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Research Article

Stability of Traveling Wave Fronts for a Three Species Predator-Prey Model with Nonlocal Dispersals

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In this paper, we consider a predator-prey model with nonlocal dispersals of two cooperative preys and one predator. We prove that the traveling wave fronts with the relatively large wave speed are exponentially stable as perturbation in some exponentially weighted spaces, when the difference between initial data and traveling wave fronts decay exponentially at negative infinity, but in other locations, the initial data can be very large. The adopted method is to use the weighted energy method and the squeezing technique with some new flavors to handle the nonlocal dispersals.

1. Introduction

In this paper, we investigate the stability of traveling wave fronts for a three species predator-prey model:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_1(x-y)[u(y,t)-u(x,t)]\mathrm{d}y + r_1 u(1-u+kv-a_1 z), & x\in\mathbb{R},\ t>0,\\ \dfrac{\partial v(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_2(x-y)[v(y,t)-v(x,t)]\mathrm{d}y + r_2 v(1-v+hu-a_2 z), & x\in\mathbb{R},\ t>0,\\ \dfrac{\partial z(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_3(x-y)[z(y,t)-z(x,t)]\mathrm{d}y + r_3 z(-1+a_1 u+a_2 v-z), & x\in\mathbb{R},\ t>0, \end{cases} \tag{1}$$

with the initial condition

$$\begin{aligned} u(x,0) &= u_0(x,0),\\ v(x,0) &= v_0(x,0),\\ z(x,0) &= z_0(x,0), \end{aligned} \tag{2}$$

where $r_i, i = 1,2,3, a_1, a_2, h,$ and k are positive constants, $u(x,t)$ and $v(x,t)$ denote the densities of two cooperative preys at time t and location x, respectively, $z(x,t)$ denotes the density of the predator at time t and location x, r_1 and r_2 are the intrinsic growth rates of $u(x,t)$ and $v(x,t)$, respectively, r_3 is the death rate of $z(x,t)$, h and k are interspecific cooperative coefficients between two preys, $r_1 a_1$ and $r_2 a_2$ are the predation rates, and $r_3 a_1$ and $r_3 a_2$ are the conversion rates. $J_i : \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1,2,3)$ are probability functions of the random dispersal of individuals and satisfy the following assumptions:

$(H_1) J_i(x) = J_i(-x) \geq 0$, $x \in \mathbb{R}$, $\int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1$ and $\int_{\mathbb{R}} J_i(y)e^{\lambda y}\mathrm{d}y < \infty$, $\lambda > 0$ is the eigenvalue of the characteristic equation of model (1)

$(H_2) 0 < hk < 1$

Wu [1] investigated the spreading speed for a predator-prey model with one predator and two preys:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 u_{xx} + r_1 u(1 - u - kv - a_1 z), & x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2 v_{xx} + r_2 v(1 - hu - v - a_2 z), & x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial z(x,t)}{\partial t} = d_3 z_{xx} + r_3 z(-1 + a_1 u + a_2 v - z), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{3}$$

with the initial condition

$$\begin{aligned} u(x,0) &= u^*, \\ v(x,0) &= v^*, \\ z(x,0) &= z_0(x), \end{aligned} \tag{4}$$

where $u(x,t)$ and $v(x,t)$ denote the densities of two competitive preys at time t and location x, respectively and $z(x,t)$ denotes the density of the predator at time t and location x. The parameter d_i, $i = 1,2,3$, is the diffusion coefficients of $u, v,$ and z, respectively, the remaining parameters are the same as model (1). Under certain conditions, the author characterized the asymptotic spreading speed by the parameters of model (3). The interaction of three species eological systems has been studied before (see [2–4]).

Yu and Pei [5] studied the stability of traveling wave fronts for the cooperative system with nonlocal dispersals:

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_1(x-y)[u_1(y,t) - u_1(x,t)]\mathrm{d}y \\ \quad + r_1 u_1(x,t)[1 - a_1 u_1(x,t) + b_1 u_2(x,t)], \\ \dfrac{\partial u_2(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_2(x-y)[u_2(y,t) - u_2(x,t)]\mathrm{d}y \\ \quad + r_2 u_2(x,t)[1 - a_2 u_2(x,t) + b_2 u_1(x,t)], \end{cases} \tag{5}$$

with the initial condition

$$\begin{aligned} u_1(x,0) &= u_{10}(x,0), \\ u_2(x,0) &= u_{20}(x,0). \end{aligned} \tag{6}$$

The authors adopt the weighted energy method and the squeezing technique to prove the stability of the traveling wave fronts. The weighted energy method for treating time-delayed reaction diffusion equations was firstly introduced by Mei et al. [6]. Then, by combining the squeezing argument, it was developed for proving the global stability of wavefronts by [7–9]. For the nonlocal model using a single integrodifferential equation, the existence, uniqueness, and stability of traveling waves have been widely studied in [10–14]. For the multicomponent nonlocal systems, the existence of traveling waves was also investigated in [15, 16], while the stability of traveling waves is less investigated and can only be found in [5, 17]. Many researchers are widely focused on the complex dynamics of biological systems such as stochastic delay population system [18] and many researchers have studied the Lotka–Volterra time delay models with two competitive preys and one predator [19]. Note that the composite population systems with stochastic effects and time delays present some complex dynamics; thus, this causes widespread researchers concern [20, 21].

Inspired by works [1, 5], we will study the stability of traveling wave fronts for problems (1) and (2). We consider the case when both preys are cooperative. In this case, there is the constant state:

$$(u,v,z) = \left(\frac{1+k}{1-hk}, \frac{1+h}{1-hk}, 0 \right) = (k_1, k_2, k_3). \tag{7}$$

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and summarize our main results. Section 3 is concerned with the proof of the main result by the technical weighted energy method and establishes the desired priori estimate, and then we get it by the squeezing technique. Finally, we make a simple summary in Section 4.

Before stating our main result, we introduce some notations.

Notations. Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0,1,2,\ldots)$ represents a specific constant. Let I be an interval, typically $I = \mathbb{R}$. $L^2(I)$ is the space of square integrable functions defined on I, and $H^k(I)$ $(k \geq 0)$ is the Sobolev space of the L^2 functions $f(x)$ defined on the interval I whose derivatives $(\mathrm{d}^i/\mathrm{d}x^i)f$ $(i = 1,\ldots,k)$ also belong to $L^2(I)$. Further, $L_w^2(I)$ be the weighted L^2 space with a weight function $w(x) > 0$, with the norm defined as

$$\|f\|_{L_w^2(I)} = \left(\int_I w(x)|f(x)|^2 \mathrm{d}x \right)^{1/2}. \tag{8}$$

$H_w^k(I)$ be the weighted Sobolev space with the norm given by

$$\|f\|_{H_w^k(I)} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i} \right|^2 \mathrm{d}x \right)^{1/2}. \tag{9}$$

Let $T > 0$ be a number and $\mathbb{B}$ be a Banach space. We denote by $C^0([0,T];\mathbb{B})$ the space of the $\mathbb{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathbb{B})$ as the space of the $\mathbb{B}$-valued L^2 functions on $[0,T]$. The corresponding spaces of the $\mathbb{B}$-valued functions on $[0,\infty)$ are defined similarly.

2. Preliminaries and Main Result

In this section, we consider the existence of traveling wave fronts for the Cauchy problems (1) and (2) by using the comparison principle and squeezing technique. Traveling waves solution of system (1) is a special solution (u, v, z) with the form $u(x,t) = \phi_1(x+ct) = \phi_1(\xi)$, $v(x,t) = \phi_2(x+ct) = \phi_2(\xi)$, and $z(x,t) = \phi_3(x+ct) = \phi_3(\xi)$ with $c > 0$ is the wave speed, where the wave profile $(\phi_1, \phi_2, \phi_3) \in C^1(\mathbb{R}, \mathbb{R}, \mathbb{R})$ and satisfies

$$\begin{cases} c\phi_1'(\xi) = \int_{\mathbb{R}} J_1(y-\xi)[\phi_1(y) - \phi_1(\xi)]\mathrm{d}y + r_1\phi_1 \\ \quad \cdot (1 - \phi_1 + k\phi_2 - a_1\phi_3), \\ c\phi_2'(\xi) = \int_{\mathbb{R}} J_2(y-\xi)[\phi_2(y) - \phi_2(\xi)]\mathrm{d}y + r_2\phi_2 \\ \quad \cdot (1 - \phi_2 + h\phi_1 - a_2\phi_3), \\ c\phi_3'(\xi) = \int_{\mathbb{R}} J_3(y-\xi)[\phi_3(y) - \phi_3(\xi)]\mathrm{d}y + r_3\phi_3 \\ \quad \cdot (-1 + a_1\phi_1 + a_2\phi_2 - \phi_3). \end{cases} \tag{10}$$

In order to state our stability result, we need to give the following two propositions in [5].

