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Research Article

Portfolio Optimization with Asset-Liability Ratio Regulation Constraints

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This paper considers both a top regulation bound and a bottom regulation bound imposed on the asset-liability ratio at the regulatory time T to reduce risks of abnormal high-speed growth of asset price within a short period of time (or high investment leverage), and to mitigate risks of low assets' return (or a sharp fall). Applying the stochastic optimal control technique, a Hamilton–Jacobi–Bellman (HJB) equation is derived. Then, the effective investment strategy and the minimum variance are obtained explicitly by using the Lagrange duality method. Moreover, some numerical examples are provided to verify the effectiveness of our results.

1. Introduction

Taking liabilities into the traditional portfolio models, Sharpe and Tint [1] put forward the asset-liability problem for pension fund management under the mean-variance framework. Kell and Müller [2] point out that liabilities affect the efficient frontier of this asset-liability problem. But early research on asset-liability problems was limited to the standard single-period mean-variance criterion; Leippold et al. [3] obtain an analytical optimal strategy and efficient frontier for asset-liability problems by using embedding technique proposed by Li and Ng [4] under a multiperiod mean-variance framework. Leippold et al. [3] also consider the multiperiod mean-variance asset-liability problem and show that the optimal strategy can be decomposed into orthogonal sets and the efficient frontier spanned by an orthogonal basis of dynamic returns. For the continuous-time asset-liability problem, Chiu and Li [5] derive the optimal policy and the mean-variance efficient frontier by applying the technique of stochastic linear-quadratic control. In comparison with the embedding technique and the stochastic linear-quadratic control method, the Lagrange duality method is more convenient to solve the mean-variance problem. Fu et al. [6] consider the continuous-time mean-variance portfolio selection under different borrowing interest rates and lending interest rates with the Lagrange duality method. Pan and Xiao [7] also investigate the continuous-time asset-liability management problem by considering the stochastic interest rates and inflation risks. Considering a financial market consists of a risk-free bond, a stock, and a derivative, Li et al. [8] give the optimal investment strategies of a continuous-time mean-variance asset-liability management in presence of stochastic volatility. For other related literature studies, you can refer to Wei and Wang [9]; Zhang et al. [10]; Duarte et al. [11]; and so on.

As everyone knows, the assets scale pursued by investment growing too fast over a relatively short period of time, which indicates that the growth rate increasing faster than the normal rate of growth, is itself accompanied by high potential risks. The most typical example in practice is the stock market crash which has happened many times almost in all countries with stock markets. The earliest stock market crash occurred in 1720, the Mississippi Stock Disaster in France and the South Sea Stock Disaster in Britain. The share price of the Mississippi company rose from around 500 livres

in May 1719, to nearly 10,000 livres in February 1720, but it declined to 500 livres in September 1721. The stock prices of South Sea Company soar from £128 in January, £175 in February, £330 in March, and £550 in May, and then, the shares leaped to £1000 per share by August 1720 and finally peaked at this level but soon it plunged and triggered an avalanche of selling. As for the most devastating stock market crashes of the United States, such as the American stock market crash in 1929 and the American stock market crash in 1987, all ended in a catastrophic decline after a period of drastic growth. When the China stock market crash broke out in 2007, the Shanghai stock exchange composite index surged from around 2700 points to the peak value 6124 points, which took only a few months. But then it went down sharply to the lowest value 1644 points. Another China stock market crash in 2015, the Shanghai stock exchange composite index rise from around 3000 points to 5178 points on June 12, but only in the next two months, it plunged to around 2800 points.

The sustained high-speed growth of stock prices within a short period of time is actually accumulating enough destructive energy which may erupt at any time. Sophisticated investors will actively sell their risky stocks in time when the stock returns reach a certain high level, instead of blindly pursuing much higher returns, so as to avoid the sharp drop before these assets are successfully sold. This high return level targeted by these traders to sell assets actively is a practical top bound used in investment management practice and determined by investment managers initiatively, which is actually the concrete example for the application of top investment regulation bound in business. Thus, it confirms the real existence of the investment regulation top bound in real investment practice. The requirement of investment regulation bottom bound is necessary for the pursuit of much higher investment returns. Only if the assets scale pursued by investment is larger than liability, then it can ensure that the investment activities are really valuable and vigorous.

