Research Article

Characterization of 2-Path Signed Network

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An edge \( ab \) in a signed network \( \Sigma \) receives a negative sign if all the paths of length two between them are negative, otherwise it receives a positive sign. A signed network is said to be if clusterable its vertex set can be partitioned into pairwise disjoint subsets, called clusters, such that every negative edge joins vertices in different clusters and every positive edge joins vertices in the same clusters. A signed network is balanced if it is clusterable with exactly two clusters. A signed network is sign-regular if the number of positive (negative) edges incident to each vertex is the same for all the vertices. We characterize the 2-path signed graphs as balanced, clusterable, and sign-regular along with their respective algorithms. The 2-path network along with these characterizations is used to develop a theoretic model for the study and control of interference of frequency in wireless communication networks.

1. Introduction

The intersection graphs or networks [1, 2] form a large family of structures which include many important network such as interval [3, 4], permutation [5, 6], chordal [7, 8], circular-arc [9, 10], circle [11], string [12], line [13, 14], and path [15, 16]. Most of these networks are of great significance not only theoretically but because of their applicability in the fields such as transportation [17], wireless networking [18], scheduling problem [19], molecular biology [20], circuit routing [21], and sociology. The 2-path network is an intersection graph of open neighborhoods [22]. Formally, a 2-path network for a given network is obtained by joining the pair of vertices which form a path of length two in the original network.

Signed networks are the networks where each edge receives a sign: positive or negative (ref. Figure 1, the edges 13, 32, 24, and 25 are negative in (b)). The concept stemmed from psychologist Heider [23, 24] who used the concept of balanced theory to model relations between individuals and society using triads. Harary formulated and restructured the signed networks by introducing structural balance theory [25] for social balance and was used for portfolio management [26], where they used signed graphs to analyse the extent of hedging in a portfolio. These networks are widely used in data clustering [27–29]. Signed networks of some intersection networks have been already studied [30–33]. Also path graphs of signed graphs are discussed in [34–38]. In this paper, we try and establish the results for signed networks defined on 2-path networks.

The study of networks has come a long way with its applicability in many fields [39–43]. One such application is in wireless networks and frequency allocation problem. The frequency allocation [44] in a wireless network [45] or radio frequency allocation [46] is one of the typical problems that we still face. The interference takes place in a network when the transmission from one station interacts with the transmission from another station. There are three dimensions to the frequency spectrum first being the space in which the emission is radiated, second is the frequency...
bandwidth, and third is the time. If these three dimensions occur simultaneously for a channel receiving transmission, then interference takes place. Many methods and models such as dipole-moment models [47], Kurtosis detection algorithm and its versions [48], and decomposition method based on reciprocity [49] have been studied for this problem. We, on the other hand, bring a model with a very different approach using the theoretic aspects of signed networks and 2-path networks.

We aim at detecting and reducing the interference by assuming the stations/channels as vertices and transmission between them as edges. Furthermore, we assume that two channels (vertices) are joined by a positive edge if and only if they are in different time or frequency bandwidth (assuming that space is always same) and by a negative edge if both are the same in time and frequency bandwidth. If $\Sigma$ is the given signed network representing a channel network, then its 2-path networks. A signed network is an ordered pair $\Sigma_{\mu} = (\Sigma, \mu)$, where $\Sigma = (\Sigma^u, \sigma)$ is a signed network and $\mu: V(\Sigma^u) \rightarrow \{+, -\}$ is a function from the vertex set $V(\Sigma^u)$ of $\Sigma^u$ into the set $\{+, -\}$, called a marking of $\Sigma$. We define $V(\Sigma)_{\mu} = \{v^+_i, v^-_i; \forall v_i \in V(\Sigma)\}$. Let $v$ be an arbitrary vertex of a network $\Sigma$. We denote the set consisting of all the vertices of $\Sigma$ adjacent with $v$ by $N(v)$. This set is called the neighborhood set of $v$ and sometimes we call it as neighborhood of $v$. Next, we define two marked neighborhoods as $\Sigma^*_{\mu}(t) = \{v^+_i \in V(\Sigma)_{\mu}; tv \text{ is an edge with sign } \mu\}$ and $\Sigma^*_{\mu}(t) = \{v^-_i \in V(\Sigma)_{\mu}; tv \text{ is an edge}\}$. For each $N(t)$ of $\Sigma^u$, there exist $N^*$ $(t)$ in $\Sigma$ and vice versa. It is important to note that the vertex marking for each vertex is neighborhood dependent, i.e., if a vertex $v$ forms a negative edge with $v_j$ and a positive edge with $v_k$, then $v^-_j$ belongs to the marked neighborhood of $v_j$, $N^*_j(v_j)$ and $v^-_k \in N^*_k(v_k)$. Also, each vertex in $N(t)$ appears with exactly one mark in $N^*_t$(t).

The clique of a network $\Sigma^u$ is a subset of vertices such that every two vertices in the subset are connected by an edge. $\delta(N(t))$ is a clique generated by vertices in $N(t)$. A cycle in a signed network $\Sigma$ is said to be positive if the product of the signs of its edges is positive or, equivalently, if the number of negative edges in it is even. A cycle which is not positive is said to be negative. A signed network is balance $d$ if all its cycles are positive. A signed network is said to be clusterable if its vertex set can be partitioned into pairwise disjoint subsets, called clusters, such that every negative edge joins vertices in different clusters and every positive edge joins vertices in the same clusters. A common neighbor of vertices $v_i$ and $v_j$ is a vertex $v$ such that $v \in N(v_i) \cap N(v_j)$.

Property 1. If $v_i, v_j, v_k \in V(\Sigma)$, then by property $P(v_i, v_j; v_k)$, we mean $\{v^-_i, v^-_j\} \subseteq N^*_i(v_k)$.

**Figure 1:** (a) Given network. (b) Its signed network.
Property 2. By property $P := P(v_i, v_j)$, we mean that if property $P(v_i, v_j ; v)$ holds for one common neighbor $v$ of $v_i$ and $v_j$, then it holds for every common neighbor $v$ of $v_i$ and $v_j$.

