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Research Article

New Stabilization Results for Semi-Markov Chaotic Systems with Fuzzy Sampled-Data Control

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This paper investigates the problem of stabilization for semi-Markov chaotic systems with fuzzy sampled-data controllers, in which the semi-Markov jump has generally uncertain transition rates. The exponential stability condition is firstly obtained by the following two main techniques: To make full use of the information about the actual sampling pattern, a novel augmented input-delay-dependent Lyapunov–Krasovskii functional (LKF) is firstly introduced. Meanwhile, a new zero-value equation is established to increase the combinations of component vectors of the resulting vector. The corresponding fuzzy sampled-data controllers are designed based on the stability condition. Finally, the validity and merits of the developed theories are shown by two numerical examples.

1. Introduction

As we all know, Takagi–Sugeno fuzzy model (T-SFM) [1] is an applicable and effective way for us to approximate complex nonlinear systems. It is often used for handling some practical nonlinear systems in many controlling engineering areas because the T-SFM could show topical linear input-output relations of a nonlinear system. For example, some famous nonlinear systems can be accurately expressed by the T-SFM, such as Chua's system [2], Lorenz system [3], and Rösler system [4]. These systems are called chaotic systems. Consequently, T-S fuzzy control has a significant role in chaotic systems, and plenty of meaningful results on T-S fuzzy chaotic systems have been proposed based on various kinds of approaches [5–9].

Because of the fast development of microelectronics technology and communication networks, sampled-data control enjoys great popularity owing to its maintenance with low cost, better reliability, and efficiency [10–16]. In general, in the realization of sampled-data control systems, only the sampling information is sent to the controller at discrete time instants, which would greatly decrease the amount of information while increasing the efficiency of bandwidth usage. Thus, to investigate the sampled-data control for fuzzy chaotic systems has considerable realistic significance. Up to now, three methods are mainly used, namely, impulsive control approach [13, 14], discrete-time technique [15], and input delay method [12, 17], to deal with the sampled-data control systems, in which the most favorite way is the input delay method. Combining the input delay method with the zero-order holder (ZOH), the sampled-data control system can be transformed as a continuous-time system with time-varying delayed control input. In recent years, according to the input delay method, lots of important stabilization results on chaotic systems with fuzzy sampled-data control were presented [18–27]. In [18, 23, 24], by designing fuzzy sampled-data control, the stabilization conditions for chaotic systems have been obtained. In [19, 20], by constructing input-delay-dependent LKFs, less conservative criteria for chaotic system have been studied. Very recently, by utilizing the fuzzy input-delay-dependent LKF approach, fewer constraint conditions for chaotic

systems have been established in [25, 26]. Most recently, by constructing a new input-delay-dependent LKF, improved results have been established for chaotic systems in [21, 27]. It is well known that the input-delay-dependent Lyapunov method is very effective for reducing conservatism because it can fully capture the information about the actual sampling pattern. However, in the most existing works, input-delay-dependent Lyapunov terms are mainly constructed based on linear function of input delay. Input-delay-dependent Lyapunov terms with high-order function of input delay are difficultly contained because their derivation may be nonlinear input delay. Then, the convex combination technique cannot be directly utilized. Thus, it is still a challenging issue to construct input-delay-dependent LKFs with high-order function of input delay to further improve the existing results for fuzzy sampled-data chaotic systems, which is the first motivation of this paper.

Moreover, it should be pointed out that the existing works [18–27] considered how to obtain less conservative stabilization conditions, while they ignored calculation load. Actually, during the application of LMI-based criteria to large-scale physical systems, the computational complexity is a very significant problem [28]. An LMI-based condition with too much complexity may be limited in many practical applications. Hence, it is important to establish stabilization criteria discussing both the conservatism and calculation complexity for fuzzy sampled-data chaotic systems, which is the second motivation of this paper.

On the contrary, in the real world, the Markov jump usually exists in chaotic systems since the abrupt changes in their structure and parameters. So, the study on chaotic systems with Markov jump parameters is of great importance [29–31]. It is noted that the transition rates in the aforementioned works are always supposed to be totally known. The transition rates are usually difficult to accurately appraise in general; thus, the Markov jump systems with generally uncertain transition rates were considered in [32–34] and the references therein. Reference [27] studied the problem of stability and stabilization for fuzzy chaotic systems with fuzzy sampled data and Markov jump of generally uncertain transition rates. Some studied Markov jump has native limitations since the Markov chain is assumed to be exponentially distributed and the transition rates are considered to be constants. As a consequence, the obtained results are conservative to some extent. In order to avoid this situation, semi-Markov jump systems were discussed in [35–39] by supposing the transition rates to be time varying instead of constant ones. Meanwhile, one can easily see that the obtained results for the Markov jump systems are special cases of those for semi-Markov jump systems. Clearly, semi-Markov jump systems are more general than Markov jump systems in applications. Therefore, it is necessary to consider semi-Markov chaotic systems with fuzzy sampled-data control. However, to the best of our knowledge, the issues have not been considered yet. On the basis of this fact, the previous stability and stabilization criteria will no longer be applicable. Hence, the second motivation of this study becomes more and more important.

Based on the above discussions, the aim of this paper is to further investigate the problem of stabilization for fuzzy chaotic systems with sampled-data control and semi-Markov jump of generally uncertain transition rates. The remainder of this article is given as follows: Many useful notations are given and the main problems are described in Section 2. The stability and stabilization conditions are provided in Section 3. In Section 4, two numerical examples are presented to show the feasibility and superiority of our results. Section 5 gives a conclusion. The advantages of this paper are summarized as follows:

(i) This discussion for chaotic systems with both fuzzy sampled-data control and semi-Markov jump of generally uncertain transition rates is for the first time to be proposed. The obtained results in practical situation are more applicable than the previous results in [18, 20–27].

(ii) To make more cross terms of vector components in the resulting vector, a novel zero equality is established.

(iii) To take full advantage of the available information with regard to the practical sampling mode, a new augmented time-dependent LKF $V_3(x(t), r_t)$ is constructed, which is firstly presented for the fuzzy sampled-data semi-Markov chaotic systems.

(iv) Compared with the existing stabilization conditions in [21, 24–26], our results are not only less computationally complex but also less conservative.

2. Problem Statement and Preliminaries

To be clear, the following symbols are firstly explained in a simple way:

T: transpose of a matrix or a vector

$\mathscr{R}^n$: n-dimensional Euclidean space

$\mathscr{R}^{n\times m}$: set of all $n \times m$ real matrices

$P > 0$: matrix P is symmetric and positive definite

$*$: symmetric terms in a symmetric matrix

I_n: identity matrix

$0_{n\times m}$: zero matrices

$\mathrm{diag}\{\cdots\}$: a block-diagonal matrix

$\mathrm{sym}\{X\}$: $X + X^T$

$\lambda_{\max}(G)$ ($\lambda_{\min}(G)$): the largest (smallest) eigenvalue of G

$\|\cdot\|$: Euclidean norm for a given vector

$(\Omega, \mathscr{F}, \mathscr{P})$: complete probability space with a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the natural filtration contains all $\mathscr{P}$-null sets and is right continuous)

$\mathbb{E}\{\cdot\}$: expectation function in regard to the given probability measure $\mathscr{P}$

Consider the semi-Markov chaotic system expressed by the following differential equation:

$$\dot{x}(t) = g(x(t), u(t, r_t), r_t), \tag{1}$$

where $x(t) \in \mathscr{R}^n$ denotes the state vector, $u(t, r_t) \in \mathscr{R}^m$ denotes the control input vector, and $g(\cdot)$ denotes a known nonlinear continuous function and satisfies $g(0, 0, r_t) = 0$. $\{r_t, t \geq 0\}$ denotes the right-continuous semi-Markov process on the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and takes values in a finite state set $\mathbb{S} = (1, 2, \ldots, s)$ with the transition rate matrix $\Pi(h) = (\pi_{pq}(h))_{s\times s}$ given by

$$\Pr\{r_{t+h} = q \mid r_t = p\} = \begin{cases} \pi_{pq}(h)h + o(h), & p \neq q, \\ 1 + \pi_{pp}(h)h + o(h), & p = q, \end{cases} \tag{2}$$

where $h > 0$ and $\lim_{h\longrightarrow 0} o(h)/h = 0$; the transition rates $\pi_{pq}(h)$ are positive, if $p \neq q$, while $\pi_{pp}(h) = -\sum_{q=1, q\neq p}^{s} \pi_{pq}(h)$ for each mode $p \in \mathbb{S}$.

Actually, not all the transition rates $\pi_{pq}(h)$ are easy to accurately estimate in the jumping process; most of them always contain the following two assumptions: (1) $\pi_{pq}(h)$ is totally uncharted and (2) $\pi_{pq}(h)$ is not exactly known, but upper and lower bounded. Based on the second condition, we can further define $\pi_{pq}(h) \in [\underline{\pi}_{pq}, \overline{\pi}_{pq}]$, in which $\underline{\pi}_{pq}$ and $\overline{\pi}_{pq}$ are known real constants to represent the lower and upper bounds of $\pi_{pq}(h)$, respectively. Furthermore, we can denote that $\pi_{pq}(h) \triangleq \pi_{pq} + \Delta\pi_{pq}(h)$, in which $\pi_{pq} = (1/2)(\underline{\pi}_{pq} + \overline{\pi}_{pq})$ and $|\Delta\pi_{pq}(h)| \leq \lambda_{pq}$ with $\lambda_{pq} = (1/2)(\overline{\pi}_{pq} - \underline{\pi}_{pq})$. Thus, the time-varying transition rate matrix $\Pi(h)$ with s jumping modes may be described as

$$\Pi(h) = \begin{bmatrix} \pi_{11} + \Delta\pi_{11}(h) & \nabla & \nabla & \cdots & \pi_{1s} + \Delta\pi_{1s}(h) \\ \nabla & \nabla & \pi_{23} + \Delta\pi_{23}(h) & \cdots & \pi_{2s} + \Delta\pi_{2s}(h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nabla & \pi_{s2} + \Delta\pi_{s2}(h) & \nabla & \cdots & \nabla \end{bmatrix}, \tag{3}$$

where ∇ is unknown time-varying transition rates. For convenience, $\forall p \in \mathbb{S}$, the set $\mathscr{G}_p$ denotes $\mathscr{G}_{p,k} \cup \mathscr{G}_{p,uk}$ with

$$\begin{aligned} \mathscr{G}_{p,k} &\triangleq \{q : \pi_{pq} \text{ can be determined for } q \in \mathbb{S}\}, \\ \mathscr{G}_{p,uk} &\triangleq \{q : \pi_{pq} \text{ is unknown for } q \in \mathbb{S}\}. \end{aligned} \tag{4}$$

Moreover, if $\mathscr{G}_{p,k} \neq \varnothing$ and $\mathscr{G}_{p,uk} \neq \varnothing$, we can obtain the following:

$$\mathscr{G}_{p,k} \triangleq \{\mathbf{k}_{p,1}, \mathbf{k}_{p,2}, \ldots, \mathbf{k}_{p,v}\}, \quad 1 \leq v < s. \tag{5}$$

By using the following IF-THEN rules, system (1) can be developed into the following systems: Plant rule i: IF $\theta_1(t)$ is α_1^{i,r_t} and ... and $\theta_l(t)$ is α_l^{i,r_t}, THEN

$$\dot{x}(t) = \mathscr{A}_{i,r_t} x(t) + \mathscr{B}_{i,r_t} u(t, r_t), \tag{6}$$

where $i \in \Theta = \{1, 2, \ldots, w\}$, $\theta_1(t), \ldots, \theta_l(t)$ represent premise variables, $\alpha_1^{i,r_t}, \ldots, \alpha_l^{i,r_t}$ represent the fuzzy sets, $\mathscr{A}_{i,r_t} \in \mathscr{R}^{n\times n}$ and $\mathscr{B}_{i,r_t} \in \mathscr{R}^{n\times m}$ represent the known constant matrices with suitable dimensions, and w shows the number of fuzzy rules.

