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*Research Article*

# Exact Solutions to a Generalized Bogoyavlensky-Konopelchenko Equation via Maple Symbolic Computations

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We aim to construct exact and explicit solutions to a generalized Bogoyavlensky-Konopelchenko equation through the Maple computer algebra system. The considered nonlinear equation is transformed into a Hirota bilinear form, and symbolic computations are made for solving both the nonlinear equation and the corresponding bilinear equation. A few classes of exact and explicit solutions are generated from different ansätze on solution forms, including traveling wave solutions, two-wave solutions, and polynomial solutions.

## 1. Introduction

One of the fundamental problems in the theory of differential equations is to determine a solution of a differential equation satisfying what are known as initial values. There are two systematical approaches in a linear world: Laplace's method for solving linear ordinary differential equations and the Fourier transform method, for linear partial differential equations [1, 2]. In the modern theory of integrable systems, the isomonodromic transform method and the inverse scattering transform method have been created for attempting initial value problems for nonlinear ordinary and partial differential equations, respectively [3, 4].

However, only the simplest differential equations, often linear, are solvable precisely. It is definitely not an easy task for us to find exact solutions to nonlinear differential equations, either ordinary or partial. Nevertheless, the Lie group method and the Hirota bilinear method are among effective approaches for finding exact solutions to nonlinear differential equations. The Lie group method is to determine Lie symmetries, which are used to solve ordinary differential equations immediately or used to reduce partial differential equations and to solve simpler reduced ones [5]. The Hirota bilinear method is to transform differential equations into bilinear counterparts and then solve the resulting bilinear ones [6, 7].

Based on Hirota bilinear forms, one can find solitons—a kind of analytic solutions exponentially localized [3, 4]. Some recent studies have also been made on another kind of interesting explicit solutions called lumps, originated from solving integrable equations [8–15]. Lump solutions are a class of analytical rational function solutions localized in all directions in space [9]. A bilinear framework for getting soliton solutions in the (2+1)-dimensional case is as follows. Suppose that a $B$ determines a Hirota bilinear form

$$B\left(D_x, D_y, D_t\right) f \cdot f = 0, \tag{1}$$

where $D_x$, $D_y$, and $D_t$ are Hirota's bilinear derivatives, for a given (2+1)-dimensional partial differential equation:

$$P\left(u_t, u_x, u_y, \cdots\right) = 0. \tag{2}$$

Through the Hirota bilinear technique, soliton solutions can be often formulated as follows:

$$u = 2\left(\ln f\right)_{xx}, \quad f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\right), \tag{3}$$

where $\sum_{\mu=0,1}$ stands for the sum over all possibilities for $\mu_1, \mu_2, \ldots, \mu_N$ taking either 0 or 1 and the wave variables and the phase shifts are determined by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \le i \le N, \tag{4}$$

and

$$e^{a_{ij}} = -\frac{B\left(k_i - k_j, l_i - l_j, \omega_j - \omega_i\right)}{B\left(k_i + k_j, l_i + l_j, \omega_j + \omega_i\right)}, \quad 1 \le i < j \le N, \tag{5}$$

with $k_i$, $l_i$, and $\omega_i$ satisfying the corresponding dispersion relation and $\xi_{i,0}$ being arbitrary translation shifts. Solitons contain various kinds of exact solutions to integrable equations, and taking long wave limits of $N$-soliton solutions can generate special lumps [16].

In this paper, we would like to look for exact solutions to a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko equation

$$\begin{aligned} P_{gBK}(u, v) &:= u_t + \alpha\left(6uu_x + u_{xxx}\right) \\ &\quad + \beta\left(u_{xxy} + 3uu_y + 3u_x v_y\right) + \gamma_1 u_x \\ &\quad + \gamma_2 u_y + \gamma_3 v_{yy} = 0, \end{aligned} \tag{6}$$

where $v_x = u$, and $\alpha, \beta, \gamma_1, \gamma_2$, and $\gamma_3$ are constant coefficients, through Maple symbolic computations. Based on its Hirota bilinear form, a few solution ansätze will be analyzed to compute exact solutions to the nonlinear equation and its bilinear counterpart. Moreover, starting from the nonlinear equation itself, we will do a thorough symbolic computation by Maple within our capacity to generate a few classes of exact and explicit solutions, including traveling wave solutions, two-wave solutions, and polynomial solutions. Conclusions and remarks will be given in the last section.