The 2-path signed network [34] $(\Sigma)_2 = (V, E, \sigma)$ of a signed network $\Sigma = (V, E, \sigma)$ is defined as follows. The vertex set is the same as the original signed network $\Sigma$, and two vertices $u, v \in V((\Sigma)_2)$ are adjacent if and only if there exist a path of length two in $\Sigma$. The edge $uv \in E((\Sigma)_2)$ is negative if and only if all the edges in all the two paths in $\Sigma$ between them are negative otherwise the edge is positive. The definition of 2-path signed network was used [53], to bring out some basic results, which was further extended and shaped in the present paper. The 2 subsets $\{v_i, v_j\}$ having property $P$ are named as $P$ pairs and the set of all $P$ pairs is denoted by $P_\Sigma$. A negative section $\Sigma$ has $P$ pairs if and only if there exists an all-negative path between 1 and 3. The resultsof which were presented in [35], give algorithms to construct 2-path signed network, detect if a network is isomorphic to a 2-path signed network, and collect $P$ pairs.

Section 2 is on the balancedness of 2-path signed networks where we characterize signed network whose 2-path signed networks are balanced. The property of balanced network is used to identify and categorize all the channels in 2-path in exactly two groups such that the negative edges are across the groups and positive are in the same class. We also provide an algorithm for the same. Section 3 is dedicated to another property of signed network known as clusterability, where we check whether we can categorize our vertices in more than two clusters so that the negative edges are across the group. We follow the property of sign-regularity in Section 4.

2.2-Path Signed Network

2.1. Characterization of 2-Path Signed Network. In this section, we give a characterization of 2-path signed network. We check if a given signed network is a 2-path signed network of some signed network. We then find its underlying signed network. Following characterization of 2-path networks was given by Acharya and Vartak.

Let $v_i, v_j$ be an edge in a signed network $\Sigma$. Let $\mu_{ij}$ denote the marking on the vertex $v_j$ in $N_s(v_i)$ such that $\mu_{ij} = \sigma(v_i, v_j)$; then, we have the following theorem.

Theorem 1. A connected signed network $\Sigma$ with vertices $v_1, v_2, \ldots, v_n$ is a 2-path signed network of some signed network $\Sigma'$ if and only if $\Sigma$ contains a collection of complete subgraphs $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ such that, for each $i, j = 1, \ldots, n$, the following hold: (i) $v_i \notin \Sigma_j$, (ii) $v_i \in G_i \iff v_j \in G_j$, and (iii) if $v_i, v_j \in G$ then there exists $\Sigma_k$ containing $v_i, v_j$.

Let $v_i, v_j$ be an edge in a signed network $\Sigma$. Let $\mu_{ij}$ denote the marking on the vertex $v_j$ in $N_s(v_i)$ such that $\mu_{ij} = \sigma(v_i, v_j)$; then, we have the following theorem.

Complexity
Proof. Necessity. Suppose $\Sigma$ is a 2-path signed network of signed network $\Sigma'$ with vertices $v_1, v_2, \ldots, v_n$. To show that there exist a collection of $n$ complete signed subgraphs $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ such that (i), (ii), and (iii) hold, consider the marked neighborhood $N_*(v_i)$ for each vertex $v_i$ in $v_1, v_2, \ldots, v_n$ in $\Sigma'$. Each $N_*(v_i)$, $i = 1$ to $n$, generates a complete signed subgraph in 2-path signed network $\Sigma$ of $\Sigma'$, since $N_*(v_i)$ consists of elements which have a common neighbor $v_i$ in $\Sigma'$. Next, $v_i \notin N_*(v_i)$ since the signed network $\Sigma'$ is simple and $v_i \notin N(v_i)$. Let $v_{ij} \in N_*(v_i)$ for some $i, j; i \neq j, 1 \leq i, j \leq n$. Then, $v_i v_j$ is an edge in $\Sigma'$ such that $\sigma(v_i v_j) = \mu_{ij}$, and thus $v_{ij} \in \Sigma$ is in the marked neighborhood $N_*(v_i)$. Therefore, for $i \neq j$, $v_{ij} \in \Sigma$ and $\mu_{ij} = \mu_{ij}$, $v_{ij} \in \Sigma$. Next, since the mark $\mu_{ij}$ depends on the sign of edge $v_i v_j$ in $\Sigma'$, the vertex $v_j \in N_*(v_i)$ gets the marking from the corresponding sign $\sigma(v_i v_j) = \mu_{ij}$; hence, $\mu_{ij} = \mu_{ij}$ and (ii) follows.

If $v_i v_j$ is an edge in 2-path signed network $\Sigma$ and thus in $\Sigma''$, then from Lemma 1, $v_i, v_j \in N(v_k)$ for some $k$. Let $v_i v_j$ be an edge in $\Sigma$ with a sign “−”, then $v_i v_k$ and $v_j v_k$ are negative edges in $\Sigma'$, by definition of 2-path signed network. Thus, by Remark 1, $\{v_i, v_j\} \in N(v_k)$ and $\{v_i, v_j\}$ is a $P$ pair. Hence, (iii) holds.

Sufficiency. Let $\Sigma$ contain a collection of complete signed subgraphs $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ with edges $v_i$ in $\Sigma_j, j \neq i$ with the marking $\mu_{ij}$, satisfying the three properties (i), (ii), and (iii). Join each vertex $v_i$ to the vertices of $\Sigma_i$, let the obtained signed network be $\Sigma'$. Let $v_j$ be in $\Sigma_i$, having the marking $\mu_{ij}$. Then, the sign of edge $v_i v_j$ in $\Sigma'$ is given by $\mu_{ij}$. From our construction of $\Sigma'$ and property (ii), as $v_{ij} \in \Sigma_i$, we have $v_{ij} \in \Sigma_i$ and $\mu_{ij} = \mu_{ij}$. Note that if $v_i v_k \in E(\Sigma)$, then by (iii), there exist $\Sigma_i$ such that $v_{ij} \in \Sigma_i$ and $v_{ik} \in \Sigma_i$. Also if sign of the edge $v_i v_k$ is negative in $\Sigma'$, then $\mu_{ij} = \mu_{ik} = -$ and $\{v_i, v_j, v_k\}$ is a $P$ pair in $\Sigma$, whereas if the edge is positive in $\Sigma'$, then the marks $\mu_{ij}$ and $\mu_{ik}$ are according to the edge sign $v_i v_j$ and $v_i v_k$ in $\Sigma$, respectively.

