}\right\} \\
\operatorname{Re}\left\{\alpha^{*} e^{-i \omega_{0} \tau^{*} s}\right\} \\
\operatorname{Re}\left\{\beta^{*} e^{-i \omega_{0} \tau^{*} s}\right\}
\end{array}\right) \\
& =\left(\begin{array}{c}
-\frac{x^{*}\left(b \cos \left(s \tau \omega_{0}\right)+\omega_{0} \sin \left(s \tau \omega_{0}\right)\right.}{b^{2}+\omega_{0}^{2}} \\
-\frac{c \cos \left(s \tau \omega_{0}\right)+\omega_{0} \sin \left(s \tau \omega_{0}\right)}{c^{2}+\omega_{0}^{2}} \\
\Psi_{2}^{*}(s)
\end{array}\right) \\
\left.=\frac{\operatorname{Im}\left\{e^{-i \omega_{0} \tau^{*} s}\right\}}{2 i}\right) \\
\operatorname{Im}\left\{\alpha e^{-i \omega_{0} \tau^{*} s}\right\} \\
\operatorname{Im}\left\{\beta e^{i \omega_{0} \tau^{*} s}\right\}
\end{array}\right),
$$

for $s \in[0,1]$. From (73), we can obtain $\left(\Psi_{1}^{*}, \Phi_{1}\right)$ and $\left(\Psi_{1}^{*}, \Phi_{2}\right)$, noting that

$$
\begin{align*}
& \left(q_{1}, p_{1}\right)=\left(\Psi_{1}^{*}, \Phi_{1}\right)-\left(\Psi_{2}^{*}, \Phi_{2}\right)+i\left[\left(\Psi_{1}^{*}, \Phi_{2}\right)+\left(\Psi_{2}^{*}, \Phi_{1}\right)\right]  \tag{79}\\
& \left(q_{1}, p_{1}\right)=1+\alpha \alpha^{*}+\beta \beta^{*}-\beta \beta^{*} \tau^{*} e^{-i \omega_{0} \tau^{*}}:=D^{*} \tag{80}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& \left(\Psi_{1}^{*}, \Phi_{1}\right)-\left(\Psi_{2}^{*}, \Phi_{2}\right)=\operatorname{Re}\left\{D^{*}\right\}  \tag{81}\\
& \left(\Psi_{1}^{*}, \Phi_{2}\right)+\left(\Psi_{2}^{*}, \Phi_{1}\right)=\operatorname{Im}\left\{D^{*}\right\}
\end{align*}
$$

Now, we define $\left(\Psi^{*}, \Phi\right)=\left(\Psi_{j}^{*}, \Phi_{k}\right)(j, k=1,2)$ and construct a new basis $\psi$ for $Q$ by

$$
\begin{equation*}
\Psi=\left(\Psi_{1}, \Psi_{2}\right)^{T}=\left(\Psi^{*}, \Phi\right)^{-1} \Psi^{*} \tag{82}
\end{equation*}
$$

Obviously, $(\Psi, \Phi)=I_{2 \times 2}$, the second-order identity matrix. In addition, define $f_{0}=\left(\xi_{0}^{1}, \xi_{0}^{2}, \xi_{0}^{3}\right)$, where

$$
\begin{align*}
& \xi_{0}^{1}=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \\
& \xi_{0}^{2}=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right),  \tag{83}\\
& \xi_{0}^{2}=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right) .
\end{align*}
$$

Let $c \cdot f_{0}$ be defined by

$$
\begin{equation*}
c \cdot f_{0}=c_{1} \xi_{0}^{1}+c_{2} \xi_{0}^{2}+c_{3} \xi_{0}^{3} \tag{84}
\end{equation*}
$$

for $c=\left(c_{1}, c_{2}, c_{3}\right)^{T}$ and $c_{j} \in R(j=1,2,3)$.
Then the center space of linear Equation (69) is given by $P_{\mathrm{CN}} \mathscr{C}$, where

$$
\begin{equation*}
P_{\mathrm{CN}} \varphi=\Phi\left(\Psi,\left\langle\varphi, f_{0}\right\rangle\right) \cdot f_{0}, \quad \varphi \in c \tag{85}
\end{equation*}
$$

and $\mathscr{C}=P_{\mathrm{CN}} \mathscr{C} \oplus P_{S} \mathscr{C}$; here $P_{S} \mathscr{C}$ denotes the complementary subspace of $P_{\mathrm{CN}} \mathscr{C}$.