## 2. One-Wave Type and Two-Wave and Polynomial Solutions

Let us consider the (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko (gBK) equation (6). Equivalently, the (2+1)-dimensional gBK equation (6) can be written as follows:

$$\begin{aligned} &v_{tx} + \alpha\left(6v_x v_{xx} + v_{xxxx}\right) \\ &\quad + \beta\left(v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y\right) + \gamma_1 v_{xx} + \gamma_2 v_{xy} \\ &\quad + \gamma_3 v_{yy} = 0. \end{aligned} \tag{7}$$

This is a generalization of the (2+1)-dimensional Bogoyavlensky-Konopelchenko (BK) equation

$$\begin{aligned} &v_{tx} + \alpha\left(6v_x v_{xx} + v_{xxxx}\right) \\ &\quad + \beta\left(v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y\right) = 0, \end{aligned} \tag{8}$$

a special case of which was introduced as a (2+1)-dimensional version of the KdV equation in [17] and described as the interaction of a long wave propagating along the $x$-axis and a Riemann wave propagating along the $y$-axis [18]. For the (2+1)-dimensional BK equation (8), a Darboux transformation has been given, together with some traveling wave solutions [19], and a few particular properties have been explored (see, e.g., [20, 21]).

By a direct computation, we can show that the (2+1)-dimensional gBK equation (6) can be written as a Hirota bilinear form [22]:

$$\begin{aligned} B_{gBK}(f) &:= \left(D_t D_x + \alpha D_x^{\,4} + \beta D_x^{\,3} D_y + \gamma_1 D_x^{\,2} \right. \\ &\quad \left. + \gamma_2 D_x D_y + \gamma_3 D_y^{\,2}\right) f \cdot f = 2\left[f_{tx} f - f_t f_x \right. \\ &\quad + \alpha\left(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^{\,2}\right) \\ &\quad + \beta\left(f_{xxxy} f - f_{xxx} f_y - 3 f_{xxy} f_x + 3 f_{xx} f_{xy}\right) \\ &\quad + \gamma_1\left(f_{xx} f - f_x^{\,2}\right) + \gamma_2\left(f_{xy} f - f_x f_y\right) \\ &\quad \left. + \gamma_3\left(f_{yy} f - f_y^{\,2}\right)\right] = 0, \end{aligned} \tag{9}$$

under the logarithmic transformations

$$\begin{aligned} u &= 2\left(\ln f\right)_{xx} = \frac{2\left(f_{xx} f - f_x^{\,2}\right)}{f^2}, \\ v &= 2\left(\ln f\right)_x = \frac{2 f_x}{f}. \end{aligned} \tag{10}$$

Such transformations play a prominent role in Bell polynomial theories for soliton equations and their generalized counterparts (see, e.g., [23, 24]). Precisely, we can have

$$P_{gBK}(u, v) = \left(\frac{B_{gBK}(f)}{f^2}\right)_x, \tag{11}$$

And, thus, when $f$ solves the bilinear gBK equation (9), $u = 2(\ln f)_{xx}$ and $v = 2(\ln f)_x$ will solve the nonlinear gBK equation (6).

Beginning with the gBK equation (6) and its Hirota bilinear form (9), we can compute various exact solutions to the gBK equation (6), through carrying out our searches via symbolic computations. For example, lower-order lump solutions have been presented to the gBK equation (6) in [22].

In what follows, we start with some special ansätze for $u$ and $f$ to construct new exact and explicit solutions to the gBK equation (6) (or equivalently (7)) and the bilinear gBK equation (9). Plugging each ansatz into the nonlinear or bilinear gBK equation leads to a system of algebraic equations on the parameters and the coefficients. Then, conduct symbolic computations with Maple to obtain solutions to the algebraic system and further exact solutions to the (2+1)-dimensional gBK equation (6).

*2.1. One-Wave Type Solutions.* First, trying one-wave type solutions, we can easily determine the following five exact and explicit solutions to the bilinear gBK equation (9):

$$\begin{aligned} f &= g_1\left(x - \frac{\alpha}{\beta}y - \frac{\alpha^2\gamma_3 - \alpha\beta\gamma_2 + \beta^2\gamma_1}{\beta^2}t + a_4\right), \\ f &= \exp\left(a_1 x + a_2 y + a_3 g_2(t) + a_4\right), \\ f &= \exp\left(-\frac{\gamma_1}{12\alpha}x^2 + a_2 y + a_3 g_3(t) + a_4\right), \\ f &= \exp\left(a_1 x^2 - \frac{12\alpha a_1^{\,2} + \gamma_1 a_1}{\gamma_3}y^2 + a_3 g_4(t) + a_4\right), \\ f &= h\Bigg(a_1 x + a_2 y \\ &\quad + \frac{4\alpha a_1^{\,4} + 4\beta a_1^{\,3} a_2 - \gamma_1 a_1^{\,2} - \gamma_2 a_1 a_2 - \gamma_3 a_2^{\,2}}{a_1}t \\ &\quad + a_4\Bigg), \end{aligned} \tag{12}$$