Next, we will show that $(\Sigma')_2 \equiv \Sigma$. Clearly, the vertex set of $\Sigma$ and $(\Sigma')_2$ is the same. We are left to show that their adjacencies along with their signs are preserved. For some
Proof. Let $v_i v_j$ be a negative edge in $\Sigma$, then by condition (iii), $\{v_i, v_j\}$ is a $P$ pair in $\Sigma$, for some $k, k \neq i, j$. By Property 2 and signing of the edges in $\Sigma'$, the edges formed by vertices incident to both $v_i$ and $v_j$ in $\Sigma'$ will all be negative, thus, the edge $v_i v_j$ in the 2-path signed network $(\Sigma')_2$ is negative. Similarly, if $\sigma(v_iv_j) = +$ in $(\Sigma')_2$, then all the $v_i v_j$-paths of length two will be all negative in $\Sigma'$. Let $v_i v_j$ be a 2-path in $\Sigma'$. As the edges incident to $v_i$ in $\Sigma'$ are forming a vertex $v_k$ with vertices in $\Sigma$ such that $\sigma(v_k v_i) = -\mu_k$, and $\sigma(v_k v_j) = -\mu_j$, where $v_k v_i, v_k v_j \in \Sigma$ and $\mu_k = \mu_j = -1$, then, all the edges in $\Sigma'$ in the 2-path between $v_i$ and $v_j$ in $\Sigma'$ are negative, then by (iii) $\sigma(v_i v_j) = -1$ in $\Sigma$. Similarly, the same holds for positive edge. Thus, $v_i v_j$ is a positive edge in $(\Sigma')_2$, there exist at least one vertex $v_k$ in $\Sigma'$, where $v_i v_j$ is a path of length two. Then, clearly, at least one of the two vertices $v_i$ and $v_j$ forms a positive edge with $v_k$ in $\Sigma$, also in the complete sub signed network $\Sigma_k$ containing both $v_i$ and $v_j$, and thus $v_i v_j \in \Sigma$ and by (iii) it cannot be a negative edge. Hence, $\sigma(v_i v_j) = +$ in $\Sigma$. Conversely, if $\sigma(v_i v_j) = +$ in $\Sigma$, then as done above for negative edge for some $\Sigma_k$, the mark of one of these two vertices would be positive, and thus there would be a heterogeneous path of length two between $v_i$ and $v_j$ in $\Sigma'$ and thus a positive edge in $(\Sigma')_2$. Hence, $(\Sigma')_2 \subseteq \Sigma$.

Before going further, we prove the following lemmas.

**Lemma 2.** If a signed network $\Sigma$ has an induced cycle of length $2k, k \geq 3$, then $(\Sigma_2)$ contains two vertex disjoint cycles of length $k$ each.

**Proof.** Let $v_1 v_2, v_2 v_3, \ldots, v_k v_1$ be an induced cycle of length $2k, k \geq 3$, in a signed network $\Sigma$. Then, there is a path of length two between each pair $v_i$ and $v_{i+1}$ for $i = 1, \ldots, k - 1$ and $v_k v_1$. Thus, $v_1 v_2, v_2 v_3, \ldots, v_k v_1$ is a cycle of length $k$ in $(\Sigma_2)$. Similarly, $t_1 t_2, t_2 t_3, \ldots, t_k t_1$ is a cycle of length $k$ in $(\Sigma_2)$. Clearly, there is no vertex in common in the two cycles. Hence, the result follows.

**Remark 2.** If the signed network $\Sigma$ consists of a cycle of length $4$, then $(\Sigma_2)$ contains two disjoint edges corresponding to the vertices of the cycle.

**Lemma 3.** For a signed network $\Sigma$, $(\Sigma_2)$ contains an induced cycle of length $k$ ($k$ is odd and $k > 3$) if either $\Sigma$ has a cycle of length $k$ or $\Sigma$ has a cycle of length $2k$.

**Proof.** First, assume that $\Sigma$ has an induced cycle $C: v_1 v_2, \ldots, v_k v_1$ of length $k$ ($k$ is odd and $k > 3$). Clearly, $C': v_1 v_2, v_2 v_3, \ldots, v_{k-1} v_1$ is a cycle of length $k$ in $(\Sigma_2)$.

Next, if $\Sigma$ has an induced cycle $v_1 v_2, \ldots, v_k v_1$ of length $2k$, where $v_i, t_i \in V(\Sigma)$ for $i = 1, 2, \ldots, k$, clearly, $v_1 v_2, v_2 v_3, \ldots, v_{k-1} v_1$ and $t_1 t_2, \ldots, t_k t_1$ are cycles of length $k$ in $(\Sigma_2)$. Hence, result follows.

**2.2. Algorithm to Detect $P$ Pairs.** Here, in Algorithm 1, we present algorithm for detection of $P$ pairs. For an input network $\Sigma$, these $P$ pairs gives the negative edges in $(\Sigma_2)$ and thus will be used as an input for all the other future algorithms.

**2.2.1. Complexity.** In Steps 5, 6, and 7, we use three different loops running up to $n$ times (number of vertices). Thus, the complexity of this fragment is equal to $O(n^3)$.

Next, we use three nested loops in Steps 12, 13, and 15. Again the complexity is $O(n^3)$.

The loops at Step 19 and Step 20 are used for the array $P$ and $Q$ with a size of $m = l - 2$ and $g = t - 2$, respectively. Since the maximum number of edges in a network is $n(n - 1)/2$ and $P$ and $Q$ collects vertices incident to edges, thus $m \leq n(n - 1)$ and $g \leq n(n - 1)$. Therefore, the complexity of this fragment is $O(n(n - 1)) \leq O(n^2)$.

Combining the above, the total complexity equals $O(n^3) + O(n^2) + O(n^2)$. Hence, the complexity involved in Algorithm 1 is $O(n^3)$.

**2.2.2. Implementation of Algorithm**

**Example 1.** In this example, we are given with a signed network, as shown in Figure 4(a), and we find all the $P$ pairs for the given signed network $\Sigma$. Let us consider adjacency matrix for the signed network which is given as follows:

$$A = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

In order to detect the $P$ pairs of $\Sigma$, we use Algorithm 1 where the inputs are the matrix $A$ and size $n = 4$. The size of array $P$ and $Q$ is initially fixed as $n^2$. After initializing the array $P$ and $Q$ as zero, $t = 1$ and $l = 1$, we enter the first, second, and third loop which run for $1$ to $4$. The three loops at Steps 5, 6, and 7 are used to obtain two distinct elements $A(i, j)$ and $A(j, k)$. Next, in Step 9 we check pair of vertices $i, j$ and $k$ if $A(i, j) = 1$ and $A(j, k) = 1$ or $A(i, j) = -1$ and $A(j, k) = -1$ or $A(i, j) = 1$ and $A(j, k) = -1$, then $i$ and $k$ enter as a consecutive entry in the array $Q$. The array $Q$ collects the vertices forming positive 2-paths in $\Sigma$. For the above matrix, as we enter the loop, then for $i = 1, j = 2, k = 4$, we find that $A(1, 2) = -1$ and $A(2, 4) = 1$. Thus, as $t = 1, Q[1] = 1$ and $Q[2] = 4$ and $t$ is incremented to $t = 3$. Similarly, after completing all the loops we get $Q$ as follows.
Input: Adjacency matrix $A$ and dimension $n$.
Output: Array $P$ which has all collection of $P$ pair and array $Q$ which has pair of vertices which are not $P$ pairs.