By using the fuzzy blending, the fuzzy semi-Markov jump system (6) can be rewritten as follows:

$$\dot{x}(t) = \sum_{i=1}^{w} \omega_{i,r_t}(\theta(t))\left(\mathscr{A}_{i,r_t} x(t) + \mathscr{B}_{i,r_t} u(t, r_t)\right), \tag{7}$$

where $\theta(t) = [\theta_1(t), \ldots, \theta_l(t)]^T$ and $\omega_{i,r_t}(\theta(t))$ is the membership function satisfying

$$\omega_{i,r_t}(\theta(t)) = \frac{\prod_{j=1}^{l} \alpha_j^{i,r_t}(\theta_j(t))}{\sum_{n=1}^{w} \prod_{j=1}^{l} \alpha_j^{n,r_t}(\theta_j(t))}, \tag{8}$$

in which $\alpha_j^{i,r_t}(\theta_j(t))$ stands for the grade of membership of $\theta_j(t)$ in α_j^{i,r_t}. For all $t > 0$, we have $\omega_{i,r_t}(\theta(t)) \geq 0$ and $\sum_{i=1}^{w} \omega_{i,r_t}(\theta(t)) = 1$.

Throughout this paper, the control signal is supposed to be generated by using a ZOH function with a series of holding moments $0 = t_0 < t_1 < \cdots < t_k < \cdots$. Accordingly, the fuzzy sampled-data controllers can be easily given as follows.

Controller rule j: IF $\theta_1(t)$ is α_1^{i,r_t} and ... and $\theta_l(t)$ is α_l^{i,r_t}, THEN

$$u(t, r_t) = \mathscr{K}_{j,r_t} x(t - \tau(t)), \quad t_k \leq t < t_{k+1}, \tag{9}$$

with $j = 1, 2, \ldots, w$, $\tau(t) = t - t_k$, and $\mathscr{K}_{j,r_t}$ $(j \in \Theta)$ denotes the controller gain matrices to be computed. The sampling interval τ_k meets the condition $0 \leq \tau(t) \leq \tau_k = t_{k+1} - t_k \leq \tau$, where τ is the upper bound of the sampling periods.

Hence, the entire fuzzy sampled-data controller can be represented as follows:

$$u(t, r_t) = \sum_{j=1}^{w} \omega_{j,r_t}(\theta(t_k)) \mathscr{K}_{j,r_t} x(t - \tau(t)). \tag{10}$$

For the sake of convenience, r_t is noted as $r_t = p\,(p \in \mathbb{S})$ hereinafter. Introducing (10) into (7), the fuzzy system (7) is further developed into

$$\begin{aligned} \dot{x}(t) = &\sum_{i=1}^{w} \sum_{j=1}^{w} \omega_{i,p}(\theta(t)) \omega_{j,p}(\theta(t_k)) \\ &\cdot \left[\mathscr{A}_{i,p} x(t) + \mathscr{B}_{i,p} \mathscr{K}_{j,p} x(t - \tau(t))\right], \quad t_k \leq t < t_{k+1}. \end{aligned} \tag{11}$$

To clearly give the main results, we firstly provide the following definition and lemmas.

Definition 1. System (11) is exponentially stable if scalars $\beta > 0$ and $\delta > 0$ satisfy

$$\mathbb{E}\|x(t)\| \le \delta e^{-\beta t} \sup_{-\tau\le\psi\le 0} \mathbb{E}\{\|x(\psi)\|, \|\dot{x}(\psi)\|\}, \tag{12}$$

where β and δ are defined as the decay rate and decay coefficient, respectively.

Lemma 1. *The system (11) holds the following conclusion:*

$$\|x(t)\|^2 \le \varepsilon \|x(t_k)\|^2, \quad t_k \le t < t_{k+1}, \tag{13}$$

with $\varepsilon_1 = \max_{i\in\Theta, p\in\mathbb{S}}(\|\mathscr{B}_{i,p}\|)$, $\varepsilon_2 = \max_{j\in\Theta, p\in\mathbb{S}}(\|\mathscr{K}_{j,p}\|)$, $\varepsilon_3 = \max_{i\in\Theta, p\in\mathbb{S}}(\|\mathscr{A}_{i,p}\|)$, *and* $\varepsilon = 3(1+\tau^2\varepsilon_1\varepsilon_2)e^{3\tau^2\varepsilon_3}$.

Proof. One can obtain for $\forall t \in [t_k, t_{k+1})$ from (11) that

$$\begin{aligned}\|x(t)\| \le \|x(t_k)\| &+ \left\|\int_{t_k}^{t} \sum_{i=1}^{w} \omega_{i,p}(\theta(\epsilon))\mathscr{A}_{i,p}x(\epsilon)\mathrm{d}\epsilon\right\| \\ &+ \left\|\int_{t_k}^{t} \sum_{i=1}^{w}\sum_{j=1}^{w} \omega_{i,p}(\theta(\epsilon))\omega_{j,p}(\theta(t_k))\mathscr{B}_{i,p}\mathscr{K}_{j,p}x(t_k)\mathrm{d}\epsilon\right\|.\end{aligned} \tag{14}$$

Using the Cauchy–Schwarz inequality, we can get from (14) that

$$\begin{aligned}\|x(t)\|^2 &\le 3\|x(t_k)\|^2 + 3\left\|\int_{t_k}^{t} \sum_{i=1}^{w} \omega_{i,p}(\theta(\epsilon))\mathscr{A}_{i,p}x(\epsilon)\mathrm{d}\epsilon\right\|^2 + 3\left\|\int_{t_k}^{t} \sum_{i=1}^{w}\sum_{j=1}^{w} \omega_{i,p}(\theta(\epsilon))\omega_{j,p}(\theta(t_k))\mathscr{B}_{i,p}\mathscr{K}_{j,p}x(t_k)\mathrm{d}\epsilon\right\|^2 \\ &\le 3\|x(t_k)\|^2 + 3\tau\int_{t_k}^{t}\left\|\sum_{i=1}^{w}\omega_{i,p}(\theta(\epsilon))\mathscr{A}_{i,p}x(\epsilon)\right\|^2\mathrm{d}\epsilon + 3\tau\int_{t_k}^{t}\left\|\sum_{i=1}^{w}\sum_{j=1}^{w}\omega_{i,p}(\theta(\epsilon))\omega_{j,p}(\theta(t_k))\mathscr{B}_{i,p}\mathscr{K}_{j,p}x(t_k)\right\|^2\mathrm{d}\epsilon \\ &\le 3\|x(t_k)\|^2 + 3\tau\varepsilon_3\int_{t_k}^{t}\|x(\varepsilon)\|^2\mathrm{d}\epsilon + 3\tau\varepsilon_1\varepsilon_2\int_{t_k}^{t}\|x(t_k)\|^2\mathrm{d}\epsilon \\ &\le 3\left(1+\tau^2\varepsilon_1\varepsilon_2\right)\|x(t_k)\|^2 + 3\tau\varepsilon_3\int_{t_k}^{t}\|x(\varepsilon)\|^2\mathrm{d}\epsilon.\end{aligned} \tag{15}$$

Utilizing the Gronwall–Bellman inequality, it is easy to see from (15) that (13) concludes. This proves Lemma 1. □

Lemma 2 (Park et al. [40]). *For a given matrix $\mathscr{V} > 0$, any differentiable function $x : [c,d] \longrightarrow \mathscr{R}^n$ satisfies the following inequality:*

$$\int_c^d \dot{x}^T(s)\mathscr{V}\dot{x}(s)\mathrm{d}s \ge \frac{\Upsilon_1^T\mathscr{V}\Upsilon_1 + 3\Upsilon_2^T\mathscr{V}\Upsilon_2}{d-c}, \tag{16}$$

with

$$\begin{aligned}\Upsilon_1 &= x(d) - x(c), \\ \Upsilon_2 &= x(d) + x(c) - \frac{2}{d-c}\int_c^d x(s)\mathrm{d}s.\end{aligned} \tag{17}$$

3. Main Results

Two important results will be established in this section: First, the exponential stability condition of the fuzzy semi-Markov jump system (11) is given for known control gain matrices $\mathscr{K}_{j,p}$. Second, by designing the control gain matrices $\mathscr{K}_{j,p}$, the exponential stabilization criterion of the fuzzy semi-Markov jump system (11) is obtained. For clarity, we need to define the following matrices and vectors:

$$\begin{aligned}z_1(t) &= \tau(t)x(t), \\ z_2(t) &= \tau(t)\dot{x}(t), \\ z_3(t) &= \tau(t)\int_{t_k}^{t} x(s)ds, \\ v(t) &= \frac{1}{t-t_k}\int_{t_k}^{t} x(s)ds, \\ \xi(t) &= \Big[x^T(t), \dot{x}^T(t), x^T(t_k), x^T(t-\tau), v^T(t), \\ &\quad \frac{1}{\tau-t+t_k}\int_{t-\tau}^{t_k} x^T(s)ds, z_1^T(t), z_2^T(t), z_3^T(t)\Big]^T, \\ \eta(t) &= \left[x^T(t) - x^T(t_k), z_1^T(t), z_3^T(t)\right]^T, \\ \mathbf{e}_i &= \left[\,0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(9-i)n}\right], \\ \Psi &= \left[\mathbf{e}_1^T, \mathbf{e}_2^T, \mathbf{e}_3^T, \mathbf{e}_4^T, \mathbf{e}_5^T, \mathbf{e}_6^T, \mathbf{e}_7^T, \mathbf{e}_8^T, \mathbf{e}_9^T\right]^T.\end{aligned} \tag{18}$$

Theorem 1. *Given scalars τ and β, the fuzzy system (11) with control gain matrices $\mathscr{K}_{j,p}$ is exponentially stable; if there exist positive definite matrices $\mathscr{P}_p$, $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{T}_{pq}$, and $\mathscr{Z}_{pq}$, any matrices $\mathscr{S}_{11}$, $\mathscr{S}_{12}$, $\mathscr{S}_{22}$, $\mathscr{H}$, $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{F}_m (m = 1, 2)$ with*

appropriate dimensions, such that the following conditions hold for all $i, j \in \Theta$, and $p \in \mathbb{S}$.

Case 1. $p \in \mathcal{G}_{p,k}$, $\forall l \in \mathcal{G}_{p,uk}$, $\mathcal{G}_{p,k} \triangleq \{\mathbf{k}_{p,1}, \mathbf{k}_{p,2}, \ldots, \mathbf{k}_{p,v1}\}$,

$$\overline{\mathcal{R}} = \begin{bmatrix} \mathcal{R} & 0 & \mathcal{S}_{11} & \mathcal{S}_{12} \\ * & 3\mathcal{R} & \mathcal{S}_{12}^T & \mathcal{S}_{22} \\ * & * & \mathcal{R} & 0 \\ * & * & * & 3\mathcal{R} \end{bmatrix} \geq 0, \tag{19}$$

$$\begin{bmatrix} \sum_1^{(1)} & \Gamma^{(1)} \\ * & -\Lambda^{(1)} \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \sum_2^{(1)} & \Gamma^{(1)} \\ * & -\Lambda^{(1)} \end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned} \sum_1^{(1)} &= \Phi(\tau(t) = 0) + \mathbf{e}_1^T T_{pq} \mathbf{e}_1, \\ \sum_2^{(1)} &= \Phi(\tau(t) = \tau) + \mathbf{e}_1^T T_{pq} \mathbf{e}_1, \\ \Gamma^{(1)} &= \left[\mathbf{e}_1^T\left(\mathcal{P}_{k_{p,1}} - \mathcal{P}_l\right), \mathbf{e}_1^T\left(\mathcal{P}_{k_{p,2}} - \mathcal{P}_l\right), \ldots, \mathbf{e}_1^T\left(\mathcal{P}_{k_{p,v1}} - \mathcal{P}_l\right)\right], \\ \Lambda^{(1)} &= \operatorname{diag}\left\{\mathcal{T}_{p,k_{p,1}}, \mathcal{T}_{p,k_{p,2}}, \ldots, \mathcal{T}_{p,k_{p,v1}}\right\}, \\ T_{pq} &= \sum_{q \in \mathcal{G}_{p,k}} \left[\frac{\left(\lambda_{pq}\right)^2}{4} \mathcal{T}_{pq} + \pi_{pq}\left(\mathcal{P}_q - \mathcal{P}_l\right)\right], \end{aligned} \tag{22}$$