Let $A_{\tau^{*}}$ be defined by

$$
\begin{equation*}
A_{\tau^{*}} \varphi(\theta)=\dot{\varphi}(\theta)+X_{0}(\theta)\left[L\left(\tau^{*}\right)(\varphi(\theta))-\dot{\varphi}(0)\right], \quad \varphi \in B \mathscr{C} \tag{86}
\end{equation*}
$$

where $X_{0}:[-1,0] \rightarrow B(X, X)$ is given by

$$
X_{0}(\theta)= \begin{cases}0, & -1 \leq \theta<0  \tag{87}\\ I, & \theta=0\end{cases}
$$

Then $A_{\tau^{*}}$ is the infinitesimal generator induced by the solution of (69) and (66) and can be rewritten as the following operator differential equation:

$$
\begin{equation*}
\dot{U}_{t}=A_{\tau^{*}} U_{t}+X_{0} F\left(U_{t}, \mu\right) \tag{88}
\end{equation*}
$$

Using the decomposition $\mathscr{C}=P_{\mathrm{CN}} \mathscr{C} \oplus P_{S} \mathscr{C}$ and (85), the solution of (66) can be rewritten as

$$
\begin{equation*}
U_{t}=\Phi\binom{x_{1}(t)}{x_{2}(t)} \cdot f_{0}+h\left(x_{1}, x_{2}, \mu\right) \tag{89}
\end{equation*}
$$

where

$$
\begin{equation*}
\binom{x_{1}(t)}{x_{2}(t)}=\left(\Psi,<U_{t}, f_{0}>\right) \tag{90}
\end{equation*}
$$

and $h\left(x_{1}, x_{2}, \mu\right) \in P_{s} c$ with $h(0,0,0)=D h(0,0,0)=0$. In particular, the solution of (66) on the center manifold is given by

$$
\begin{equation*}
U_{t}^{*}=\Phi\binom{x_{1}(t)}{x_{2}(t)} \cdot f_{0}+h\left(x_{1}, x_{2}, 0\right) \tag{91}
\end{equation*}
$$

Setting $z=x_{1}-i x_{2}$ and noticing that $p_{1}=\Phi_{1}+i \Phi_{2}$, then (91) can be rewritten as

$$
\begin{equation*}
U_{t}^{*}=\frac{1}{2} \Phi\binom{z+\bar{z}}{i(z-\bar{z})} \cdot f_{0}+w(z, \bar{z})=\frac{1}{2}\left(p_{1} z+\bar{p}_{1} \bar{z}\right) \cdot f_{0}+W(z, \bar{z}), \tag{92}
\end{equation*}
$$

where $W(z, \bar{z})=h((z+\bar{z}) / 2,-(z-\bar{z}) / 2 i, 0)$. Moreover, by [26], $z$ satisfies

$$
\begin{equation*}
\dot{z}=i \omega_{0} \tau^{*} z+g(z, \bar{z}) \tag{93}
\end{equation*}
$$

where

$$
\begin{equation*}
g(z, \bar{z})=\left(\Psi_{1}(0)-i \Psi_{2}(0)\right)<F\left(U_{t}^{*}, 0\right), f_{0}>. \tag{94}
\end{equation*}
$$

Let

$$
\begin{gather*}
W(z, \bar{z})=W_{20} \frac{z^{2}}{2}+W_{11} z \bar{z}+W_{02} \frac{\bar{z}^{2}}{2}+\cdots  \tag{95}\\
g(z, \bar{z})=g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+\cdots \tag{96}
\end{gather*}
$$