Process:
1. Enter the order $n$ and adjacency matrix $A$ for the signed network $\Sigma$.
2. for $i = 1$ to $n^2$ do
   3. $P[i] = 0; Q[i] = 0,$
   4. $t = 1; l = 1,$
   5. for $i = 1$ to $n$ do
      6. for $j = 1$ to $n$ do
         7. for $k = 1$ to $n$ do
            8. if $(i \neq j \neq k)$ then
               9. if $((A(i, j) = 1) \& \& (A(k, j) = -1)) \& \& ((A(i, j) = -1) \& \& (A(k, j) = 1))$ then
                  10. $Q[t] = i; Q[t + 1] = k;$
                  11. $t = t + 2;$
                  12. for $i = 1$ to $n$ do
                     13. for $j = 1$ to $n$ do
                        14. if $(A(i, j) = -1)$ then
                           15. for $k = 1$ to $n$ do
                              16. if $(A(k, j) = -1) \& \& (i \neq j)$ then
                                 17. $P[l] = i; P[l + 1] = k;$
                                 18. $l = l + 2;$
                                 19. for $i = 1$ to $l - 2$, $i = i + 2$ do
                                    20. for $j = 1$ to $l - 2$, $j = j + 2$ do
                                       21. if $(P[i] = Q[j]) \& \& (P[i + 1] = Q[j + 1])$ then
                                          22. $P[i] = 0;$
                                          23. $P[i + 1] = 0;$

Algorithm 1: An algorithm to detect and collect $P$ pairs.

One can note that the number of elements in array $Q$ is given by the $t - 2$ (since every time we enter Step 9, $t$ is incremented by 2). In the given example, $t = 16$, and thus size of array $Q$ is 14. Next, we enter loop at Step 13, to find all the edges $ij$ and $jk$ such that $A(i, j) = -1$ and $A(j, k) = -1$. Once such pair is obtained then $i$ and $k$ are collected in Array $P$ as done for array $Q$. For the given example, array $P$ comes out to be

$$P = [1, 3, 1, 2, 2, 2, 2, 3].$$

The size of $P$ is given by $l - 2$ and in our example $l = 10$. Finally, we compare the elements of $P$ and $Q$ pairwise to find if there are pairs of vertices which is common in both and then we remove these pairs from $P$. This is done by using two loops one moves from 1 to $t - 2$ (the size of array $Q$) and the other moves from 1 to $l - 2$ (size of array $P$). For $i = 5$ and $j = 5$, we see that $P[5] = Q[5]$ and $P[6] = Q[6]$. Therefore, $P[5] = 0$ and $P[6] = 0$. Proceeding in the same way, we obtain $P$ for the given example as:

$$P = [1, 3, 1, 2, 2, 0, 0, 0, 0].$$

Thus, the $P$ pair are 1,3 and 1,2. From Figure 4 and Remark 1, we can verify that the $P$ pair obtained in the algorithm is the same as that for the given signed network.

Remark 3. Each pair $Q[i], Q[i + 1]$ in array $Q$, for $i = 1$ to $l - 1$, is a pair of vertices in $\Sigma$, which have a path of length two between them in $\Sigma$. Since they do not form a $P$ pair, they form a positive edge in $(\Sigma)_2$ (by Remark 1).

2.3. Algorithm to Construct 2-Path Signed Network. Next in Algorithm 2, we obtain the 2-path signed network $(\Sigma)_2$ for a given signed network $\Sigma$. We use the adjacency matrix $A$ of $\Sigma$. We use Algorithm 1, to obtain the vertices forming negative edges (collected in array $P$) and positive edges (collected in array $Q$) of 2-path signed network. The adjacency matrix of 2-path signed network obtained is saved in matrix $B$.

2.3.1. Complexity. In Step 2, we use two loops to initialize the matrix $B$ zero, and thus the complexity of these steps is $O(n^2)$.

In Step 5, we use a loop which runs up to $m - 1$, $m \leq n(n - 1)$ (from previous algorithm). Thus, the complexity of this step is $O(n(n - 1)) = O(n^2)$. Similarly, in Step 10, we use a loop which runs up to $g - 1$, $g \leq n(n - 1)$. Again complexity of this step is $O(n(n - 1)) = O(n^2)$.

The total complexity is $O(n^2) + O(n^2) + O(n^2) = O(n^2)$.

2.3.2. Implementation of the Algorithm

Example 2. We are given with a signed network $\Sigma$, as shown in Figure 4(a), with the adjacency matrix $A$ as in Example 1. We have to find the adjacency matrix $B$ of the 2-path signed network $(\Sigma)_2$. In Steps 2 and 3, we use two loops, each running from 1 to 4 as $n = 4$ and assign zero to all the entries.
of matrix $B$. Next, with the help of Algorithm 1, we obtain $P$ as $Q$ as

$$
P = \begin{bmatrix} 1 & 3 & 1 & 2 & 0 & 0 & 0 & 0 \\
Q = \begin{bmatrix} 1 & 4 & 2 & 4 & 2 & 3 & 4 & 3 & 2 & 4 & 3 & 4 & 2 
\end{bmatrix}
\]$$