Proposition 1. *Assume that (H_1)-(H_2) hold. Then there exists a positive number c^* such that, for any $c > c^*$, (1) and (2) admits a traveling wavefront $(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ satisfying*

$$\begin{aligned} &\lim_{t\longrightarrow+\infty}(\phi_1, \phi_2, \phi_3) = (k_1, k_2, k_3) \triangleq \mathbf{K}, \\ &\lim_{t\longrightarrow-\infty}(\phi_1, \phi_2, \phi_3) = (0, 0, 0) \triangleq \mathbf{0}. \end{aligned} \tag{11}$$

Proposition 2. *Assume that (H_1)-(H_2) hold. Let $c_0(x,0) = (u_0(x,0), v_0(x,0), z_0(x,0)) \in [\mathbf{0}, \mathbf{K}]_X$, where $X = B \cup C(\mathbb{R}, \mathbb{R}^3)$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^3$ with the usual supremum norm $\|\cdot\|_X$. Then, (1) and (2) admit a unique, bounded, and nonnegative solution $c(x,t,c_0) = (u(x,t,c_0), v(x,t,c_0), z(x,t,c_0))$ on $(x,t) \in \mathbb{R} \times [0,+\infty)$ with the initial data $(u(x,0,c_0), v(x,0,c_0), z(x,0,c_0))$, and $\mathbf{0} \le c(x,t,c_0) \le \mathbf{K}$ for any $x \in \mathbb{R}$ and $t > 0$. Furthermore, letting $(u^-(x,t), v^-(x,t), z^-(x,t))$ and $(u^+(x,t), v^+(x,t), z^+(x,t))$ be the solutions of (1) with no diffusion term with the initial data $(u^-(x,0), v^-(x,0), z^-(x,0))$ and $(u^+(x,0), v^+(x,0), z^+(x,0))$, respectively, and if*

$$\begin{aligned} \mathbf{0} &\le (u^-(x,0), v^-(x,0), z^-(x,0)) \\ &\le (u^+(x,0), v^+(x,0), z^+(x,0)) \le \mathbf{K}, \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{12}$$

then

$$\begin{aligned} \mathbf{0} &\le (u^-(x,t), v^-(x,t), z^-(x,t)) \\ &\le (u^+(x,t), v^+(x,t), z^+(x,t)) \le \mathbf{K}, \quad x \in \mathbb{R},\ t \ge 0. \end{aligned} \tag{13}$$

Propositions 1 and 2 are based on reference [5] and are generalized from two dimension to three dimension. Their proofs are similar to the ones of Theorem 3.5 in [22] and Lemma 3.2 in [23], respectively.

In order to state our stability result, we need to add the following conditions:

(H_3) $0 < r_2 h < \min\{1/k_2(r_1 - r_1 k k_1 - (1/4))$ $1/k_2(r_2 - r_1 k k_1 - 1/4)\}$,

(H_4) $0 < r_3 h < \min\{1/a_2(r_3 - (1/8))\}$,

(H_5) $0 < r_3 k < \min\{1/a_1(r_3 - (1/8))\}$.

Define three functions on η as follows:

$$\begin{aligned} f_1(\eta) &= 2r_1 - 2r_1 k k_1 - 2r_2 h k_2 - \int_{-\infty}^0 J_1(y) e^{-\eta y}\mathrm{d}y, \\ f_2(\eta) &= 2r_2 - 2r_2 h k_2 - 2r_1 k k_1 - \int_{-\infty}^0 J_2(y) e^{-\eta y}\mathrm{d}y, \\ f_3(\eta) &= 4r_3 - 2a_1 r_3 k - 2a_2 r_3 h - \int_{-\infty}^0 J_3(y) e^{-\eta y}\mathrm{d}y. \end{aligned} \tag{14}$$

According to (H_1)-(H_5), it is easily checked

$$\begin{aligned} f_1(0) &= 2r_1 - 2r_1 k k_1 - 2r_2 h k_2 - \frac{1}{2} > 0, \\ f_2(0) &= 2r_2 - 2r_2 h k_2 - 2r_1 k k_1 - \frac{1}{2} > 0, \\ f_3(0) &= 4r_3 + 2a_1 r_3 k + 2a_2 r_3 h - \frac{1}{2} > 0. \end{aligned} \tag{15}$$

Therefore, by continuity, there exists $\eta_0 > 0$ such that $f_1(\eta_0) > 0$, $f_2(\eta_0) > 0$ and $f_3(\eta_0) > 0$.

In addition, define three functions on ξ as follows:

$$\begin{aligned} g_1(\xi) &= -2r_1 + 4r_1\phi_1 - 4r_1 k k_2 - 2r_1 k k_1 - 2r_2 h k_2 \\ &\quad - \int_{-\infty}^0 J_1(y) e^{-\eta_0 y}\mathrm{d}y, \\ g_2(\xi) &= -2r_2 + 4r_2\phi_2 - 4r_2 h k_1 - 2r_2 h k_2 - 2r_1 k k_1 \\ &\quad - \int_{-\infty}^0 J_2(y) e^{-\eta_0 y}\mathrm{d}y, \\ g_3(\xi) &= -4r_3 + 4r_3\phi_3 - 4r_3 k_3 - 2a_1 k r_3 - 2a_2 h r_3 \\ &\quad - \int_{-\infty}^0 J_3(y) e^{-\eta_0 y}\mathrm{d}y, \end{aligned} \tag{16}$$

where (ϕ_1, ϕ_2, ϕ_3) is a traveling wave front of (10). We can easily prove

$$\begin{aligned} &\lim_{\xi\longrightarrow+\infty} g_1(\xi) = f_1(\eta_0) > 0, \\ &\lim_{\xi\longrightarrow+\infty} g_2(\xi) = f_2(\eta_0) > 0, \\ &\lim_{\xi\longrightarrow+\infty} g_3(\xi) = f_3(\eta_0) > 0, \end{aligned} \tag{17}$$

which imply that

$$\begin{cases} g_1(\xi_0) = -2r_1 + 4r_1\phi_1 - 4r_1kk_2 - 2r_1kk_1 - 2r_2hk_2 \\ \quad - \int_{-\infty}^{0} J_1(y)e^{-\eta_0 y}\mathrm{d}y > 0, \\ g_2(\xi_0) = -2r_2 + 4r_2\phi_2 - 4r_2hk_1 - 2r_2hk_2 - 2r_1kk_1 \\ \quad - \int_{-\infty}^{0} J_2(y)e^{-\eta_0 y}\mathrm{d}y > 0, \\ g_3(\xi_0) = -4r_3 + 4r_3\phi_3 - 4r_3k_3 - 2a_1kr_3 - 2a_2hr_3 \\ \quad - \int_{-\infty}^{0} J_3(y)e^{-\eta_0 y}\mathrm{d}y > 0, \end{cases} \tag{18}$$

where ξ_0 is chosen to be large enough.

Define a weight function by

$$w(\xi) = \begin{cases} e^{-\eta_0(\xi-\xi_0)}, & \xi \le \xi_0, \\ 1, & \xi > \xi_0. \end{cases} \tag{19}$$

Now, we state the stability result.

Theorem 1. *Assume that (H_1)-(H_5) hold. If any traveling wave front $(\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct))$ of (10) with the wave speed $c > \max\{c^*, (1/\eta_0)\max\{c_1, c_2, c_3\}\}$, where*

$$\begin{aligned} c_1 &= 2r_1 + 4r_1kk_2 + 2r_1kk_1 + 2r_2hk_2 - \frac{1}{2} \\ &\quad - \int_{-\infty}^{0} J_1(y)e^{-\eta_0 y}\mathrm{d}y, \\ c_2 &= 2r_2 + 4r_2hk_1 + 2r_2hk_2 + 2r_1kk_1 - \frac{1}{2} \\ &\quad - \int_{-\infty}^{0} J_2(y)e^{-\eta_0 y}\mathrm{d}y, \\ c_3 &= 4r_3 + 4r_3k_3 + 2a_1kr_3 + 2a_2hr_3 - \frac{1}{2} \\ &\quad - \int_{-\infty}^{0} J_3(y)e^{-\eta_0 y}\mathrm{d}y, \end{aligned} \tag{20}$$

the initial data satisfy

$$(0,0,0) \le (u_0(x,0), v_0(x,0), z_0(x,0)) \le (k_1, k_2, k_3), \quad x \in \mathbb{R}, \tag{21}$$

and the initial perturbation satisfies

$$\begin{aligned} u_0(x,0) - \phi_1(x) &\in H_w^1(\mathbb{R}), \\ v_0(x,0) - \phi_2(x) &\in H_w^1(\mathbb{R}), \\ z_0(x,0) - \phi_3(x) &\in H_w^1(\mathbb{R}). \end{aligned} \tag{22}$$

Then, the nonnegative solution of problems (1) and (2) exists uniquely and satisfies

$$\begin{aligned} &(0,0,0) \le (u(x,t), v(x,t), z(x,t)) \\ &\quad \le (k_1, k_2, k_3), \quad x \in \mathbb{R}, t > 0, \\ &\quad u(x,t) - \phi_1(x+ct), v(x,t) - \phi_2(x+ct), z(x,t) \\ &\quad - \phi_3(x+ct) \in C\left([0,+\infty); H_w^1(\mathbb{R}) \cap L^2\right. \\ &\quad \cdot \left.([0,+\infty)); H_w^1(\mathbb{R})\right), \end{aligned} \tag{23}$$

where $w(x)$ is defined by (19). Moreover, $(u(x,t), v(x,t), z(x,t))$ converges to the traveling wave front $(\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct))$ exponentially in time t, i.e.,

$$\begin{aligned} \sup_{x\in\mathbb{R}} \left| u(x,t) - \phi_1(x+ct) \right| &\le Ce^{-\mu t}, \\ \sup_{x\in\mathbb{R}} \left| v(x,t) - \phi_2(x+ct) \right| &\le Ce^{-\mu t}, \\ \sup_{x\in\mathbb{R}} \left| z(x,t) - \phi_3(x+ct) \right| &\le Ce^{-\mu t}, \end{aligned} \tag{24}$$

for all $t > 0$, where C and μ are some positive constants.

