Research Article

Relaxation Oscillations and Dynamical Properties in Two Time-Delay Slow-Fast Modified Leslie-Gower Models

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In this paper, we consider two kinds of time-delay slow-fast modified Leslie-Gower models. For the first system, we prove the existence and uniqueness of relaxation oscillation cycle through the geometric singular perturbation theory and entry-exit function. For the second system, we put forward a conjecture that the relaxation oscillation of the system is unique. Numerical simulation also verifies our results for the systems.

1. Introduction

In 2003, Alaoui and Okiye proposed the next modified Leslie-Gower model:

$$\begin{aligned}\frac{dx}{dT} &= \left(r_1 - b_1 x - \frac{a_1 y}{x + k_1} \right) x, \\ \frac{dy}{dT} &= \left(r_2 - \frac{a_2 y}{x + k_2} \right) y,\end{aligned}\quad (1)$$

where x is the density of a prey, y is the density of a predator, and the parameters $r_1, k_1, a_1, b_1, r_2, k_2$, and a_2 are positive, which play biology roles.

Consider the rescaling,

$$\begin{aligned}x &= \frac{r_1}{b_1} \bar{x}, \\ y &= \frac{r_1 r_2}{a_2 b_1} \bar{y}, \\ T &= \frac{1}{r_1} t.\end{aligned}\quad (2)$$

Substitute (2) into (1), and we have $\bar{x}$ and $\bar{y}$ in terms of x and y , then we have

$$\begin{aligned}\frac{dx}{dt} &= x' = x(1-x) - \frac{axy}{x+e_1}, \\ \frac{dy}{dt} &= y' = \varepsilon y \left(1 - \frac{y}{x+e_2} \right),\end{aligned}\quad (3)$$

where $a = (a_1 r_2 / a_2 r_1)$, $e_1 = (b_1 k_1 / r_1)$, $e_2 = (b_1 k_2 / r_1)$, and $\varepsilon = (r_2 / r_1)$. The parameter ε can be regarded as a small parameter in some special models, such as prey hares and predator lynx.

Consider the case where ε is small enough, system (3) could be treated as a slow-fast system which has the fast variable x and the slow one y . Applying the geometric singular perturbation theory, we could study the slow-fast systems [4, 5].

The geometric singular perturbation theory is mathematically rigorous that can analyze dynamics of some slow-fast systems. The study of invariant manifolds in Fenichel's theory [3] is the basis of geometric singular perturbation theory, and the theory guarantees that any compact normally hyperbolic submanifold So of the critical manifold Co could perturb a locally invariant manifold S_ε ($0 < \varepsilon \ll 1$), which is $O(\varepsilon)$, close to So . In recent years, geometric singular perturbation theory contains a fairly broad class of geometric points used to study the slow-fast systems, see, for example, [6–12]. And, for the exchange lemma, see [13, 14].

Recently, Wang and Zhang proved the existence and uniqueness of relaxation oscillation cycle of the slow-fast modified Leslie-Gower model [15]. Valery et al. studied the global dynamics in the Leslie-Gower model with the Allee Effect [16]. Du et al. considered two delays induced Hopf bifurcation and double Hopf bifurcation in a diffusive Leslie-Gower predator-prey system [17]. Karl et al. discussed the bifurcation of critical sets and relaxation oscillations in singular fast-slow systems [18]. Wang and Zhang considered the stability loss delay and smoothness of the return map in slow-fast systems [19]. Ambrosio et al. addressed the canard phenomenon in a slow-fast modified Leslie-Gower model [20]. Xia et al. discussed relaxation oscillations and the mechanism in a periodically excited vector field with pitchfork-Hopf bifurcation [21]. Atabaigi and Barati studied relaxation oscillations and canard explosion in a predator-prey system of Holling and Leslie types [22]. Ai and Sadhu considered the entry-exit theorem and relaxation oscillations in slow-fast planar systems [23].

The rest of the paper is organized as follows: Section 2 introduces the entry-exit function theory. Section 3 studies the first time-delay slow-fast modified Leslie-Gower model. Section 4 investigates another kind of time-delay slow-fast modified Leslie-Gower model.

2. Entry-Exit Function

We are going to consider a slow-fast vector field in the form of

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, \varepsilon)x, \\ \frac{dy}{dt} &= \varepsilon g(x, y, \varepsilon),\end{aligned}\quad (4)$$

where $(x, y) \in \mathbb{R}^2$ are state-space variables and the parameter ε represents the ratio of time scales. We define the functions f and g satisfying

$$\begin{aligned}f(0, y, 0) &< 0, \quad \text{for } y > 0, \\ f(0, y, 0) &> 0, \quad \text{for } y < 0, \\ g(0, y, 0) &< 0.\end{aligned}\quad (5)$$