In Step 5, we enter the loop running up to $m = 7$ for each pair $P[t], P[t + 1]$ which saves the vertices forming negative edge in $(\Sigma)_2$. As an edge $jk$ is given by entry $j$th row and $k$th column and by $k$th row and $j$th in the adjacency matrix, for each pair $P[t] = j$ and $P[t + 1] = k$, $B(j, k) = B(k, j) = -1$. Thus, for $t = 1$, $B(1, 3) = B(3, 1) = -1$, similarly, for $t = 3$, $B[1][2] = B[2][1] = -1$. Thus, the matrix $B$ after Step 9 is

\[
\begin{bmatrix}
0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Next, in Step 10, we enter the loop to find the vertices forming positive edges in $(\Sigma)_2$, which is given by pair of vertices in array $Q$. For $t = 1$, $Q[1] = 1$, $Q[2] = 4$ and so $B(1, 4) = B(1, 4) = 1$. After running the loop for $g = 13$, matrix $B$ is given by

\[
\begin{bmatrix}
0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Next, in Step 5, we enter the loop running up to $m = 7$ for each pair $P[t], P[t + 1]$ which saves the vertices forming negative edge in $(\Sigma)_2$. As an edge $jk$ is given by entry $j$th row and $k$th column and by $k$th row and $j$th in the adjacency matrix, for each pair $P[t] = j$ and $P[t + 1] = k$, $B(j, k) = B(k, j) = -1$. Thus, for $t = 1$, $B(1, 3) = B(3, 1) = -1$, similarly, for $t = 3$, $B[1][2] = B[2][1] = -1$. Thus, the matrix $B$ after Step 9 is
\[
A = \begin{bmatrix}
0 & -1 & -1 & 1 \\
-1 & 0 & 1 & 1 \\
-1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix},
\]
which is the adjacency matrix for the 2-path signed network \((\Sigma)_2\), as shown in Figure 4(b).

3. Balance in 2-Path Signed Network

3.1. Characterization of Balanced 2-Path Signed Network. In this section, we provide with a characterization of balanced 2-path signed network. The characterization helps us to identify the groups in the interference network (2-path signed network) and find the balanced. We refer the following lemma given by Zaslavsky.

**Lemma 4** (see [54]). A signed network in which every chordless cycle is positive and is balanced.

**Theorem 2.** For a signed network \(\Sigma\) of order \(n\), the following statements are equivalent:

(i) \((\Sigma)_2\) is balanced.

(ii)

(a) Each homogeneous cycle in \(\Sigma\) is either positive or is a \(P\) cycle of length \(4k\), for some positive integer \(k\).

(b) Each heterogeneous odd cycle in \(\Sigma\) either does not contain \(P\) section or contains even number of \(P\) section of even length.

(c) For each heterogeneous even cycle (length greater than 4), the following conditions hold:

(c1) If it has \(P\) sections of odd length \(l\), then either \(l \equiv 1 \pmod{4}\) or if \(l \equiv 1 \pmod{4}\) then, there are even number of such \(P\) sections each separated by positive section of even length.

(c2) If it has \(P\) sections of even length, then there are even number of such \(P\) sections each separated by positive section of even length.

(d) For each vertex \(u\) in \(\Sigma\) with \(d(u) \geq 3\), \(N_u(\Sigma)\) does not contain any \(P\) pairs.

**Proof.** (i) \(\Rightarrow\) (ii) Let \((\Sigma)_2\) be balanced. If a cycle in \(\Sigma\) is a positive homogeneous cycle, then the cycle (or cycles in case the cycle in \(\Sigma\) is of even length) formed in its 2-path signed graph \((\Sigma)_2\) will be a positive cycle. Let \(C\) be an all-negative cycle (\(P\) cycle), with length \(\neq 4k\) in \(\Sigma\). Here, the length of \(C\) is either odd or 2\(t\), where \(t\) is odd. By Lemma 3, the corresponding cycle or cycles formed in \((\Sigma)_2\) will be odd length all-negative, since cycle \(C\) is a \(P\) cycle in \(\Sigma\). Thus, by Lemma 4, \((\Sigma)_2\) is not balanced, which is a contradiction. Hence, \(\Sigma\) does not contain a \(P\) cycle with length \(\neq 4k\) as \((\Sigma)_2\) is balanced. Thus, (a) of (ii) follows.

Next, let there exist a heterogeneous cycle \(C_1: v_1v_2 \ldots v_pv_1, p\) being odd in \(\Sigma\), with a \(P\) section \(v_1v_2, \ldots, v_pv_1, k\) being odd. Clearly, the length of \(P\) section is even. The cycle \(C_1\) in \(\Sigma\) will generate an odd cycle \(C_1^*: v_1v_2, \ldots, v_kv_{k-1}v_1\) in \(\Sigma\). Now, \(v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\) are odd number of negative edges in \(C_1^\prime\) of \((\Sigma)_2\) and also \(v_3v_4, \ldots, v_{k-3}v_{k-2}\) are even number of negative edges in the cycle \(C_1^\prime\) of \((\Sigma)_2\), see (b) in Figure 5, here \(k = 3\). Thus, the cycle \(C_1\) in \((\Sigma)_2\) will be a negative cycle and hence \((\Sigma)_2\) is not balanced, which is a contradiction. Similarly, for odd number of such \(P\) sections, the signed network \((\Sigma)_2\) is unbalanced, whereas if there are even number of such \(P\) sections in the cycle of \(\Sigma\), then, there would be even number of negative edges in the corresponding cycle of \((\Sigma)_2\).

Next, let us consider a heterogeneous even cycle \(C_1: v_1v_2, \ldots, v_pv_1\) in \(\Sigma\), \(p\) even and \(v_1v_2, \ldots, v_1\) be a \(P\) section of odd length \(l\). From Lemma 2, we know that a cycle of even length \(p\) in \(\Sigma\) gives rise to two cycles in \((\Sigma)_2\) of length \(p/2\) each. Now, the \(P\) section \(v_1v_2, \ldots, v_1\) would correspond to \((l-1)/2\) negative edges in both the cycles, clearly as \((\Sigma)_2\) is balanced, that is, each of these cycles have even number of negative edges which is possible if \((l-1)/2\) is an even number or there is another negative edge which is due to some other \(P\) section in the same cycle \(\Sigma\). If \((l-1)/2\) is even or \(l \equiv 1 \pmod{4}\), then we are done, if not then we prove that there are even number of such \(P\) section. Clearly, \(l \equiv 1 \pmod{4}\). Let \(v_0, \ldots, v_s\), where \(i \neq k\) and \(i, k \in \{l+1, l+2, \ldots, p\}\), be another \(P\) section of length \(s\), such that \(s \equiv 1 \pmod{4}\). Then, again as proved above there would be \((s-1)/2\) negative edges in the cycles of \((\Sigma)_2\). Now, as each cycle has odd number of negative edges in \((\Sigma)_2\) due to \(P\) sections \(v_1, \ldots, v_p\) and \(v_1, \ldots, v_k\) in \(\Sigma\), therefore there are now even number of edges in each of the cycles generated by cycle \(C_1\) in \((\Sigma)_2\), which makes these cycles balanced.