These kinds of investment risks of asset price collapse resulting from the abnormally rapid rise within a limited time period have extremely destructive, which may even be the real manifestation of systemic financial risks. As for systemic financial risks, Guerra et al. [12] and Souza et al. [13] conduct in-depth and innovative research studies and propose a novel methodology to measure systemic risk in networks composed of financial institutions. They define the bank's probability of default and calculate this probability using Merton's method inspired by Black and Scholes [14] and Merton [15]. Interestingly, the probability of default defined by Guerra et al. [12] and Souza et al. [13] can also be used to research the asset-liability management problem with unbearable collapse risk resulted from the abnormally rapid rise of asset price within a limited time period. For instance, you can take the horizon on which the entity (firm) may default to be short enough and then large declines in assets or large increases in liabilities can be researched similar to the probability of default calculated by Merton's method. However, based on different considerations, we realize our ideas about the asset-liability management problem with unbearable investment risk using a widely used mathematical method in this paper.

Financial leverage needs to be considered because it is ubiquitous in practice, and the excessive leverage is an issue tackled in the Basel III requirements. In the narrow sense, financial leverage refers to a measure of operating large-scale business with less money, which is widely used in various economic activities. For example, the transaction relying on futures margin is commonly used in futures markets and the margin rate usually varies between 5% and 15% of the total contract value. Only a down payment of 5% of the full contract value is required, and an investor can carry out the financial transaction equivalent to 20 times the amount of the margin. Another example is the way to invest in real estate with installment repayment, and investors only need to pay the down payment, usually 10% to 30%, but they can leverage the investment scales of the total market value. Quite evidently, investors generally use smaller self-owned funds to operate businesses of great market value with financial leverage. If these assets appreciate, investors can immediately obtain considerable investment returns. However, once the value of these assets shrinks, the losses are magnified and the corresponding leverage multiples are catastrophic and unbearable. Therefore, the essence of financial leverage lies in magnifying the gains and losses to meet the needs of investors who have insufficient funds but want to do large-scale businesses. There is a large body in the literature of financial stability which estimates systemic risk and explicitly takes into account excessive leverage. Related works can be referred to the study bySilva et al. [16] and many others.

However, financial markets themselves also have the function of enlarging profits and losses. As long as investors invest in risky assets in financial markets, they are actually using the financial market to enlarge their scale of assets, which is essentially consistent with the core meaning of financial leverage in narrow sense. Therefore, in a broad sense, a financial market itself can be regarded as a financial leverage. In other words, the financial leverage has already been used by investors as long as they invest in financial markets in pursuit of higher investment returns, so it is also necessary to consider leverage regulation even if you only invest in ordinary financial markets.

Different from the previous research on the modeling method of the asset-liability problem, both a bottom regulation bound and a top regulation bound are imposed on the asset-liability ratio to control the variation range so as to prevent the risks of asset price collapse resulted from an abnormal high-speed growth (such as stock market crash), asset shrink, and the collapse of investment leverage. The dynamic process of asset-liability ratio is defined as an asset process being divided by a liability process after eliminating the influence of inflation. A stochastic optimal control model is formulated under the framework of mean variance with the variance being minimized, given a determined expectation of asset-liability ratio at regulatory time T. Using the Lagrange multiplier method, the original problem is transformed into an unconstrained optimization problem, and then a Hamilton–Jacobi–Bellman (HJB) equation is established by adopting technique of stochastic control. At last, using Lagrange duality between the original problem

and the unconstrained problem, the minimum variance and the effective investment strategy are obtained.

The remainder of this paper is organized as follows. Section 2 describes the models of asset-liability ratio and the constrained control problem. In Section 3, a Lagrange unconstrained problem is solved by technique of stochastic optimal control, and the results of the original problem are obtained according to Lagrange duality. Meanwhile, a special case is also solved at the end of this section. Some interpretations of main results are presented in Section 4, and numerical examples are illustrated in Section 5. At last, Section 6 gives a conclusion of the research work.

2. Formulation of the Model

Throughout this paper, $(\Omega, \mathscr{F}, P, \{F_t\}_{0\le t\le T})$ denotes a complete probability space satisfying the usual condition. A finite constant $T>0$ represents the preselected investment regulatory time; $\mathscr{F}_t$ is the smallest σ–field generated by all random information available until time t, and all random variables and stochastic processes involved in this article are $\mathscr{F}_t$ measurable for every $t\in[0, T]$.