3. Stability

In this section, the existence of the solution to problems (1) and (2) can be proved via the upper and lower solutions method (see [24]). We first examine the case when two species cooperate while remaining in the same area without diffusive movement:

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = r_1u_1(x,t)\left[1 - a_1u_1(x,t) + b_1u_2(x,t)\right], \\ \dfrac{\partial u_2(x,t)}{\partial t} = r_2u_2(x,t)\left[1 - a_2u_2(x,t) + b_2u_1(x,t)\right], \end{cases} \tag{25}$$

where all parameters are positive and have four constant equilibria $(0,0)$, $(0, 1/a_2)$, and $(1/a_1, 0)$ and coexistence equilibrium $((a_2+b_1)/(a_1a_2-b_1b_2), (a_1+b_2)/(a_1a_2-b_1b_2)) = (k_1, k_2)$ provided that $a_1a_2 > b_1b_2$.

Li and Lin [22] investigated the existence of traveling wave fronts connecting $(0,0)$ to the constant equilibrium (k_1, k_2) for system (5) by using the known theory [25]. Integrodifferential system (5) is related to the classic Laplacian diffusion system, for example,

$$J(x) = \delta(x) + \delta''(x), \tag{26}$$

where δ is the Dirac delta (see, Medlock et al. [26]), and then (5) reduces to

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + r_1u_1(x,t)\left[1 - a_1u_1(x,t) + b_1u_2(x,t)\right], \\ \dfrac{\partial u_2(x,t)}{\partial t} = \dfrac{\partial^2 u_2(x,t)}{\partial x^2} + r_2u_2(x,t)\left[1 - a_2u_2(x,t) + b_2u_1(x,t)\right]. \end{cases} \tag{27}$$

Because time delay often affects the evolutionary process, the system was incorporated into discrete and nonlocal delays while the existence of traveling wave fronts was obtained (see, [27–29]). The existence of traveling wave solutions of system (1) is similar to them. That is, the initial data satisfies

$$(0,0,0) \le (u_0(x,0), v_0(x,0), z_0(x,0)) \le (k_1, k_2, k_3), \quad x \in \mathbb{R}, \tag{28}$$

and the initial perturbation satisfies

$$\begin{aligned} &u_0(x,0)-\phi_1(x)\in H_w^1(\mathbb{R}),\\ &v_0(x,0)-\phi_2(x)\in H_w^1(\mathbb{R}),\\ &z_0(x,0)-\phi_3(x)\in H_w^1(\mathbb{R}),\end{aligned}\tag{29}$$

and then the nonnegative solution of problems (1) and (2) exists uniquely and satisfies

$$\begin{aligned}(0,0,0)&\le(u(x,t),v(x,t),z(x,t))\\ &\le(k_1,k_2,k_3),\quad x\in\mathbb{R},t>0,\\ &\quad u(x,t)-\phi_1(x+ct),v(x,t)-\phi_2(x+ct),z(x,t)\\ &\quad-\phi_3(x+ct)\in C\big([0,+\infty);H_w^1(\mathbb{R})\big)\cap L^2\\ &\quad\cdot\big([0,+\infty);H_w^1(\mathbb{R})\big).\end{aligned}\tag{30}$$

Here, we omit the details.

Define

$$\begin{cases} u_0^-(x,0)=\min\{u_0(x,0),\phi_1(x)\}, & x\in\mathbb{R},\\ u_0^+(x,0)=\max\{u_0(x,0),\phi_1(x)\}, & x\in\mathbb{R},\\ v_0^-(x,0)=\min\{v_0(x,0),\phi_2(x)\}, & x\in\mathbb{R},\\ v_0^+(x,0)=\max\{v_0(x,0),\phi_2(x)\}, & x\in\mathbb{R},\\ z_0^-(x,0)=\min\{z_0(x,0),\phi_3(x)\}, & x\in\mathbb{R},\\ z_0^+(x,0)=\max\{z_0(x,0),\phi_3(x)\}, & x\in\mathbb{R},\end{cases}\tag{31}$$

and then we obtain

$$\begin{cases} 0\le u_0^-(x,0)\le u_0(x,0)\le u_0^+(x,0)\le k_1, & x\in\mathbb{R},\\ 0\le u_0^-(x,0)\le \phi_1(x)\le u_0^+(x,0)\le k_1, & x\in\mathbb{R},\\ 0\le v_0^-(x,0)\le v_0(x,0)\le v_0^+(x,0)\le k_2, & x\in\mathbb{R},\\ 0\le v_0^-(x,0)\le \phi_2(x)\le v_0^+(x,0)\le k_2, & x\in\mathbb{R},\\ 0\le z_0^-(x,0)\le z_0(x,0)\le z_0^+(x,0)\le k_3, & x\in\mathbb{R},\\ 0\le z_0^-(x,0)\le \phi_3(x)\le z_0^+(x,0)\le k_3, & x\in\mathbb{R}.\end{cases}\tag{32}$$

Therefore, it follows from the comparison principle that

$$\begin{cases} 0\le u^-(x,t)\le u(x,t)\le u^+(x,t)\le k_1, & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ 0\le u^-(x,t)\le \phi_1(x+ct)\le u^+(x,t)\le k_1, & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ 0\le v^-(x,t)\le v(x,t)\le v^+(x,t)\le k_2, & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ 0\le v^-(x,t)\le \phi_2(x+ct)\le v^+(x,t)\le k_2, & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ 0\le z^-(x,t)\le z(x,t)\le z^+(x,t)\le k_3, & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ 0\le z^-(x,t)\le \phi_3(x+ct)\le z^+(x,t)\le k_3, & (x,t)\in\mathbb{R}\times\mathbb{R}_+.\end{cases}\tag{33}$$

Letting

$$\begin{aligned} &U_1(\xi,t)=u^+(x,t)-\phi_1(\xi),\\ &U_{10}(\xi,0)=u_0^+-\phi_1(\xi),\\ &\qquad(x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ &U_2(\xi,t)=v^+(x,t)-\phi_2(\xi),\\ &U_{20}(\xi,0)=v_0^+-\phi_2(\xi),\\ &\qquad(x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ &U_3(\xi,t)=z^+(x,t)-\phi_3(\xi),\\ &U_{30}(\xi,0)=z_0^+-\phi_3(\xi),\\ &\qquad(x,t)\in\mathbb{R}\times\mathbb{R}_+,\end{aligned}\tag{34}$$

where $\xi=x+ct$, and it follows from (32) and (33) that

$$\begin{aligned}&(0,0,0)\le\big(U_{10}(\xi,0),U_{20}(\xi,0),U_{30}(\xi,0)\big)\le(k_1,k_2,k_3),\\ &(0,0,0)\le\big(U_1(\xi,t),U_2(\xi,t),U_3(\xi,t)\big)\le(k_1,k_2,k_3).\end{aligned}\tag{35}$$

3.1. A Prior Estimate. This section is devoted to establishing a prior estimate, which is the core of this paper. The approach is the weighted energy method.

Define

$$B_{\mu,w}^i(\xi,t)=A_w^i(\xi,t)-2\mu,\quad i=1,2,3,4,5,6,\tag{36}$$

where

$$\begin{aligned}A_w^1(\xi,t)={}&-c\frac{w'}{w}-2r_1+4r_1\phi_1-2r_1kU_2-2r_1k\phi_2\\ &+2a_1r_1U_3+2a_1r_1\phi_3-kr_1\phi_1+a_1r_1\phi_1-hr_2\phi_2\\ &+r_2a_2\phi_2+1-\int_{\mathbb{R}}J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y,\end{aligned}\tag{37}$$

$$\begin{aligned}A_w^2(\xi,t)={}&-c\frac{w'}{w}-2r_2+4r_2\phi_2-2r_2hU_1-2r_2h\phi_1\\ &+2a_2r_2U_3+2a_2r_2\phi_3-hr_2\phi_2+a_2r_2\phi_2-kr_1\phi_1\\ &+r_1a_1\phi_1+1-\int_{\mathbb{R}}J_2(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y,\end{aligned}\tag{38}$$

$$\begin{aligned}A_w^3(\xi,t)={}&-c\frac{w'}{w}+2r_3+4r_3\phi_3-2a_1r_3U_1-2a_1r_3\phi_1\\ &-2a_1r_3\phi_1-2a_2r_3U_2-2a_2r_3\phi_2-a_1r_3\phi_1-a_2r_3\phi_3\\ &-kr_1\phi_1-hr_2\phi_2+1-\int_{\mathbb{R}}J_3(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y,\end{aligned}\tag{39}$$

$$A_w^4(\xi,t) = -c\frac{w'}{w} - 2r_1 + 4r_1\phi_1 - 2r_1kU_2 - 2r_1k\phi_2 + 4r_1U_1 + 2a_1r_1U_3 + 2a_1r_1\phi_3 - kr_1U_1 - kr_1\phi_1 + a_1r_1U_1 + a_1r_1\phi_1 + hr_2U_2 - hr_2\phi_2 + r_2a_2\phi_2 + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y, \tag{40}$$

$$A_w^5(\xi,t) = -c\frac{w'}{w} - 2r_2 + 4r_2\phi_2 - 2r_2hU_1 - 2r_2h\phi_1 + 4r_2U_2 + 2a_2r_2U_3 + 2a_2r_2\phi_3 - kr_1U_1 - kr_1\phi_1 + a_2r_2U_2 + a_2r_2\phi_2 - r_2hU_2 - r_2h\phi_2 + r_1a_1\phi_1 + 1 - \int_{\mathbb{R}} J_2(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y, \tag{41}$$

$$A_w^6(\xi,t) = -c\frac{w'}{w} + 2r_3 + 4r_3\phi_3 - 2a_1r_3U_1 - 2a_1r_3\phi_1 - 2a_2r_3U_2 - 2a_2r_3\phi_2 + 4r_3U_3 - a_1r_3U_3 - a_1r_3\phi_3 - a_2r_3U_3 - a_2r_3\phi_3 - kr_1\phi_1 - hr_2\phi_2 - r_3\phi_3 + 1 - \int_{\mathbb{R}} J_3(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y. \tag{42}$$

In order to get the basic estimate, we must prove that $A_w^i(\xi,t) \geq C > 0$, $(i = 1,2,3,4,5,6)$ and $B_w^i(\xi,t) \geq C > 0$, $(i = 1,2,3,4,5,6)$, for some constant C. We need the following key lemma.