where $g_i, 1 \le i \le 4$, are arbitrary functions, $h = \sin$ or $\cos$, and the constant parameters $a_i$, $1 \le i \le 4$, are arbitrary provided that every term in the solutions makes sense. Though all classes of solutions above are interesting solutions to the bilinear gBK equation (9), only the first class of solutions can lead to nontrivial exact solutions to the nonlinear gBK equation (6). Amazingly, the first class of solutions presents traveling wave solutions involving an arbitrary function for the (2+1)-dimensional gBK equation (6), which is a special characteristic of the the gBK equation (6) [22]. Also from the above first class of exact solutions, we can easily formulate various lump-type solutions (i.e., rational and analytical function solutions that are localized in almost all directions in space, under the Lebesgue measure) to the gBK equation (6), by taking $g_1$ to be positive polynomial functions.

*2.2. Two-Wave Solutions.* To search for two-wave solutions, let us now set

$$\begin{aligned} \xi_1 &= a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 &= a_5 x + a_6 y + a_7 t + a_8, \end{aligned} \tag{13}$$

where $a_i$, $1 \le i \le 8$, are constant parameters to be determined. Try an ansatz of two-wave solutions to the bilinear gBK equation (9):

$$f = e^{\xi_1} + e^{\xi_2} + a_9, \tag{14}$$

where $a_9$ is another constant parameter to be determined. and we can show that the resulting system of algebraic equations has two classes of explicit solutions:

$$\begin{aligned} a_3 &= -\frac{1}{a_1 - a_5}\Big(\alpha a_1^{\,4} - 4\alpha a_1^{\,3} a_5 + 6\alpha a_1^{\,2} a_5^{\,2} - 4\alpha a_1 a_5^{\,3} \\ &\quad + \alpha a_5^{\,4} + \beta a_1^{\,3} a_2 - \beta a_1^{\,3} a_6 - 3\beta a_1^{\,2} a_2 a_5 \\ &\quad + 3\beta a_1^{\,2} a_5 a_6 + 3\beta a_1 a_2 a_5^{\,2} - 3\beta a_1 a_5^{\,2} a_6 - \beta a_2 a_5^{\,3} \\ &\quad + \beta a_5^{\,3} a_6 + \gamma_1 a_1^{\,2} + \gamma_2 a_1 a_2 - 2\gamma_1 a_1 a_5 - \gamma_2 a_1 a_6 \\ &\quad + \gamma_3 a_2^{\,2} - \gamma_2 a_2 a_5 - 2\gamma_3 a_2 a_6 + \gamma_1 a_5^{\,2} + \gamma_2 a_5 a_6 \\ &\quad + \gamma_3 a_6^{\,2} - a_1 a_7 + a_5 a_7\Big), \\ a_9 &= 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} a_2 &= \frac{a_1 b}{a_5}, \\ a_3 &= -\frac{a_1}{a_5^{\,2}}\Big(3\beta a_1^{\,2} a_5 b - 3\beta a_1 a_5^{\,2} b + \beta a_5^{\,3} b + 4\alpha a_1^{\,2} a_5^{\,2} \\ &\quad - 6\alpha a_1 a_5^{\,3} + 3\alpha a_5^{\,4} + \beta a_1^{\,2} a_5 a_6 - 3\beta a_1 a_5^{\,2} a_6 \\ &\quad + 2\beta a_5^{\,3} a_6 + \gamma_2 a_5 b + 2\gamma_3 a_6 b + \gamma_1 a_5^{\,2} - \gamma_3 a_6^{\,2}\Big), \\ a_7 &= -\frac{\alpha a_5^{\,4} + \beta a_5^{\,3} a_6 + \gamma_1 a_5^{\,2} + \gamma_2 a_5 a_6 + \gamma_3 a_6^{\,2}}{a_5}; \end{aligned} \tag{16}$$

where the constant $b$ satisfies

$$\begin{aligned} &\gamma_3 b^2 - \left(2\beta a_1^{\,2} a_5 - 3\beta a_1 a_5^{\,2} + \beta a_5^{\,3} + 2\gamma_3 a_6\right) b \\ &\quad - 3\alpha a_1^{\,2} a_5^{\,2} + 6\alpha a_1 a_5^{\,3} - 3\alpha a_5^{\,4} - \beta a_1^{\,2} a_5 a_6 \\ &\quad + 3\beta a_1 a_5^{\,2} a_6 - 2\beta a_5^{\,3} a_6 + \gamma_3 a_6^{\,2} = 0, \end{aligned} \tag{17}$$

and the other parameters could be arbitrary provided that the solutions of $u$ and $v$ presented by (10) will be well defined.

