Hindawi
Complexity
Volume 2019, Article ID 7952871, 9 pages
https://doi.org/10.1155/2019/7952871

Research Article

Fractional Soliton Dynamics and Spectral Transform of Time-Fractional Nonlinear Systems: A Concrete Example

Sheng Zhang,[1,2,3] Yuanyuan Wei,[1] and Bo Xu[4]

[1] *School of Mathematics and Physics, Bohai University, Jinzhou 121013, China*
[2] *Department of Mathematics, Hohhot Minzu College, Hohhot 010051, China*
[3] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*
[4] *School of Education and Sports, Bohai University, Jinzhou 121013, China*

Correspondence should be addressed to Sheng Zhang; szhangchina@126.com

Received 25 March 2019; Accepted 19 June 2019; Published 6 August 2019

Academic Editor: Mohamed Boutayeb

In this paper, the spectral transform with the reputation of nonlinear Fourier transform is extended for the first time to a local time-fractional Korteweg-de vries (tfKdV) equation. More specifically, a linear spectral problem associated with the KdV equation of integer order is first equipped with local time-fractional derivative. Based on the spectral problem with the equipped local time-fractional derivative, the local tfKdV equation with Lax integrability is then derived and solved by extending the spectral transform. As a result, a formula of exact solution with Mittag-Leffler functions is obtained. Finally, in the case of reflectionless potential the obtained exact solution is reduced to fractional n-soliton solution. In order to gain more insights into the fractional n-soliton dynamics, the dynamical evolutions of the reduced fractional one-, two-, and three-soliton solutions are simulated. It is shown that the velocities of the reduced fractional one-, two-, and three-soliton solutions change with the fractional order.

1. Introduction

Since the increasing interest on fractional calculus and its applications, dynamical processes and dynamical systems of fractional orders have attracted much attention. In 2010, Fujioka et al. [1] investigated soliton propagation of an extended nonlinear Schröinger equation with fractional dispersion term and fractional nonlinearity term. In 2014, Yang et al. [2] used a local fractional KdV equation to model fractal waves on shallow water surfaces.

In the field of nonlinear mathematical physics, the spectral transform [3] with the reputation of nonlinear Fourier transform is a famous analytical method for constructing exact and explicit n-soliton solutions of nonlinear partial differential equations (PDEs). Since put forward by Gardner et al. in 1967, the spectral transform method has achieved considerable developments [4–26]. With the close attentions of fractional calculus and its applications [27–53], some of the natural questions are whether the existing methods like those in [54–70] in soliton theory can be extended to nonlinear PDEs of fractional orders and what about the fractional soliton dynamics and integrability of fractional PDEs. As far as we know there are no research reports on the spectral transform for nonlinear PDEs of fractional orders. This paper is motivated by the desire to extend the spectral transform to nonlinear fractional PDEs and then gain more insights into the fractional soliton dynamics of the obtained solutions. For such a purpose, we consider the following local tfKdV equation:

$$D_t^{(\alpha)} u + 6uu_x + u_{xxx} = 0, \quad 0 < \alpha \leq 1. \tag{1}$$

Here we note that if $\alpha = 1$ then eq. (1) becomes the celebrated KdV equation $u_t + 6uu_x + u_{xxx} = 0$. In eq. (1), the local time-fractional derivative $D_t^{\alpha} u$ at the point $t = t_0$ is defined as [30]

$$\begin{aligned} D_t^{(\alpha)} u(x, t_0) &= \left.\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right|_{t=t_0} \\ &= \lim_{t \longrightarrow t_0} \frac{\Delta^{\alpha} \left(u(x,t) - u(x,t_0)\right)}{(t - t_0)^{\alpha}}, \end{aligned} \tag{2}$$

where $\Delta^{\alpha}(u(x,t)-u(x,t_0)) \cong \Gamma(1+\alpha)(u(x,t)-u(x,t_0))$; some useful properties [33] of the local time-fractional derivative have been used in this paper.

The rest of this paper is organized as follows. In Section 2, we derive the local tfKdV eq. (1) by introducing a linear spectral problem equipped with local time-fractional derivative. In Section 3, we construct fractional n-soliton solution of the local tfKdV eq. (1) by extending the spectral transform method. In Section 4, we investigate the dynamical evolutions of the obtained fractional one-soliton solution, two-soliton solution, and three-soliton solution. In Section 5, we conclude this paper.

2. Derivation of the Local tfKdV Equation

For the local tfKdV eq. (1), we have the following Theorem 1.

Theorem 1. *The local tfKdV eq. (1) is a Lax system, which can be derived from the linear spectral problem equipped with a local time-fractional evolution equation:*

$$\phi_{xx} = (\eta - u)\phi, \quad \eta = -k^2, \tag{3}$$

$$D_t^{(\alpha)}\phi = (u_x + \vartheta)\phi + \left(4k^2 - 2u\right)\phi_x, \tag{4}$$

where $\phi = \phi(x,t)$ and $u = u(x,t)$ are all differentiable functions with respect to x and t, the spectral parameter η is independent of t, and ϑ is an arbitrary constant.

Proof. Taking the time-fractional derivative of eq. (3) yields

$$D_t^{(\alpha)}\phi_{xx} = -\phi D_t^{(\alpha)}u - \left(u + k^2\right)D_t^{(\alpha)}\phi, \tag{5}$$

Substituting eq. (4) into eq. (5), we have

$$\begin{aligned} D_t^{(\alpha)}\phi_{xx} = &-\left[D_t^{(\alpha)}u + (u_x + \vartheta)\left(u + k^2\right)\right]\phi \\ &- \left(4k^2 - 2u\right)\left(u + k^2\right)\phi_x, \end{aligned} \tag{6}$$

Taking the derivative of eq. (4) with respect to x twice gives

$$\left(D_t^{(\alpha)}\phi\right)_{xx} = u_{xxx}\phi - (3u_x - \vartheta)\phi_{xx} + \left(4k^2 - 2u\right)\phi_{xxx}, \tag{7}$$