For system (4), the y -axis consists of equilibria when $\varepsilon = 0$ and $Y^+ := \{(x, y) \mid x = 0, y > 0\}$ attracting and $Y^- := \{(x, y) \mid x = 0, y < 0\}$ repelling (Figure 1(a)). The Figure 1(b) is the case for $\varepsilon > 0$, since the orbit of system (4) is attracted by the y -axis, the orbit starting at (x_0, y_0) with $y_0 > 0$ and $x_0 > 0$ slightly moves toward the downside at the speed of ε . After the orbit passes positive x -axis, the y -axis begins repelling, then the orbit tends to move away from the negative y -axis at the point $(0, p_0(y_0))$, and then it intersects the line $x = x_0$ at the point whose y -coordinate saying $p_\varepsilon(y_0)$ satisfies $\lim_{\varepsilon \rightarrow 0} p_\varepsilon(y_0) = p_0(y_0)$, where $p_0(y_0)$ is determined by

$$\int_{y_0}^{p_0(y_0)} \frac{f(0, y, 0)}{g(0, y, 0)} dy = 0. \quad (6)$$

3. System with Two Positive Time Delays

Consider a system with time delays τ_1 and τ_2 , which are sufficiently small constants:

$$\begin{aligned}\frac{dx}{dt} &= x' = x(1-x) - \frac{ay}{x+e_1}x(t-\tau_1), \\ \frac{dy}{dt} &= y' = \varepsilon y \left(1 - \frac{y(t-\tau_2)}{x+e_2}\right).\end{aligned}\quad (7)$$

Using the Taylor formula,

$$\begin{aligned}x(t-\tau_1) &\approx x(t) - x'(t)\tau_1, \\ y(t-\tau_2) &\approx y(t) - y'(t)\tau_2.\end{aligned}\quad (8)$$

Substitute (8) into system (7), it yields

$$\begin{aligned}\frac{dx}{dt} &= x' = \left(x(1-x) - \frac{axy}{x+e_1}\right) \frac{x+e_1}{x+e_1-ay\tau_1} \\ &:= xf(x, y, \varepsilon), \\ \frac{dy}{dt} &= y' = \varepsilon y \left(1 - \frac{y}{x+e_2}\right) \frac{x+e_2}{x+e_2-\varepsilon y\tau_2} := \varepsilon g(x, y, \varepsilon).\end{aligned}\quad (9)$$

3.1. Equilibria

Lemma 1. For system (9), we have the following result: system (9) and system (3) have the same equilibria. So, that system (9) has equilibria $E_1(0, 0)$, $E_2(0, e_2)$, and $E_3(1, 0)$.

Moreover, if $ae_2 - e_1 < 0$, system (9) has a unique positive equilibrium $E_*(x_*, y_*)$ with $x_* = ((-(a+e_1-1) + \sqrt{(a+e_1-1)^2 - 4(ae_2-e_1)})/2)$, $y_* = x_* + e_2$.

3.2. Hopf Bifurcation. We consider the bifurcation of system (9) at the unique positive equilibrium $E_*(x_*, y_*)$ about parameter τ_1 . For convenience, we denote

$$\begin{aligned}F(x, y) &= x \left((1-x) - \frac{axy}{x+e_1} \right) \frac{x+e_1}{x+e_1-ay\tau_1}, \\ G(x, y) &= \varepsilon y \left(1 - \frac{y}{x+e_2} \right) \frac{x+e_2}{x+e_2-\varepsilon y\tau_2},\end{aligned}\quad (10)$$

so the linearized system of system (9) at the unique positive equilibrium $E_*(x_*, y_*)$ could be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (11)$$

with

FIGURE 1: (a) For $\varepsilon = 0$, the orbits of system (4) is shown and the y -axis consists of equilibria. (b) For $0 < \varepsilon \ll 1$, a symbolic orbit of system (4) starts at (x_0, y_0) and ends at $\{x = x_0, y_0 < 0\}$.

$$\begin{aligned} F_x &= \frac{-2x_*^2 - (e_1 - 1)x_*}{x_* + e_1 - ay_*\tau_1}, \\ F_y &= \frac{-ax_*}{x_* + e_1 - ay_*\tau_1}, \\ G_x &= \frac{\varepsilon(x_* + e_2)}{x_* + e_2 - \varepsilon y_*\tau_2}, \\ G_y &= \frac{\varepsilon - 2}{1 - \varepsilon\tau_2}. \end{aligned} \quad (12)$$

Then, the characteristic equation of system (11) is

$$\lambda^2 - (F_x + G_y)\lambda - G_x F_y = 0. \quad (13)$$

Lemma 2. For equation (13), we have the following result. If $\tau_1 = (1 - \varepsilon\tau_2)/((2 - \varepsilon)ae_2)(1 + (2 - \varepsilon/(1 - \varepsilon\tau_2))(1 - a) - e_1 - 2a)x_* + (e_1/ae_2)$, then equation (13) has a pair of pure imaginary roots.

Proof. The condition of equation (13) has a pair of pure imaginary roots:

$$F_x + G_y = 0, \quad G_x F_y < 0. \quad (14)$$

Consider ε, τ_1 , and τ_2 are sufficiently small constants. We can easily get $G_x > 0, F_y < 0$, so $G_x F_y < 0$ holds.

According to the equation $F_x + G_y = 0$, we reach

$$\frac{-2x_*^2 - (e_1 - 1)x_*}{x_* + e_1 - ay_*\tau_1} + \frac{\varepsilon - 2}{1 - \varepsilon\tau_2} = 0. \quad (15)$$

Taking $y_* = x_* + e_2$ into equation (15), then we get

$$\frac{2x_*^2 + (e_1 - 1)x_*}{x_* + e_1 - a(x_* + e_2)\tau_1} = \frac{\varepsilon - 2}{1 - \varepsilon\tau_2}, \quad (16)$$

i.e.,

$$\tau_1 = \frac{1 - \varepsilon\tau_2}{(2 - \varepsilon)ae_2} \left(1 + \frac{2 - \varepsilon}{1 - \varepsilon\tau_2} (1 - a) - e_1 - 2a \right) x_* + \frac{e_1}{ae_2}. \quad (17)$$

The proof is completed.