Lastly, let \(u\) be a vertex in \(\Sigma\) with degree greater than or equal to 3. If neighborhood \(N_u(\Sigma)\) contains exactly one \(P\) pair, say \(\{v_1, v_2\}\), then there exist at least one vertex \(v_3\) in \(N_u(\Sigma)\) which does not form \(P\) pair with vertices \(v_1, v_2\). Thus, by definition of 2-path, \(v_1v_2v_3\) will be a cycle in \((\Sigma)_2\) with exactly one negative edge \(v_2v_3\). 2-path signed network is not balanced. Similarly, if neighborhood \(N_u(\Sigma)\) contains more than one \(P\) pair and there exist at least one vertex in \(N_u(\Sigma)\) which does not form \(P\) pair, then again there is a cycle of length three with exactly one negative edge. Next, if all the vertices in \(N_u(\Sigma)\) are \(P\) pairs, then any three vertices \(v_1, v_2, v_3\) in \(N_u(\Sigma)\) give rise to a negative cycle of length three in \((\Sigma)_2\) as \(\{v_1, v_2\}, \{v_2, v_3\}\) and \(\{v_1, v_3\}\) are \(P\) pairs in \(N_u(\Sigma)\). Thus, again making \((\Sigma)_2\) unbalanced, which is a contradiction to our hypothesis. Therefore, no neighborhood of a vertex of degree greater than three in \(\Sigma\) contains a \(P\) pair.

(ii) \(\Rightarrow\) (i) Let, if possible, (a), (b), (c), and (d) hold. Let us consider the neighborhood for each vertex \(v_i\) in \(\Sigma\). If the neighborhood consist of a single vertex, then it does not contribute to the edges in \((\Sigma)_2\). Next, if the neighborhood contains two vertices say \(v, v_2\), then either \(v, v_2\) is an edge in \((\Sigma)_2\), which is part of a cycle or a tree. If the edge \(v, v_2\) is part of the tree, then it is balanced by default. Next, if it is part of a cycle in \((\Sigma)_2\), then by Lemma 2 and Lemma 3, we know that this cycle in \((\Sigma)_2\) is due to a cycle in \(\Sigma\). If the corresponding cycle in \(\Sigma\) is homogeneous, then by condition (a) either it is
positive or a P cycle of length 4k, k being odd. A positive cycle in Σ gives rise to a positive cycle in (Σ)_2, thus making the cycle balanced, else a cycle of even length greater than 4 in Σ gives rise to two cycles in (Σ)_2 of equal length. Thus, the cycles formed in (Σ)_2 due to the P cycle in Σ are of even length and thus balanced. If the cycle is heterogeneous in Σ, then the following cases arise:

1. If cycle in Σ is of odd length, then by (b) the cycle either does not contain P section or contains even number of P section of even length. If the cycle does not contain a P section then cycle in (Σ)_2 is also positive hence balanced. Next, if the cycle in Σ contains odd number of P section of even length then the cycle in (Σ)_2 contain even number of negative edges (as an even P section in Σ gives rise to odd negative edges in cycle of (Σ)_2 as the negative even section in Figure 5(b) gives rise to three negative edges 13, 35, and 24 in 2-path signed network). Hence, again giving cycle to a balanced cycle.

2. If cycle in Σ is of even length greater than 4, then by (c) either it has P sections of odd length l such that l ≡ 1 (mod 4) and if l ≠ 1 (mod 4) then there are even number of such P sections each separated by positive section of even length or as P sections of even length then there are even number of such P sections each separated by positive section of even length. In both the cases, even number of negative edges appear in both the cycles of (Σ)_2, thus making these cycles balanced.

Next, if the neighborhood of v_i for some i = 1 to n contain more than two vertices, they give rise to a clique in (Σ)_2. Now, the negative edges of these cliques are due to the P pairs in each neighborhood but from (d) no neighborhood contains a P pair; thus, all the cliques are positive. Therefore, by virtue of the conditions, all cycles (and cliques) in (Σ)_2 will be positive, and thus by Lemma 4, (Σ)_2 will be balanced.

4. Clusterability

4.1. Clusterability in 2-Path Signed Networks. In this section, we discuss the clusterability of a 2-path signed network.

Lemma 5 (see [55]). A signed network Σ is clusterable if and only if Σ contains no cycle with exactly one negative edge.

**Theorem 3.** For a given signed network Σ of order n, the following conditions are equivalent:

1. (Σ)_2 is clusterable.
2. For all sequence of vertices v_1, v_2, . . . , v_r; 1 ≤ r ≤ n in Σ such that v_1, v_2 ∈ N(t_1); v_2, v_3 ∈ N(t_2); . . . ; v_r, v_t ∈ N(t_r) for some t_1, t_2, . . . , t_r ∈ V(Σ). If there exist a pair of vertices in sequence v_i, v_i+1 ∈ N(t_l) having property P, then the sequence has at least one pair of vertices v_i, v_i+1 ∈ N(t_l), l ≠ i satisfying property P, for some l, 1 ≤ l ≤ r.
3. (a) If Σ contains a heterogeneous cycle, then no even cycle in Σ contains exactly one P section of length <5 and no odd cycle contains exactly one P section of length 2.
   (b) Each neighborhood of vertex v in Σ with d(v) ≥ 3 either does not contain a P pair or all vertices are P pairs.

Proof. (i)⇒(ii) Let for a given signed network Σ, (Σ)_2 be clusterable. Then, no cycle in (Σ)_2 has exactly one negative edge. Let v_1, v_2, . . . , v_r; 1 ≤ r ≤ n be a sequence of vertices in Σ such that v_1, v_2 ∈ N(t_1); v_2, v_3 ∈ N(t_2); . . . ; v_r, v_t ∈ N(t_r) for some t_1, t_2, . . . , t_r ∈ V(Σ). Let v_i, v_i+1 ∈ N(t_l) be a pair in sequence v_i, v_i+1, v_t for some r, 1 ≤ r ≤ n having property P. Now, C: v_i v_2 . . . v_t v_i is a cycle in (Σ)_2, due to the sequence N(t_1), . . . , N(t_r) in Σ. Clearly, v_i, v_i+1 will form a single negative edge in cycle C in (Σ)_2, which is not possible. Thus, there is at least one more pair of vertices v_i, v_i+1 ∈ N(t_l), l ≠ i satisfying property P.