From (92), we have

$$
\begin{align*}
& <F\left(U_{t}^{*}, 0\right), f_{0}> \\
& =\frac{\tau^{*} z^{2}}{4}\left(\begin{array}{l}
1 \\
\alpha \\
0
\end{array}\right)+\frac{\tau^{*} z \bar{z}}{4}\left(\begin{array}{c}
2 \\
\alpha+\bar{\alpha} \\
0
\end{array}\right)+\frac{\tau^{*} \bar{z}^{2}}{4}\left(\begin{array}{c}
1 \\
\alpha^{2} \\
0
\end{array}\right) \\
& \quad+\frac{\tau^{*}}{4}\left(\begin{array}{c}
\left\langle 4 w_{11}^{(1)}(0)+2 w_{20}^{(1)}(0), 1\right\rangle \\
\left.\left\langle 2 w_{11}^{(2)}(0)+w_{20}^{(2)}(0)+2 \alpha w_{11}^{(1)}(0)+\bar{\alpha} w_{20}^{(1)}(0), 1\right\rangle\right) z^{2} \bar{z} \\
0
\end{array}\right) \\
& \quad+\cdots, \tag{97}
\end{align*}
$$

where

$$
\begin{equation*}
<W_{i j}^{n}(\theta), 1>=\frac{1}{\pi} \int_{0}^{\pi} W_{i j}^{n}(\theta)(x) d x, \quad i+j=2, n \in \mathbb{N} \tag{98}
\end{equation*}
$$

Let $\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=\Psi_{1}(0)-i \Psi_{2}(0)$. Then by (94), (95), and (96), we can obtain the following quantities:

$$
\begin{align*}
g_{20}= & \frac{\tau^{*}}{2}\left[\psi_{1}+\alpha \psi_{2}\right] \\
g_{11}= & \frac{\tau^{*}}{4}\left[2 \psi_{1}+(\alpha+\bar{\alpha}) \psi_{2}\right], \\
g_{02}= & \frac{\tau^{*}}{2}\left[\bar{\psi}_{1}+\alpha^{2} \bar{\psi}_{2}\right], \\
g_{21}= & \frac{\tau^{*}}{2}\left[\left\langle 4 w_{11}^{(1)}(0)+2 w_{20}^{(1)}(0), 1\right\rangle \psi_{1}\right. \\
& \left.+\left\langle 2 w_{11}^{(2)}(0)+w_{20}^{(2)}(0)+2 \alpha w_{11}^{(1)}(0)+\bar{\alpha} w_{20}^{(1)}(0), 1\right\rangle \psi_{2}\right] . \tag{99}
\end{align*}
$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in[-1,0]$ appear in $g_{21}$, we still need to compute them. It follows easily from (95) that

$$
\begin{align*}
& \dot{W}(z, \bar{z})=W_{20} z \dot{z}+W_{11}(\dot{z} \bar{z}+z \overline{\bar{z}})+W_{02} \bar{z} \bar{z}+\cdots  \tag{100}\\
& A_{\tau^{*}} W=A_{\tau^{*}} W_{20} \frac{z^{2}}{2}+A_{\tau^{*}} W_{11} z \bar{z}+A_{\tau^{*}} W_{02} \frac{\bar{z}^{2}}{2}+\cdots . \tag{101}
\end{align*}
$$

In addition, by [26], $W(z(t)$ and $\bar{z}(t)$ satisfy

$$
\begin{equation*}
\dot{W}=A_{\tau^{*}} W+H(z, \bar{z}) \tag{102}
\end{equation*}
$$

where

$$
\begin{align*}
H(z, \bar{z})= & H_{20} \frac{z^{2}}{2}+H_{11} z \bar{z}+H_{02} \frac{\bar{z}^{2}}{2}+\cdots=X_{0} F\left(U_{t}^{*}, 0\right) \\
& -\Phi\left(\Psi,<X_{0} F\left(U_{t}^{*}, 0\right), f_{0}>\right) \cdot f_{0} \tag{103}
\end{align*}
$$

with $H_{i j} \in P_{S} \mathscr{C}, i+j=2$. Thus, from (92), (100), (101), and (102), we can obtain that

$$
\left\{\begin{array}{l}
\left(2 i \omega_{0} \tau^{*}-A_{\tau^{*}}\right) W_{20}=H_{20}  \tag{104}\\
-A_{\tau^{*}} W_{11}=H_{11}
\end{array}\right.
$$