Lemma 1. *Assume that (H_1)-(H_5) hold. For any $c > \max\{c^*, (1/\eta_0)\max\{c_1,c_2,c_3\}\}$, there exists some positive constant C such that*

$$A_w^i(\xi,t) \geq C, \quad i = 1,2,3, \text{ for all } \xi \in \mathbb{R},\ t \geq 0. \tag{43}$$

Proof. We prove $A_w^1(\xi,t) \geq C_1$ for some positive constant C_1. Since $c > \max\{c^*, (1/\eta_0)\max\{c_1,c_2,c_3\}\}$, we have $c\eta_0 \geq c_1, c\eta_0 \geq c_2$ and $c\eta_0 \geq c_3$, that is,

$$\begin{aligned} &c\eta_0 - 2r_1 - 6r_1k - 2r_2h + \frac{1}{2} - \int_{\mathbb{R}} J_1(y)e^{\eta_0 y}\mathrm{d}y > 0, \\ &c\eta_0 - 2r_2 - 6r_2h - 2r_1k + \frac{1}{2} - \int_{\mathbb{R}} J_2(y)e^{\eta_0 y}\mathrm{d}y > 0, \\ &c\eta_0 - 4r_3 - 2a_1r_3 - 2a_2r_3 + \frac{1}{2} - \int_{\mathbb{R}} J_3(y)e^{\eta_0 y}\mathrm{d}y > 0. \end{aligned} \tag{44}$$

□

Case 1 $(\xi \leq \xi_0)$. Since, $w(\xi) = e^{-\eta_0(\xi-\xi_0)}$ and $w(\xi)$ is non-increasing, it follows from (37) that

$$\begin{aligned} A_w^1(\xi,t) &= -c\frac{w'}{w} - 2r_1 + 4r_1\phi_1 - 2r_1kU_2 - 2r_1k\phi_2 \\ &\quad + 2a_1r_1U_3 + 2a_1r_1\phi_3 - kr_1\phi_1 \\ &\quad + a_1r_1\phi_1 - hr_2\phi_2 + r_2a_2\phi_2 \\ &\quad + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y \\ &\geq c\eta_0 - 2r_1 - 4r_1kk_2 - r_1kk_1 - hr_2k_2 + 1 \\ &\quad - \int_{-\infty}^{\xi_0-\xi} J_1(y)e^{-\eta_0 y}\mathrm{d}y - \int_{\xi_0-\xi}^{+\infty} J_1(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y \\ &\geq c\eta_0 - 2r_1 - 4r_1kk_2 - r_1kk_1 - hr_2k_2 + 1 \\ &\quad - \int_{\mathbb{R}} J_1(y)e^{\eta_0 y}\mathrm{d}y - \int_0^{+\infty} J_1(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y \\ &\geq c\eta_0 - 2r_1 - 4r_1kk_2 - r_1kk_1 - hr_2k_2 + \frac{1}{2} \\ &\quad - \int_{\mathbb{R}} J_1(y)e^{\eta_0 y}\mathrm{d}y \\ &> r_1k + r_2h > 0. \end{aligned} \tag{45}$$

Case 2 $(\xi > \xi_0)$. In this case, $w(\xi) = 1$ and $(w'(\xi)/w(\xi)) = 0$. Thus, it follows from (37) that

$$\begin{aligned} A_w^1(\xi,t) &= -c\frac{w'}{w} - 2r_1 + 4r_1\phi_1 - 2r_1kU_2 - 2r_1k\phi_2 + 2a_1r_1U_3 \\ &\quad + 2a_1r_1\phi_3 - kr_1\phi_1 + a_1r_1\phi_1 - hr_2\phi_2 \\ &\quad + r_2a_2\phi_2 + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y \\ &\geq -2r_1 + 4r_1\phi_1(\xi_0) - 4r_1kk_2 - r_1kk_1 - hr_2k_2 \\ &\quad - \int_{-\infty}^0 J_1(y)e^{-\eta_0 y}\mathrm{d}y > r_1k + r_2h > 0. \end{aligned} \tag{46}$$

Next, setting $C_1 = r_1k + r_2h > 0$, we have $A_w^1(\xi,t) \geq C_1$. We can prove $A_w^2(\xi,t) \geq C_2$ and $A_w^3(\xi,t) \geq C_3$ in the same way.

Lemma 2. *Assume that (H_1)-(H_5) hold. For any $c > \max\{c^*, (1/\eta_0)\max\{c_1,c_2,c_3\}\}$, there exists some positive constant C such that*

$$A_w^i(\xi,t) \geq C, \quad i = 4,5,6, \text{ for all } \xi \in \mathbb{R},\ t \geq 0. \tag{47}$$

Proof. We prove $A_w^4(\xi,t) \geq C_7$ for some positive constant C_7. □

Case 3 $(\xi \leq \xi_0)$. It follows from $w(\xi) = e^{-\eta_0(\xi-\xi_0)}$ and the monotonicity of $w(\xi)$ that

$$\begin{aligned}
A_w^4(\xi,t) &= -c\frac{w'}{w} - 2r_1 + 4r_1\phi_1 - 2r_1kU_2 - 2r_1k\phi_2 + 4r_1U_1 + 2a_1r_1U_3 + 2a_1r_1\phi_3 - kr_1U_1 - kr_1\phi_1 \\
&\quad + a_1r_1U_1 + a_1r_1\phi_1 + hr_2U_2 - hr_2\phi_2 + r_2a_2\phi_2 + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y \\
&\geq c\eta_0 - 2r_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y \\
&\geq c\eta_0 - 2r_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 + 1 \\
&\quad - \int_{-\infty}^{\xi_0-\xi} J_1(y)e^{-\eta_0 y}\mathrm{d}y - \int_{\xi_0-\xi}^{+\infty} J_1(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y \\
&\geq c\eta_0 - 2r_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 + 1 - \int_{\mathbb{R}} J_1(y)e^{\eta_0 y}\mathrm{d}y - \int_0^{+\infty} J_1(y)e^{\eta_0(\xi-\xi_0)}\mathrm{d}y \\
&\geq c\eta_0 - 2r_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 + \frac{1}{2} - \int_{\mathbb{R}} J_1(y)e^{\eta_0 y}\mathrm{d}y \\
&= C_7 > 0.
\end{aligned} \tag{48}$$

Case 4 $(\xi > \xi_0)$. According to $w(\xi) = 1$ and $(w'(\xi)/w(\xi)) = 0$, we know

$$\begin{aligned}
A_w^4(\xi,t) &= -c\frac{w'}{w} - 2r_1 + 4r_1\phi_1 - 2r_1kU_2 - 2r_1k\phi_2 + 4r_1U_1 + 2a_1r_1U_3 + 2a_1r_1\phi_3 - kr_1U_1 - kr_1\phi_1 \\
&\quad + a_1r_1U_1 + a_1r_1\phi_1 + hr_2U_2 - hr_2\phi_2 + r_2a_2\phi_2 + 1 - \int_{\mathbb{R}} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y \\
&\geq -2r_1 + 4r_1\phi_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 + 1 \\
&\quad - \int_{-\infty}^{\xi_0-\xi} J_1(y)e^{-\eta_0(\xi+y-\xi_0)}\mathrm{d}y - \int_{\xi_0-\xi}^{+\infty} J_1(y)\mathrm{d}y \\
&\geq -2r_1 + 4r_1\phi_1 - 4r_1kk_2 - 2r_1kk_1 - 2hr_2k_2 - \int_{-\infty}^{0} J_1(y)e^{-\eta_0 y}\mathrm{d}y \\
&= C_8 > 0.
\end{aligned} \tag{49}$$

In view of the above argument and letting $C_4 := \min\{C_7, C_8\}$, we have $A_w^4(\xi,t) \geq C_4$. We can prove $A_w^5(\xi,t) \geq C_5$ and $A_w^6(\xi,t) \geq C_6$ in the same way. This completes the proof.

We easily check the following result.

Lemma 3. *Assume that* (H_1)-(H_5) *hold. For any* $c > \max\{c^*, (1/\eta_0)\max\{c_1, c_2, c_3\}\}$, *there exists some positive constant* C *such that*

$$B_{\mu,w}^i(\xi,t) \geq C, \quad i = 1,2,3,4,5,6, \tag{50}$$

hold for all $\xi \in \mathbb{R}$, *and* $0 < \mu < (1/2)\min_{i=1,2,3,4,5,6}\{C_i\}$.

Now we are going to derive the energy estimates for $U_1(t,\xi), U_2(t,\xi)$, *and* $U_3(t,\xi)$ *in the weighted Sobolev space* $H_w^1(\mathbb{R})$.