(ii)⇒(i) Let v_1, v_2, . . . , v_r; 1 ≤ r ≤ n be an arbitrary sequence of vertices in Σ such that v_1, v_2 ∈ N(t_1); v_2, v_3 ∈ N(t_2); . . . ; v_r, v_t ∈ N(t_r) for some t_1, t_2, . . . , t_r ∈ V(Σ). If there exist a pair of vertices v_i, v_i+1 ∈ N(t_l) for some i having property P, then the sequence has at least one other pair of vertices v_i, v_i+1 ∈ N(t_l) for some l ∈ {1, . . . , r} satisfying property P. This sequence of vertices generates cycles in (Σ)_2 such that no cycle has exactly one negative edge. Thus, by Lemma 5, (Σ)_2 is clusterable.

(i)⇒(iii) Let (Σ)_2 be clusterable. To prove that (iii)(a) and (iii)(b) hold, let, if possible, (a) does not hold. This implies, there exist an even heterogeneous cycle C in Σ of length 2k, with P section of length <5. C will give rise to two
cycles $C_1$ and $C_2$ of length $k$ each in $(\Sigma)_2$, and clearly, at least one of the cycles will contain exactly one negative edge (see Figure 6), which is a contradiction to the hypothesis by Lemma 5. Next, if there exist an odd heterogeneous cycle with $P$ section of length 2, then it will correspond to a single negative edge in the cycle of $(\Sigma)_2$, which is again not possible. Hence, (iii) (a) and (iii) (b) hold.

Next, we assume that (b) in (iii) does not hold. Let $N_+(u)$ be a neighborhood of a vertex $u$ in $\Sigma$ with $d(u) \geq 3$ and containing at least one $P$ pair. Let $v_1$ be a vertex in $N_+(u)$, which does not form a $P$ pair with any other vertex in $N_+(u)$. Clearly, as $d(u) \geq 3$, $\exists v_2, v_3$ such that $\{v_2, v_3\}$ is a $P$ pair. The 2-path signed network $(\Sigma)_2$, will now contain a cycle $v_1v_2v_3v_1$ with exactly one negative edge $v_2v_3$, since each neighborhood $N(u)$ gives rise to clique $\delta(N(u))$. By Lemma 5, it is a contradiction to the hypothesis, whereas if all vertices in $N_+(u)$ form $P$ pairs, then all the edges in the corresponding clique will be negative, thus $(\Sigma)_2$ remains clusterable.

(iii) $\implies$ (i) Assume that condition (a) and (b) in (iii) hold. We have to show that $(\Sigma)_2$ is clusterable. By Theorem 1, we know that $(\Sigma)_2$ is obtained by taking the union of cliques generated by the neighborhood of vertices of $\Sigma$. Thus, each cycle in $(\Sigma)_2$ is either due to cliques generated by $N(t)$ for $t \in V(\Sigma)$ such that $|N(t)| \geq 3$ or due to induced cycles in $\Sigma$. Now, by condition (a), no heterogeneous even cycle in $\Sigma$ contains exactly one $P$ section of length $<5$. Thus, each cycle formed in $(\Sigma)_2$ has either no negative edge or has more than one negative edge. Also, if no odd cycle in $\Sigma$ contains exactly one $P$ section of length 2, then its corresponding cycle in $(\Sigma)_2$ does not contain exactly one negative edge. From (b), it is clear that no clique in $(\Sigma)_2$ contains exactly one negative edge as the clique formed here is homogeneous; hence, by Lemma 5 $(\Sigma)_2$ is clusterable.

5. Sign-Regularity

5.1. Property of Sign-Regularity in 2-Path Signed Networks. In this section, we establish a characterization of a sign-regular 2-path signed network. Note that

$$\rho_{v_i} = \bigcup_{v_j \in V(\Sigma)} \{N(t): v_i \in V(\Sigma)\},$$

$$|v_i| = \left|\left\{v_i, v_j\right\}: v_j \in V(\Sigma)\right|.$$  (5)

Theorem 4. For a signed network $\Sigma$ of order $n$, $(\Sigma)_2$ is sign-regular if and only if

(i) For all vertices $v_i, v_j \in V(\Sigma)$ such that $v_i \in N(v_j)$, $\rho_{v_i}$ is identical for every $v_i \in V(\Sigma)$

(ii) If $P_*$ is collection of all $P$ pairs, then $|v_i|$ is identical for all $1 \leq i \leq n$

Proof. Necessity. Let for a given signed network $\Sigma$, $(\Sigma)_2$ be sign-regular. Then, number of positive and negative edges incident to each vertex in $(\Sigma)_2$ is identical. Since $(\Sigma)_2$ is obtained by taking the union of cliques generated by the neighborhood of vertices of $\Sigma$, thus the total number of vertices in each neighborhood containing $v_i$ is the same for each vertex $v_i \in V(\Sigma)$. Also, $\rho_{v_i}$ gives the total number of
Input: Adjacency matrix $A$ of $\Sigma$, number of vertices $n$ and array $P$ from Algorithm 1.

Output: Whether the 2-path of a given network is sign-regular or not.

Process:
1. Enter the adjacency matrix $A$ with elements $A(i,j)$ for a given signed network $\Sigma$ along with the order $n$ of signed network $\Sigma$.
2. $h = 1$;
3. for $i = 1$ to $n$ do
   4. $\text{count}[i] = 0$;
   5. $\text{count1}[i] = 0$;
   6. $\text{countrow}[i] = 0$;
   7. for $i = 1$ to $n$ do
      8. for $j = 1$ to $n$ do
         9. if $(A(i,j) \neq 0)$ then
            10. $\text{countrow}[i] = \text{countrow}[i] + 1$;
      11. for $i = 0$ to $n$ do
         12. for $j = 1$ to $n$ do
            13. if $(A(i,j) \neq 0)$ then
               14. $\text{count}[j] = \text{count}[j] + \text{countrow}[i] - 1$;
      15. for $i = 1$ to $n$ do
         16. for $j = 1$ to $n$ do
            17. if $(\text{count}[i] \neq \text{count}[j])$ then
               18. Print The 2-path network is not regular.
               19. $h = 0$; break;
      20. for $i = 1$ to $n$ do
         21. for $j = 1$ to $n$ do
            22. if $(i = P[j])$ do
               23. $\text{count1}[i] = \text{count1}[i] + 1$;
      24. for $i = 1$ to $n$ do
         25. for $j = 1$ to $n$ do
            26. if $\text{count1}[i] \neq \text{count1}[j]$ then
               27. Print "The two path network is not regular."
               28. $h = 0$;
               29. break;
      30. if $(h \neq 0)$ then
         31. Print "The network is sign-regular."