Let $\lambda(\tau) = p(\tau) \pm iq(\tau)$ be the pair of pure imaginary roots of equation (13) that are satisfying $p(\tau_1) = 0, q(\tau_1) \neq 0$. Then the following transversality condition holds. $\square$

Lemma 3. If $a \neq \sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)}$ holds, then $\lambda'(\tau_1) \neq 0$.

Proof. The condition of equation (13) has a pair of pure imaginary roots:

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda}{d\tau_1}\right) &= \frac{F_x + G_y}{2a} = \frac{1}{2a} \frac{dF_x}{d\tau_1} = \frac{x_*}{(x_* + e_1 - ay_*\tau_1)^2} \\ &\cdot (2x_* + e_1 - 1) = \frac{x_*}{(x_* + e_1 - ay_*\tau_1)^2} \\ &\cdot \left(-a + \sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} \right) \neq 0. \end{aligned} \quad (18)$$

The proof is completed.

Denote $\tau_1' = ((1 - \varepsilon\tau_2)/((2 - \varepsilon)ae_2))(1 + (2 - \varepsilon/(1 - \varepsilon\tau_2))(1 - a) - e_1 - 2a)x_* + (e_1/ae_2)$. Combining Lemma 2 and Lemma 3, we conclude the following. $\square$

Theorem 1. For system (9), if $\tau_1 = \tau_1'$ and $a \neq \sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)}$, then the equilibrium $E_*(x_*, y_*)$ is a Hopf bifurcation point and a limit cycle occurs.

3.3. Dynamics of Limit Systems. Consider the limit systems of system (9). Setting $\varepsilon = 0$ in (9), we reach the fast subsystem:

$$\begin{aligned}\frac{dx}{dt} &= x' = \left(x(1-x) - \frac{axy}{x+e_1} \right) \frac{x+e_1}{x+e_1-ay\tau_1}, \\ \frac{dy}{dt} &= y' = 0.\end{aligned}\quad (19)$$

Consider the slow time scale $\tau = \varepsilon t$ and take the singular again, we get the slow subsystem:

$$\begin{aligned}0 &= \left(x(1-x) - \frac{axy}{x+e_1} \right) \frac{x+e_1}{x+e_1-ay\tau_1}, \\ \frac{dy}{d\tau} &= \dot{y} = y \left(1 - \frac{y}{x+e_2} \right) \frac{x+e_2}{x+e_2-\varepsilon y\tau_2},\end{aligned}\quad (20)$$

which is a differential-algebraic equation on the critical set:

$$C_0 = \left\{ (x, y) \mid x = 0 \text{ or } y = \frac{(1-x)(x+e_1)}{a} := F(x) \right\}. \quad (21)$$

We restrict the parameters a, e_1 , and e_2 to a subset

$$U = \left\{ (a, e_1, e_2) \mid \sqrt{(a+e_1-1)^2 - 4(ae_2-e_1)} < a < \frac{e_1}{e_2} \right\}. \quad (22)$$

Under this restriction, $E_*(x_*, y_*)$ and $E_2(0, e_2)$ are located in the exclusion part of C_0 . Some studies have shown that the branch $\{(x, y) \mid y = F(x), x > 0, y > 0\}$ of C_0 has a unique generic fold point $D(x_M, y_M) = ((1-e_1/2), F(1-e_1/2))$. And, the two branches of C_0 cross transversally at the transcritical point $E(x, y) = (0, (e_1/a))$ (Figure 2). Then, the critical set C_0 is divided into four parts by points D and E :

$$\begin{aligned}C_0^a &= \left\{ (x, y) \mid \frac{(1-e_1)}{2} < x < 1, y = F(x) \right\}, C_0^+ \\ &= \left\{ (x, y) \mid x = 0, y > \frac{e_1}{a} \right\}, \\ C_0^r &= \left\{ (x, y) \mid 0 < x < \frac{(1-e_1)}{2}, y = F(x) \right\}, C_0^- \\ &= \left\{ (x, y) \mid x = 0, 0 < y < \frac{e_1}{a} \right\},\end{aligned}\quad (23)$$

where C_0^r and C_0^- are normally hyperbolic repelling and C_0^a and C_0^+ are normally hyperbolic attracting.

In fact, when the parameter is limited to U , $E_*(x_*, y_*)$ is on C_0^r and $E_2(0, e_2)$ is on C_0^- .