With the help of eq. (3), from eq. (7) we have

$$\begin{aligned} \left(D_t^{(\alpha)}\phi\right)_{xx} = &\left[u_{xxx} + \left(5u - k^2\right)u_x - \vartheta\left(u + k^2\right)\right]\phi \\ &- \left(4k^2 - 2u\right)\left(u + k^2\right)\phi_x. \end{aligned} \tag{8}$$

On the other hand, at the aribitrary point $t = t_0$ we have

$$\begin{aligned} D_t^{\alpha}\phi_{xx}(x,t_0) &= \lim_{t\longrightarrow t_0}\frac{\Delta^{\alpha}\left(\phi_{xx}(x,t) - \phi_{xx}(x,t_0)\right)}{(t-t_0)^{\alpha}}, \\ \left(D_t^{\alpha}\phi(x,t_0)\right)_{xx} &= \lim_{t\longrightarrow t_0}\frac{\Delta^{\alpha}\left(\phi(x,t) - \phi(x,t_0)\right)_{xx}}{(t-t_0)^{\alpha}}. \end{aligned} \tag{9}$$

Finally, using eqs. (6), (8), and (9) we arrive at eq. (1). Thus, we finish the proof. The process of proof shows that eq. (1) is a Lax integrable system. □

3. Local Fractional Spectral Transform

Since the local tfKdV eq. (1) is a local time-fractional system, the orders of the derivatives with respect to space variable x are integers. So, all the existing results about the spectral problem (3), a part of the Lax pair for the classical KdV equation, can be translated to the local tfKdV eq. (1).

For the direct scattering problem, we translate some necessary results and definitions [9] to the local tfKdV eq. (1).

Lemma 2. *If the real potential $u(x)$ defined in the whole real axis $-\infty < x < \infty$ and its various derivatives are differentiable functions which vanishes rapidly as $x \longrightarrow \pm\infty$ and satisfies*

$$\int_{-\infty}^{+\infty}\left|x^j u(x)\right|\mathrm{d}x < +\infty, \quad (j = 0,1,2), \tag{10}$$

then the linear spectral problem (3) has a set of basic solutions called Jost solutions $\phi^{+}(x,k)$ and $\phi^{-}(x,k)$, and they are not only bounded for all values of x but also analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$ and have the following asymptotic properties:

$$\phi^{+}(x,k) \longrightarrow \mathrm{e}^{\mathrm{i}kx}, \quad (x \longrightarrow +\infty), \tag{11}$$

$$\phi^{-}(x,k) \longrightarrow \mathrm{e}^{-\mathrm{i}kx}, \quad (x \longrightarrow -\infty). \tag{12}$$

Lemma 3. *Define the Wronskian*

$$\begin{aligned} &W\left(\phi^{+}(x,k), \phi^{-}(x,k)\right) \\ &\quad = \phi_x^{+}(x,k)\phi^{-}(x,k) - \phi^{+}(x,k)\phi_x^{-}(x,k), \end{aligned} \tag{13}$$

and let

$$\phi^{-}(x,k) = a(k)\phi^{+}(x,-k) + b(k)\phi^{+}(x,k). \tag{14}$$

Then

$$\begin{aligned} a(k) &= \frac{1}{2\mathrm{i}k}W\left(\phi^{-}(x,k), \phi^{+}(x,k)\right), \\ b(k) &= \frac{1}{2\mathrm{i}k}W\left(\phi^{+}(x,-k), \phi^{-}(x,k)\right), \end{aligned} \tag{15}$$

where $a(k)$ is analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$, $b(k)$ is defined only on the real axis $\operatorname{Im} k = 0$, and the analytic function $a(k)$ has a finite number of simple zeros $k_m = \mathrm{i}\kappa_m$ ($\kappa_m > 0, m = 1, 2, \ldots, n$).

Lemma 4. *For the linear spectral problem (3), there exists a constant b_m, such that*

$$\phi^{-}(x,\mathrm{i}\kappa_m) = b_m\phi^{+}(x,\mathrm{i}\kappa_m), \tag{16}$$

$$\int_{-\infty}^{+\infty} c_m^2\phi^{+2}(x,\mathrm{i}\kappa_m)\,\mathrm{d}x = 1, \quad c_m^2 = -\frac{\mathrm{i}b_m}{a_k(\mathrm{i}\kappa_m)}. \tag{17}$$

Definition 5. The constant c_m satisfying eq. (17) is named the normalization constant for the eigenfunction $\phi^{+}(x,\mathrm{i}\kappa_m)$, and $c_m\phi^{+}(x,\mathrm{i}\kappa_m)$ is named normalization eigenfunction.

Definition 6. The set

$$\left\{k\,(\operatorname{Im} k = 0)\,,\; R(k) = \frac{b(k)}{a(k)},\; i\kappa_m, c_m,\; m = 1, 2, \ldots, n\right\} \tag{18}$$

is named the scattering data of the linear spectral problem (3).

Lemma 7. *If the eigenfunction $\phi(x,k)$ satisfies the linear spectral problem (3), then*

$$P = D_t^{(\alpha)}\phi - (u_x + \vartheta)\,\phi - \left(4k^2 - 2u\right)\phi_x \tag{19}$$

solves eq. (3) as well.

Proof. A direct computation on eq. (19) tells that

$$\begin{aligned} P_{xx} &= D_t^{(\alpha)}\phi_{xx} - u_{xxx}\phi - \left(4k^2 - 2u\right)\phi_{xxx} \\ &\quad + (3u_x - \vartheta)\,\phi_{xx}. \end{aligned} \tag{20}$$

With the help of eqs. (3) and (20), we have

$$P_{xx} + \left(k^2 + u\right)P = -\phi\left(D_t^{(\alpha)}u + 6uu_x + u_{xxx}\right) = 0. \tag{21}$$

Thus, the proof is finished. □

For the time dependence of the scattering data, we have the following Theorem 8.