Noticing that $A_{\tau^{*}}$ has only two eigenvalues $\pm i \omega_{0} \tau^{*}$ with zero real parts, (102), therefore, has a unique solution $W_{i j}(i+j=2)$ in $P_{S} \mathscr{C}$ given by

$$
\left\{\begin{array}{l}
W_{20}=\left(2 i \omega_{0} \tau^{*}-A_{\tau^{*}}\right)^{-1} H_{20}  \tag{105}\\
W_{11}=-A_{\tau^{*}}^{-1} H_{11}
\end{array}\right.
$$

From (103), we know that for $-1 \leq \theta<0$,

$$
\begin{align*}
H(z, \bar{z})= & -\Phi(\theta) \Psi(0)<F\left(U_{t}^{*}, 0\right), f_{0}>\cdot f_{0} \\
= & -\left(\frac{p_{1}(\theta)+p_{2}(\theta)}{2}, \frac{p_{1}(\theta)-p_{2}(\theta)}{2 i}\right)\left(\Psi_{1}(0) \Psi_{2}(0)\right) \\
& \times<F\left(U_{t}^{*}, 0\right), f_{0}>\cdot f_{0}=-\frac{1}{2}\left[p_{1}(\theta)\left(\Psi_{1}(0)-i \Psi_{2}(0)\right)\right. \\
& \left.+p_{2}(\theta)\left(\Psi_{1}(0)+i \Psi_{2}(0)\right)\right] \times<F\left(U_{t}^{*}, 0\right), f_{0}>\cdot f_{0} \\
= & -\frac{1}{4}\left[g_{20} p_{1}(\theta)+\bar{g}_{02} p_{2}(\theta)\right] z^{2} \cdot f_{0}-\frac{1}{2}\left[g_{11} p_{1}(\theta)\right. \\
& \left.+\bar{g}_{11} p_{2}(\theta)\right] z \bar{z} \cdot f_{0}+\cdots . \tag{106}
\end{align*}
$$

Therefore, for $-1 \leq \theta<0$,

$$
\begin{align*}
H_{20}(\theta) & =-\frac{1}{2}\left[g_{20} p_{1}(\theta)+\bar{g}_{02} p_{2}(\theta)\right] \cdot f_{0},  \tag{107}\\
H_{11}(\theta) & =-\frac{1}{2}\left[g_{11} p_{1}(\theta)+\bar{g}_{11} p_{2}(\theta)\right] \cdot f_{0},  \tag{108}\\
H(z, \bar{z})(0) & =F\left(U_{t}^{*}, 0\right)-\Phi\left(\Psi,<F\left(U_{t}^{*}, 0\right), f_{0}>\right) \cdot f_{0}, \\
H_{20}(0) & =\frac{\tau^{*}}{2}\left(\begin{array}{l}
1 \\
\alpha \\
0
\end{array}\right)-\frac{1}{2}\left[g_{20} p_{1}(0)+\bar{g}_{02} p_{2}(0)\right] \cdot f_{0}, \\
H_{11}(0) & =\frac{\tau^{*}}{4}\left(\begin{array}{c}
2 \\
\alpha+\bar{\alpha} \\
0
\end{array}\right)-\frac{1}{2}\left[g_{11} p_{1}(0)+\bar{g}_{11} p_{2}(0)\right] \cdot f_{0} . \tag{109}
\end{align*}
$$

By the definition of $A_{\tau^{*}}$, we get from (105) that

$$
\begin{align*}
\dot{W}_{20}(\theta)= & 2 i \omega_{0} \tau^{*} W_{20}(\theta) \\
& +\frac{1}{2}\left[g_{20} p_{1}(\theta)+\bar{g}_{02} p_{2}(\theta)\right] \cdot f_{0}, \quad-1 \leq \theta<0 . \tag{110}
\end{align*}
$$

Noting that $p_{1}(\theta)=p_{1}(0) e^{i \omega_{0} \tau^{*}},-1 \leq \theta \leq 0$. Hence,

$$
\begin{equation*}
W_{20}(\theta)=\frac{i}{2}\left[\frac{g_{20}}{\omega_{0} \tau^{*}} p_{1}(\theta)+\frac{\bar{g}_{02}}{3 \omega_{0} \tau^{*}} p_{2}(\theta)\right] \cdot f_{0}+E e^{2 i \omega_{0} \tau^{*} \theta} \tag{111}
\end{equation*}
$$

$$
\begin{equation*}
E=W_{20}(0)-\frac{i}{2}\left[\frac{g_{20}}{\omega_{0} \tau^{*}} p_{1}(0)+\frac{\bar{g}_{02}}{3 \omega_{0} \tau^{*}} p_{2}(0)\right] \cdot f_{0} \tag{112}
\end{equation*}
$$