If we try another ansatz for two-wave solutions:

$$f = e^{\xi_1} + h(\xi_2) + a_9, \quad h = \sin \text{ or } \cos, \tag{18}$$

and then we can have

$$\begin{aligned} a_2 &= \frac{b}{a_5}, \\ a_3 &= -\frac{1}{a_5^2}\Big(4\alpha a_1^{\,3} a_5^{\,2} + \beta a_1^{\,3} a_5 a_6 + 3\beta a_1^{\,2} a_5 b \\ &\quad + \beta a_1 a_5^{\,3} a_6 - \beta a_5^{\,3} b + \gamma_1 a_1 a_5^{\,2} - \gamma_3 a_1 a_6^{\,2} + \gamma_2 a_5 b \\ &\quad + 2\gamma_3 a_6 b\Big), \\ a_7 &= \frac{4\alpha a_5^{\,4} + 4\beta a_5^{\,3} a_6 - \gamma_1 a_5^{\,2} - \gamma_2 a_5 a_6 - \gamma_3 a_6^{\,2}}{a_5}, \\ a_9 &= 0, \end{aligned} \tag{19}$$

where the constant $b$ needs to satisfy a quadratic equation

$$\gamma_3 b^2 - 2a_1 \left(\beta a_1{}^2 a_5 + \beta a_5{}^3 + \gamma_3 a_6\right) b - 3\alpha a_1{}^4 a_5{}^2 - 6\alpha a_1{}^2 a_5{}^4 - 3\alpha a_5{}^6 - \beta a_1{}^4 a_5 a_6 - 4\beta a_1{}^2 a_5{}^3 a_6 - 3\beta a_5{}^5 a_6 + \gamma_3 a_1{}^2 a_6{}^2 = 0. \quad (20)$$

This is different from (17). When $\gamma_3 \neq 0$, these are two quadratic equations and so each of them has two roots for $b$; but when $\gamma_3 = 0$, they become linear and so only one solution for $b$ is possible.

Then, through the transformations in (10), we can obtain diverse classes of exact and explicit two-wave solutions to the (2+1)-dimensional gBK equation (6).

*2.3. Polynomial Solutions.* Let us thirdly try an ansatz on polynomial solutions for $v$:

$$v = \sum_{i,j,k=0}^{n} a_{i,j,k} x^i y^j t^k, \quad (21)$$

where $n \in \mathbb{N}$ and $a_{i,j,k}$, $1 \leq i,j,k \leq n$, are constants coefficients to be determined. By symbolic computations for the case of $n = 2$, we can get the following two classes of exact polynomial solutions to the gBK equation (7):

$$v = -a_{0,2,1}\gamma_3 x t^2 + a_{0,2,1} y^2 t + a_{0,1,2} y t^2 - 2a_{0,2,0}\gamma_3 x t + a_{0,2,0} y^2 + a_{0,1,1} y t + a_{0,0,2} t^2 + a_{1,0,0} x + a_{0,1,0} y + a_{0,0,1} t + a_{0,0,0}, \quad (22)$$

and

$$v = \frac{3a_{2,0,0}\left(2\alpha a_{2,0,0} b - \beta^3 a_{0,1,1}\right)}{\beta^2} x t^2 - \frac{3\alpha a_{2,0,0}\left(2\alpha a_{2,0,0} b - \beta^3 a_{0,1,1}\right)}{\beta^3} y t^2 + a_{2,0,0} x^2 - \frac{2\alpha a_{2,0,0}}{\beta} x y - \frac{2a_{2,0,0} b}{\beta^2} x t + \frac{\alpha^2 a_{2,0,0}}{\beta^2} y^2 + a_{0,1,1} y t + a_{0,0,2} t^2 + a_{1,0,0} x + a_{0,1,0} y + a_{0,0,1} t + a_{0,0,0}, \quad (23)$$

where the constant $b$ is given by

$$b = 3\alpha\beta^2 a_{1,0,0} + 3\beta^3 a_{0,1,0} + \alpha^2\gamma_3 - \alpha\beta\gamma_2 + \beta^2\gamma_1, \quad (24)$$

and the other involved parameters are arbitrary.