3.4. Relaxation Oscillation

Lemma 4. For system (9), there exists a unique $\bar{y}^* (e_2 < \bar{y}^* < (e_1/a))$ such that,

$$\int_{\bar{y}^*}^{y_M} \frac{f(0, y, 0)}{g(0, y, 0)} dy = 0. \quad (24)$$

Proof. Considering the system (9), for $\bar{y} \in (e_2, (e_1/a))$,

$$I(\bar{y}) = \int_{\bar{y}}^{y_M} \frac{f(0, y, 0)}{g(0, y, 0)} dy = \int_{\bar{y}}^{y_M} \frac{1 - (ay/e_1)}{y(1 - (y/e_2))} \cdot \frac{e_1}{e_1 - ay\tau_1} dy. \quad (25)$$

So that

$$I(\bar{y}) \longrightarrow -\infty \text{ as } \bar{y} \longrightarrow e_2^+. \quad (26)$$

Furthermore

$$\begin{aligned}I\left(\frac{e_1}{a}\right) &= \int_{(e_1/a)}^{y_M} \frac{f(0, y, 0)}{g(0, y, 0)} dy \\ &= \int_{(e_1/a)}^{y_M} \frac{1 - (ay/e_1)}{y(1 - (y/e_2))} \cdot \frac{e_1}{e_1 - ay\tau_1} dy > 0.\end{aligned}\quad (27)$$

Then, we combine (25) and (27) and conclude that there exist a unique $\bar{y}^* (e_2 < \bar{y}^* < (e_1/a))$ such that,

$$\int_{\bar{y}^*}^{y_M} \frac{f(0, y, 0)}{g(0, y, 0)} dy = 0. \quad (28)$$

The proof is completed.

Let us define x_r to be the x -coordinate of the intersection point of $y = \bar{y}^*$ and C_0^a . Then, define a singular slow-fast cycle γ_0 , which contains two slow segments on C_0^a from $(x_r, \bar{y}^*)$ to (x_M, y_M) and on the positive y -axis from $(0, y_M)$ to $(0, \bar{y}^*)$ and the two fast connections from (x_M, y_M) to $(0, y_M)$ and $(0, \bar{y}^*)$ to $(x_r, \bar{y}^*)$, respectively. The next theorem explains the existence and uniqueness of the relaxation oscillation. $\square$

Theorem 2. For system (9), restrict the parameters to subset U and let V be a tubular neighborhood of γ_0 . Then, for each fixed $\varepsilon > 0$ sufficiently small, system (9) has a unique limit cycle $\gamma_\varepsilon \in V$, which is strongly attracting. Moreover, the cycle γ_ε is the unique limit cycle that converges to γ_0 in the Hausdorff distance [3] as $\varepsilon \rightarrow 0$.

Proof. By Fenichel's theory, the critical submanifold C_0^a perturbs to a nearby slow manifold C_ε^a , which is $O(\varepsilon)$ near C_0^a . By Theorem 2.1 of [9] on the analysis of a jump point, the slow manifold C_ε^a can be continued and passes the fold point $D(x_M, y_M)$, and then it jumps to another attracting branch C_0^+ .

Let x_l be a sufficiently small positive number. Define two vertical sections Δ^{in} and Δ^{out} as shown in Figure 2:

$$\Delta^{\text{in}} := \{(x_l, y) \in V \mid y \in I_{\text{in}}\}, \quad \Delta^{\text{out}} := \{(x_l, y) \in V \mid y \in I_{\text{out}}\}, \quad (29)$$

where I_{in} and I_{out} are closed intervals centered at y_M and $y_{\text{out}} = p_0(y_M)$, respectively. Through the flow of system (4), we define the transition map $\Pi: \Delta^{\text{in}} \rightarrow \Delta^{\text{in}}$, which is a composition of the next two maps:

$$\Pi_1: \Delta^{\text{in}} \rightarrow \Delta^{\text{out}}, \quad \Pi_2: \Delta^{\text{out}} \rightarrow \Delta^{\text{in}}. \quad (30)$$

Then, $\Pi: \Delta^{\text{in}} \rightarrow \Delta^{\text{in}}$ is given by the composition $\Pi = \Pi_2 \circ \Pi_1$.

FIGURE 2: Illustration for the relaxation oscillations of system (9). The critical manifold C_0 contains repelling part C_0^r and attracting part C_0^a . The dashed black line is the predator isocline, the red dot is the fold point $D(x_M, y_M)$, the solid blue line is the slow-fast cycle γ_0 , the double and single arrows are fast and slow flows, and the solid red is the attracting relaxation oscillation γ_ε .

Now, we analyze the properties of these two maps Π_1 and Π_2 .

- (a) *Analysis of Π_1 .* Use the same proof as Lemma 4, for each $y_0 \in I_{in}$, we can define $p_0(y_0)$, with $e_2 < p_0(y_0) < (e_1/a)$ through the following formula:

$$\int_{p_0(y_0)}^{y_0} \frac{f(0, y, 0)}{g(0, y, 0)} dy = 0. \quad (31)$$

- (b) *Analysis of Π_2 .* Consider two orbits γ_ε^1 and γ_ε^2 starting on Δ^{out} , from Fenichel's theory, γ_ε^1 and γ_ε^2 will be attracted to C_ε^a at the exponential rate $O(e^{-1/\varepsilon})$. By Theorem 2.1 of [9], γ_ε^1 and γ_ε^2 pass by the generic fold point $D(x_M, y_M)$ contracting exponentially toward each other. Then, they fly to Δ^{in} .

Therefore, according to the results of (a) and (b), we obtain that the transition map $\Pi: \Delta^{in} \rightarrow \Delta^{in}$ is a contraction at the exponential rate $O(e^{-1/\varepsilon})$. Then, it follows from the contraction mapping theorem that Π has a unique fixed point in Δ^{in} , which must be stable. This fixed point provides a unique relaxation oscillation cycle $\gamma_\varepsilon \subset V$ of system (9) passing Δ^{in} for each $0 < \varepsilon \ll 1$.