with

$$\begin{aligned} \Phi(\tau(t)) &= \Phi_0(\tau(t)) + \Phi_1(\tau(t)), \\ \Phi_0(\tau(t)) &= \operatorname{sym}\left\{\Psi^T \mathcal{X} \begin{bmatrix} \tau(t)\mathbf{e}_1 - \mathbf{e}_7 \\ \tau(t)\mathbf{e}_2 - \mathbf{e}_8 \\ \tau^2(t)\mathbf{e}_5 - \mathbf{e}_9 \end{bmatrix}\right\} + \operatorname{sym}\left\{\left(\mathbf{e}_1^T \mathcal{F}_1^T + \mathbf{e}_2^T \mathcal{F}_2^T\right)\left(\mathcal{A}_{i,p}\mathbf{e}_1 + \mathcal{B}_{i,p}\mathcal{K}_{j,p}\mathbf{e}_3 - \mathbf{e}_2\right)\right\}, \\ \Phi_1(\tau(t)) &= \operatorname{sym}\left\{\mathbf{e}_1^T \mathcal{P}_p \mathbf{e}_2\right\} + 2\beta \mathbf{e}_1^T \mathcal{P}_p \mathbf{e}_1 + \mathbf{e}_1^T \mathcal{Q} \mathbf{e}_1 - e^{-2\beta\tau} \mathbf{e}_4^T \mathcal{Q} \mathbf{e}_4 + \tau^2 \mathbf{e}_2^T \mathcal{R} \mathbf{e}_2 - e^{-2\beta\tau} \Pi_1^T \overline{\mathcal{R}} \Pi_1 \\ &\quad - \operatorname{sym}\left\{\Pi_2^T \mathcal{H} \Pi_2\right\} + (\tau - \tau(t))\operatorname{sym}\left\{2\beta \Pi_2^T \mathcal{H} \Pi_2\right\} + (\tau - \tau(t))\operatorname{sym}\left\{\begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_1 + \mathbf{e}_8 \\ \tau(t)\mathbf{e}_5 + \mathbf{e}_7 \end{bmatrix}^T \mathcal{H} \Pi_2\right\} \\ &\quad + 2\beta(\tau - \tau(t))\tau(t)\mathbf{e}_5^T \mathcal{Y} \mathbf{e}_5 + (\tau - 2\tau(t))\mathbf{e}_5^T \mathcal{Y} \mathbf{e}_5 + \operatorname{sym}\left\{(\tau - \tau(t))\mathbf{e}_5^T \mathcal{Y}\left(\mathbf{e}_1 - \mathbf{e}_5\right)\right\}, \\ \Pi_1 &= \begin{bmatrix} \mathbf{e}_1 - \mathbf{e}_3 \\ \mathbf{e}_1 + \mathbf{e}_3 - 2\mathbf{e}_5 \\ \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{e}_3 + \mathbf{e}_4 - 2\mathbf{e}_6 \end{bmatrix}, \\ \Pi_2 &= \begin{bmatrix} \mathbf{e}_1 - \mathbf{e}_3 \\ \mathbf{e}_7 \\ \mathbf{e}_9 \end{bmatrix}. \end{aligned} \tag{23}$$

Case 2. $p \in \mathcal{G}_{p,uk}$, $\forall l \in \mathcal{G}_{p,uk}$, $\mathcal{G}_{p,k} \triangleq \{\mathbf{k}_{p,1}, \mathbf{k}_{p,2}, \ldots, \mathbf{k}_{p,v2}\}$, $l \neq p$,

$$\overline{\mathcal{R}} = \begin{bmatrix} \mathcal{R} & 0 & \mathcal{S}_{11} & \mathcal{S}_{12} \\ * & 3\mathcal{R} & \mathcal{S}_{12}^T & \mathcal{S}_{22} \\ * & * & \mathcal{R} & 0 \\ * & * & * & 3\mathcal{R} \end{bmatrix} \geq 0, \tag{24}$$

$$\mathcal{P}_p - \mathcal{P}_l \geq 0, \tag{25}$$

$$\begin{bmatrix} \sum_1^{(2)} & \Gamma^{(2)} \\ * & -\Lambda^{(2)} \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} \sum_2^{(2)} & \Gamma^{(2)} \\ * & -\Lambda^{(2)} \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned}
\sum\nolimits_1^{(2)} &= \Phi(\tau(t)=0) + \mathbf{e}_1^T Z_{pq}\mathbf{e}_1,\\
\sum\nolimits_2^{(2)} &= \Phi(\tau(t)=\tau) + \mathbf{e}_1^T Z_{pq}\mathbf{e}_1,\\
\Gamma^{(2)} &= \left[\mathbf{e}_1^T\left(\mathscr{P}_{k_{p,1}} - \mathscr{P}_l\right), \mathbf{e}_1^T\left(\mathscr{P}_{k_{p,2}} - \mathscr{P}_l\right), \ldots, \mathbf{e}_1^T\left(\mathscr{P}_{k_{p,v2}} - \mathscr{P}_l\right)\right],\\
\Lambda^{(2)} &= \operatorname{diag}\left\{\mathscr{Z}_{p,k_{p,1}}, \mathscr{Z}_{p,k_{p,2}}, \ldots, \mathscr{Z}_{p,k_{p,v2}}\right\},\\
Z_{pq} &= \sum_{q\in\mathscr{G}_{p,k}}\left[\frac{\left(\lambda_{pq}\right)^2}{4}\mathscr{Z}_{pq} + \pi_{pq}\left(\mathscr{P}_q - \mathscr{P}_l\right)\right].
\end{aligned} \tag{28}$$

Proof. Noticing the fact that $\tau(t)x(t) - z_1(t) = 0$, $\tau(t)\dot{x}(t) - z_2(t) = 0$, and $\tau(t)\int_{t_k}^{t} x(s)\mathrm{d}s - z_3(t) = 0$, for any matrix $\mathscr{X}$ with suitable dimensions, the following zero equality holds:

$$2e^{2\beta t}\begin{bmatrix} \tau(t)x(t) - z_1(t)\\ \tau(t)\dot{x}(t) - z_2(t)\\ \tau(t)\displaystyle\int_{t_k}^{t} x(s)\mathrm{d}s - z_3(t)\end{bmatrix}^T \mathscr{X}^T\xi(t) = 0. \tag{29}$$

Consider an LKF candidate as

$$V(x(t), r_t) = \sum_{m=1}^{4} V_m(x(t), r_t), \quad t\in[t_k, t_{k+1}), \tag{30}$$

with

$$\begin{aligned}
V_1(x(t), r_t) &= e^{2\beta t}x^T(t)\mathscr{P}(r_t)x(t),\\
V_2(x(t), r_t) &= \int_{t-\tau}^{t} e^{2\beta s}x^T(s)\mathscr{Q}x(s)\mathrm{d}s\\
&\quad + \tau\int_{t-\tau}^{t}\int_{\theta}^{t} e^{2\beta s}\dot{x}^T(s)\mathscr{R}\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta,\\
V_3(x(t), r_t) &= 2e^{2\beta t}(\tau - \tau(t))\eta^T(t)\mathscr{H}\eta(t),\\
V_4(x(t), r_t) &= e^{2\beta t}(\tau - \tau(t))\tau(t)v^T(t)\mathscr{Y}v(t).
\end{aligned} \tag{31}$$

Let $\mathscr{L}$ be the weak infinitesimal operator for the semi-Markov process acting on $V(x(t), r_t)$ as follows:

$$\mathscr{L}V(x(t), r_t) = \lim_{\iota\longrightarrow 0}\frac{\mathbb{E}\{V(x(t+\iota), r_{t+\iota}) \mid x(t), r_t\} - V(x(t), r_t)}{\iota}, \tag{32}$$

where $\iota > 0$ is a scalar. Then, similar to the computation in [35, 36], we have

$$\mathscr{L}V_1(x(t), r_t) = e^{2\beta t}\left\{2\beta x^T(t)\mathscr{P}_p x(t) + 2x^T(t)\mathscr{P}_p\dot{x}(t) + x^T(t)\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q x(t)\right\}, \tag{33}$$

$$\mathscr{L}V_2(x(t), r_t) \le e^{2\beta t}\left\{x^T(t)\mathscr{Q}x(t) - e^{-2\beta\tau}x^T(t-\tau)\mathscr{Q}x(t-\tau) + \tau^2\dot{x}^T(t)\mathscr{R}\dot{x}(t) - \tau e^{-2\beta\tau}\int_{t-\tau}^{t}\dot{x}^T(s)\mathscr{R}\dot{x}(s)\mathrm{d}s\right\}, \tag{34}$$

$$\mathscr{L}V_3(x(t), r_t) = e^{2\beta t}\left\{4\beta(\tau - \tau(t))\eta^T(t)\mathscr{H}\eta(t) - 2\eta^T(t)\mathscr{H}\eta(t) + 2(\tau - \tau(t))\begin{bmatrix}\dot{x}(t)\\ x(t) + z_2(t)\\ \displaystyle\int_{t_k}^{t} x(s)\mathrm{d}s + z_1(t)\end{bmatrix}^T\mathscr{H}\eta(t)\right\}, \tag{35}$$

$$\mathscr{L}V_4(x(t), r_t) = e^{2\beta t}\left\{2\beta(\tau - \tau(t))\tau(t)v^T(t)\mathscr{Y}v(t) + 2(\tau - \tau(t))(x(t) - v(t))^T\mathscr{Y}v(t) + (\tau - 2\tau(t))v^T(t)\mathscr{Y}v(t)\right\}. \tag{36}$$

Since $\overline{\mathscr{R}} > 0$, by Lemma 2 and the reciprocally convex approach in [41], we have

$$-\tau e^{-2\beta\tau}\int_{t-\tau}^{t}\dot{x}^T(s)\mathscr{R}\dot{x}(s)\mathrm{d}s \le -e^{-2\beta\tau}\xi^T(t)\Pi_1^T\overline{\mathscr{R}}\Pi_1\xi(t). \tag{37}$$

According to the fuzzy system (11), for any appropriate dimensional matrices $\mathscr{F}_m\,(m = 1, 2)$, the following zero equality holds:

$$0 = 2e^{2\beta t}\sum_{i=1}^{w}\sum_{j=1}^{w}\omega_{i,p}(\theta(t))\omega_{j,p}(\theta(t_k))\Upsilon(t)\left[-\dot{x}(t) + \mathscr{A}_{i,p}x(t) + \mathscr{B}_{i,p}\mathscr{K}_{j,p}x(t-\tau(t))\right], \tag{38}$$

with $\Upsilon(t) = x^T(t)\mathscr{F}_1^T + \dot{x}^T(t)\mathscr{F}_2^T$.

Merging (29) and (33)–(38), we have for $t_k \le t < t_{k+1}$ that

$$\mathbb{E}\mathscr{L}V(x(t), r_t) \le \mathbb{E}e^{2\beta t}\mathscr{W}(t), \tag{39}$$

where $\mathscr{W}(t) = \sum_{i=1}^{w}\sum_{j=1}^{w}\omega_{i,p}(\theta(t))\omega_{j,p}(\theta(t_k))\xi^T(t)\Xi(\tau(t))\xi(t)$ and $\Xi(\tau(t)) = \Phi(\tau(t)) + \mathbf{e}_1^T\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q\mathbf{e}_1$.

On the contrary, since $\pi_{pq}(h)$ is generally uncertain transition rates in this paper, the proof needs further derivation by the following two cases (i.e., $p \in \mathscr{G}_{p,k}$ and $p \in \mathscr{G}_{p,uk}$).