Lemma 4. *Assume that* (H_1)-(H_5) *hold. For any* $c > \max\{c^*, (1/\eta_0)\max\{c_1, c_2, c_3\}\}$, *it holds*

$$\begin{aligned}&\|U_1(t)\|_{L_w^2}^2+\|U_2(t)\|_{L_w^2}^2+\|U_3(t)\|_{L_w^2}^2\\&\quad+\int_0^t e^{-2\mu(t-s)}\left(\|U_1(s)\|_{L_w^2}^2+\|U_2(s)\|_{L_w^2}^2+\|U_3(s)\|_{L_w^2}^2\right)\mathrm{d}s\\&\quad\le Ce^{-2\mu t}\left(\|U_{10}(0)\|_{L_w^2}^2+\|U_{20}(0)\|_{L_w^2}^2+\|U_{30}(0)\|_{L_w^2}^2\right),\end{aligned}\tag{51}$$

for some positive constant C.

Proof. From (1) and (10), $(U_1(\xi,t),U_2(\xi,t),U_3(\xi,t))$ satisfies

$$\begin{cases}U_{1t}+cU_{1\xi}=\int_{\mathbb{R}}J_1(y)[U_1(\xi-y,t)-U_1(\xi,t)]\mathrm{d}y+r_1k\phi_1U_2-r_1a_1\phi_1U_3\\ \qquad+U_1(\xi,t)(r_1-2r_1\phi_1+r_1kU_2+r_1k\phi_2-r_1a_1U_3-r_1a_1\phi_3)-r_1U_1^2,\\ U_{2t}+cU_{2\xi}=\int_{\mathbb{R}}J_2(y)[U_2(\xi-y,t)-U_2(\xi,t)]\mathrm{d}y+r_2h\phi_2U_1-r_2a_2\phi_2U_3\\ \qquad+U_2(\xi,t)(r_2-2r_2\phi_2+r_2hU_1+r_2h\phi_1-r_2a_2U_3-r_2a_2\phi_3)-r_2U_2^2,\\ U_{3t}+cU_{3\xi}=\int_{\mathbb{R}}J_3(y)[U_3(\xi-y,t)-U_3(\xi,t)]\mathrm{d}y+a_1r_3\phi_3U_1+a_2r_3\phi_3U_2\\ \qquad+U_3(\xi,t)(-r_3-2r_3\phi_3+a_1r_3U_1+a_1r_3\phi_1+a_2r_3U_2+a_2r_3\phi_2)-r_3U_3^2,\end{cases}\tag{52}$$

with the initial data $U_1(\xi,0)=U_{10}(\xi,0),U_2(\xi,0)=U_{20}(\xi,0)$, and $U_3(\xi,0)=U_{30}(\xi,0),\xi\in\mathbb{R}$. Multiplying the three equations of (52) by $e^{2\mu t}w(\xi)U_1(\xi,t)$, $e^{2\mu t}w(\xi)U_2(\xi,t)$, and $e^{2\mu t}w(\xi)U_3(\xi,t)$, respectively, where $\mu>0$ is defined like that, we have

$$\begin{aligned}&\left(\frac{1}{2}e^{2\mu t}wU_1^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_1^2\right)_\xi-e^{2\mu t}wU_1\int_{\mathbb{R}}J_1(y)[U_1(\xi-y,t)-U_1(\xi,t)]\mathrm{d}y\\&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu-r_1+2r_1\phi_1-r_1kU_2-r_1k\phi_2+r_1a_1U_3+r_1a_1\phi_3\right)e^{2\mu t}wU_1^2\\&\quad=-r_1e^{2\mu t}wU_1^3+kr_1\phi_1e^{2\mu t}wU_1U_2-r_1a_1\phi_1e^{2\mu t}wU_1U_3,\end{aligned}\tag{53}$$

$$\begin{aligned}&\left(\frac{1}{2}e^{2\mu t}wU_2^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_2^2\right)_\xi-e^{2\mu t}wU_2\int_{\mathbb{R}}J_2(y)[U_2(\xi-y,t)-U_2(\xi,t)]\mathrm{d}y\\&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu-r_2+2r_2\phi_2-r_2hU_1-r_2h\phi_1+r_2a_2U_3+r_2a_2\phi_3\right)e^{2\mu t}wU_2^2\\&\quad=-r_2e^{2\mu t}wU_2^3+hr_2\phi_2e^{2\mu t}wU_1U_2-r_2a_2\phi_2e^{2\mu t}wU_2U_3,\end{aligned}\tag{54}$$

$$\begin{aligned}&\left(\frac{1}{2}e^{2\mu t}wU_3^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_3^2\right)_\xi-e^{2\mu t}wU_3\int_{\mathbb{R}}J_3(y)[U_3(\xi-y,t)-U_3(\xi,t)]\mathrm{d}y\\&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu+r_3+2r_3\phi_3-a_1r_3U_1-a_1r_3\phi_1-r_3a_2U_2-r_3a_2\phi_2\right)e^{2\mu t}wU_3^2\\&\quad=-r_3e^{2\mu t}wU_3^3+a_1r_3\phi_3e^{2\mu t}wU_1U_3+a_2r_3\phi_3e^{2\mu t}wU_2U_3.\end{aligned}\tag{55}$$

Integrating (53)–(55) over $\mathbb{R}\times[0,t]$ with respect to ξ and t and noting the vanishing term at far field

$$\begin{aligned}&\left\{\frac{c}{2}e^{2\mu t}wU_1^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0,\\&\left\{\frac{c}{2}e^{2\mu t}wU_2^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0,\\&\left\{\frac{c}{2}e^{2\mu t}wU_3^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0,\end{aligned}\tag{56}$$

we obtain

$$\begin{aligned}
&e^{2\mu t}\|U_1(t)\|_{L_w^2}^2 - 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_1(\xi,s)\int_{\mathbb{R}} J_1(y)[U_1(\xi-y,t)-U_1(\xi,t)]\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\left(-c\frac{w'}{w}-2\mu-2r_1+4r_1\phi_1-2r_1kU_2-2r_1k\phi_2+2r_1a_1U_3+2r_1a_1\phi_3\right)w(\xi)U_1^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\|U_{10}(0)\|_{L_w^2}^2+2kr_1\int_0^t\int_{\mathbb{R}}\phi_1 e^{2\mu s}w(\xi)U_1(\xi,s)U_2(\xi,s)\mathrm{d}\xi\,\mathrm{d}s\\
&\quad-2r_1a_1\int_0^t\int_{\mathbb{R}}\phi_1 e^{2\mu s}w(\xi)U_1(\xi,s)U_3(\xi,s)\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}\tag{57}$$

$$\begin{aligned}
&e^{2\mu t}\|U_2(t)\|_{L_w^2}^2 - 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_2(\xi,s)\int_{\mathbb{R}} J_2(y)[U_2(\xi-y,t)-U_2(\xi,t)]\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\left(-c\frac{w'}{w}-2\mu-2r_2+4r_2\phi_2-2r_2hU_1-2r_2h\phi_1+2r_2a_2U_3+2r_2a_2\phi_3\right)w(\xi)U_2^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\|U_{20}(0)\|_{L_w^2}^2+2hr_2\int_0^t\int_{\mathbb{R}}\phi_2 e^{2\mu s}w(\xi)U_1(\xi,s)U_2(\xi,s)\mathrm{d}\xi\,\mathrm{d}s\\
&\quad-2r_2a_2\int_0^t\int_{\mathbb{R}}\phi_2 e^{2\mu s}w(\xi)U_2(\xi,s)U_3(\xi,s)\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}\tag{58}$$

$$\begin{aligned}
&e^{2\mu t}\|U_3(t)\|_{L_w^2}^2 - 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_3(\xi,s)\int_{\mathbb{R}} J_3(y)[U_3(\xi-y,t)-U_3(\xi,t)]\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\left(-c\frac{w'}{w}-2\mu+2r_3+4r_3\phi_3-2a_1r_3U_1-2a_1r_3\phi_1-2a_2r_3U_2-2a_2r_3\phi_2\right)w(\xi)U_3^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\|U_{30}(0)\|_{L_w^2}^2+2a_1r_3\int_0^t\int_{\mathbb{R}}\phi_3 e^{2\mu s}w(\xi)U_1(\xi,s)U_3(\xi,s)\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2a_2r_3\int_0^t\int_{\mathbb{R}}\phi_3 e^{2\mu s}w(\xi)U_2(\xi,s)U_3(\xi,s)\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}\tag{59}$$

By using the Cauchy–Schwarz inequality $2xy\le x^2+y^2$, we get

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_i(\xi,s)\int_{\mathbb{R}} J_i(y)[U_i(\xi-y,s)-U_i(\xi,s)]\mathrm{d}y\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)U_i^2(\xi,t)\left[\int_{\mathbb{R}} J_i\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y-1\right]\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}\tag{60}$$

Similarly, we obtain

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)U_1U_2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)\left(U_1^2+U_2^2\right)\mathrm{d}\xi\,\mathrm{d}s,\quad i=1,2,
\end{aligned}\tag{61}$$

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)U_2U_3\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)\left(U_2^2+U_3^2\right)\mathrm{d}\xi\,\mathrm{d}s,\quad i=2,3,
\end{aligned}\tag{62}$$

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)U_1U_3\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\int_0^t\int_{\mathbb{R}}\phi_i e^{2\mu s}w(\xi)\left(U_1^2+U_3^2\right)\mathrm{d}\xi\,\mathrm{d}s,\quad i=1,3.
\end{aligned}\tag{63}$$