According to Fenichel's theory and Theorem 2.1 of [9], we obtain that the relaxation oscillation cycle γ_ε converges to the slow-fast cycle γ_0 as $\varepsilon \rightarrow 0$ in the Hausdorff distance.

So, we can get that the relaxation oscillation cycle γ_ε is a unique limit cycle of system (9) located in V for each $0 < \varepsilon \ll 1$.

The theorem is proved. $\square$

3.5. Numerical Simulation. First of all, we provide an example to illustrate Theorem 2.

Select the parameter values $(a, e_1, e_2, \varepsilon, \tau_1, \tau_2) = (1, 0.2, 0.022, 0.01, 0.2, 0.2)$, and it is easy to state that the system (9) has a unique positive equilibrium point $E_*(0.33, 0.36)$. Numerical simulation shows that it has a unique limit cycle. Figure 3 shows the phase portrait and the time series of the relaxation oscillation cycle.

- (a) The orbit is shown in red in phase space
- (b) Time series with the black dashed curve (prey) and red full curve (predator)

In addition, we compare the phase portrait and time series of system (9) with system (7). Figure 4 presents the phase portrait of system (9) and system (7). We can see that the two curves are very close to each other. Figure 5 shows the time series of system (9) and system (7).

- (a) Time series for the relaxation oscillation of the predator with the red dashed curve (system (7)) and the blue full curve (system (9))
- (b) Time series for the relaxation oscillation of the prey with the blue dashed curve (system (7)) and the red full curve (system (9)). Time series for the relaxation oscillation of prey with the blue dashed curve (system (4)) and the red full curve (system (6)).

4. System with Two Time Delays

In this section, we consider time delays τ_1 and τ_2 in system (3), then we have

$$\begin{aligned} \frac{dx}{dt} &= x' = x(1-x) - \frac{ax}{x+e_1}y(t-\tau_1), \\ \frac{dy}{dt} &= y' = \varepsilon y \left(1 - \frac{y}{x(t-\tau_2)+e_2} \right), \end{aligned} \quad (32)$$

where τ_1 stands for delayed maturation of the predator and τ_2 stands for the time needed to digest the prey.

4.1. Equilibria and Hopf Bifurcation. System (32) has the same equilibria with system (9). So, system (32) has a unique positive equilibrium $E_*(x_*, y_*)$ with $x_* = ((-(a+e_1-1)+\sqrt{(a+e_1-1)^2-4(ae_2-e_1)})/2)$, $y_* = x_*+e_2$.

In order to get the conditions for Hopf bifurcation, we reach the linearized system of system (32) at the equilibrium $E_*(x_*, y_*)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + B \begin{pmatrix} x(t-\tau_1) \\ y(t-\tau_1) \end{pmatrix} + C \begin{pmatrix} x(t-\tau_2) \\ y(t-\tau_2) \end{pmatrix}, \quad (33)$$

where

FIGURE 3: Numerical simulation of system (9).

FIGURE 4: Phase space: the dashed blue curve shows the orbit of system (9) and the red full curve shows the orbit of system (7).

$$\begin{aligned}
 A &= \begin{pmatrix} 1 - 2x_* - \frac{ae_1}{(x+e_1)^2} y(t-\tau_1) & 0 \\ 0 & \varepsilon \left(1 - \frac{2y_*}{x(t-\tau_2)+e_2} \right) \end{pmatrix}, \\
 B &= \begin{pmatrix} 0 & \frac{-ax_*}{x_*+e_1} \\ 0 & 0 \end{pmatrix}, \\
 C &= \begin{pmatrix} 0 & 0 \\ \frac{\varepsilon y_*^2}{(x(t-\tau_2)+e_2)^2} & 0 \end{pmatrix}.
 \end{aligned} \tag{34}$$

Substituting $y_* = x_* + e_2$ into (34) yields

$$\begin{aligned}
 A &= \begin{pmatrix} 1 - 2x_* - \frac{ae_1}{(x+e_1)^2} y(t-\tau_1) & 0 \\ 0 & \varepsilon \left(1 - \frac{2(x_*+e_2)}{x(t-\tau_2)+e_2} \right) \end{pmatrix}, \\
 B &= \begin{pmatrix} 0 & \frac{-ax_*}{x_*+e_1} \\ 0 & 0 \end{pmatrix}, \\
 C &= \begin{pmatrix} 0 & 0 \\ \frac{\varepsilon(x_*+e_2)^2}{(x(t-\tau_2)+e_2)^2} & 0 \end{pmatrix},
 \end{aligned} \tag{35}$$

i.e.,

$$\begin{aligned}
 A &= \begin{pmatrix} \frac{x_*}{x_*+e_1} (2x_*+e_1-1) & 0 \\ 0 & -\varepsilon \end{pmatrix}, \\
 B &= \begin{pmatrix} 0 & \frac{-ax_*}{x_*+e_1} \\ 0 & 0 \end{pmatrix}, \\
 C &= \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}.
 \end{aligned} \tag{36}$$

Denote $\tau = \tau_1 + \tau_2$, the associated characteristic equation of system (33) can be obtained as follows:

$$\begin{aligned}
 \lambda^2 - \left(\frac{x_*}{x_*+e_1} (2x_*+e_1-1) + \varepsilon \right) \lambda + \varepsilon \\
 \cdot \left(\frac{ax_*}{x_*+e_1} e^{-\lambda\tau} + \frac{x_*}{x_*+e_1} (2x_*+e_1-1) \right) = 0.
 \end{aligned} \tag{37}$$

FIGURE 5: Time series of system (9) and system (7).