When $p \in \mathscr{G}_{p,k}$, let $\lambda_{p,k} \triangleq \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)$. According to $\mathscr{G}_{p,uk} \neq \varnothing$, we have $\lambda_{p,k} < 0$. Then, the formula $\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q$ can be represented as

$$\begin{aligned}\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q &= \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\mathscr{P}_q + \sum_{q\in\mathscr{G}_{p,uk}}\pi_{pq}(h)\mathscr{P}_q \\ &= \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\mathscr{P}_q - \lambda_{p,k}\sum_{q\in\mathscr{G}_{p,uk}}\frac{\pi_{pq}(h)}{-\lambda_{p,k}}\mathscr{P}_q.\end{aligned} \tag{40}$$

It is clear that $0 \le (\pi_{pq}(h)/(-\lambda_{p,k})) \le 1\,(q \in \mathscr{G}_{p,uk})$ and $\sum_{q\in\mathscr{G}_{p,uk}}\pi_{pq}(h)/(-\lambda_{p,k}) = 1$. Thus, for $\forall l \in \mathscr{G}_{p,uk}$, one can obtain

$$\Xi(\tau(t)) = \sum_{q\in\mathscr{G}_{p,uk}}\frac{\pi_{pq}(h)}{-\lambda_{p,k}}\left[\Phi(\tau(t)) + \mathbf{e}_1^T\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right)\mathbf{e}_1\right]. \tag{41}$$

Accordingly, for $0 \le \pi_{pq}(h) \le -\lambda_{p,k}$, $\Xi(\tau(t)) < 0$ is equivalent to

$$\Phi(\tau(t)) + \mathbf{e}_1^T\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right)\mathbf{e}_1 < 0. \tag{42}$$

Then, we can find from the above inequality (42) that

$$\begin{aligned}\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right) &= \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}\left(\mathscr{P}_q - \mathscr{P}_l\right) \\ &\quad + \sum_{q\in\mathscr{G}_{p,k}}\Delta\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right).\end{aligned} \tag{43}$$

From the inequality $c(M + M^T) \le c^2S + MS^{-1}M^T$ [42], for any $\mathscr{T}_{pq} > 0$, we have

$$\begin{aligned}\sum_{q\in\mathscr{G}_{p,k}}\Delta\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right) &= \sum_{q\in\mathscr{G}_{p,k}}\left[\frac{1}{2}\Delta\pi_{pq}(h)\left(\left(\mathscr{P}_q - \mathscr{P}_l\right) + \left(\mathscr{P}_q - \mathscr{P}_l\right)\right)\right] \\ &\le \sum_{q\in\mathscr{G}_{p,k}}\left[\left(\frac{\lambda_{pq}}{2}\right)^2\mathscr{T}_{pq} + \left(\mathscr{P}_q - \mathscr{P}_l\right)\mathscr{T}_{pq}^{-1}\left(\mathscr{P}_q - \mathscr{P}_l\right)\right].\end{aligned} \tag{44}$$

Combining (40)–(44) and using Schur complement, it can be concluded (19)–(21) guarantee $\Xi(\tau(t)) < 0$ for all $p \in \mathscr{G}_{p,k}$.

When $p \in \mathscr{G}_{p,uk}$, define $\lambda_{p,k} \triangleq \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)$. Because $\mathscr{G}_{p,k} \neq \varnothing$, we have $\lambda_{p,k} > 0$. So, the expression $\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q$ can be rewritten as

$$\begin{aligned}\sum_{q=1}^{s}\pi_{pq}(h)\mathscr{P}_q &= \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\mathscr{P}_q + \pi_{pp}(h)\mathscr{P}_p \\ &\quad + \sum_{q\in\mathscr{G}_{p,uk},q\neq p}\pi_{pq}(h)\mathscr{P}_q \\ &= \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\mathscr{P}_q + \pi_{pp}(h)\mathscr{P}_p \\ &\quad -\left(\pi_{pp}(h) + \lambda_{p,k}\right)\sum_{q\in\mathscr{G}_{p,uk},q\neq p}\frac{\pi_{pq}(h)\mathscr{P}_q}{-\pi_{pp}(h) - \lambda_{p,k}}.\end{aligned} \tag{45}$$

It is obvious that $0 \le (\pi_{pq}(h)/(-\pi_{pp}(h) - \lambda_{p,k})) \le 1\,(q \in \mathscr{G}_{p,uk})$ and $\sum_{q\in\mathscr{G}_{p,uk},q\neq p}\pi_{pq}(h)/(-\pi_{pp}(h) - \lambda_{p,k}) = 1$. Consequently, for $\forall l \in \mathscr{G}_{p,uk}$, $l \neq p$, we get

$$\begin{aligned}\Xi(\tau(t)) = \sum_{q\in\mathscr{G}_{p,uk},q\neq p}\frac{\pi_{pq}(h)}{-\pi_{pp}(h) - \lambda_{p,k}}\Big[&\Phi(\tau(t)) \\ &+ \mathbf{e}_1^T\pi_{pp}(h)\left(\mathscr{P}_p - \mathscr{P}_l\right)\mathbf{e}_1 + \mathbf{e}_1^T\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right)\mathbf{e}_1\Big].\end{aligned} \tag{46}$$

Therefore, for $0 \le \pi_{pq}(h) \le -\pi_{pp}(h) - \lambda_{p,\mathbf{k}}$, the equivalent form of $\Xi(\tau(t)) < 0$ can be described as follows:

$$\Phi(\tau(t)) + \mathbf{e}_1^T\pi_{pp}(h)\left(\mathscr{P}_p - \mathscr{P}_l\right)\mathbf{e}_1 + \mathbf{e}_1^T\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right)\mathbf{e}_1 < 0. \tag{47}$$

Since $\pi_{pp}(h) < 0$, the inequality (47) holds if we have

$$\begin{cases}\mathscr{P}_p - \mathscr{P}_l \ge 0, \\ \Phi(\tau(t)) + \mathbf{e}_1^T\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right)\mathbf{e}_1 < 0.\end{cases} \tag{48}$$

In addition, similar to (43) and (44), for any $\mathscr{Z}_{pq} > 0$, we obtain

$$\begin{aligned}\sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}(h)\left(\mathscr{P}_q - \mathscr{P}_l\right) &\le \sum_{q\in\mathscr{G}_{p,k}}\pi_{pq}\left(\mathscr{P}_q - \mathscr{P}_l\right) + \sum_{q\in\mathscr{G}_{p,k}} \\ &\quad \cdot\left[\left(\frac{\lambda_{pq}}{2}\right)^2\mathscr{Z}_{pq} + \left(\mathscr{P}_q - \mathscr{P}_l\right)\mathscr{Z}_{pq}^{-1}\left(\mathscr{P}_q - \mathscr{P}_l\right)\right].\end{aligned} \tag{49}$$

From (45)–(49), by Schur complement and inequalities (24)–(27), it is known that $\Xi(\tau(t)) < 0$ for all $p \in \mathcal{G}_{p,uk}$. In summary, based on the above discussions, one can obtain

$$\mathbb{E}\mathscr{L}V(x(t), r_t) \le 0, \quad t \in [t_k, t_{k+1}). \tag{50}$$

Following this, we illustrate $V(x(t), r_t)$ is continuous at sampling times. By (30), we can obtain

$$\lim_{t \longrightarrow t_k} V_m(x(t), r_t) = V_m\left(x(t_k), r_{t_k}\right) \ge 0, \quad m = 1, 2, \tag{51}$$

$$\begin{aligned} \lim_{t \longrightarrow t_k^-} V_n(x(t), r_t) &= \lim_{t \longrightarrow t_k^+} V_n(x(t), r_t) \\ &= V_n\left(x(t_k), r_{t_k}\right) = 0, \quad n = 3, 4. \end{aligned} \tag{52}$$

From (51) and (52), one can get

$$\lim_{t \longrightarrow t_k} V(x(t), r_t) = V\left(x(t_k), r_{t_k}\right) \ge 0. \tag{53}$$

Hence, $V(x(t), r_t)$ is continuous at sampling times. Besides, from (50)–(53), since $\xi(t) \ne 0$, we have for $t \in [t_k, t_{k+1})$ that

$$V(x(t), r_t) > V\left(x(t_{k+1}), r_{t_{k+1}}\right) > 0, \tag{54}$$

which implies that $V(x(t), r_t) > 0$ at the sampling times.

Combining Dynkin's formula with (50), we have

$$\mathbb{E}V(x(t), r_t) \le \mathbb{E}V(x(0), r_0). \tag{55}$$

Moreover, it is followed from (30) that

$$\begin{aligned} \mathbb{E}V(x(0), r_0) &= \sum_{m=1}^{4} \mathbb{E}V_m(x(0), r_0) \\ &\le \max_{p \in \mathbb{S}}\left\{\lambda_{\max}\left(\mathscr{P}_p\right)\right\}\mathbb{E}\|x(0)\|^2 \\ &\quad + \frac{1 - e^{-2\beta\tau}}{2\beta}\lambda_{\max}(\mathscr{Q}) \sup_{-\tau \le \psi \le 0} \mathbb{E}\left\{\|x(\psi)\|^2\right\} \\ &\quad + \frac{\tau\left(e^{-2\beta\tau} + 2\beta\tau - 1\right)}{4\beta^2}\lambda_{\max}(\mathscr{R}) \sup_{-\tau \le \psi \le 0} \mathbb{E}\left\{\|\dot{x}(\psi)\|^2\right\} \\ &\le \mathscr{M}\mathbb{E}\left\{\sup_{-\tau \le \psi \le 0}\{\|x(\psi)\|, \|\dot{x}(\psi)\|\}\right\}^2, \end{aligned} \tag{56}$$

where

$$\begin{aligned} \mathscr{M} &= \max_{p \in \mathbb{S}}\left\{\lambda_{\max}\left(\mathscr{P}_p\right)\right\} + \frac{1 - e^{-2\beta\tau}}{2\beta}\lambda_{\max}(\mathscr{Q}) \\ &\quad + \frac{\tau\left(e^{-2\beta\tau} + 2\beta\tau - 1\right)}{4\beta^2}\lambda_{\max}(\mathscr{R}). \end{aligned} \tag{57}$$

On the contrary, on the basis of Lemma 1 and (55), for $\forall t \in [t_k, t_{k+1})$, one can derive

$$\begin{aligned} \mathbb{E}\|x(t)\|^2 &\le \varepsilon\mathbb{E}\left\|x(t_k)\right\|^2 \\ &= \frac{\varepsilon e^{2\beta t_k}}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}e^{2\beta t_k}} \min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}\mathbb{E}\left\|x(t_k)\right\|^2 \\ &\le \frac{\varepsilon}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}e^{2\beta t_k}}\mathbb{E}\left(e^{2\beta t_k}x^T(t_k)\mathscr{P}\left(r_{t_k}\right)x(t_k)\right) \\ &= \frac{\varepsilon}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}e^{2\beta t_k}}\mathbb{E}V_1\left(x(t_k), r_{t_k}\right) \\ &\le \frac{\varepsilon}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}e^{2\beta t_k}}\mathbb{E}V\left(x(t_k), r_{t_k}\right) \\ &\le \frac{\varepsilon}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}}e^{-2\beta t_k}\mathbb{E}V(x(0), r_0) \\ &= \frac{\varepsilon}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}}e^{-2\beta t}e^{2\beta(t - t_k)}\mathbb{E}V(x(0), r_0) \\ &\le \frac{\varepsilon e^{2\beta\tau}}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}}e^{-2\beta t}\mathbb{E}V(x(0), r_0), \end{aligned} \tag{58}$$

which implies that

$$\mathbb{E}\|x(t)\|^2 \le \frac{\varepsilon e^{2\beta\tau}\mathscr{M}e^{-2\beta t}}{\min_{p \in \mathbb{S}}\left\{\lambda_{\min}\left(\mathscr{P}_p\right)\right\}}\mathbb{E}\left\{\sup_{-\tau \le \psi \le 0}\{\|x(\psi)\|, \|\dot{x}(\psi)\|\}\right\}^2. \tag{59}$$

Therefore, we have

$$\mathbb{E}\|x(t)\| \le \mathscr{U}e^{-\beta t} \sup_{-\tau \le \psi \le 0} \mathbb{E}\{\|x(\psi)\|, \|\dot{x}(\psi)\|\}, \tag{60}$$

with $\mathscr{U} = \sqrt{\varepsilon e^{2\beta\tau}\mathscr{M}/\min_{p \in \mathbb{S}}\left\{\lambda_{\min}(\mathscr{P}_p)\right\}}$.

Thus, the semi-Markov chaotic system (11) is exponentially stable with exponential convergence rate β according to Definition 1. This proves Theorem 1. □

Remark 1. Noting that the aforementioned system has been studied in [27], the modeling and method in handling uncertain transition rates are completely dissimilar to those in this paper. Firstly, we investigated a more common subject on fuzzy sampled-data chaotic systems with semi-Markov jump. Secondly, reference [27] provided lots of conservative stipulations in modeling these uncertain transition rates, for example, the upper and lower bounds of uncertain transition rates are often assumed to be the same when the absolute values are taken but, however, different from the study in [27], where a

more general approach takes the average of the upper and lower bounds of jumping time-dependent transition rates. Thirdly, from the perspective of dealing with uncertain transition rates which also are not alike for the proof, our approach is more universal than that in [27] and also can be used to deal with the case of uncertain transition rates with constant numbers. Nevertheless, the approach presented in [27] is very particular, and many assumptions in coping with uncertain transition rates are much conservative, which are not applicable in this paper.