Theorem 8. *If the time evolution of $u(x,t)$ obeys the local tfKdV eq. (1), then the scattering data (18) for the linear spectral problem (3) possess the following time dependences:*

$$\begin{aligned} D_t^{(\alpha)}\kappa_m(t) &= 0, \\ D_t^{(\alpha)}c_m(t) &= 4\kappa_m^3(t)\,c_m(t)\,, \end{aligned} \tag{22}$$

$$\begin{aligned} D_t^{(\alpha)}a(k) &= 0, \\ D_t^{(\alpha)}b(k) &= 8ik^3(t)\,b(k)\,. \end{aligned} \tag{23}$$

Proof. Substituting eqs. (12) and (14) into eq. (3) and using the asymptotic properties of eqs. (11) and (12) as $x \longrightarrow +\infty$ and $x \longrightarrow -\infty$, respectively, we have

$$\left(\vartheta - 4ik^3\right)e^{-ikx} = 0, \tag{24}$$

$$\begin{aligned} &D_t^{(\alpha)}a(k)\,e^{-ikx} + D_t^{(\alpha)}b(k)\,e^{ikx} \\ &\quad = \vartheta\left(a(k)\,e^{-ikx} + b(k)\,e^{ikx}\right) \\ &\qquad + 4k^2\left(-ika(k)\,e^{-ikx} + ikb(k)\,e^{ikx}\right). \end{aligned} \tag{25}$$

Namely,

$$\vartheta = 4ik^3, \tag{26}$$

$$D_t^{(\alpha)}a(k) = a(k)\left(\vartheta - 4ik^3\right) = 0, \tag{27}$$

$$D_t^{(\alpha)}b(k) = b(k)\left(\vartheta + 4ik^3\right) = 8ik^3 b(k)\,. \tag{28}$$

It is easy to see that all the zeros of $a(k)$ are independent of t because of $D_t^{(\alpha)}a(k) = 0$. Therefore, we arrive at $D_t^{(\alpha)}\kappa_m(t) = 0$.

Similarly, substituting eq. (16) into eq. (3) and using the asymptotic property of eq. (11) as $x \longrightarrow +\infty$, we have

$$D_t^{(\alpha)}b_m(t) = \left(\vartheta + 4\kappa_m^3(t)\right)b_m(t) = 8\kappa_m^3(t)\,b_m(t)\,. \tag{29}$$

In view of eqs. (17) and (28), we obtain

$$D_t^{(\alpha)}\left(c_m^2(t)\right) = -\frac{D_t^{(\alpha)}\left(ib_m(t)\right)}{a_k\left(i\kappa_m\right)} = 8\kappa_m^3(t)\,c_m^2(t)\,, \tag{30}$$

which can be finally reduced to the second term of eq. (22). Then we finish the proof. □

For the inverse scattering problem, we have the following Theorem 9.

Theorem 9. *The local tfKdV eq. (1) has an exact solution of the form*

$$u(x,t) = 2\frac{d}{dx}K(x,x,t)\,, \tag{31}$$

where $K(x,y,t)$ satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:

$$\begin{aligned} &K(x,y,t) + F(x+y,t) \\ &\quad + \int_x^{\infty} K(x,z,t)\,F(x+z,t)\,dz = 0, \end{aligned} \tag{32}$$

with

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\,e^{ikx}dk + \sum_{m=1}^{n} c_m^2 e^{i\kappa_m x}, \tag{33}$$

and $R(k,t)$, c_m, and κ_m are determined by eqs. (22) and (23).

Proof. The process of the proof of Theorem 9 is similar to that of the classical KdV equation [9] with integer order, and the only differences are the scattering data. To avoid unnecessary repetition, we omit it here. □

For the fractional n-soliton solution, we have the following Theorem 10.

Theorem 10. *In the case of reflectionless potential, the local tfKdV eq. (1) has fractional n-soliton solution of the form*

$$u(x,t) = 2\frac{d^2}{dx^2}\ln\left(\det D(x,t)\right), \tag{34}$$

where

$$\begin{aligned} D(x,t) &= \left(d_{jm}(x,t)\right)_{n\times n}, \\ d_{jm}(x,t) &= \delta_{jm} + \frac{c_j(t)\,c_m(t)}{\kappa_j(0) + \kappa_m(0)}e^{-(\kappa_j(0)+\kappa_m(0))x}, \end{aligned} \tag{35}$$

$$\begin{aligned} c_j(t) &= c_j(0)\,E_\alpha\left(4\kappa_j^3(0)\,t^\alpha\right), \\ c_m(t) &= c_m(0)\,E_\alpha\left(4\kappa_m^3(0)\,t^\alpha\right). \end{aligned} \tag{36}$$

In eq. (36), $E_\alpha(\cdot)$ is the Mittag-Leffler function [33].

Proof. Firstly, we further determine the scattering data. Solving eqs. (22) and (23) yields

$$\begin{aligned}\kappa_m(t) &= \kappa_m(0),\\ c_m(t) &= c_m(0)\,E_\alpha\left(4\kappa_m^3(0)\,t^\alpha\right),\end{aligned} \tag{37}$$

$$\begin{aligned}a(k,t) &= a(k,0),\\ b(k,t) &= b(k,0)\,E_\alpha\left(8\mathrm{i}\kappa^3(0)\,t^\alpha\right).\end{aligned} \tag{38}$$

Secondly, we let $R(k,t)=0$. In this case of reflectionless, eq. (32) reduces to

$$\begin{aligned}&K(x,y,t)+\sum_{m=1}^{n}c_m^2(t)\,\mathrm{e}^{-\kappa_m(0)(x+y)}\\ &\quad+\sum_{m=1}^{n}c_m^2(t)\,\mathrm{e}^{-\kappa_m(0)y}\int_x^{\infty}K(x,z,t)\,\mathrm{e}^{-\kappa_m(0)z}\mathrm{d}z\\ &=0.\end{aligned} \tag{39}$$

Suppose that eq. (39) has a separation solution

$$K(x,y,t)=\sum_{j=1}^{n}c_j(t)\,h_j(x)\,\mathrm{e}^{-\kappa_j(0)y}, \tag{40}$$

where $h_j(x)$ is an undetermined function which can be determined by the substitution of eq. (40) into eq. (39). With the determined function $h_j(x)$, we have

$$K(x,y,t)=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\det D(x,t)\right). \tag{41}$$

Finally, from eqs. (31), (37), (38), and (41) we obtain eq. (34). Therefore, the proof is over. □

4. Fractional Soliton Dynamics

In order to gain more insights into the soliton dynamics of the obtained fractional n-soliton solution (34), we consider the cases of $n=1,2,3$.