Using the definition of $A_{\tau^{*}}$ and combining (105) and (112) we get

$$
\begin{align*}
& 2 i \omega_{0} \tau^{*}\left[\frac{i g_{20}}{2 \omega_{0} \tau^{*}} p_{1}(0) \cdot f_{0}+\frac{i \bar{g}_{02}}{6 \omega_{0} \tau^{*}} p_{2}(0) \cdot f_{0}+E\right] \\
& -L\left(\tau^{*}\right)\left[\frac{i g_{20}}{2 \omega_{0} \tau^{*}} p_{1}(\theta) \cdot f_{0}+\frac{i \bar{g}_{02}}{6 \omega_{0} \tau^{*}} p_{2}(\theta) \cdot f_{0}+E e^{2 i \omega_{0} \tau^{*} \theta}\right] \\
& \quad=\frac{\tau^{*}}{2}\left(\begin{array}{l}
1 \\
\alpha \\
0
\end{array}\right)-\frac{1}{2}\left[g_{20} p_{1}(0)+\bar{g}_{02} p_{2}(0)\right] \cdot f_{0} \tag{113}
\end{align*}
$$

Notice that

$$
\left\{\begin{array}{l}
L\left(\tau^{*}\right)\left[p_{1}(\theta) \cdot f_{0}\right]=i \omega_{0} \tau^{*} p_{1}(0) \cdot f_{0}  \tag{114}\\
L\left(\tau^{*}\right)\left[p_{2}(\theta) \cdot f_{0}\right]=-i \omega_{0} \tau^{*} p_{2}(0) \cdot f_{0}
\end{array}\right.
$$

Then, we have

$$
2 i \omega_{0} \tau^{*} E-\tau^{*} D \Delta E-L\left(\tau^{*}\right)\left(E e^{2 i \omega_{0} \tau^{*} \theta}\right)=\frac{\tau^{*}}{2}\left(\begin{array}{l}
1  \tag{115}\\
\alpha \\
0
\end{array}\right)
$$

From the above expression, we can see easily that

$$
E=\frac{1}{2}\left(\begin{array}{ccc}
2 i \omega_{0}-y^{*}+a & -x^{*} & e^{-2 i \omega_{0} \tau^{*}}  \tag{116}\\
2 x^{*} & 2 i \omega_{0}+b & 0 \\
1 & 0 & 2 i \omega_{0}+c
\end{array}\right)^{-1} \times\left(\begin{array}{l}
1 \\
\alpha \\
0
\end{array}\right)
$$

By the similar way, we have

$$
\begin{align*}
& \dot{W}_{11}(\theta)=\frac{1}{2}\left[g_{11} p_{1}(\theta)+\bar{g}_{11} p_{2}(\theta)\right] \cdot f_{0}, \quad-1 \leq \theta<0  \tag{117}\\
& W_{11}(\theta)=\frac{i}{2 \omega_{0} \tau^{*}}\left[-g_{11} p_{1}(\theta)+\bar{g}_{11} p_{2}(\theta)\right] \cdot f_{0}+F \tag{118}
\end{align*}
$$



Figure 1: The equilibrium $P_{0}$ is locally asymptotically stable with $\tau=10$.


Figure 2: The equilibrium $P_{0}$ is locally asymptotically stable for $\tau=0.03$.

Similar to the above, we can obtain that

$$
F=\frac{1}{4}\left(\begin{array}{ccc}
a-y^{*} & -x^{*} & 1  \tag{119}\\
2 \mathrm{x}^{*} & b & 0 \\
1 & 0 & c
\end{array}\right)^{-1} \times\left(\begin{array}{c}
2 \\
\alpha+\bar{\alpha} \\
0
\end{array}\right) .
$$

So far, $W_{20}(\theta)$ and $W_{11}(\theta)$ have been expressed by the parameters of system (4). Therefore, $g_{21}$ can be expressed explicitly.