Lemma 5. For equation (37), if $\sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} \geq 2a$, then equation (37) has no pairs of pure imaginary roots. If $\sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} < 2a$, then equation (37) has a pair of pure imaginary roots.

Proof. Let $\lambda = \pm\omega i$ be the pair of pure imaginary roots of equation (37), then we take $\lambda = \omega i$ into equation (37) with $e^{-\lambda\tau} = e^{-\omega i\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$, it yields

$$\begin{aligned} & -\omega^2 - \left(\frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) + \varepsilon \right) \omega i, \\ & + \varepsilon \left(\frac{ax_*}{x_* + e_1} (\cos(\omega\tau) - i\sin(\omega\tau)) + \frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) \right) = 0. \end{aligned} \quad (38)$$

Separating the real and imaginary parts of equation (38), then we get

$$\begin{aligned} & -\omega^2 + \frac{\varepsilon x_*}{x_* + e_1} (2x_* + e_1 - 1) + \frac{a\varepsilon x_*}{x_* + e_1} \cos(\omega\tau) = 0, \\ & \omega \left(\frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) + \varepsilon \right) - \frac{a\varepsilon x_*}{x_* + e_1} \sin(\omega\tau) = 0. \end{aligned} \quad (39)$$

So that

$$\cos(\omega\tau) = \frac{\omega^2 - (\varepsilon x_*/x_* + e_1)(2x_* + e_1 - 1)}{a\varepsilon x_*/(x_* + e_1)} \triangleq Q_0, \quad (40)$$

$$\sin(\omega\tau) = \frac{\omega((x_*/x_* + e_1)(2x_* + e_1 - 1) + \varepsilon)}{a\varepsilon x_*/(x_* + e_1)}.$$

According to $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, we have

$$\omega^4 + D\omega^2 + E = 0, \quad (41)$$

with

$$\begin{aligned} D &= \left(\frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) + \varepsilon \right)^2 - \frac{2\varepsilon x_*}{x_* + e_1} (2x_* + e_1 - 1) \\ &= \left(\frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) \right)^2 + \varepsilon^2 > 0, \\ E &= \left(\frac{\varepsilon x_*}{x_* + e_1} (2x_* + e_1 - 1) \right)^2 - \left(\frac{a\varepsilon x_*}{x_* + e_1} \right)^2 = \left(\frac{\varepsilon x_*}{x_* + e_1} \right)^2 \\ &\quad \cdot ((2x_* + e_1 - 1)^2 - a^2), \\ &= \left(\frac{\varepsilon x_*}{x_* + e_1} \right)^2 (2x_* + e_1 - 1 + a)(2x_* + e_1 - 1 - a). \end{aligned} \quad (42)$$

Since

$$\begin{aligned} 2x_* + e_1 - 1 + a &= \sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} > 0, \\ 2x_* + e_1 - 1 - a &= \sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} - 2a. \end{aligned} \quad (43)$$

We can conclude that if $\sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} \geq 2a$, that $E \geq 0$, which implies that Equation (41) has no positive roots, so equation (37) has no pairs of pure imaginary roots. Besides, if $\sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} < 2a$, then $E < 0$ and equation (41) has a positive root:

$$\omega = \sqrt{\frac{-D + \sqrt{D^2 - 4E}}{2}}, \quad (44)$$

which implies equation (37) has a pair of pure imaginary roots when

$$\tau_{0j} = \frac{\arccos(Q_0) + 2j\pi}{\omega}, \quad j = 0, 1, 2, \dots \quad (45)$$

The proof is completed.

Denote $\lambda(\tau) = p(\tau) \pm iq(\tau)$ be the pair of pure imaginary roots of equation (37) that are satisfying $p(\tau_{0j}) = 0$, $q(\tau_{0j}) = \omega$. $\square$

Lemma 6. (transversality condition) If $\sqrt{(a + e_1 - 1)^2 - 4(ae_2 - e_1)} < 2a$ holds, then $p'(\tau_{0j}) > 0$, $j = 0, 1, 2, \dots$

Proof. Differentiating both sides of equation (37) with respect to τ , we reach

$$2\lambda \frac{d\lambda}{d\tau} + \left(\frac{x_*}{x_* + e_1} (2x_* + e_1 - 1) + \varepsilon \right) \frac{d\lambda}{d\tau} - \lambda \varepsilon \frac{ax_*}{x_* + e_1} e^{-\lambda\tau} = 0, \quad (46)$$

then

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + (x_*/x_* + e_1)(2x_* + e_1 - 1) + \varepsilon}{\lambda \varepsilon (ax_*/x_* + e_1) e^{-\lambda\tau}}. \quad (47)$$

So

$$\begin{aligned} \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1}_{\tau=\tau_{0j}} &= \operatorname{Re} \left[\frac{2\lambda + (x_*/x_* + e_1)(2x_* + e_1 - 1) + \varepsilon}{\lambda \varepsilon (ax_*/x_* + e_1) e^{-\lambda\tau}} \right]_{\tau=\tau_{0j}} \\ &= \frac{2\omega \cos \omega\tau + [(ax_*/x_* + e_1)(2x_* + e_1 - 1) + \varepsilon] \sin \omega\tau}{(\omega \varepsilon ax_*/x_* + e_1)} \\ &= \frac{2\omega^2 + D}{(\varepsilon ax_*/x_* + e_1)^2} > 0. \end{aligned} \quad (48)$$

To sum up, we get $p'(\tau_{0j}) > 0$.