When $n=1$, we have

$$\det D(x,t)=1+\frac{c_1^2(0)\,E_\alpha\left(8\kappa_1^3(0)\,t^\alpha\right)}{2\kappa_1(0)}\mathrm{e}^{-2\kappa_1(0)x}, \tag{42}$$

and we, hence, obtain, from eq. (34), the fractional one-soliton solution:

$$\begin{aligned}u &= 2\kappa_1^2(0)\\ &\quad\cdot\operatorname{sech}^2\left[\kappa_1(0)\,x-\frac{1}{2}\ln\frac{c_1^2(0)\,E_\alpha\left(8\kappa_1^3(0)\,t^\alpha\right)}{2\kappa_1(0)}\right].\end{aligned} \tag{43}$$

Similarly, when $n=2$ we obtain the fractional two-soliton solution:

$$\begin{aligned}u = 2\ln\Bigg[&1+\frac{c_1^2(0)\,E_\alpha\left(8\kappa_1^3(0)\,t^\alpha\right)}{2\kappa_1(0)}\mathrm{e}^{-2\kappa_1(0)x}\\ &+\frac{c_2^2(0)\,E_\alpha\left(8\kappa_2^3(0)\,t^\alpha\right)}{2\kappa_2(0)}\mathrm{e}^{-2\kappa_2(0)x}\\ &+\frac{c_1^2(0)\,c_2^2(0)\left(\kappa_1(0)-\kappa_2(0)\right)^2E_\alpha\left(8\kappa_1(0)\,t^\alpha\right)E_\alpha\left(8\kappa_2^3(0)\,t^\alpha\right)}{4\kappa_1(0)\,\kappa_2(0)\left(\kappa_1(0)+\kappa_2(0)\right)^2}\\ &\cdot\mathrm{e}^{-2(\kappa_1(0)+\kappa_2(0))x}\Bigg]_{xx}.\end{aligned} \tag{44}$$

When $n=3$, we obtain the fractional three-soliton solution:

$$\begin{aligned}u = 2\ln\Bigg[&1+\frac{c_1^2(0)\,E_\alpha\left(8\kappa_1^3(0)\,t^\alpha\right)}{2\kappa_1(0)}\mathrm{e}^{-2\kappa_1(0)x}+\frac{c_2^2(0)\,E_\alpha\left(8\kappa_2^3(0)\,t^\alpha\right)}{2\kappa_2(0)}\mathrm{e}^{-2\kappa_2(0)x}+\frac{c_3^2(0)\,E_\alpha\left(8\kappa_3^3(0)\,t^\alpha\right)}{2\kappa_3(0)}\mathrm{e}^{-2\kappa_3(0)x}\\ &+\frac{c_1^2(0)\,c_2^2(0)\left(\kappa_1(0)-\kappa_2(0)\right)^2E_\alpha\left(8\kappa_1(0)\,t^\alpha\right)E_\alpha\left(8\kappa_2^3(0)\,t^\alpha\right)}{4\kappa_1(0)\,\kappa_2(0)\left(\kappa_1(0)+\kappa_2(0)\right)^2}\mathrm{e}^{-2(\kappa_1(0)+\kappa_2(0))x}\\ &+\frac{c_1^2(0)\,c_3^2(0)\left(\kappa_1(0)-\kappa_3(0)\right)^2E_\alpha\left(8\kappa_1(0)\,t^\alpha\right)E_\alpha\left(8\kappa_3^3(0)\,t^\alpha\right)}{4\kappa_1(0)\,\kappa_3(0)\left(\kappa_1(0)+\kappa_3(0)\right)^2}\mathrm{e}^{-2(\kappa_1(0)+\kappa_3(0))x}\\ &+\frac{c_2^2(0)\,c_3^2(0)\left(\kappa_2(0)-\kappa_3(0)\right)^2E_\alpha\left(8\kappa_2(0)\,t^\alpha\right)E_\alpha\left(8\kappa_3^3(0)\,t^\alpha\right)}{4\kappa_2(0)\,\kappa_3(0)\left(\kappa_2(0)+\kappa_3(0)\right)^2}\mathrm{e}^{-2(\kappa_2(0)+\kappa_3(0))x}\\ &+\frac{c_1^2(0)\,c_2^2(0)\,c_3^2(0)\left(\kappa_1(0)-\kappa_2(0)\right)^2\left(\kappa_1(0)-\kappa_3(0)\right)^2\left(\kappa_2(0)-\kappa_3(0)\right)^2}{8\kappa_1(0)\,\kappa_2(0)\,\kappa_3(0)\left(\kappa_1(0)+\kappa_2(0)\right)^2\left(\kappa_1(0)+\kappa_3(0)\right)^2\left(\kappa_2(0)+\kappa_3(0)\right)^2}E_\alpha\left(8\kappa_1(0)\,t^\alpha\right)E_\alpha\left(8\kappa_2(0)\,t^\alpha\right)E_\alpha\left(8\kappa_3^3(0)\right.\\ &\left.\cdot\,t^\alpha\right)\mathrm{e}^{-2(\kappa_1(0)+\kappa_2(0)+\kappa_3(0))x}\Bigg]_{xx}.\end{aligned} \tag{45}$$

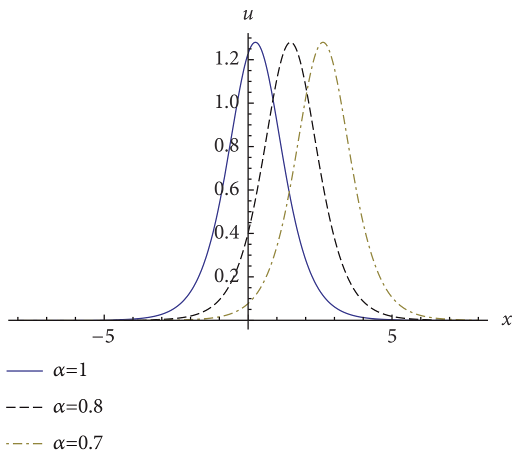


FIGURE 1: Fractional one-solitons determined by solution (43) at time $t = 1$.