Remark 2. Recently, the input-delay-dependent LKF approach has been widely used in some reported works (e.g., see [21, 24, 25]). However, we find that the constructed LKFs are simple and not easy to involve the high-order function of the sampling delay $\tau(t)$. Thus, in order to further solve this difficulty, we take $\tau(t)x(t)$, $\tau(t)\dot{x}(t)$, and $\tau(t)\int_{t_k}^{t} x(s)\mathrm{d}s$ into the intermediate vector $\xi(t)$ such that the cubic function of $\tau(t)$ is successfully presented in the novel additional term $V_3(x(t), r_t)$. Meanwhile, the equality (29) is used to increase the combination among the vectors of $x(t)$, $\dot{x}(t)$, $x(t_k)$, $(1/t-t_k)\int_{t_k}^{t} x(s)\mathrm{d}s$, $\tau(t)x(t)$, $\tau(t)\dot{x}(t)$, and $\tau(t)\int_{t_k}^{t} x(s)\mathrm{d}s$. It is also worth noting that $\Xi(\tau(t))$ in Theorem 1 is nonlinear matrix inequalities since it contains the quadratic function of the sampling input delay $\tau(t)$. So, it cannot be directly solved by using Matlab LMI Toolbox. Fortunately, by applying Lemma 7 in [43], the nonlinear matrix inequalities can be further transformed as general LMIs. Moreover, in comparison with the mentioned results in [21, 24, 25], our less conservative condition is built, which will be verified by examples in Numerical Simulation.

Theorem 2. *Given scalars τ, β, and μ, the system (11) can be exponentially stabilized; if there exist positive definite matrices $\widehat{\mathscr{P}}_p$, $\widehat{\mathscr{Q}}$, $\widehat{\mathscr{R}}$, $\widehat{\mathscr{T}}_{pq}$, and $\widehat{\mathscr{Z}}_{pq}$, any matrices $\widehat{\mathscr{S}}_{11}$, $\widehat{\mathscr{S}}_{12}$, $\widehat{\mathscr{S}}_{22}$, $\widetilde{\mathscr{H}}$, $\widetilde{\mathscr{X}}$, $\widetilde{\mathscr{Y}}$, $\mathscr{F}$, and $\widetilde{\mathscr{K}}_{j,p}$ with appropriate dimensions, such that the following predonditions hold for all i, $j \in \Theta$, $p \in \mathbb{S}$.*

Case 1. $p \in \mathscr{G}_{p,k}$, $\forall l \in \mathscr{G}_{p,uk}$, $\mathscr{G}_{p,k} \triangleq \{\mathbf{k}_{p,1}, \mathbf{k}_{p,2}, \ldots, \mathbf{k}_{p,v1}\}$,

$$\widehat{\overline{\mathscr{R}}} = \begin{bmatrix} \mathrm{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} & \widehat{\mathscr{S}} \\ * & \mathrm{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} \end{bmatrix} \geq 0, \tag{61}$$

$$\begin{bmatrix} \widehat{\Sigma}_1^{(1)} & \widehat{\Gamma}^{(1)} \\ * & -\widehat{\Lambda}^{(1)} \end{bmatrix} < 0, \tag{62}$$

$$\begin{bmatrix} \widehat{\Sigma}_2^{(1)} & \widehat{\Gamma}^{(1)} \\ * & -\widehat{\Lambda}^{(1)} \end{bmatrix} < 0, \tag{63}$$

with

$$\begin{aligned}
\widehat{\mathscr{S}} &= \begin{bmatrix} \widehat{\mathscr{S}}_{11} & \widehat{\mathscr{S}}_{12} \\ * & \widehat{\mathscr{S}}_{22} \end{bmatrix}, \\
\widehat{\Sigma}_1^{(1)} &= \widehat{\Phi}(\tau(t)=0) + \mathbf{e}_1^T \widehat{T}_{pq} \mathbf{e}_1, \\
\widehat{\Sigma}_2^{(1)} &= \widehat{\Phi}(\tau(t)=\tau) + \mathbf{e}_1^T \widehat{T}_{pq} \mathbf{e}_1, \\
\widehat{\Gamma}^{(1)} &= \Big[\mathbf{e}_1^T\Big(\widehat{\mathscr{P}}_{k_{p,1}} - \widehat{\mathscr{P}}_l\Big), \mathbf{e}_1^T\Big(\widehat{\mathscr{P}}_{\mathbf{k}_{p,2}} - \widehat{\mathscr{P}}_l\Big), \ldots, \mathbf{e}_1^T \\
&\quad \cdot \Big(\widehat{\mathscr{P}}_{k_{p,v1}} - \widehat{\mathscr{P}}_l\Big)\Big], \\
\widehat{\Lambda}^{(1)} &= \mathrm{diag}\Big\{\widehat{\mathscr{T}}_{p,k_{p,1}}, \widehat{\mathscr{T}}_{p,k_{p,2}}, \ldots, \widehat{\mathscr{T}}_{p,k_{p,v1}}\Big\}, \\
\widehat{T}_{pq} &= \sum_{q \in \mathscr{G}_{p,k}} \left[\frac{(\lambda_{pq})^2}{4}\widehat{\mathscr{T}}_{pq} + \pi_{pq}\Big(\widehat{\mathscr{P}}_q - \widehat{\mathscr{P}}_l\Big)\right], \\
\widehat{\Phi}(\tau(t)) &= \widehat{\Phi}_0(\tau(t)) + \widehat{\Phi}_1(\tau(t)), \\
\widehat{\Phi}_0(\tau(t)) &= \mathrm{sym}\left\{\Psi^T \widehat{\mathscr{X}} \begin{bmatrix} \tau(t)\mathbf{e}_1 - \mathbf{e}_7 \\ \tau(t)\mathbf{e}_2 - \mathbf{e}_8 \\ \tau^2(t)\mathbf{e}_5 - \mathbf{e}_9 \end{bmatrix}\right\} \\
&\quad + \mathrm{sym}\Big\{\Big(\mu\mathbf{e}_1^T + \mathbf{e}_2^T\Big)\Big(\mathscr{A}_{i,p}\mathscr{F}\mathbf{e}_1 + \mathscr{B}_{i,p}\widetilde{\mathscr{K}}_{j,p}\mathbf{e}_3 - \mathscr{F}\mathbf{e}_2\Big)\Big\}, \\
\widehat{\Phi}_1(\tau(t)) &= \mathrm{sym}\Big\{\mathbf{e}_1^T\widehat{\mathscr{P}}_p\mathbf{e}_2\Big\} + 2\beta\mathbf{e}_1^T\widehat{\mathscr{P}}_p\mathbf{e}_1 + \mathbf{e}_1^T\widehat{\mathscr{Q}}\mathbf{e}_1 \\
&\quad - e^{-2\beta\tau}\mathbf{e}_4^T\widehat{\mathscr{Q}}\mathbf{e}_4 + \tau^2\mathbf{e}_2^T\widehat{\mathscr{R}}\mathbf{e}_2 - e^{-2\beta\tau}\Pi_1^T\widehat{\overline{\mathscr{R}}}\Pi_1 \\
&\quad + 2\beta(\tau-\tau(t))\tau(t)\mathbf{e}_5^T\widehat{\mathscr{Y}}\mathbf{e}_5 \\
&\quad + (\tau-2\tau(t))\mathbf{e}_5^T\widehat{\mathscr{Y}}\mathbf{e}_5 + \mathrm{sym}\Big\{(\tau-\tau(t))\mathbf{e}_5^T\widehat{\mathscr{Y}}(\mathbf{e}_1-\mathbf{e}_5)\Big\} \\
&\quad - \mathrm{sym}\Big\{\Pi_2^T\widehat{\mathscr{H}}\Pi_2\Big\} + (\tau-\tau(t))\mathrm{sym}\Big\{2\beta\Pi_2^T\widehat{\mathscr{H}}\Pi_2\Big\} \\
&\quad + (\tau-\tau(t))\mathrm{sym}\left\{\begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_1+\mathbf{e}_8 \\ \tau(t)\mathbf{e}_5+\mathbf{e}_7 \end{bmatrix}^T \widehat{\mathscr{H}}\Pi_2\right\}, \\
\Pi_1 &= \begin{bmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ \mathbf{e}_1+\mathbf{e}_3-2\mathbf{e}_5 \\ \mathbf{e}_3-\mathbf{e}_4 \\ \mathbf{e}_3+\mathbf{e}_4-2\mathbf{e}_6 \end{bmatrix}, \\
\Pi_2 &= \begin{bmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ \mathbf{e}_7 \\ \mathbf{e}_9 \end{bmatrix}.
\end{aligned} \tag{64}$$

Case 2. $p \in \mathscr{G}_{p,uk}$, $\forall l \in \mathscr{G}_{p,uk}$, $\mathscr{G}_{p,k} \triangleq \{\mathbf{k}_{p,1}, \mathbf{k}_{p,2}, \ldots, \mathbf{k}_{p,v2}\}$, $l \neq p$,

$$\widehat{\overline{\mathscr{R}}} = \begin{bmatrix} \operatorname{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} & \widehat{\mathscr{S}} \\ * & \operatorname{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} \end{bmatrix} \geq 0, \tag{65}$$

$$\widehat{\mathscr{P}}_p - \widehat{\mathscr{P}}_l \geq 0, \tag{66}$$

$$\begin{bmatrix} \widehat{\Sigma}_1^{(2)} & \widehat{\Gamma}^{(2)} \\ * & -\widehat{\Lambda}^{(2)} \end{bmatrix} < 0, \tag{67}$$

$$\begin{bmatrix} \widehat{\Sigma}_2^{(2)} & \widehat{\Gamma}^{(2)} \\ * & -\widehat{\Lambda}^{(2)} \end{bmatrix} < 0, \tag{68}$$

with

$$\begin{aligned}
\widehat{\Sigma}_1^{(2)} &= \widehat{\Phi}(\tau(t) = 0) + \mathbf{e}_1^T \widehat{Z}_{pq} \mathbf{e}_1, \\
\widehat{\Sigma}_2^{(2)} &= \widehat{\Phi}(\tau(t) = \tau) + \mathbf{e}_1^T \widehat{Z}_{pq} \mathbf{e}_1, \\
\widehat{\Gamma}^{(2)} &= \left[\mathbf{e}_1^T\left(\widehat{\mathscr{P}}_{k_{p,1}} - \widehat{\mathscr{P}}_l\right), \mathbf{e}_1^T\left(\widehat{\mathscr{P}}_{k_{p,2}} - \widehat{\mathscr{P}}_l\right), \ldots, \mathbf{e}_1^T\left(\widehat{\mathscr{P}}_{k_{p,v2}} - \widehat{\mathscr{P}}_l\right)\right], \\
\widehat{\Lambda}^{(2)} &= \operatorname{diag}\left\{\widehat{\mathscr{Z}}_{p,k_{p,1}}, \widehat{\mathscr{Z}}_{p,k_{p,2}}, \ldots, \widehat{\mathscr{Z}}_{p,k_{p,v2}}\right\}, \\
\widehat{Z}_{pq} &= \sum_{q \in \mathscr{G}_{p,k}} \left[\frac{(\lambda_{pq})^2}{4} \widehat{\mathscr{Z}}_{pq} + \pi_{pq}\left(\widehat{\mathscr{P}}_q - \widehat{\mathscr{P}}_l\right)\right].
\end{aligned} \tag{69}$$

Moreover, the controller gain matrices in (10) are designed as

$$\mathscr{K}_{j,p} = \widehat{\mathscr{K}}_{j,p} \mathscr{F}^{-1}. \tag{70}$$

Proof. Define $\mathscr{F}_2 = \mathscr{F}^{-1}$, $\mathscr{F}_1 = \mu\mathscr{F}^{-1}$, $\widehat{\mathscr{K}}_{j,p} = \mathscr{K}_{j,p}\mathscr{F}$, $\mathscr{J}_n = \operatorname{diag}\left\{\underbrace{\mathscr{F}, \mathscr{F}, \ldots, \mathscr{F}}_{n}\right\}$, $\widehat{\mathscr{P}}_p = \mathscr{F}^T \mathscr{P}_p \mathscr{F}$, $\widehat{\mathscr{Q}} = \mathscr{F}^T \mathscr{Q} \mathscr{F}$, $\widehat{\mathscr{R}} = \mathscr{F}^T \mathscr{R} \mathscr{F}$, $\widehat{\mathscr{T}}_{pq} = \mathscr{F}^T \mathscr{T}_{pq} \mathscr{F}$, $\widehat{\mathscr{Z}}_{pq} = \mathscr{F}^T \mathscr{Z}_{pq} \mathscr{F}$, $\widehat{\mathscr{X}} = \mathscr{J}_9^T \mathscr{X} \mathscr{J}_3$, $\widehat{\mathscr{H}} = \mathscr{J}_3^T \mathscr{H} \mathscr{J}_3$, $\widehat{\mathscr{Y}} = \mathscr{F}^T \mathscr{Q} \mathscr{F}$, $\widehat{\mathscr{S}}_{11} = \mathscr{F}^T \mathscr{S}_{11} \mathscr{F}$, $\widehat{\mathscr{S}}_{12} = \mathscr{F}^T \mathscr{S}_{12} \mathscr{F}$, and $\widehat{\mathscr{S}}_{22} = \mathscr{F}^T \mathscr{S}_{22} \mathscr{F}$. By pre- and postmultiplying $\mathscr{J}_4^T$ and $\mathscr{J}_4$ with (19) and (24), the inequalities (61) and (65) are easily obtained. By pre- and postmultiplying $\mathscr{J}_{v1+9}^T$ and $\mathscr{J}_{v1+9}$ with (20) and (21), the inequalities (62) and (63) hold. By pre- and postmultiplying $\mathscr{J}_1^T$ and $\mathscr{J}_1$ with (25), the inequality (66) can be easily obtained. By pre- and postmultiplying $\mathscr{J}_{v2+9}^T$ and $\mathscr{J}_{v2+9}$ with (26) and (27), the inequalities (67) and (68) hold. This proves Theorem 2.