The proof is completed. $\square$

Theorem 3. For system (32), if $\tau_1 + \tau_2 = \tau_{0j}$, $j = 0, 1, 2, \dots$, system (32) undergoes a Hopf bifurcation and τ_{0j} are the Hopf bifurcation values.

Proof. If $\tau_1 + \tau_2 = \tau_{0j}$, $j = 0, 1, 2, \dots$ then the equation (37) has a pair of purely imaginary roots which satisfy the transversality condition. Thus system (32) undergoes a Hopf bifurcation and τ_{0j} are the Hopf bifurcation values.

The theorem is proved. $\square$

Remark 1. If $0 < \tau_1 + \tau_2 < \tau_{00}$, then $E_*(x_*, y_*)$ is locally asymptotically stable and if $\tau_1 + \tau_2 > \tau_{00}$, then $E_*(x_*, y_*)$ is unstable.

4.2. Analysis of Limit Systems. In order to get rid of the delay τ_1 and τ_2 , we denote $x = x_1$, $x(t - \tau_2) = x_2$, $y = y_1$, $y(t - \tau_1) = y_2$. Then, we get

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 - x_1) - \frac{ax_1}{x_1 + e_1} y_2, \\ \frac{dx_2}{dt} &= x_2(1 - x_2) - \frac{ax_2}{x_2 + e_1} y(t - \tau_1 - \tau_2), \\ \frac{dy_1}{dt} &= \varepsilon y_1 \left(1 - \frac{y_1}{x_2 + e_2} \right), \\ \frac{dy_2}{dt} &= \varepsilon y_2 \left(1 - \frac{y_2}{x(t - \tau_1 - \tau_2) + e_2} \right), \end{aligned} \quad (49)$$

i.e.,

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 - x_1) - \frac{ax_1}{x_1 + e_1} y_2, \\ \frac{dx_2}{dt} &= x_2(1 - x_2) - \frac{ax_2}{x_2 + e_1} y_1(t - \tau), \\ \frac{dy_1}{dt} &= \varepsilon y_1 \left(1 - \frac{y_1}{x_2 + e_2} \right), \\ \frac{dy_2}{dt} &= \varepsilon y_2 \left(1 - \frac{y_2}{x_1(t - \tau) + e_2} \right). \end{aligned} \quad (50)$$

When $\tau = 0$, system (50) goes to

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 - x_1) - \frac{ax_1}{x_1 + e_1} y_2, \\ \frac{dx_2}{dt} &= x_2(1 - x_2) - \frac{ax_2}{x_2 + e_1} y_1, \\ \frac{dy_1}{dt} &= \varepsilon y_1 \left(1 - \frac{y_1}{x_2 + e_2} \right), \\ \frac{dy_2}{dt} &= \varepsilon y_2 \left(1 - \frac{y_2}{x_1 + e_2} \right). \end{aligned} \quad (51)$$

Consider the case where ε is small enough, system (51) could be viewed as a slow-fast system which has the fast variable $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the slow variable $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. We have

$$f(X, Y, \varepsilon) = (f_1, f_2)$$

$$= \left(x_1(1 - x_1) - \frac{ax_1}{x_1 + e_1} y_2, x_2(1 - x_2) - \frac{ax_2}{x_2 + e_1} y_1 \right). \quad (52)$$

Then, we arrive at the critical set:

$$C_0 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid f(X, Y, 0) = 0\} = C_0^1 \cup C_0^2 \cup C_0^3 \cup C_0^4. \quad (53)$$

with

$$\begin{aligned}
 C_0^1 &= \{x_1 = 0, x_2 = 0\}, \\
 C_0^2 &= \left\{x_1 = 0, y_1 = \frac{1}{a}(1-x_2)(x_2+e_1)\right\}, \\
 C_0^3 &= \left\{y_2 = \frac{1}{a}(1-x_1)(x_1+e_1), x_2 = 0\right\}, \\
 C_0^4 &= \left\{y_2 = \frac{1}{a}(1-x_1)(x_1+e_1), y_1 = \frac{1}{a}(1-x_2)(x_2+e_1)\right\}.
 \end{aligned} \tag{54}$$

Since

$$D_X f = \begin{pmatrix} 1 - 2x_1 - \frac{ae_1 y_2}{(x_1+e_1)^2} & 0 \\ 0 & 1 - 2x_2 - \frac{ae_1 y_1}{(x_2+e_1)^2} \end{pmatrix}, \tag{55}$$

Then, the eigenvalues of matrix (55) are

$$\begin{aligned}
 \mu_1 &= 1 - 2x_1 - \frac{ae_1 y_2}{(x_1+e_1)^2}, \\
 \mu_2 &= 1 - 2x_2 - \frac{ae_1 y_1}{(x_2+e_1)^2}.
 \end{aligned} \tag{56}$$