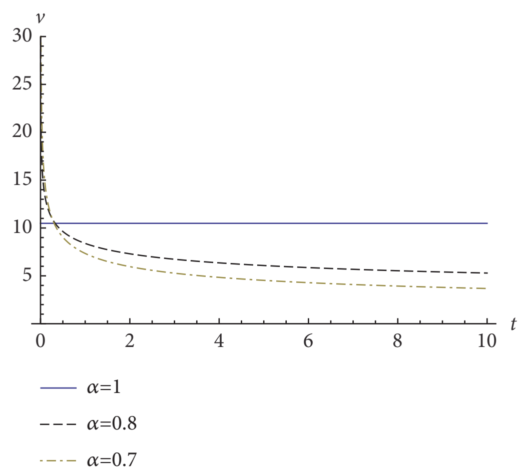


FIGURE 2: Velocity images of fractional one-solitons determined by solution (43).

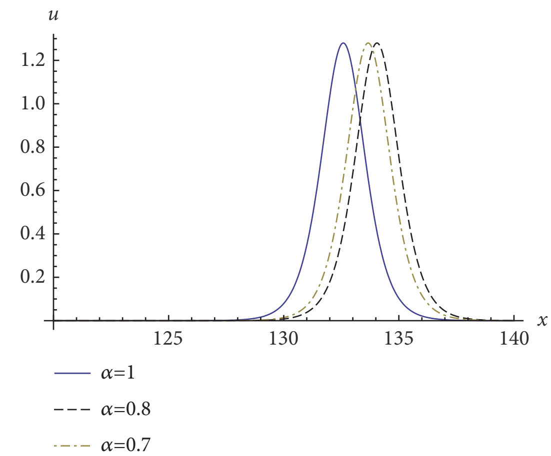


FIGURE 3: Fractional one-solitons determined by solution (43) at time $t = 80$.

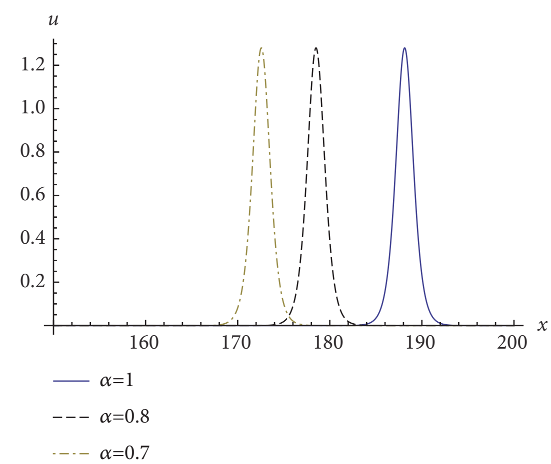


FIGURE 4: Fractional one-solitons determined by solution (43) at time $t = 195$.

In Figure 1, we simulate the fractional one-soliton solution (43) with different values of α, where the parameters are selected as $\kappa_1(0) = 0.8$ and $c_1(0) = 0.2$. With the help of velocity images in Figure 2 and the formula of velocity

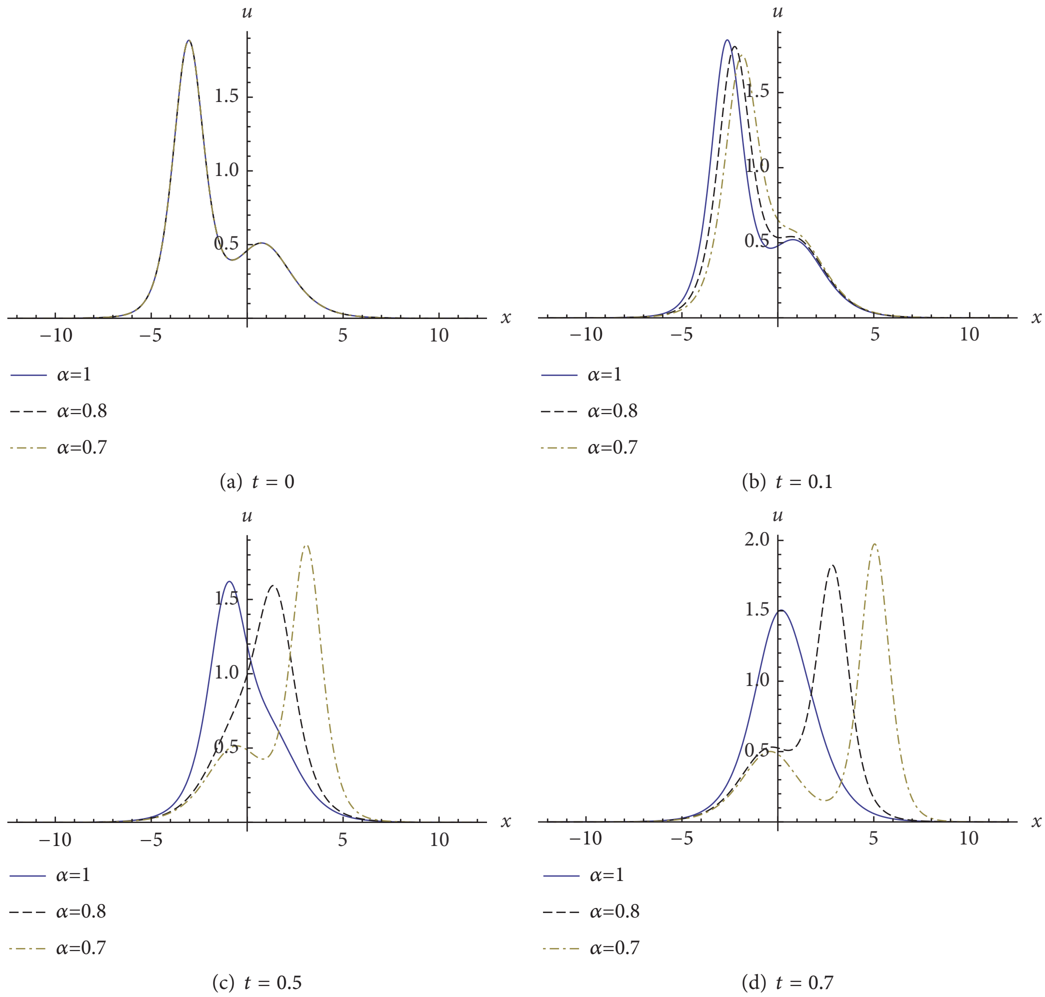
$$v = 32\alpha\kappa_1^5(0)\,t^{\alpha-1}, \tag{46}$$

we can see that the bell-shaped solitons have different velocities depending on the values of α. At the initial stage, the smaller the value of α is selected, the faster the soliton propagates. But soon it was the opposite; for more details see Figures 3 and 4.