Specially, when there are no semi-Markov jump parameters considered, the semi-Markov process $\{r_t, t \geq 0\}$ only takes account of $\mathbb{S} = \{1\}$, and the fuzzy sampled-data controller (10) and the system (11) are rewritten as

$$u(t) = \sum_{j=1}^{w} \omega_j(\theta(t_k)) \mathscr{K}_j x(t - \tau(t)), \quad t \in [t_k, t_{k+1}), \tag{71}$$

$$\dot{x}(t) = \sum_{i=1}^{w} \sum_{j=1}^{w} \omega_i(\theta(t)) \omega_j(\theta(t_k)) \left[\mathscr{A}_i x(t) + \mathscr{B}_i \mathscr{K}_j x(t - \tau(t))\right], \quad t \in [t_k, t_{k+1}), \tag{72}$$

which have been extensively discussed in [18–27]. Choosing $\mathscr{P}(r_t) = \mathscr{P}$ in (30), one can obtain the following corollary from Theorem 2. □

Corollary 1. *For given scalars τ, β, and μ, the chaotic system (72) is exponentially stabilized; if there exist $\widehat{\mathscr{P}} > 0$, $\widehat{\mathscr{Q}} > 0$, and $\widehat{\mathscr{R}} > 0$, any matrices $\widehat{\mathscr{S}}_{11}$, $\widehat{\mathscr{S}}_{12}$, $\widehat{\mathscr{S}}_{22}$, $\widehat{\mathscr{H}}$, $\widehat{\mathscr{X}}$, $\widehat{\mathscr{Y}}$, $\mathscr{F}$, and $\widehat{\mathscr{K}}_j$ with appropriate dimensions, such that the following conditions hold for all i, $j \in \Theta$.*

$$\widehat{\overline{\mathscr{R}}} = \begin{bmatrix} \operatorname{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} & \widehat{\mathscr{S}} \\ * & \operatorname{diag}\{\widehat{\mathscr{R}}, 3\widehat{\mathscr{R}}\} \end{bmatrix} \geq 0, \tag{73}$$

$$\overline{\Phi}(\tau(t) = 0) < 0, \tag{74}$$

$$\overline{\Phi}(\tau(t) = \tau) < 0, \tag{75}$$

where

$$\begin{aligned}
\overline{\Phi}(\tau(t)) &= \overline{\Phi}_0(\tau(t)) + \widehat{\Phi}_1(\tau(t)), \\
\overline{\Phi}_0(\tau(t)) &= \operatorname{sym}\left\{\Psi^T \widehat{\mathscr{X}} \begin{bmatrix} \tau(t)\mathbf{e}_1 - \mathbf{e}_7 \\ \tau(t)\mathbf{e}_2 - \mathbf{e}_8 \\ \tau^2(t)\mathbf{e}_5 - \mathbf{e}_9 \end{bmatrix}\right\} + \operatorname{sym} \\
&\quad \cdot \left\{\left(\mu\mathbf{e}_1^T + \mathbf{e}_2^T\right)\left(\mathscr{A}_i \mathscr{F}\mathbf{e}_1 + \mathscr{B}_i \widehat{\mathscr{K}}_j \mathbf{e}_3 - \mathscr{F}\mathbf{e}_2\right)\right\}.
\end{aligned} \tag{76}$$

The controller gain matrices in (71) are provided as $\mathscr{K}_j = \widehat{\mathscr{K}}_j \mathscr{F}^{-1}$.

Remark 3. It should be pointed out that Theorem 2 and Corollary 1 involve the regulation parameter μ. Accordingly, to find a suitable value of this parameter is a very importance issue. Taking Theorem 2 as an example, in line with the grid search algorithm [22, 44], we can look for the optimal tuning parameter μ in Theorem 2. Similarly, this algorithm can also be used for Corollary 1 to find an optimal value of parameter μ.

4. Numerical Simulation

In this section, we will give two conventional models to show the merits and effectiveness of our results obtained in this paper.

Example 1. Inspect the dynamic model of the Lorenz system [21, 24, 27] as follows:

$$\begin{cases} \dot{x}_1(t) = -a_1 x_1(t) + a_1 x_2(t) + u_1(t), \\ \dot{x}_2(t) = a_3 x_1(t) - x_2(t) - x_1(t) x_3(t), \\ \dot{x}_3(t) = x_1(t) x_2(t) - a_2 x_3(t). \end{cases} \tag{77}$$

Assuming that $x_1(t) \in [-a_4, a_4]$, the system (77) can be expressed as the fuzzy system (11) with

$$\begin{aligned} \mathscr{A}_{1,p} &= \begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & -a_4 \\ 0 & a_4 & -a_2 \end{bmatrix}, \\ \mathscr{A}_{2,p} &= \begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & a_4 \\ 0 & a_4 & -a_2 \end{bmatrix}, \\ \mathscr{B}_{1,p} &= \mathscr{B}_{2,p} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \end{aligned} \tag{78}$$

and the membership functions are

$$\begin{aligned} \omega_{1,p}(x_1(t)) &= \frac{a_4 + x_1(t)}{2a_4}, \\ \omega_{2,p}(x_1(t)) &= 1 - \omega_{1,p}(x_1(t)). \end{aligned} \tag{79}$$

a_1, a_2, a_3, and a_4 follow three jumping modes with the values provided in Table 1, and the generally uncertain transition rate matrix for unlike modes is given as follows:

$$\Pi(h) = \begin{bmatrix} -1.5 + \Delta\pi_{11}(h) & \nabla & \nabla \\ \nabla & \nabla & 0.8 + \Delta\pi_{23}(h) \\ 0.5 + \Delta\pi_{31}(h) & \nabla & -1 + \Delta\pi_{33}(h) \end{bmatrix}, \tag{80}$$

with $\Delta\pi_{pq}(h) \le \lambda_{pq} \overleftarrow{=} |0.1 * \pi_{pq}|$.

Now, we start to check the effectiveness and advantages of Theorem 2 from three aspects. At first, if $\mathscr{K}_{j,p} = 0$, the initial conditions are set as $x(0) = [12, 15, -13]^T$ regarding mode 1, $x(0) = [2, -6, -2]^T$ regarding mode 2, and $x(0) = [-3, 6, 9]^T$ regarding mode 3, and the relevant chaotic actions of (77) are shown in Figures 1–3, respectively.

Then, if $\mathscr{K}_{j,p} \neq 0$, choosing the adjustable parameters $\mu = 90$ and $\beta = 0$ and resolving the LMIs (61)–(68) in Theorem 2 by Matlab LMI Control Toolbox, the maximum value of the sampling interval $\tau = 0.0537$ can be obtained. Simultaneously, we can obtain the following state feedback gain matrices:

$$\begin{aligned} \mathscr{K}_{1,1} &= [-29.5405 \quad -17.3695 \quad 8.2551], \\ \mathscr{K}_{2,1} &= [-30.1036 \quad -17.2315 \quad -8.1225], \\ \mathscr{K}_{1,2} &= [-28.5657 \quad -17.1105 \quad 8.1691], \\ \mathscr{K}_{2,2} &= [-29.9845 \quad -17.1972 \quad -8.1155], \\ \mathscr{K}_{1,3} &= [-29.5317 \quad -17.2559 \quad 8.2034], \\ \mathscr{K}_{2,3} &= [-30.0220 \quad -17.1328 \quad -8.0798]. \end{aligned} \tag{81}$$

Table 1: Values of a_1, a_2, a_3, and a_4 for different modes in example 1.

Mode p	a_1	a_2	a_3	a_4
1	10	8/3	28	25
2	10.1	2.7	28.1	25.2
3	9.9	2.6	27.8	24.8

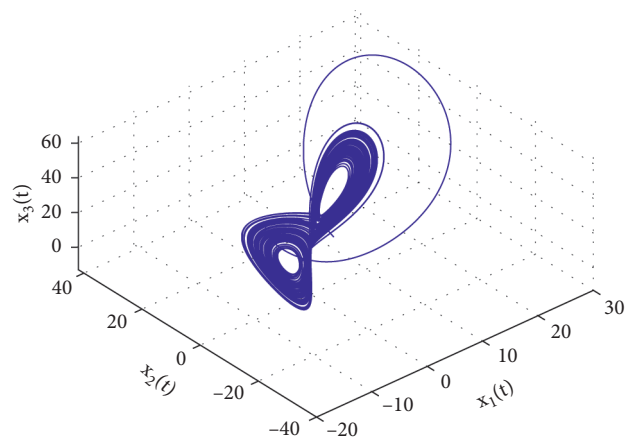

Figure 1: Behavior of (77) with $x(0) = [12, 15, -13]^T$ for mode 1.

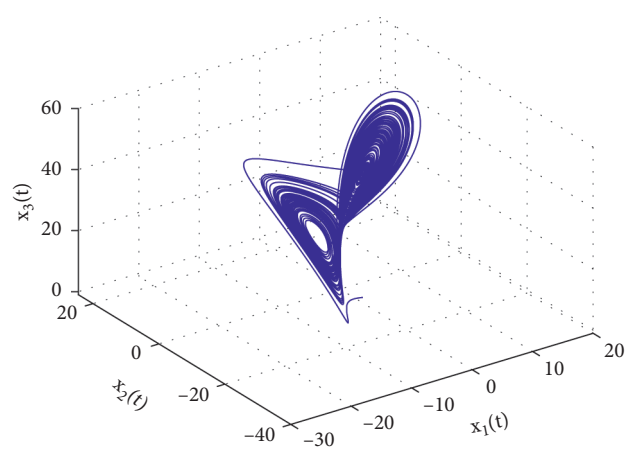

Figure 2: Curve of (77) with $x(0) = [2, -6, -2]^T$ for mode 2.

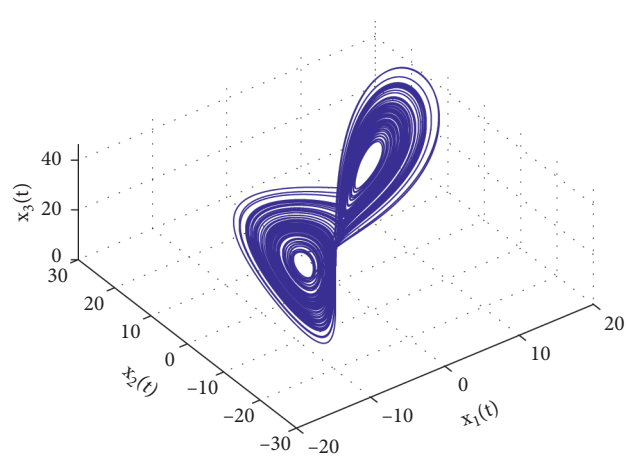

Figure 3: Trajectory of (77) with $x(0) = [-3, 6, 9]^T$ for mode 3.