Thus, we reach the attractive part and the repulsive part of C_0^1 are, respectively,

$$\begin{aligned}
 C_0^{1a} &= \left\{x_1 = 0, x_2 = 0, y_1 > \frac{e_1}{a}, y_2 > \frac{e_1}{a}\right\}, \\
 C_0^{1r} &= \left\{x_1 = 0, x_2 = 0, y_1 < \frac{e_1}{a}, y_2 < \frac{e_1}{a}\right\}.
 \end{aligned} \tag{57}$$

The attractive part and the repulsive part of C_0^2 are, respectively,

$$\begin{aligned}
 C_0^{2a} &= \left\{x_1 = 0, x_2 > \frac{1-e_1}{2}, y_1 = \frac{1}{a}(1-x_2)(x_2+e_1), y_2 > \frac{e_1}{a}\right\}, \\
 C_0^{2r} &= \left\{x_1 = 0, 0 < x_2 < \frac{1-e_1}{2}, y_1 = \frac{1}{a}(1-x_2)(x_2+e_1), y_2 < \frac{e_1}{a}\right\}, \\
 C_0^{3a} &= \left\{x_1 > \frac{1-e_1}{2}, x_2 = 0, y_1 > \frac{e_1}{a}, y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}, \\
 C_0^{3r} &= \left\{0 < x_1 < \frac{1-e_1}{2}, x_2 = 0, y_1 < \frac{e_1}{a}, y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}.
 \end{aligned} \tag{58}$$

FIGURE 6: Choose $(\tau_1, \tau_2) = (3, 1)$ satisfying $\tau_1 + \tau_2 = 4 < \tau_{00}$, then the equilibrium $E_*(x_*, y_*)$ of system (32) is asymptotically stable. Time series with the black dashed curve (prey) and red full curve (predator).

$$\begin{aligned}
 C_0^{3a} &= \left\{x_1 > \frac{1-e_1}{2}, x_2 = 0, y_1 > \frac{e_1}{a}, y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}, \\
 C_0^{3r} &= \left\{0 < x_1 < \frac{1-e_1}{2}, x_2 = 0, y_1 < \frac{e_1}{a}, y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}.
 \end{aligned} \tag{59}$$

The attractive part and the repulsive part of C_0^3 are, respectively,

$$\begin{aligned}
 C_0^{4a} &= \left\{x_1 > \frac{1-e_1}{2}, x_2 > \frac{1-e_1}{2}, y_1 = \frac{1}{a}(1-x_2)(x_2+e_1), y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}, \\
 C_0^{4r} &= \left\{0 < x_1 < \frac{1-e_1}{2}, 0 < x_2 < \frac{1-e_1}{2}, y_1 = \frac{1}{a}(1-x_2)(x_2+e_1), y_2 = \frac{1}{a}(1-x_1)(x_1+e_1)\right\}.
 \end{aligned} \tag{60}$$

and the attractive part and the repulsive part of C_0^4 are, respectively,

4.3. Conjecture of the System. In this section, we combine the result of Subsection 4.1 and Subsection 4.2 and propose a conjecture.

Conjecture 1. *If $x_* > (1 - e_1/2)$ and $(x_*/x_* + e_1)(2x_* + e_1 - 1) + \varepsilon = 0$, then the equilibrium $E_*(x_*, y_*)$ is a Hopf bifurcation point of system (51) and a periodic solution is bifurcated from $E_*(x_*, y_*)$. Moreover, the relaxation oscillation cycle of the system (51) is unique.*

4.4. Numerical Simulation. We show some numerical simulations in this part to verify our theoretical results.

For system (32), we choose the parameter values $(a, e_1, e_2, \varepsilon) = (0.805, 0.2, 0.022, 0.01)$, then we get $(x_*, y_*) = (0.422, 0.444)$. A series of calculations gives us that $\tau_{00} \approx 11.50$. Thus, we choose $(\tau_1, \tau_2) = (3, 1)$, and then the equilibrium $E_*(x_*, y_*)$ of system (32) is asymptotically stable (Figure 6). Besides, we choose $(\tau_1, \tau_2) = (3, 13)$ and get the periodic solutions of system (32) (Figure 7).

For system (51), we choose the parameter values $(a, e_1, e_2, \varepsilon) = (1, 0.2, 0.022, 0.01)$. Figure 8 shows the phase portrait and the time series of the relaxation oscillation cycle.

FIGURE 7: Choose $(\tau_1, \tau_2) = (3, 13)$ satisfying $\tau_1 + \tau_2 = 16 > \tau_{00}$, and then reach the periodic solutions of system (32) (time series with the black dashed curve (prey) and red full curve (predator)).

FIGURE 8: Numerical simulation of system (51). (a) The orbit is shown in red in the phase space. (b) Time series with the black dashed curve (prey) and red full curve (predator).

5. Conclusion

In this paper, we mainly study two kinds of time-delay slow-fast modified Leslie-Gower models. For the first system, through the geometric singular perturbation theory and entry-exit function, we prove the existence and uniqueness of relaxation oscillation cycle. For another system, when $\tau = \tau_1 + \tau_2$ crosses some critical values τ_{0j} , the Hopf bifurcation occurs. Meanwhile, we put forward a conjecture that the relaxation oscillation cycle of the system is unique when $\tau = 0$. Numerical simulation verified our theoretical results and indicated that our method is effective. Our results show that delay affects the stability of the positive equilibrium and produces more complex dynamics than the model without delay.[1, 2]

Data Availability

All data, models, and code generated or used during the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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