Theorem 4. System (4) has the following Poincaré normal form

$$
\begin{equation*}
\dot{\xi}=i \omega_{0} \tau^{*} \xi+c_{1}(0) \xi|\xi|^{2}+o\left(|\xi|^{5}\right) \tag{120}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{1}(0)=\frac{i}{2 \omega_{0} \tau^{*}}\left[g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|^{2}}{3}\right]+\frac{g_{21}}{2} \tag{121}
\end{equation*}
$$

so we can compute the following results:

$$
\begin{align*}
& \sigma_{2}=-\frac{\operatorname{Re}\left(c_{1}(0)\right)}{\operatorname{Re}\left(\lambda^{\prime}\left(\tau^{*}\right)\right)}, \\
& \beta_{2}=2 \operatorname{Re}\left(c_{1}(0)\right)  \tag{122}\\
& T_{2}=-\frac{\operatorname{Im}\left(c_{1}(0)\right)+\sigma_{2} \operatorname{Im}\left(\lambda^{\prime}\left(\tau^{*}\right)\right)}{\omega_{0} \tau^{*}},
\end{align*}
$$



Figure 3: The periodic solutions emerge from the equilibrium $P_{0}$ with $\tau=0.06$.


FIgure 4: (a) $P_{1}$ and $P_{2}$ are both stable with $\tau=0.7$; (b) the periodic solutions emerge from the equilibria $P_{1}$ and $P_{2}$ with $\tau=0.71$.


Figure 5: (a) Maximum and minimum of $x$; (b) the two limit cycles emerging from the equilibria $P_{1}$ and $P_{2}$ appear to overlap with $\tau=1$.


Figure 6: Lyapunov exponent spectrum of system (4) when the parameter value of $c$ is changed continuously with $\tau=0$.


Figure 7: Bifurcation diagrams of system (4) when the parameter value of $c$ is changed continuously with $\tau=0$.


Figure 8: (a) Time trajectories of system (4) with $\tau=0$; (b) phase diagram of system (4) with $\tau=0$.


Figure 9: Lyapunov exponent spectrum of system (4) when the parameter value of $\tau$ is changed continuously.


Figure 10: Bifurcation diagrams of system (4) when the parameter value of $\tau$ is changed continuously.
which determine the properties of bifurcating periodic solutions at the critical values $\tau^{*}$, i.e., $\sigma_{2}$ determines the directions of the Hopf bifurcation: if $\sigma_{2}>0\left(\sigma_{2}<0\right)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tau^{*} ; \beta_{2}$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are stable (unstable), if $\beta_{2}<0\left(\beta_{2}>0\right)$; and $T_{2}$ determines the period of the bifurcating periodic solutions: the periodic increase (decrease), if $T_{2}>0\left(T_{2}<0\right)$.

## 4. Numerical Simulation

In this section, we present numerical simulations of some examples to illustrate our theoretical results.
4.1. Stability of Equilibrium $P_{0}$ for All $\tau \geq 0$. Consider system (4) with the following parameters: $a=3, b=0.7$, and $c=0.9$. By a direct calculation, we obtain that system (4)


Figure 11: Phase diagrams showing chaos disappearing via inverse period-doubling: (a) $\tau=0.4$; (b) $\tau=0.44$; (c) $\tau=0.45$; (d) $\tau=0.5$; (e) $\tau=0.54$; (f) $\tau=0.56$.
has only an equilibrium $P_{0}=(0,1.4286,0)$ and the parameters satisfy the conditions of (H2)-(H4). According to Theorem 1 , the system is locally asymptotically stable at $P_{0}$ for all $\tau \geq 0$; see Figure 1. However, in this case, the interest rate and the price index are all zero; this is impractical.
4.2. Hopf Bifurcation at Equilibrium $P_{0}$. Consider system (4) with the following parameters: $a=0.5, b=0.8$, and $c=0.8$. By a simple calculation, we obtain that system (4) has only an equilibrium $P_{0}=(0,1.25,0)$. Obviously, (H2), (H3), and (H5) are satisfied. By (28), we obtain $\tau_{0}=0.05$. According
to Theorem 2 , system (4) is locally asymptotically stable at $P_{0}$ for $\tau=0.03 \in\left[0, \tau_{0}\right)$ (see Figure 2) and Hopf bifurcation occurs at $\tau=0.06>\tau_{0}$, as shown in Figure 3.