Algorithm 3: Algorithm to check sign-regularity of 2-path signed network.

edges incident to a vertex $v_i$, $i \in \{1, \ldots, n\}$. Hence, $\rho_v$ is identical $V_N \in V(\Sigma)_2$. Therefore, (i) holds. Since $P_*$ consists of all pair of vertices satisfying property $P$, hence number of negative edges incident to each vertex in $(\Sigma)_2$ must be equal. Thus, the vertex $v_i$ appearing in the number of $P$ pairs in $P_*$, $|v_i|$ is the same for every $i \leq i \leq n$.

Sufficiency. Let (i) and (ii) hold. Now, (i) suggests that, for all the vertices $v_i$ in $V(\Sigma)$ such that $v_i \in N(v_j)$, the union of all these neighborhoods which gives the total vertices adjacent to $v_i$ in $(\Sigma)_2$ have the same cardinality. Thus, each vertex in $(\Sigma)_2$ is adjacent to the same number of vertices. Next, we know that the elements of $P_*$ generate all the negative edges of $(\Sigma)_2$ and $|v_i|$ gives the number of negative edges incident to $v_i$ in $(\Sigma)_2$. By (ii), the cardinality $|v_i|$ is the same for each vertex $v_i, i = 1, \ldots, n$. Thus, the number of negative edges incident to $v_i$ is the same for each $i = 1$ to $n$. Since the number of total edges and negative edges incident to vertex $v_i$ is the same for each $i$, the number of positive edges will also be same. Therefore, $(\Sigma)_2$ is sign-regular.

5.2. Algorithm to Detect if 2-Path Signed Network of a Signed Network Is Sign-Regular. Algorithm 3 detects if, for a given signed network $\Sigma$, its 2-path $(\Sigma)_2$ is sign-regular, by using results of Theorem 4. Consider the adjacency matrix $A$ and its order $n$ as input. The vector $\text{countrow}$ gives the number of nonzero elements in each row (degree of the vertices). The array $\text{count}$ gives the number of edges for each vertex $v$ present in some neighborhood of a vertex $u$. Also, vector $\text{count1}$ counts the number of negative edges in $(\Sigma)_2$ for each vertex $u$.

5.2.1. Complexity. From Steps 3 to 6, initialization of the vector arrays uses a single loop which runs up to $n$. Thus, the complexity of these steps = $O(n)$.

In Steps 7 and 8, two loops are used up to Step 10. Also, at Steps 11, 15, and 24, again two loops running up to $n$ are used independent of each other. Thus, they have combined complexity = $O(n^2)$.

In Step 21, the $P$ pairs are collected, and we know that the complexity of this step is of order $n^3$. Finally, the two loops in
Steps 20 and 21 go up to \( n \) and \( l \), respectively. Thus, the complexity of this step is \( O(nl) \). Total complexity is \( O(n) + O(n^2) + O(n^3) + O(nl) = O(n^3) \).}

5.2.2. Implementation of Algorithm

**Example 3.** In this example, we implement Algorithm 3, for a signed network, as shown Figure 4(a). We want to check whether, for a given signed network, its corresponding 2-path signed network is sign-regular. This is done by the adjacency matrix (as in Example 1) of the given signed network \( \Sigma \). Now, in the given signed network, \( n = 4 \), so the vector arrays \( \text{count} \) and \( \text{count}1 \) are initialized as zero along with variable \( h \) which is initialized as 1. After we enter the loop at Step 7 and Step 8, we fetch each nonzero value of the matrix. For \( i = 1 \) and \( j = 2 \), \( \text{count}[2] = 1 \) (as 2 and 3 are in the neighborhood of 1 and \( \text{count}[2] = 0 + 2 - 1 \)) and \( \text{count}[3] = 1 \). Next, for \( i = 2 \), \( \text{count}[1] = 0 + 3 - 1 = 2 \), \( \text{count}[3] = 3 \) and \( \text{count}[4] = 2 \). Similarly, moving further, when \( i = 3 \), we get \( \text{count}[1] = 2 + 3 - 1 = 4 \), \( \text{count}[2] = 1 + 3 - 1 = 3 \), and \( \text{count}[4] = 2 + 3 - 1 = 4 \). For \( i = 4 \), \( \text{count}[2] = 4 \) and \( \text{count}[3] = 4 \). In Step 14, we check that if there exist \( i, j \), such that \( \text{count}[1][i] \neq \text{count}[1][j] \), but since this is not true we move to Step 20 and collect all \( P \) pairs. Next, in Step 21, we check whether the number of negative edges incident to each vertex is identical for all the vertices. We count the number of appearance of each vertex in array \( P \) of Algorithm 1; this is done by array \( \text{count}1 \) as \( P \) is the following:

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
\end{array}
\]

Thus, \( \text{count}[1][1] = 2 \), \( \text{count}[2] = 1 \), \( \text{count}[3] = 1 \), and \( \text{count}[4] = 0 \). Next, in Step 26, we check these entries of array \( \text{count}1 \) and initially find that \( \text{count}[1][1] \neq \text{count}[2] \). Hence, the number of negative edges incident to vertex 1 and 2 is not the same; thus, the 2-path signed network is not sign-regular. The same is clear from Figure 4(b), hence the proof.

6. Conclusion

In this paper, we studied various characterizations of 2-path signed networks and other related properties such as balancedness, clusterability, and sign-regularity. We designed a model using the 2-path signed networks on radio frequency interference. We assumed that the vertices represented channel/stations and transmission between them was represented by edges. The negative sign in signed network \( \Sigma \) was given to the edge \( uv \) if the transmission from \( u \) and \( v \) takes place at the same time and frequency, otherwise \( uv \) was given a positive sign. The paper explored both theoretical and applicable aspect of 2-path signed networks. In our work, we not only focused on the characterization and the results due to the signing but also the algorithms which can be readily used in real world problems.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

All the authors declare that they have no conflicts of interest regarding the publication of this paper.

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