For the fractional two-solitons and three-solitons determined, respectively, by solutions (44) and (45), similar features shown in Figures 5–7 are observed. In Figures 5 and 6, we select the parameters as $\kappa_1(0) = 0.5$, $c_1(0) = 1.5$, $\kappa_2(0) = 1$, and $c_2(0) = -0.2$. While the parameters in Figure 7 are selected as $\kappa_1(0) = 1$, $c_1(0) = 1$, $\kappa_2(0) = 1.1$, $c_2(0) = 0.5$, $\kappa_3(0) = 1.3$, $c_3(0) = 2$.

5. Conclusion

In summary, we have derived and solved the local tfKdV eq. (1) in the fractional framework of the spectral transform method. This is due to the linear spectral problem (3) equipped with the local time-fractional evolution (4). As for the fractional derivatives, there are many definitions [33] except the local fractional derivative, such as Grünwald–Letnikov fractional derivative, Riemann-Liouville fractional derivative and Caputo's fractional derivative. Generally speaking, whether or not the spectral transform can be extended to some other nonlinear evolution equations with another type of fractional derivative depends on whether fractional derivative has the good properties required by the spectral transform method. To the best of our knowledge, combined with the Mittag-Leffler functions the obtained exact solution (31), the fractional n-soliton solution (34), and its special cases, the fractional one-, two-, and three-soliton solutions (43)-(45), are all new, and they have not been reported in literature. It is graphically shown that the fractional order of the local tfKdV eq. (1) influences



(a) $t = 0$

(b) $t = 0.1$

(c) $t = 0.5$

(d) $t = 0.7$

Figure 5: Dynamical evolutions of fractional two-solitons determined by solution (44) at different times.

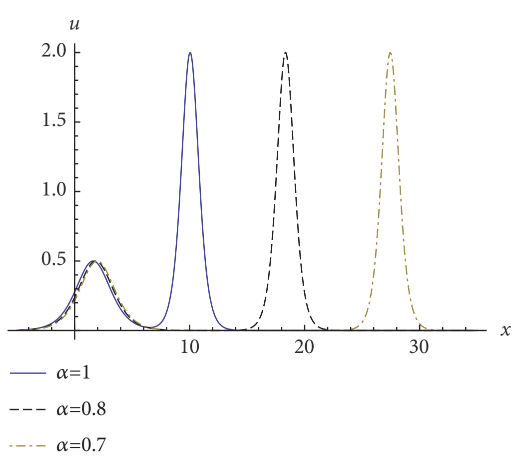


Figure 6: Fractional two-solitons determined by solution (44) at $t = 3$.

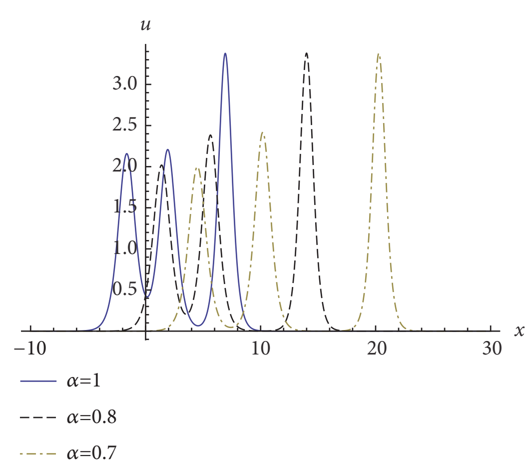


Figure 7: Fractional three-solitons determined by solution (45) at $t = 1$.

the velocity of the fractional one-soliton solution (43) with Mittag-Leffler function in the process of propagations. More importantly, the fractional scheme of the spectral transform presented in this paper for constructing n-soliton solution of the local tfKdV eq. (1) can be extended to some other integrable local time-fractional PDEs.

Data Availability

The data in the paper are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This work was supported by the Natural Science Foundation of China (11547005), the Natural Science Foundation of Liaoning Province of China (20170540007), the Natural Science Foundation of Education Department of Liaoning Province of China (LZ2017002), and Innovative Talents Support Program in Colleges and Universities of Liaoning Province (LR2016021).

References

[1] J. Fujioka, A. Espinosa, and R. Rodríguez, "Fractional optical solitons," *Physics Letters A*, vol. 374, no. 9, pp. 1126–1134, 2010.

[2] X.-J. Yang, J. Hristov, H. M. Srivastava, and B. Ahmad, "Modelling fractal waves on shallow water surfaces via local fractional korteweg-de vries equation," *Abstract and Applied Analysis*, vol. 2014, Article ID 278672, 10 pages, 2014.

[3] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for solving the Korteweg-deVries equation," *Physical Review Letters*, vol. 19, no. 19, pp. 1095–1097, 1967.

[4] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "The inverse scattering transform-Fourier analysis for nonlinear problems," *Studies in Applied Mathematics*, vol. 53, no. 4, pp. 249–315, 1974.

[5] H. H. Chen and C. S. Liu, "Solitons in nonuniform media," *Physical Review Letters*, vol. 37, no. 11, pp. 693–697, 1976.

[6] R. Hirota and J. Satsuma, "N-soliton solutions of the K-dV equation with loss and nonuniformity terms," *Journal of the Physical Society of Japan*, vol. 41, no. 6, pp. 2141-2142, 1976.

[7] F. Calogero and A. Degasperis, "Coupled nonlinear evolution equations solvable via the inverse spectral transform, and solitons that come back: the boomeron," *Lettere Al Nuovo Cimento*, vol. 16, no. 14, pp. 425–433, 1976.

[8] W. L. Chan and K.-S. Li, "Nonpropagating solitons of the variable coefficient and nonisospectral Korteweg-de Vries equation," *Journal of Mathematical Physics*, vol. 30, no. 11, pp. 2521–2526, 1989.