When $\tau=\tau_{0}=0.05$, we can compute $c_{1}(0)=-0.0023+$ $0.0042 i, \sigma_{2}=-\operatorname{Re}\left(c_{1}(0)\right) / \operatorname{Re}\left(\lambda^{\prime}\left(\tau^{*}\right)\right)=0.0047>0$, and $\beta_{2}=$ $2 \operatorname{Re}\left(c_{1}(0)\right)=-0.0047<0$. Therefore, from the discussions in Section 3, we know that the bifurcated periodic solutions are orbitally asymptotically stable on the center manifold. In addition, from Theorem 4, we know that system (4) has a stable center manifold near the equilibrium $P_{0}$ for $\tau$ near $\tau_{0}=0.05$. Therefore, the center manifold theory implies
that the bifurcated periodic solutions of system (4) when $\tau_{0}=0.05$ in the whole phase space are orbitally asymptotically stable, and the Hopf bifurcation is supercritical for $\sigma_{2}>0$.
4.3. Hopf Bifurcation at Equilibria $P_{1}$ and $P_{2}$. Choose the parameters of system (4) as $a=0.5, b=0.8$, and $c=1.5$. By a simple calculation, it is easy to obtain that $P_{1}=(0.2582,1$. $1667,-0.1721)$ and $P_{2}=(-0.2582,1.1667,0.1721)$. Obviously, the parameters satisfy (H6) and (H7). By (50), we obtain the critical value $\tau_{0}=0.7307$. According to Theorem 3, $P_{1}$ and $P_{2}$ are both stable with $\tau=0.68 \in\left[0 \tau_{0}\right)$, and with $\tau=0.71>\tau_{0}$, two limit cycles emerge from the equilibria $P_{1}$ and $P_{2}$, as shown in Figure 4.

In addition, when $\tau=\tau_{0}=0.7307$, at equilibrium $P_{1}$, we get $c_{1}(0)=-0.0316+0.056 i, \sigma_{2}=-\operatorname{Re}\left(c_{1}(0)\right) / \operatorname{Re}\left(\lambda^{\prime}\left(\tau^{*}\right)\right)=$ $0.0898>0$, and $\beta_{2}=2 \operatorname{Re}\left(c_{1}(0)\right)=-0.0632<0$. At equilibrium $P_{2}$, we get $c_{1}(0)=-0.0366+0.0675 i, \sigma_{2}=-\operatorname{Re}\left(c_{1}(0)\right) /$ $\operatorname{Re}\left(\lambda^{\prime}\left(\tau^{*}\right)\right)=0.0104>0$, and $\beta_{2}=2 \operatorname{Re}\left(c_{1}(0)\right)=-0.0733<$ 0 . According to Theorem 4 in Section 3, the bifurcated periodic solutions of system (4) when $\tau_{0}=0.7307$ in the whole phase space are both orbitally asymptotically stable, and the Hopf bifurcations are supercritical for $\sigma_{2}>0$.

However, with increasing delay $\tau$, the two limit cycles emerging form the equilibria $P_{1}$ and $P_{2}$ and appear to overlap, as shown in Figure 5. Figure 5(a) shows that the maximum and minimum of $x$ varies with $\tau$ under two groups of different initial values. It shows that two lines about maxima and minima appear to overlap with increasing delay $\tau$, which mean that two limit cycles overlap; see Figure 5(b).
4.4. Chaos Vanishes by Delay $\tau$. According to [10, 15, 27], system (4) is chaotic for appropriate parameters. Figure 6 shows the Lyapunov exponents' spectrum of system (4) with the increasing of parameter $c$, where $a=2$ and $b=0.1$. Figure 7 shows the bifurcation diagram of system (4) in the $c-y$ plane. Let $c=1.1$, and the chaotic attractor of system (4) is shown in Figure 8.