Substituting (60)–(63) into (57)–(59), respectively, we have

$$
\begin{aligned}
& e^{2\mu t}\left\|U_1(t)\right\|_{L_w^2}^2+\int_0^t \int_{\mathbb{R}} e^{2\mu s}\left(-c \frac{w'}{w}-2\mu-2r_1+4r_1\phi_1-2r_1kU_2-2r_1k\phi_2+2r_1a_1U_3+2r_1a_1\phi_3-kr_1\phi_1+a_1r_1\phi_1+1\right. \\
& \left.\quad-\int_{\mathbb{R}} J_1(y) \frac{w(\xi+y)}{w(\xi)} \mathrm{d}y\right) w(\xi) U_1^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s \\
& \quad \leq\left\|U_{10}(0)\right\|_{L_w^2}^2+kr_1 \int_0^t \int_{\mathbb{R}} \phi_1 e^{2\mu s} w(\xi) U_2^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s-r_1a_1 \int_0^t \int_{\mathbb{R}} \phi_1 e^{2\mu s} w(\xi) U_3^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s,
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
& e^{2\mu t}\left\|U_2(t)\right\|_{L_w^2}^2+\int_0^t \int_{\mathbb{R}} e^{2\mu s}\left(-c \frac{w'}{w}-2\mu-2r_2+4r_2\phi_2-2r_2hU_1-2r_2h\phi_1+2r_2a_2U_3+2r_2a_2\phi_3-r_2h\phi_2+a_2r_2\phi_2+1\right. \\
& \left.\quad-\int_{\mathbb{R}} J_2(y) \frac{w(\xi+y)}{w(\xi)} \mathrm{d}y\right) w(\xi) U_2^2 \mathrm{d}\xi \, \mathrm{d}s \\
& \quad \leq\left\|U_{20}(0)\right\|_{L_w^2}^2+hr_2 \int_0^t \int_{\mathbb{R}} \phi_2 e^{2\mu s} w(\xi) U_1^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s-r_2a_2 \int_0^t \int_{\mathbb{R}} \phi_2 e^{2\mu s} w(\xi) U_3^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s,
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
& e^{2\mu t}\left\|U_3(t)\right\|_{L_w^2}^2+\int_0^t \int_{\mathbb{R}} e^{2\mu s}\left(-c \frac{w'}{w}-2\mu+2r_3+4r_3\phi_3-2a_1r_3U_1-2a_1r_3\phi_1-2a_2r_3U_2-2a_2r_3\phi_2-a_1r_3\phi_3-a_2r_3\phi_3+1\right. \\
& \left.\quad-\int_{\mathbb{R}} J_3(y) \frac{w(\xi+y)}{w(\xi)} \mathrm{d}y\right) w(\xi) U_3^2 \mathrm{d}\xi \, \mathrm{d}s \\
& \quad \leq\left\|U_{30}(0)\right\|_{L_w^2}^2+a_1r_3 \int_0^t \int_{\mathbb{R}} \phi_3 e^{2\mu s} w(\xi) U_1^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s+a_2r_3 \int_0^t \int_{\mathbb{R}} \phi_3 e^{2\mu s} w(\xi) U_2^2(\xi, s) \mathrm{d}\xi \, \mathrm{d}s.
\end{aligned} \tag{66}
$$

Adding the three inequalities (64), (65), and (66), we obtain

$$
\begin{aligned}
& e^{2\mu t}\left(\left\|U_1(t)\right\|_{L_w^2}^2+\left\|U_2(t)\right\|_{L_w^2}^2+\left\|U_3(t)\right\|_{L_w^2}^2\right) \\
& \quad+\int_0^t \int_{\mathbb{R}} e^{2\mu s}\left[B_{\mu,w}^1 w(\xi) U_1^2(\xi, s)+B_{\mu,w}^2 w(\xi) U_2^2(\xi, s)\right. \\
& \quad\left.+B_{\mu,w}^3 w(\xi) U_3^2(\xi, s)\right] \mathrm{d}\xi \, \mathrm{d}s \\
& \quad \leq\left\|U_{10}(0)\right\|_{L_w^2}^2+\left\|U_{20}(0)\right\|_{L_w^2}^2+\left\|U_{30}(0)\right\|_{L_w^2}^2,
\end{aligned} \tag{67}
$$

where $B_{\mu,w}^1, B_{\mu,w}^2$, and $B_{\mu,w}^3$ are defined in (36). According to (36), we can verify

$$
\begin{aligned}
& \left\|U_1(t)\right\|_{L_w^2}^2+\left\|U_2(t)\right\|_{L_w^2}^2+\left\|U_3(t)\right\|_{L_w^2}^2 \\
& \quad+\int_0^t e^{-2\mu(t-s)}\left(\left\|U_1(s)\right\|_{L_w^2}^2+\left\|U_2(s)\right\|_{L_w^2}^2+\left\|U_3(s)\right\|_{L_w^2}^2\right) \mathrm{d}\xi \, \mathrm{d}s \\
& \quad \leq C e^{-2\mu t}\left(\left\|U_{10}(0)\right\|_{L_w^2}^2+\left\|U_{20}(0)\right\|_{L_w^2}^2+\left\|U_{30}(0)\right\|_{L_w^2}^2\right),
\end{aligned} \tag{68}
$$

for some positive constant C. This completes the proof. □

Lemma 5. *Assume that (H_1)-(H_5) hold. For any $c>\max\{c^*, (1/\eta_0)\max\{c_1, c_2, c_3\}\}$, it holds*

$$
\begin{aligned}
& \left\|U_{1\xi}(t)\right\|_{L_w^2}^2+\left\|U_{2\xi}(t)\right\|_{L_w^2}^2+\left\|U_{3\xi}(t)\right\|_{L_w^2}^2 \\
& \quad+\int_0^t e^{-2\mu(t-s)}\left(\left\|U_{1\xi}(s)\right\|_{L_w^2}^2+\left\|U_{2\xi}(s)\right\|_{L_w^2}^2+\left\|U_{3\xi}(s)\right\|_{L_w^2}^2\right) \mathrm{d}s \\
& \quad \leq C e^{-2\mu t}\left(\left\|U_{10}(0)\right\|_{H_w^1}^2+\left\|U_{20}(0)\right\|_{H_w^1}^2+\left\|U_{30}(0)\right\|_{H_w^1}^2\right),
\end{aligned} \tag{69}
$$

for some positive constant C.

Proof. Differentiating (52) with respect to ξ, we have

$$
\left\{
\begin{aligned}
& U_{1t\xi}+cU_{1\xi\xi}=\int_{\mathbb{R}} J_1(y)\left[U_{1\xi}(\xi-y, t)-U_{1\xi}(\xi, t)\right] \mathrm{d}y \\
& \quad+U_{1\xi}(\xi, t)\left(r_1-2r_1\phi_1+r_1kU_2+r_1k\phi_2-r_1a_1U_3-r_1a_1\phi_3\right) \\
& \quad-2r_1U_1U_{1\xi}+r_1k\phi_1'U_2+kr_1\phi_1U_{2\xi}-r_1a_1\phi_1'U_3-r_1a_1\phi_1U_{3\xi} \\
& \quad+U_1\left(-2r_1\phi_1'+r_1kU_{2\xi}+r_1k\phi_2'-r_1a_1U_{3\xi}-r_1a_1\phi_3'\right), \\
& U_{2t\xi}+cU_{2\xi\xi}=\int_{\mathbb{R}} J_2(y)\left[U_{2\xi}(\xi-y, t)-U_{2\xi}(\xi, t)\right] \mathrm{d}y \\
& \quad+U_{2\xi}(\xi, t)\left(r_2-2r_2\phi_2+r_2hU_1+r_2h\phi_1-r_2a_2U_3-r_2a_2\phi_3\right) \\
& \quad-2r_2U_2U_{2\xi}+r_2h\phi_2'U_1+r_2h\phi_2U_{1\xi}-r_2a_2\phi_2'U_3-r_2a_2\phi_2U_{3\xi} \\
& \quad+U_2\left(-2r_2\phi_2'+r_2hU_{1\xi}+r_2h\phi_1'-r_2a_2U_{3\xi}-r_2a_2\phi_3'\right), \\
& U_{3t\xi}+cU_{3\xi\xi}=\int_{\mathbb{R}} J_3(y)\left[U_{3\xi}(\xi-y, t)-U_{3\xi}(\xi, t)\right] \mathrm{d}y \\
& \quad+U_{3\xi}(\xi, t)\left(-r_3-2r_3\phi_3+a_1r_3U_1+a_1r_3\phi_1+a_2r_3U_2+a_2r_3\phi_2\right) \\
& \quad-2r_3U_3U_{3\xi}+a_1r_3\phi_3'U_1+a_1r_3\phi_3U_{1\xi}+a_2r_3\phi_3'U_2+a_2r_3\phi_3U_{2\xi} \\
& \quad+U_3\left(-2r_3\phi_3'+a_1r_3U_{1\xi}+a_1r_3\phi_1'+a_2r_3U_{2\xi}+a_2r_3\phi_2'\right).
\end{aligned}
\right. \tag{70}
$$