With the above controller gain matrices and the initial conditions $x(0) = [-3, 6, 9]^T$ and $r_0 = 1$, we have the relevant numerical simulation, as shown in Figures 4–6. The semi-Markov jump process is illustrated in Figure 4. Figure 5 demonstrates the state response of the fuzzy semi-Markov jump chaotic system (11). Figure 6 shows the fuzzy sampled-data control input. It is easily seen from Figure 5 that the system (11) can be stabilized by the presented fuzzy controller (10), which verifies the feasibility of Theorem 2.

Third, when $r_t = \{1\}$, taking $a_1 = 10$, $a_2 = 8/3$, $a_3 = 28$, $a_4 = 25$, $\mu = 90$, and $\beta = 0$, and by utilizing Matlab LMI Control Toolbox to calculate the LMIs (73)–(75), we can get the largest value $\tau = 0.0729$. Actually, in [21, 24, 25, 27], the largest sampling intervals are 0.0347, 0.0412, 0.0438, and 0.0450, respectively. And those results are listed in Table 2. It is clearly seen that our proposed approach gives importantly better results than those obtained in [21, 24, 25, 27]. Additionally, in comparison with the number of variables in [21, 24], the number of variables in our presented approach in Corollary 1 is much smaller. Hence, it is easy to see that our proposed method gives a less conservative and less computationally complex result than those obtained in [21, 24], which checks the availability of Remark 2 discussed.

When the largest sampling period τ reaches 0.0729, the relevant controller gains are computed as

$$\begin{aligned} \mathscr{K}_1 &= \left[\, -21.8910 \quad -17.5823 \quad 4.7085 \,\right], \\ \mathscr{K}_2 &= \left[\, -21.8910 \quad -17.5823 \quad -4.7085 \,\right]. \end{aligned} \tag{82}$$

Based on the above fuzzy sampled-data control gain matrices, the simulations are provided in Figures 7 and 8 with the initial condition $x(0) = [-1.5, -1, 2]^T$. Figure 7 depicts the state trajectory of the system (72). Figure 8 shows the fuzzy sampled-data control input variables. From Figure 7, it is clearly seen that system (72) is exponentially stabilized, which shows the usefulness of our method provided in this paper.

Example 2. Consider the system (72) with the following parameters [24–26]:

$$\begin{aligned} \mathscr{A}_1 &= \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -25 \\ 0 & 25 & \dfrac{-8}{3} \end{bmatrix}, \\ \mathscr{A}_2 &= \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 25 \\ 0 & 25 & \dfrac{-8}{3} \end{bmatrix}, \\ \mathscr{B}_1 = \mathscr{B}_2 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{83}$$

and the fuzzy membership functions are

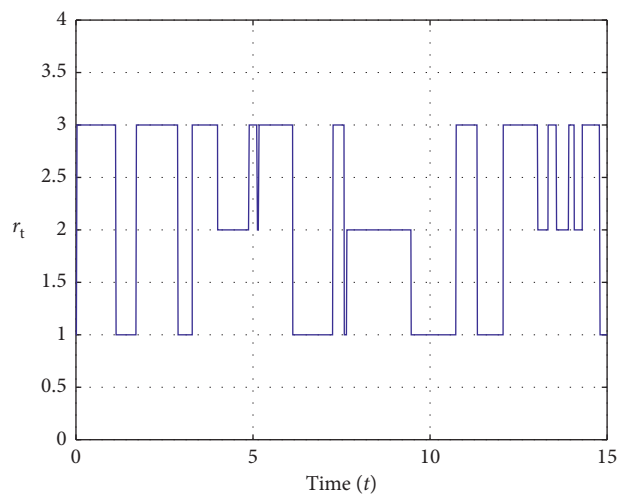

Figure 4: Semi-Markov jump signal.

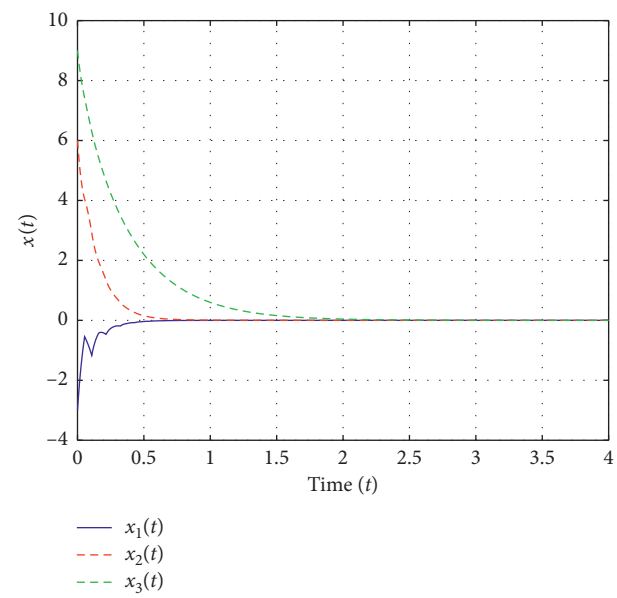

Figure 5: State response of the fuzzy semi-Markov chaotic system (11) with control input.

$$\begin{aligned} \omega_1(x_1(t)) &= \frac{1}{2}\left(1 + \frac{x_1(t)}{25}\right), \\ \omega_2(x_1(t)) &= 1 - \omega_1(x_1(t)). \end{aligned} \tag{84}$$

In this example, similar to that in [24–26], we choose the same tuning parameter $\mu = 1$ and take $\beta = 0$; the comparisons on the allowable maximum sampling period by different methods are shown in Table 3. From Table 3, we can observe that a larger sampling period has been obtained by our approach. Moreover, to verify the effectiveness of Remark 3 discussed, by specifying the range $\mu \in [0, 10]$ and increments $\Delta\mu = 1$ for μ and $\Delta\tau = 0.0001$ for τ and then by applying a similar algorithm in [26], we can find the optimal tuning parameter $\mu = 8$, and the corresponding allowable sampling period τ is 0.0689, which improves the former one 20.5%. As a

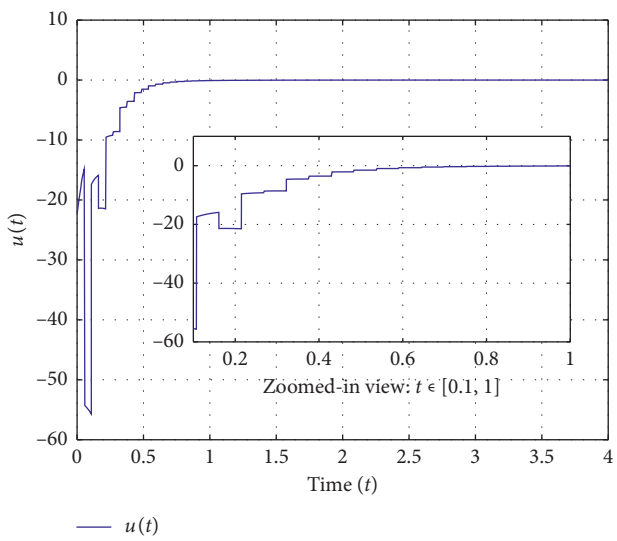

Figure 6: Fuzzy sampled-data control input $u(t)$.

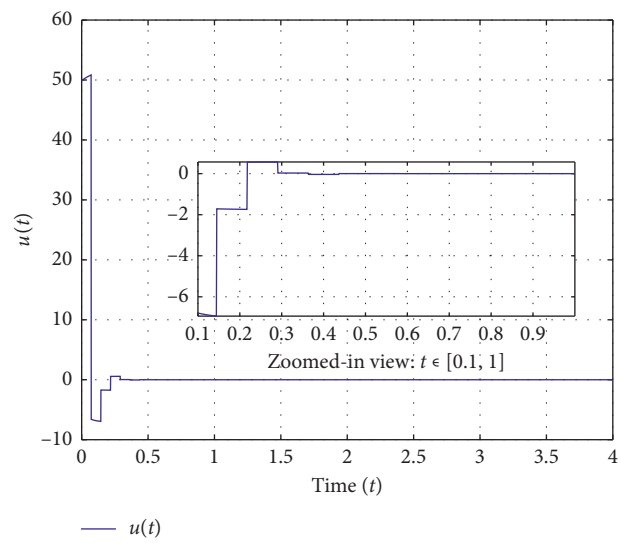

Figure 8: Fuzzy sampled-data controller $u(t)$.

Table 2: Largest sampling interval τ with different approaches.

Approach	τ	Number of variables
[25]	0.0347	$24n^2 + 3n + 2mn$
[24]	0.0412	$513.5n^2 + 111.5n + 2mn$
[21]	0.0438	$48.5n^2 + 8.5n + 2mn$
[27]	0.0450	$19.5n^2 + 1.5n + 2mn$
Corollary 1	0.0729	$40.5n^2 + 1.5n + 2mn$

Table 3: Maximum value of the sampling period τ.

Method	τ	Degree of improvement (%)
Corollary 1	0.0572	
[25]	0.0016	3475
[24]	0.0022	2500
[26]	0.0253	126.1

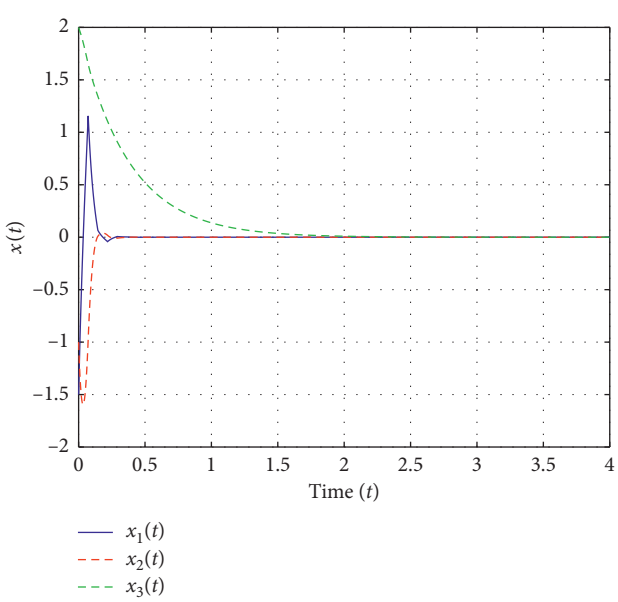

Figure 7: State trajectory of the fuzzy chaotic system (72) with control input.

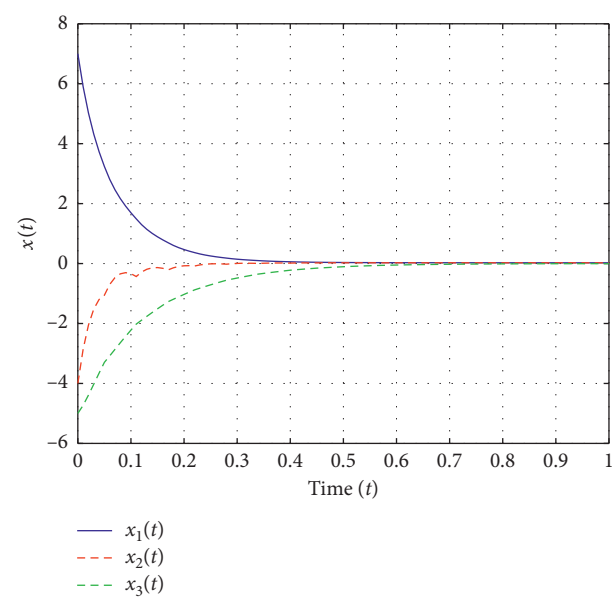

Figure 9: State path of the fuzzy chaotic system (72) with control input.

result, finding an optimal tuning parameter is very important for us to obtain the maximum value of the sampling period τ. On the contrary, choosing $\tau = 0.0572$, and by utilizing Matlab LMI Control Toolbox to solve the conditions (73)–(75), the related controller gain matrices are obtained as follows:

$$\mathscr{K}_1 = \begin{bmatrix} -17.7941 & -9.3902 & 24.1495 \\ -0.9946 & -23.2460 & -3.5656 \end{bmatrix},$$
$$\mathscr{K}_2 = \begin{bmatrix} -17.7941 & -9.3902 & -24.1495 \\ 0.9946 & 23.2460 & -3.5656 \end{bmatrix}. \quad (85)$$

Under the controller (68) with above gain matrices, taking the initial condition $x(0) = [7, -4, -5]^T$, the state path of system (72) is shown in Figure 9. The simulation result shows that the designed fuzzy sampled-data controller achieves exponential stabilization successfully with a larger sampling period than that in [24–26].