The first class of polynomial solutions does not depend on any coefficient of the nonlinear terms in the gBK equation (7), but the second class depends critically on the coefficient $\beta$ of the second group containing nonlinear terms in the gBK equation (7) and involves two more monomials: $x^2$ and $xy$. It is also surprising that the cases of $n \geq 3$ are just too complicated for us to work out any nontrivial explicit polynomial solutions.

Taking $a_{0,2,1} = a_{0,1,2} = 0$ in the first solution (22) and

$$a_{0,1,1} = \frac{2\alpha a_{2,0,0} b}{\beta^3} \quad (25)$$

in the second solution (23), we obtain two quadratic function solutions immediately:

$$v = -2a_{0,2,0}\gamma_3 x t + a_{0,2,0} y^2 + a_{0,1,1} y t + a_{0,0,2} t^2 + a_{1,0,0} x + a_{0,1,0} y + a_{0,0,1} t + a_{0,0,0}, \quad (26)$$

and

$$v = a_{2,0,0} x^2 - \frac{2\alpha a_{2,0,0}}{\beta} x y - \frac{2a_{2,0,0} b}{\beta^2} x t + \frac{\alpha^2 a_{2,0,0}}{\beta^2} y^2 + \frac{2\alpha a_{2,0,0} b}{\beta^3} y t + a_{0,0,2} t^2 + a_{1,0,0} x + a_{0,1,0} y + a_{0,0,1} t + a_{0,0,0}, \quad (27)$$

where $b$ is defined by (24).

*2.4. An Illustrative Example.* Let us now take

$$\alpha = 1, \quad \beta = -1, \quad \gamma_1 = 2, \quad \gamma_2 = 1, \quad \gamma_3 = -1. \quad (28)$$

Then, from (7), we obtain the following specific gBK equation

$$v_{tx} + \left(6v_x v_{xx} + v_{xxxx}\right) - \left(v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y\right) + 2v_{xx} + v_{xy} - v_{yy} = 0. \quad (29)$$

Based on our presented solutions, this nonlinear equation has a traveling wave solution

$$v = -\frac{2\sinh\left(-x - y + 2t - 5\right)}{\cosh\left(-x - y + 2t - 5\right) + 10}, \quad (30)$$

a two-wave solution

$$v = \frac{2e^{x-2y-3t-1} + 4e^{2x-3y-5t+2}}{e^{x-2y-3t-1} + e^{2x-3y-5t+2}}, \quad (31)$$

and a polynomial solution

$$v = \left(-33x - 33y + 5\right) t^2 + \left(14x + 3y + 1\right) t + 2xy + y^2 - x + 2y + 1. \quad (32)$$

All the solutions computed above via symbolic computations supplement the solution theories available on soliton solutions and dromion-type solutions, generated through powerful existing approaches such as the Hirota perturbation method and symmetry constraints including symmetry reductions (see, e.g., [25–30]).

## 3. Concluding Remarks

We have constructed a few classes of exact and explicit solutions, including two-wave solutions and polynomial solutions, to a (2+1)-dimensional generalized Bogoyavlensky-Konopelchenko (gBK) equation. Symbolic computations with Maple are the adopted technique and the Hirota bilinear form is a basis for getting one-wave type and two-wave solutions.

The obtained result enriches the existing studies on the (2+1)-dimensional Bogoyavlensky-Konopelchenko equation [19–22]. We remark that there are plenty of interaction solutions to integrable equations (see, e.g., [31]), particularly between lumps and other kinds of exact solutions to (2+1)-dimensional nonlinear integrable equations (see, e.g., [32–35] for lump-kink interaction solutions and [36–39] for lump-soliton interaction solutions).

However, we couldn't find any interaction solutions between lump (or lump-type) solutions and kink (or soliton) solutions, and any nontrivial $N$-wave solutions, including $N$-soliton solutions, where $N \geq 3$, for the (2+1)-dimensional gBK equation (6). We believe that the existence of interaction solutions and three-wave solutions to partial differential equations should be a characteristic property of complete integrability for the differential equations under consideration. It is definitely interesting to search for exact solutions and interaction solutions to partial differential equations in both (1+1)- and (2+1)-dimensions. Another important problem is to identify nonlinear partial differential equations that possess soliton solutions and interaction solutions, to understand integrable properties of nonlinear equations.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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