Multiplying the three equations of (70) by $e^{2\mu t}w(\xi)U_{1\xi}(\xi,t), e^{2\mu t}w(\xi)U_{2\xi}(\xi,t)$, and $e^{2\mu t}w(\xi)U_{3\xi}(\xi,t)$, respectively, it holds

$$\begin{aligned}
&\left(\frac{1}{2}e^{2\mu t}wU_{1\xi}^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_{1\xi}^2\right)_\xi-e^{2\mu t}wU_{1\xi}\int_{\mathbb{R}}J_1(y)\left[U_{1\xi}(\xi-y,t)-U_{1\xi}(\xi,t)\right]\mathrm{d}y\\
&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu-r_1+2r_1\phi_1-r_1kU_2-r_1k\phi_2+r_1a_1U_3+r_1a_1\phi_3+2r_1U_1\right)e^{2\mu t}wU_{1\xi}^2\\
&\quad=\left(-2r_1\phi_1'+r_1kU_{2\xi}+r_1k\phi_2'-r_1a_1U_{3\xi}-r_1a_1\phi_3'\right)e^{2\mu t}wU_1U_{1\xi}+kr_1\phi_1'e^{2\mu t}wU_2U_{1\xi}\\
&\quad+kr_1\phi_1e^{2\mu t}wU_{2\xi}U_{1\xi}-r_1a_1\phi_1'e^{2\mu t}wU_3U_{1\xi}-r_1a_1\phi_1e^{2\mu t}wU_{1\xi}U_{3\xi},
\end{aligned}\tag{71}$$

$$\begin{aligned}
&\left(\frac{1}{2}e^{2\mu t}wU_{2\xi}^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_{2\xi}^2\right)_\xi-e^{2\mu t}wU_{2\xi}\int_{\mathbb{R}}J_2(y)\left[U_{2\xi}(\xi-y,t)-U_{2\xi}(\xi,t)\right]\mathrm{d}y\\
&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu-r_2+2r_2\phi_2-r_2hU_1-r_2h\phi_1+r_2a_2U_3+r_2a_2\phi_3+2r_2U_2\right)e^{2\mu t}wU_{2\xi}^2\\
&\quad=\left(-2r_2\phi_2'+r_2hU_{1\xi}+r_2h\phi_1'-r_2a_2U_{3\xi}-r_2a_2\phi_3'\right)e^{2\mu t}wU_2U_{2\xi}+r_2h\phi_2'e^{2\mu t}wU_1U_{2\xi}\\
&\quad+hr_2\phi_2e^{2\mu t}wU_{1\xi}U_{2\xi}-r_2a_2\phi_2'e^{2\mu t}wU_3U_{2\xi}-a_2r_2\phi_2e^{2\mu t}wU_{2\xi}U_{3\xi},
\end{aligned}\tag{72}$$

$$\begin{aligned}
&\left(\frac{1}{2}e^{2\mu t}wU_{3\xi}^2\right)_t+\left(\frac{c}{2}e^{2\mu t}wU_{3\xi}^2\right)_\xi-e^{2\mu t}wU_{3\xi}\int_{\mathbb{R}}J_3(y)\left[U_{3\xi}(\xi-y,t)-U_{3\xi}(\xi,t)\right]dy\\
&\quad+\left(-\frac{c}{2}\frac{w'}{w}-\mu+r_3+2r_3\phi_3-a_1r_3U_1-a_1r_3\phi_1-r_3a_2U_2-r_3a_2\phi_2+2r_3U_3\right)e^{2\mu t}wU_{3\xi}^2\\
&\quad=\left(-2r_3\phi_3'+a_1r_3U_{1\xi}+a_1r_3\phi_1'+a_2r_3U_{2\xi}+a_2r_3\phi_2'\right)e^{2\mu t}wU_3U_{3\xi}+a_1r_3\phi_3'e^{2\mu t}wU_1U_{3\xi}\\
&a_1r_3\phi_3e^{2\mu t}wU_{1\xi}U_{3\xi}+a_2r_3\phi_3'e^{2\mu t}wU_2U_{3\xi}+a_2r_3\phi_3e^{2\mu t}wU_{2\xi}U_{3\xi}.
\end{aligned}\tag{73}$$

According to $U_1\in H_w^1, U_2\in H_w^1$, and $U_3\in H_w^1$, we have

$$\begin{aligned}
&\left\{\frac{c}{2}e^{2\mu t}wU_{1\xi}^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0,\\
&\left\{\frac{c}{2}e^{2\mu t}wU_{2\xi}^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0,\\
&\left\{\frac{c}{2}e^{2\mu t}wU_{3\xi}^2\right\}\Big|_{\xi\longrightarrow-\infty}^{\infty}=0.
\end{aligned}\tag{74}$$

Integrating (71)–(73) over $\mathbb{R}\times[0,t]$ with respect to ξ and t and according to the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&e^{2\mu t}\left\|U_{1\xi}(t)\right\|_{L_w^2}^2+\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-c\frac{w'}{w}-2\mu-2r_1+4r_1\phi_1-2r_1kU_2-2r_1k\phi_2+2r_1a_1U_3+2r_1a_1\phi_3+4r_1U_1\right.\\
&\quad\left.-\int_{\mathbb{R}}J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y+1-r_1kU_1+a_1r_1U_1-kr_1\phi_1+a_1r_1\phi\right)w(\xi)U_{1\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\left\|U_{10\xi}(t)\right\|_{L_{wa}^2}^2+kr_1\int_0^t\int_{\mathbb{R}}(U_1+\phi_1)e^{2\mu s}w(\xi)U_{2\xi}^2\mathrm{d}\xi\,\mathrm{d}s-a_1r_1\int_0^t\int_{\mathbb{R}}(U_1+\phi_1)e^{2\mu s}w(\xi)U_{3\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2kr_1\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_1'w(\xi)U_2U_{1\xi}\mathrm{d}\xi\,\mathrm{d}s-2a_1r_1\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_1'w(\xi)U_3U_{1\xi}\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-2r_1\phi_1'+kr_1\phi_2'-a_1r_1\phi_3'\right)w(\xi)U_1U_{1\xi}\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}\tag{75}$$

$$\begin{aligned}
&e^{2\mu t}\left\|U_{2\xi}(t)\right\|_{L_w^2}^2+\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-c\frac{w'}{w}-2\mu-2r_2+4r_2\phi_2-2r_2hU_1-2r_2h\phi_1+2r_2a_2U_3+2r_2a_2\phi_3+4r_2U_2\right.\\
&\quad\left.-\int_{\mathbb{R}}J_2(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y+1-r_2hU_2+a_2r_2U_2-r_2h\phi_2+a_2r_2\phi_2\right)w(\xi)U_{2\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\left\|U_{20\xi}(0)\right\|_{L_w^2}^2+hr_2\int_0^t\int_{\mathbb{R}}(U_2+\phi_2)e^{2\mu s}w(\xi)U_{1\xi}^2\mathrm{d}\xi\,\mathrm{d}s-a_2r_2\int_0^t\int_{\mathbb{R}}(U_2+\phi_2)e^{2\mu s}w(\xi)U_{3\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2r_2h\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_2'w(\xi)U_1U_{2\xi}\mathrm{d}\xi\,\mathrm{d}s-2a_2r_2\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_2'w(\xi)U_3U_{2\xi}\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-2r_2\phi_2'+r_2h\phi_1'-r_2a_2\phi_3'\right)w(\xi)U_2U_{2\xi}\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}\tag{76}$$

$$\begin{aligned}
&e^{2\mu t}\left\|U_{3\xi}(t)\right\|_{L_w^2}^2+\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-c\frac{w'}{w}-2\mu+2r_3+4r_3\phi_3-2a_1r_3U_1-2a_1r_3\phi_1-2a_2r_3U_2-2a_2r_3\phi_2+4r_3U_3\right.\\
&\quad\left.-\int_{\mathbb{R}}J_2(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y+1-a_1r_3U_3-a_2r_3U_3-a_1r_3\phi_3-a_2r_3\phi_3\right)w(\xi)U_{3\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\left\|U_{30\xi}(0)\right\|_{L_w^2}^2+a_1r_3\int_0^t\int_{\mathbb{R}}(U_3+\phi_3)e^{2\mu s}w(\xi)U_{1\xi}^2\mathrm{d}\xi\,\mathrm{d}s+a_2r_3\int_0^t\int_{\mathbb{R}}(U_3+\phi_3)e^{2\mu s}w(\xi)U_{2\xi}^2\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2a_1r_3\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_3'w(\xi)U_1U_{3\xi}\mathrm{d}\xi\,\mathrm{d}s+2a_2r_3\int_0^t\int_{\mathbb{R}}e^{2\mu s}\phi_3'w(\xi)U_2U_{3\xi}\mathrm{d}\xi\,\mathrm{d}s\\
&\quad+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left(-2r_3\phi_3'+a_1r_3\phi_3'+a_2r_3\phi_2'\right)w(\xi)U_3U_{3\xi}\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}\tag{77}$$

Adding the three inequalities (75), (76), and (77), we obtain

$$\begin{aligned}
&e^{2\mu t}\left(\left\|U_{1\xi}(t)\right\|_{L_w^2}^2+\left\|U_{2\xi}(t)\right\|_{L_w^2}^2+\left\|U_{3\xi}(t)\right\|_{L_w^2}^2\right)\\
&\quad+\int_0^t\int_{\mathbb{R}}e^{2\mu s}\left[B_{\mu,w}^4w(\xi)U_{1\xi}^2(\xi,s)+B_{\mu,w}^5w(\xi)U_{2\xi}^2(\xi,s)+B_{\mu,w}^6w(\xi)U_{3\xi}^2(\xi,s)\right]\mathrm{d}\xi\,\mathrm{d}s\\
&\quad\le\left\|U_{10\xi}(0)\right\|_{L_w^2}^2+\left\|U_{20\xi}(0)\right\|_{L_w^2}^2+\left\|U_{30\xi}(0)\right\|_{L_w^2}^2+2\int_0^t\int_{\mathbb{R}}Q(\xi,s)w(\xi)e^{2\mu s}\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}\tag{78}$$

where $B_{\mu,w}^4$, $B_{\mu,w}^5$, and $B_{\mu,w}^6$ are given by (36), and

$$\begin{aligned}
Q(\xi,t)&=kr_1\phi_1'U_2U_{1\xi}-r_1a_1\phi_1'U_3U_{1\xi}+\left(-2r_1\phi_1'+kr_1\phi_2'-a_1r_1\phi_3'\right)w(\xi)U_1U_{1\xi}\\
&\quad+r_2h\phi_2'U_1U_{2\xi}-a_2r_2\phi_2'U_3U_{2\xi}+\left(-2r_2\phi_2'+hr_2\phi_1'-a_2r_2\phi_3'\right)w(\xi)U_2U_{2\xi}\\
&\quad+a_1r_3\phi_3'U_1U_{3\xi}+a_2r_3\phi_3'U_2U_{3\xi}+\left(-2r_3\phi_3'+a_1r_3\phi_1'+a_2r_3\phi_2'\right)w(\xi)U_3U_{3\xi}.
\end{aligned}\tag{79}$$