[9] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, UK, 1991.

[10] B. Z. Xu and S. Q. Zhao, "Inverse scattering transformation for the variable coefficient sine-Gordon type equation," *Applied Mathematics-A Journal of Chinese Universities*, vol. 9, no. 4, pp. 331–337, 1994.

[11] Y. Zeng, W. Ma, and R. Lin, "Integration of the soliton hierachy with selfconsistent sources," *Journal of Mathematical Physics*, vol. 41, no. 8, pp. 5453–5489, 2000.

[12] V. N. Serkin and A. Hasegawa, "Novel soliton solutions of the nonlinear Schrodinger equation model," *Physical Review Letters*, vol. 85, no. 21, pp. 4502–4505, 2000.

[13] V. N. Serkin and T. L. Belyaeva, "Optimal control of optical soliton parameters: Part 1. The Lax representation in the problem of soliton management," *Quantum Electronics*, vol. 31, no. 11, pp. 1007–1015, 2001.

[14] T.-k. Ning, D.-y. Chen, and D.-j. Zhang, "The exact solutions for the nonisospectral AKNS hierarchy through the inverse scattering transform," *Physica A: Statistical Mechanics and its Applications*, vol. 339, no. 3-4, pp. 248–266, 2004.

[15] V. N. Serkin, A. Hasegawa, and T. L. Belyaeva, "Nonautonomous solitons in external potentials," *Physical Review Letters*, vol. 98, no. 7, Article ID 074102, 2007.

[16] B. Guo and L. Ling, "Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation," *Journal of Mathematical Physics*, vol. 53, no. 7, Article ID 073506, 20 pages, 2012.

[17] S. Zhang, B. Xu, and H.-Q. Zhang, "Exact solutions of a KdV equation hierarchy with variable coefficients," *International Journal of Computer Mathematics*, vol. 91, no. 7, pp. 1601–1616, 2014.

[18] S. Zhang and D. Wang, "Variable-coefficient nonisospectral Toda lattice hierarchy and its exact solutions," *Pramana—Journal of Physics*, vol. 85, no. 6, pp. 1143–1156, 2015.

[19] S. Zhang and X.-D. Gao, "Mixed spectral AKNS hierarchy from linear isospectral problem and its exact solutions," *Open Physics*, vol. 13, no. 1, pp. 310–322, 2015.

[20] S. Randoux, P. Suret, and G. El, "Inverse scattering transform analysis of rogue waves using local periodization procedure," *Scientific Reports*, vol. 6, Article ID 29238, 10 pages, 2016.

[21] S. Zhang and X. Gao, "Exact solutions and dynamics of a generalized AKNS equations associated with the nonisospectral depending on exponential function," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 6, pp. 4529–4541, 2016.

[22] S. Zhang and J. Li, "Soliton solutions and dynamical evolutions of a generalized AKNS system in the framework of inverse scattering transform," *Optik - International Journal for Light and Electron Optics*, vol. 137, pp. 228–237, 2017.

[23] S. Zhang and S. Hong, "Lax integrability and soliton solutions for a nonisospectral integro-differential system," *Complexity*, vol. 2017, Article ID 9457078, 10 pages, 2017.

[24] S. Zhang and S. Hong, "On a generalized Ablowitz–Kaup–Newell–Segur hierarchy in inhomogeneities of media: soliton solutions and wave propagation influenced from coefficient functions and scattering data," *Waves in Random and Complex Media*, vol. 28, no. 3, pp. 435–452, 2017.

[25] S. Zhang and S. Hong, "Lax Integrability and Exact Solutions of a Variable-Coefficient and Nonisospectral AKNS Hierarchy," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 19, no. 3-4, pp. 251–262, 2018.

[26] Z. Kang, T. Xia, and X. Ma, "Multi-solitons for the coupled Fokas–Lenells system via Riemann–Hilbert approach," *Chinese Physics Letters*, vol. 35, no. 7, Article ID 070201, 5 pages, 2018.

[27] S. Zhang and H. Q. Zhang, "Fractional sub-equation method and its applications to nonlinear fractional PDEs," *Physics Letters A*, vol. 375, no. 7, pp. 1069–1073, 2011.

[28] İ. Aslan, "Analytic investigation of a reaction-diffusion brusselator model with the time-space fractional derivative," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 15, no. 2, pp. 149–155, 2014.

[29] İ. Aslan, "An analytic approach to a class of fractional differential-difference equations of rational type via symbolic

computation," *Mathematical Methods in the Applied Sciences*, vol. 38, no. 1, pp. 27–36, 2015.

[30] İ. Aslan, "Symbolic computation of exact solutions for fractional differential-difference equation models," *Nonlinear Analysis: Modelling and Control*, vol. 20, no. 1, pp. 132–144, 2015.

[31] Y. Wang, L. Liu, X. Zhang, and Y. Wu, "Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection," *Applied Mathematics and Computation*, vol. 258, pp. 312–324, 2015.

[32] X. Yang and H. Srivastava, "An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 29, no. 1-3, pp. 499–504, 2015.

[33] X.-J. Yang, D. Baleanu, and H. M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, London, UK, 2015.

[34] G.-C. Wu, D. Baleanu, Z.-G. Deng, and S.-D. Zeng, "Lattice fractional diffusion equation in terms of a Riesz-Caputa difference," *Physica A: Statistical Mechanics and its Applications*, vol. 438, pp. 335–339, 2015.

[35] Y. Hu and J. He, "On fractal space-time and fractional calculus," *THERMAL SCIENCE*, vol. 20, no. 3, pp. 773–777, 2016.

[36] L. Guo, L. Liu, and Y. Wu, "Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions," *Nonlinear Analysis, Modelling and Control*, vol. 21, no. 5, pp. 635–650, 2016.

[37] S. Zhang, M. Liu, and L. Zhang, "Variable separation for time fractional advection-dispersion equation with initial and boundary conditions," *THERMAL SCIENCE*, vol. 20, no. 3, pp. 789–792, 2016.