5. Conclusion

The issue of stabilization for semi-Markov jump chaotic systems based on fuzzy sampled-data control has been considered in this paper. The semi-Markov jump has generally uncertain transition rates. A new augmented LKF with the cubic function of the sampling input delay $\tau(t)$ has been constructed in which the available information with regard to the practical sampling mode is fully used. At the same time, we have also constructed a novel zero equality to make more cross terms of vector components in the resulting vector. In view of this, less conservative criteria with reduced computational complexity have been established to guarantee the stability and stabilization of the fuzzy sampled-data semi-Markov chaotic systems. Furthermore, according to the results obtained, the desired fuzzy sampled-data controller gain matrices have been given during larger sampling periods compared to the existing results. Finally, the validity and practicality of the derived results have been demonstrated by two interesting numerical examples. In the future, the employed results and methods can be extended to several cases including those studied extensively in the literature studies such as event-triggered sampling control [45, 46], quantized sampled-data control [47, 48], and asynchronous control [49, 50].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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References

[1] X. Guan and C. Chen, "Adaptive fuzzy control for chaotic systems with H_∞ tracking performance," *Fuzzy Sets and Systems*, vol. 139, no. 1, pp. 81–93, 2003.

[2] W. Xu, H. Shen, D. Hu, and A. Lei, "Impulse tuning of Chua chaos," *International Journal of Engineering Science*, vol. 43, no. 3-4, pp. 275–280, 2005.

[3] W. Yu, "Passive equivalence of chaos in Lorenz system," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, no. 7, pp. 876–878, 1999.

[4] M. Rafikov and J. M. Balthazar, "On an optimal control design for Rössler system," *Physics Letters A*, vol. 333, no. 3-4, pp. 241–245, 2004.

[5] Q. Zhong, J. Bao, Y. Yu, and X. Liao, "Impulsive control for T–S fuzzy model-based chaotic systems," *Mathematics and Computers in Simulation*, vol. 79, no. 3, pp. 409–415, 2008.

[6] L.-M. Tam, H.-K. Chen, and S.-Y. Li, "Adaptive synchronization of complicated chaotic systems with uncertainties via fuzzy modeling-based control strategy," *Information Sciences*, vol. 427, pp. 18–31, 2018.

[7] Y. Liu and S. Zhao, "T–S fuzzy model-based impulsive control for chaotic systems and its application," *Mathematics and Computers in Simulation*, vol. 81, no. 11, pp. 2507–2516, 2011.

[8] H. Wand and K. Tanaka, "An LMI-based stable fuzzy control of nonlinear systems and its applications to control of chaos," *Physical Review Letters*, vol. 339, pp. 325–332, 2005.

[9] K. Tanaka, T. Ikeda, and H. O. Wang, "A unified approach to controlling chaos via an LMI-based fuzzy control system design," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 45, no. 10, pp. 1021–1040, 1998.

[10] H.-B. Zeng, K. L. Teo, Y. He, H. Xu, and W. Wang, "Sampled-data synchronization control for chaotic neural networks subject to actuator saturation," *Neurocomputing*, vol. 260, pp. 25–31, 2017.

[11] H. K. Lam and F. H. F. Leung, "Stabilization of chaotic systems using linear sampled-data controller," *International Journal of Bifurcation and Chaos*, vol. 17, no. 6, pp. 2021–2031, 2007.

[12] E. Fridman, "A refined input delay approach to sampled-data control," *Automatica*, vol. 46, no. 2, pp. 421–427, 2010.

[13] C. Briat, "Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints," *Automatica*, vol. 49, no. 11, pp. 3449–3457, 2013.

[14] C. Briat, "Stability analysis and stabilization of stochastic linear impulsive, switched and sampled-data systems under dwell-time constraints," *Automatica*, vol. 74, pp. 279–287, 2016.

[15] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Automatica*, vol. 48, no. 1, pp. 177–182, 2012.

[16] R. Zhang, D. Zeng, S. Zhong, K. Shi, and J. Cui, "New approach on designing stochastic sampled-data controller for exponential synchronization of chaotic Lur'e systems," *Nonlinear Analysis: Hybrid Systems*, vol. 29, pp. 303–321, 2018.

[17] Y. Wang, X. Yang, and H. Yan, "Reliable fuzzy tracking control of near-space hypersonic vehicle using aperiodic measurement information," *IEEE Transactions on Industrial Electronics*, vol. 66, no. 12, pp. 9439–9447, 2019.

[18] C. Ge, H. Wang, Y. Liu, and J. H. Park, "Stabilization of chaotic systems under variable sampling and state quantized controller," *Fuzzy Sets and Systems*, vol. 344, no. 1, pp. 129–144, 2018.

[19] R. Zhang, D. Zeng, J. H. Park, Y. Liu, and S. Zhong, "A new approach to stabilization of chaotic systems with nonfragile fuzzy proportional retarded sampled-data control," *IEEE Transactions on Cybernetics*, vol. 49, no. 9, pp. 3218–3229, 2019.

[20] Y. Liu, J. H. Park, B.-Z. Guo, and Y. Shu, "Further results on stabilization of chaotic systems based on fuzzy memory sampled-data control," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 2, pp. 1040–1045, 2018.

[21] T. H. Lee and J. H. Park, "New methods of fuzzy sampled-data control for stabilization of chaotic systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 48, no. 12, pp. 2026–2034, 2018.

[22] X.-L. Zhu, B. Chen, D. Yue, and Y. Wang, "An improved input delay approach to stabilization of fuzzy systems under variable sampling," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 2, pp. 330–341, 2012.

[23] G. B. Koo, J. B. Park, and Y. H. Joo, "An improved digital redesign for sampled-data fuzzy control systems: fuzzy Lyapunov function approach," *Information Sciences*, vol. 406-407, pp. 71–86, 2017.

[24] Z. P. Wang and H. N. Wu, "On fuzzy sampled-data control of chaotic systems via a time-dependent Lyapunov functional approach," *IEEE Transactions on Cybernetics*, vol. 45, no. 4, pp. 819–829, 2015.

[25] Z.-G. Wu, P. Shi, H. Su, and J. Chu, "Sampled-data fuzzy control of chaotic systems based on a T–S fuzzy model," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 1, pp. 153–163, 2014.

[26] Y. Wang, Y. Xia, and P. Zhou, "Fuzzy-model-based sampled-data control of chaotic systems: a fuzzy time-dependent Lyapunov–Krasovskii functional approach," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 6, pp. 1672–1684, 2017.

[27] R. Zhang, X. Liu, D. Zeng, S. Zhong, and K. Shi, "A novel approach to stability and stabilization of fuzzy sampled-data Markovian chaotic systems," *Fuzzy Sets and Systems*, vol. 344, no. 1, pp. 108–128, 2018.

[28] C. K. Zhang, Y. He, L. Jiang, and M. Wu, "Stability analysis for delayed neural networks considering both conservativeness and complexity," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 7, pp. 1486–1501, 2016.

[29] T. Wu, J. Cao, L. Xiong, and B. Ahmad, "Exponential passivity conditions on neutral stochastic neural networks with leakage delay and partially unknown transition probabilities in Markovian jump," *Advances in Difference Equations*, vol. 2018, no. 1, p. 317, 2018.

[30] H. Shen, F. Li, H. Yan, H. R. Karimi, and H.-K. Lam, "Finite-time event-triggered H_∞ control for T–S fuzzy Markov jump systems," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 5, pp. 3122–3135, 2018.

[31] T. Wu, L. Xiong, J. Cao, Z. Liu, and H. Zhang, "New stability and stabilization conditions for stochastic neural networks of neutral type with Markovian jumping parameters," *Journal of the Franklin Institute*, vol. 355, no. 17, pp. 8462–8483, 2018.

[32] L. Xiong, J. Tian, and X. Liu, "Stability analysis for neutral Markovian jump systems with partially unknown transition probabilities," *Journal of the Franklin Institute*, vol. 349, no. 6, pp. 2193–2214, 2012.

[33] Y. Guo and Z. Wang, "Stability of Markovian jump systems with generally uncertain transition rates," *Journal of the Franklin Institute*, vol. 350, no. 9, pp. 2826–2836, 2013.

[34] L. L. Xiong, H. Y. Zhang, Y. K. Li, and Z. X. Liu, "Improved stability and H_∞ performance for neutral systems with uncertain Markovian jump," *Nonlinear Analysis: Hybrid Systems*, vol. 19, pp. 13–25, 2016.

[35] F. Li and H. Shen, "Finite-time H_∞ synchronization control for semi-Markov jump delayed neural networks with randomly occurring uncertainties," *Neurocomputing*, vol. 166, pp. 447–454, 2015.

[36] L. Zhang, T. Yang, P. Shi, and M. Liu, "Stability and stabilization of a class of discrete-time fuzzy systems with semi-Markov stochastic uncertainties," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 46, no. 12, pp. 1642–1653, 2016.

[37] H. Shen, F. Li, Z. G. Wu, J. H. Park, and V. Sreeram, "Fuzzy-model-based non-fragile control for nonlinear singularly perturbed systems with semi-Markov jump parameters," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3428–3439, 2018.

[38] H. Zhang, Z. Qiu, and L. Xiong, "Stochastic stability criterion of neutral-type neural networks with additive time-varying delay and uncertain semi-Markov jump," *Neurocomputing*, vol. 333, pp. 395–406, 2019.

[39] H. Yan, Y. Tian, H. Li, H. Zhang, and Z. Li, "Input-output finite-time mean square stabilization of nonlinear semi-Markovian jump systems," *Automatica*, vol. 104, pp. 82–89, 2019.

[40] P. Park, W. Lee, and S. Lee, "Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems," *Journal of the Franklin Institute*, vol. 352, no. 4, pp. 1378–1396, 2015.

[41] P. G. Park, J. W. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, 2011.

[42] J. W. Yi, Y. W. Wang, J. W. Xiao, and Y. Huang, "Exponential synchronization of complex dynamical networks with Markovian jump parameters and stochastic delays and its application to multi-agent systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 5, pp. 1175–1192, 2013.

[43] X. M. Zhang and Q. L. Han, "Global asymptotic stability analysis for delayed neural networks using a matrix-based quadratic convex approach," *Neural Networks*, vol. 54, pp. 57–69, 2014.

[44] J. Yoneyama, "Robust H_∞ control of uncertain fuzzy systems under time-varying sampling," *Fuzzy Sets and Systems*, vol. 161, no. 6, pp. 859–871, 2010.

[45] R. Zhang, D. Zeng, S. Zhong, and Y. Yu, "Event-triggered sampling control for stability and stabilization of memristive neural networks with communication delays," *Applied Mathematics and Computation*, vol. 310, pp. 57–74, 2017.

[46] Y. Wang, Y. Lei, T. Bian, and Z. Guan, "Distributed control of nonlinear multi-agent systems with unknown and non-identical control directions via event-triggered communication," *IEEE Transactions on Cybernetics*, pp. 1–13, 2019.

[47] R. Zhang, D. Zeng, X. Liu, S. Zhong, and K. Shi, "A new method for quantized sampled-data synchronization of delayed chaotic Lur'e systems," *Applied Mathematical Modelling*, vol. 70, pp. 471–489, 2019.

[48] R. Zhang, D. Zeng, and S. Zhong, "Novel master-slave synchronization criteria of chaotic Lur'e systems with time delays using sampled-data control," *Journal of The Franklin Institute*, vol. 354, pp. 4930–4954, 2017.

[49] Y. Liu, F. Fang, J. H. Park, H. Kim, and X. Yi, "Asynchronous output feedback dissipative control of Markovian jump systems with input time delay and quantized measurements," *Nonlinear Analysis: Hybrid Systems*, vol. 31, pp. 109–122, 2019.

[50] Y. Liu, B. Z. Guo, and J. H. Park, "Non-fragile H_∞ filtering for delayed Takagi–Sugeno fuzzy systems with randomly occurring gain variations," *Fuzzy Sets and Systems*, vol. 316, pp. 99–116, 2017.