In the following, in order to investigate the effect of delay $\tau$ on system (4), we fix $a=2, b=0.1$, and $c=1.1$, and choosing $\tau$ as a parameter, the Lyapunov exponent spectrum and the detailed bifurcation scenarios of system (4) are shown in Figures 9 and 10. It can be seen that chaos disappears through a cascade of inverse period-doubling; see Figure 11. This observation indicates that the delay is a sensitive factor for system bifurcation and chaos and that chaos can be suppressed by delay $\tau$.
4.5. Chaos Induced by Delay $\tau$. Consider system (4) with the following parameters $a=3, b=0.2$, and $c=6$. Obviously, parameters satisfy condition (H6). Therefore, according to Lemma $5, P_{1,2}=( \pm 0.6055,3.1667, \mp 0.1009)$ are both locally stable with $\tau=0$. However, with increasing delay $\tau$, system (4) presents strong nonlinear phenomena such as periodic motion, double-periodic motion, and chaotic motion and the bifurcation diagram of system (4) with increasing delay $\tau$, which can be seen from the bifurcation diagram and maximum Lyapunov exponent with the parameter value


Figure 12: Bifurcation diagram of system (4) with the parameter value of $\tau$ is changed continuously.


Figure 13: Maximum Lyapunov exponent of system (4) when the parameter value of $\tau$ is changed continuously.

Table 1: The dynamic behaviors of system (4) for different $\tau$.

| $\tau$ | Dynamics of system (4) | Figure |
| :--- | :---: | :---: |
| $[0,1.52)$ | $P_{1}$ and $P_{2}$ are both stable | Figure 14(a) |
| $(1.52,3.18]$ | System (4) is chaotic | Figure 14(b) |
| $(3.18,5.52]$ | System (4) exhibits period 1 motion | Figure 14(c) |
| $(5.52,5.61]$ | System (4) is chaotic | Figure 14(d) |
| $(5.61,8.32]$ | $P_{1}$ and $P_{2}$ are both stable | Figure 14(e) |
| $(8.32,9.46]$ | System (4) exhibits period 1 motion | Figure 14(f) |
| $(9.46,9.68]$ | System (4) exhibits period 2 motion | Figure 14(g) |
| $(9.68,9.71]$ | System (4) exhibits period 4 motion | Figure 14(h) |
| $(9.71,12]$ | System (4) is chaotic | Figure 14(i) |

of $\tau$ changed continuously, as shown in Figures 12 and 13. We list dynamic behaviors of system (4) corresponding to different delays in Table 1 and Figure 14. The route to chaos in finance system (4) was shown to be via classical period-doubling bifurcations (see Figures 14(e)-14(i)).


Figure 14: Continued.

(i)

Figure 14: Nonlinear phenomena of system (4) induced by delay $\tau$ : (a) $\tau=1$ (stable); (b) $\tau=2$ (chaos); (c) $\tau=4.2$ (period 1); (d) $\tau=5.58$ (chaos); (e) $\tau=7$ (stable); (f) $\tau=8.5($ period 1$) ;(\mathrm{g}) \tau=9.5($ period 2$) ;$ (h) $\tau=9.69($ period 4$) ;$ (i) $\tau=12$ (chaos).

## 5. Conclusions

In this study, we have investigated dynamical behaviors such as stability, Hopf bifurcation, and chaos for a delayed finance system.

Firstly, we took delay $\tau$ as the bifurcation parameters to study the Hopf bifurcation of system (4). We have proved theoretically that the discrete delay is responsible for the stability switch of the model and that a Hopf bifurcation occurs as the delays increase through a certain threshold.

Secondly, by the normal form method and center manifold theorem, we have derived the normal forms of Hopf bifurcation.

Finally, by numerical simulations, we have given the Hopf bifurcation (Figures 3 and 4) that was induced by delay. We have also given the bifurcation diagram (Figures 10 and 12) and the corresponding Lyapunov exponents' spectrum (Figures 6 and 13). All these show that delay $\tau$ can cause the system to exhibit strong nonlinear phenomena such as periodic motion, double-periodic motion, and chaotic motion (Figure 14).

The study will help in understanding the role of financial policies and interpreting economics phenomena in theory.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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