Note that $(0,0,0)\le(U_1(\xi,t),U_2(\xi,t),U_3(\xi,t))\le(k_1,k_2,k_3)$, and ϕ_1',ϕ_2',ϕ_3' are bounded on $\mathbb{R}$. There exists a positive constant C_0 such that

$$\begin{aligned}
&\left|kr_1\phi_1'\right|\le C_0,\\
&\left|r_1a_1\phi_1'\right|\le C_0,\\
&\left|-2r_1\phi_1'+kr_1\phi_2'-a_1r_1\phi_3'\right|\le C_0,\\
&\left|r_2h\phi_2'\right|\le C_0,\\
&\left|a_2r_2\phi_2'\right|\le C_0,\\
&\left|-2r_2\phi_2'+hr_2\phi_1'-a_2r_2\phi_3'\right|\le C_0,\\
&\left|a_1r_3\phi_3'\right|\le C_0,\\
&\left|a_2r_3\phi_3'\right|\le C_0,\\
&\left|-2r_3\phi_3'+a_1r_3\phi_1'+a_2r_3\phi_2'\right|\le C_0.
\end{aligned}\tag{80}$$

According to Lemma 4, we obtain the following inequality

$$\begin{aligned}&\int_0^t e^{2\mu s}\left(\|U_1(t)\|_{L_w^2}^2+\|U_2(t)\|_{L_w^2}^2+\|U_3(t)\|_{L_w^2}^2\right)ds\\&\quad\le C\left(\|U_{10}(0)\|_{L_w^2}^2+\|U_{20}(0)\|_{L_w^2}^2+\|U_{30}(0)\|_{L_w^2}^2\right).\end{aligned}\tag{81}$$

By using the Young inequality $2xy\le\eta x^2+(1/\eta)y^2$ and (51) we have

$$\begin{aligned}&2\int_0^t\int_{\mathbb{R}}e^{2\mu s}Q(\xi,s)w(\xi)d\xi\,ds\\&\le C_0\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)\left[\frac{1}{\eta}\left(U_1^2(\xi,s)+U_2^2(\xi,s)+U_3^2(\xi,s)\right)\right.\\&\left.+\eta\left(U_{1\xi}^2(\xi,s)+U_{2\xi}^2(\xi,s)+U_{3\xi}^2(\xi,s)\right)\right]d\xi\,ds\\&\le\frac{C_0}{\eta}\int_0^t e^{2\mu s}\left(\|U_1(s)\|_{L_w^2}^2+\|U_2(s)\|_{L_w^2}^2+\|U_3(s)\|_{L_w^2}^2\right)ds\\&+\eta C_0\int_0^t e^{2\mu s}\left(\|U_{1\xi}(s)\|_{L_w^2}^2+\|U_{2\xi}(s)\|_{L_w^2}^2+\|U_{3\xi}(s)\|_{L_w^2}^2\right)ds\\&\le\frac{C_0C}{\eta}\left(\|U_{10}(0)\|_{L_w^2}^2+\|U_{20}(0)\|_{L_w^2}^2+\|U_{30}(0)\|_{L_w^2}^2\right)\\&+\eta C_0\int_0^t e^{2\mu s}\left(\|U_{1\xi}(s)\|_{L_w^2}^2+\|U_{2\xi}(s)\|_{L_w^2}^2+\|U_{3\xi}(s)\|_{L_w^2}^2\right)ds.\end{aligned}\tag{82}$$

Choosing $\eta>0$ such that $\eta C_0<(1/2)\min\{c_4,c_5,c_6\}$ and combining with (81), we have

$$\begin{aligned}&\|U_{1\xi}(t)\|_{L_w^2}^2+\|U_{2\xi}(t)\|_{L_w^2}^2+\|U_{3\xi}(t)\|_{L_w^2}^2\\&+\int_0^t e^{-2\mu(t-s)}\left(\|U_{1\xi}(s)\|_{L_w^2}^2+\|U_{2\xi}(s)\|_{L_w^2}^2+\|U_{3\xi}(s)\|_{L_w^2}^2\right)ds\\&\le Ce^{-2\mu t}\left(\|U_{10}(0)\|_{H_w^1}^2+\|U_{20}(0)\|_{H_w^1}^2+\|U_{30}(0)\|_{H_w^1}^2\right),\end{aligned}\tag{83}$$

for some positive constant C. This completes the proof.

Combining Lemma 4 and Lemma 5 and noting that $w(\xi)\ge 1$ on R, we obtain the following result. □

Lemma 6. *Assume that* (H_1)-(H_5) *hold. For any* $c>\max\{c^*,(1/\eta_0)\max\{c_1,c_2,c_3\}\}$, *it holds*

$$\begin{aligned}&\|U_1(t)\|_{H_w^1}\le Ce^{-\mu t}\left[\|U_{10}(0)\|_{H_w^1}^2+\|U_{20}(0)\|_{H_w^1}^2+\|U_{30}(0)\|_{H_w^1}^2\right]^{1/2},\\&\|U_2(t)\|_{H_w^1}\le Ce^{-\mu t}\left[\|U_{10}(0)\|_{H_w^1}^2+\|U_{20}(0)\|_{H_w^1}^2+\|U_{30}(0)\|_{H_w^1}^2\right]^{1/2},\\&\|U_3(t)\|_{H_w^1}\le Ce^{-\mu t}\left[\|U_{10}(0)\|_{H_w^1}^2+\|U_{20}(0)\|_{H_w^1}^2+\|U_{30}(0)\|_{H_w^1}^2\right]^{1/2},\end{aligned}\tag{84}$$

for some positive constant C *and all* $t>0$.

3.2. Proof of Theorem 1. Using Sobolev embedding inequality $H^1(\mathbb{R})\hookrightarrow C(\mathbb{R})$ and $w(\xi)\ge 1$, we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}}|U_1(\xi,t)|\le C\|U_1(t)\|_{H^1}\le C\|U_1(t)\|_{H_w^1}\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|U_2(\xi,t)|\le C\|U_2(t)\|_{H^1}\le C\|U_2(t)\|_{H_w^1}\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|U_3(\xi,t)|\le C\|U_3(t)\|_{H^1}\le C\|U_3(t)\|_{H_w^1}\le Ce^{-\mu t}.\end{aligned}\tag{85}$$

In view of Lemma 6, we obtain the following results

$$\begin{aligned}&\sup_{x\in\mathbb{R}}|u^+-\phi_1(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_1(\xi,t)|\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|v^+-\phi_2(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_2(\xi,t)|\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|z^+-\phi_3(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_3(\xi,t)|\le Ce^{-\mu t},\end{aligned}\tag{86}$$

for $t>0$. Similarly, for $t>0$, we can also verify

$$\begin{aligned}&\sup_{x\in\mathbb{R}}|u^--\phi_1(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_1(\xi,t)|\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|v^--\phi_2(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_2(\xi,t)|\le Ce^{-\mu t},\\&\sup_{x\in\mathbb{R}}|z^--\phi_3(x+ct)|=\sup_{\xi\in\mathbb{R}}|U_3(\xi,t)|\le Ce^{-\mu t}.\end{aligned}\tag{87}$$

Hence, by the squeezing technique, we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}}|u(x,t)-\phi_1(x+ct)|\le Ce^{-\mu t},\quad t>0,\\&\sup_{x\in\mathbb{R}}|v(x,t)-\phi_2(x+ct)|\le Ce^{-\mu t},\quad t>0,\\&\sup_{x\in\mathbb{R}}|z(x,t)-\phi_3(x+ct)|\le Ce^{-\mu t},\quad t>0.\end{aligned}\tag{88}$$

This completes the proof of Theorem 1.

4. Conclusions

Inspired by the works of [1, 5], in this paper, we consider a predator-prey model with nonlocal dispersals of two cooperative preys and one predator. Since the predation models of the three species add to the computational difficulty, it takes a lot of effort to the priority estimation, and the key point in proving the stability of the traveling front is devoted to establishing a prior estimate by using the weighted energy method. By the standard Sobolev embedding inequality and the squeezing technique, we prove the stability of the traveling wave solution.

In recent years, there has been great progress in modeling and analysis dynamical behavior of predator-prey population involving both time delay and spatial diffusion. In a pioneer work, some researchers have studied a scalar reaction diffusion equation with a single discrete delay by using the phase-plan technique; we can take more attention and initiate the study of traveling wave solutions to delayed reaction diffusion systems on the basis of this paper. Futhermore, we will consider the existence and stability of traveling wave fronts for three-dimensional diffusion systems with convolution delay by the comparison principle and squeezing technique in the future. Another interesting and difficulty problem is the stability of the traveling wave solution under quasi-monotone or nonquasi-monotone assumptions. We leave these issues for future research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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