[38] L. L. Liu, X. Zhang, L. Liu, and Y. Wu, "Iterative positive solutions for singular nonlinear fractional differential equation with integral boundary conditions," *Advances in Difference Equations*, Paper No. 154, 13 pages, 2016.

[39] B. Zhu, L. Liu, and Y. Wu, "Local and global existence of mild solutions for a class of nonlinear fractional reaction-diffusion equations with delay," *Applied Mathematics Letters*, vol. 61, pp. 73–79, 2016.

[40] H. Liu and F. Meng, "Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications," *Journal of Inequalities and Applications*, vol. 2016, no. 1, article no. 213, 7 pages, 2016.

[41] İ. Aslan, "Exact Solutions for a Local Fractional DDE Associated with a Nonlinear Transmission Line," *Communications in Theoretical Physics*, vol. 66, no. 3, pp. 315–320, 2016.

[42] İ. Aslan, "Exact solutions for fractional DDEs via auxiliary equation method coupled with the fractional complex transform," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 18, pp. 5619–5625, 2016.

[43] J. Wu, X. Zhang, L. Liu, and Y. Wu, "Twin iterative solutions for a fractional differential turbulent flow model," *Boundary Value Problems*, Article ID 98, 9 pages, 2016.

[44] R. Xu and F. Meng, "Some new weakly singular integral inequalities and their applications to fractional differential equations," *Journal of Inequalities and Applications*, vol. 2016, Article ID 78, 16 pages, 2016.

[45] X. Yang, J. A. Tenreiro Machado, D. Baleanu, and C. Cattani, "On exact traveling-wave solutions for local fractional Korteweg-de Vries equation," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 26, no. 8, Article ID 084312, 5 pages, 2016.

[46] X. Yang, F. Gao, and H. Srivastava, "Exact travelling wave solutions for the local fractional two-dimensional Burgers-type equations," *Computers & Mathematics with Applications*, vol. 73, no. 2, pp. 203–210, 2017.

[47] X.-J. Yang and J. A. Machado, "A new fractional operator of variable order: application in the description of anomalous diffusion," *Physica A: Statistical Mechanics and its Applications*, vol. 481, pp. 276–283, 2017.

[48] Y. Wang and J. Jiang, "Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian," *Advances in Difference Equations*, Paper No. 337, 19 pages, 2017.

[49] K. M. Zhang, "On a sign-changing solution for some fractional differential equations," *Boundary Value Problems*, vol. 2017, no. 59, 8 pages, 2017.

[50] X. Du and A. Mao, "Existence and multiplicity of nontrivial solutions for a class of semilinear fractional schrödinger equations," *Journal of Function Spaces*, vol. 2017, Article ID 3793872, 7 pages, 2017.

[51] Y. Wang and L. Liu, "Positive solutions for a class of fractional 3-point boundary value problems at resonance," *Advances in Difference Equations*, Paper No. 7, 13 pages, 2017.

[52] X. Zhang, L. Liu, Y. Wu, and B. Wiwatanapataphee, "Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion," *Applied Mathematics Letters*, vol. 66, pp. 1–8, 2017.

[53] J.-H. He, "Fractal calculus and its geometrical explanation," *Results in Physics*, vol. 10, pp. 272–276, 2018.

[54] R. Hirota, "Exact solution of the korteweg—de vries equation for multiple Collisions of solitons," *Physical Review Letters*, vol. 27, no. 18, pp. 1192–1194, 1971.

[55] M. Wang, "Exact solutions for a compound KdV-Burgers equation," *Physics Letters A*, vol. 213, no. 5-6, pp. 279–287, 1996.

[56] E. G. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Physics Letters A*, vol. 277, no. 4-5, pp. 212–218, 2000.

[57] Z. Yan and H. Zhang, "New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water," *Physics Letters A*, vol. 285, no. 5-6, pp. 355–362, 2001.

[58] E. G. Fan, "Travelling wave solutions in terms of special functions for nonlinear coupled evolution systems," *Physics Letters A*, vol. 300, no. 2-3, pp. 243–249, 2002.

[59] S. Zhang and T. Xia, "A generalized F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equations," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1190–1200, 2006.

[60] J. He and X. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 30, no. 3, pp. 700–708, 2006.

[61] I. Aslan, "Multi-wave and rational solutions for nonlinear evolution equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 11, no. 8, pp. 619–623, 2010.

[62] I. Aslan, "Analytic investigation of the (2 + 1)-dimensional Schwarzian Korteweg–de Vries equation for traveling wave solutions," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 6013–6017, 2011.

[63] Y. Wang and Y. Chen, "Binary Bell polynomial manipulations on the integrability of a generalized (2+1)-dimensional Korteweg–de Vries equation," *Journal of Mathematical Analysis and Applications*, vol. 400, no. 2, pp. 624–634, 2013.

[64] S. Y. Lou, "Consistent Riccati expansion for integrable systems," *Studies in Applied Mathematics*, vol. 134, no. 3, pp. 372–402, 2015.

[65] S. F. Tian, "Initial-boundary value problems for the coupled modified Korteweg-de Vries equation on the interval," *Communications on Pure and Applied Analysis*, vol. 17, no. 3, pp. 923–957, 2018.

[66] P. Zhao and E. Fan, "On quasiperiodic solutions of the modified Kadomtsev–Petviashvili hierarchy," *Applied Mathematics Letters*, vol. 97, pp. 27–33, 2019.

[67] Z. Yan, "The NLS(n, n) equation: Multi-hump compactons and their stability and interaction scenarios," *Chaos, Solitons & Fractals*, vol. 112, pp. 25–31, 2019.

[68] A.-M. Wazwaz, "Compacton solutions of higher order nonlinear dispersive KdV-like equations," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 449–460, 2004.

[69] L.-Q. Kong, J. Liu, D.-Q. Jin, D.-J. Ding, and C.-Q. Dai, "Soliton dynamics in the three-spine α-helical protein with inhomogeneous effect," *Nonlinear Dynamics*, vol. 87, no. 1, pp. 83–92, 2017.

[70] B. Zhang, X.-L. Zhang, and C.-Q. Dai, "Discussions on localized structures based on equivalent solution with different forms of breaking soliton model," *Nonlinear Dynamics*, vol. 87, no. 4, pp. 2385–2393, 2017.