2.1. Financial Market. Similar to the previous research work, this paper considers an inflation-affected financial market in continuous time, which consists of one risk-free bond, one inflation-linked index bond, and one risky stock. The inflation rate $P(t)$ can be regarded as the Consumer Price Index, which is described by a price level process as follows:

$$P(t) = \exp\left\{\int_0^t\left(\widetilde{\mu}_s - \frac{1}{2}\widetilde{\sigma}_s^2\right)\mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}\widetilde{W}(s)\right\}, \quad (1)$$

where $\widetilde{\mu}_t$ is the expected inflation rate at time t; $\widetilde{\sigma}_t>0$ is the volatility of inflation rate; and $\widetilde{W}(t)$ is a standard Brownian motion on the probability space $(\Omega, \mathscr{F}, P)$.

The price process of the risk-free bond is modeled as $B(t) = e^{\widetilde{r}t}$, where the constant $\widetilde{r}\ge 0$ represents the nominal interest rate.

The inflation-linked index bond $\{I(t), t\ge 0\}$ has the same risk source with the price level process, and thus, it can be expressed as

$$I(t) = \exp\left\{\int_0^t\left(\left(r+\widetilde{\mu}_s\right) - \frac{1}{2}\widetilde{\sigma}_s^2\right)\mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}\widetilde{W}(s)\right\}, \quad (2)$$

where r is the real interest rate at time t.

The price process of risky stock is formulated as

$$S(t) = S_0 \exp\left\{\int_0^t\left(\widehat{\mu}_s - \frac{1}{2}\widehat{\sigma}_s^2\right)\mathrm{d}s + \int_0^t \widehat{\sigma}_s \mathrm{d}\widehat{W}(s)\right\}, \quad (3)$$

where $\widehat{\mu}_t$ is the investment return rate satisfying $\widehat{\mu}_t > \widetilde{r}$ and $\widehat{\sigma}_t$ is the volatility of the risky asset.

2.2. Asset Process and Liability Process. A control vector $(\pi_0(t), \pi_1(t), \pi_2(t))$ represents the investment strategy need to be found, where $\pi_0(t)$ denotes the investment share of risk-free bond, $\pi_1(t)$ represents the investment share of inflation-linked index bond, and $\pi_2(t)$ signifies the investment share of risky stock, and they always satisfy the equation $\pi_0(t)+\pi_1(t)+\pi_2(t)=1$.

The company's accumulated wealth is allowed to invest in the financial market; thus, the asset process can be described as an investment process. Assuming that $\pi := \{(\pi_0(t), \pi_1(t), \pi_2(t))\}_{0\le t\le T}$ is an adaptive control variable, the dynamics of asset process is governed by the following equation:

$$\begin{cases} \dfrac{\mathrm{d}X^\pi(t)}{X^\pi(t)} = \pi_0(t)\dfrac{\mathrm{d}B(t)}{B(t)} + \pi_1(t)\dfrac{\mathrm{d}I(t)}{I(t)} + \pi_2(t)\dfrac{\mathrm{d}S(t)}{S(t)} \\ = \left[\widetilde{r} + \pi_1(t)\left(\widetilde{\mu}_t + r - \widetilde{r}\right) + \pi_2(t)\left(\widehat{\mu}_t - \widetilde{r}\right)\right]\mathrm{d}t + \pi_1(t)\widetilde{\sigma}_t\mathrm{d}\widetilde{W}(t) + \pi_2(t)\widehat{\sigma}_t\mathrm{d}\widehat{W}(t), \\ X^\pi(0) = x_0. \end{cases} \quad (4)$$

To avoid the influence of control variable on the liability process $L(t)$, we also assume the liability process is governed by a geometric Brownian motion as follows:

$$L(t) = L_0 \exp\left\{\int_0^t\left(\mu_s - \frac{1}{2}\sigma_s^2\right)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W(s)\right\}, \quad L_0>0. \quad (5)$$

The real liability process $\widehat{L}(t)$ after eliminating inflation is defined as $\widehat{L}(t) = L(t)/P(t)$. It is easy to get the following form of expression using Itô's formula:

$$\begin{aligned} \frac{\mathrm{d}\widehat{L}(t)}{\widehat{L}(t)} &= \left(\mu_t - \widetilde{\mu}_t + \widetilde{\sigma}_t^2\right)\mathrm{d}t - \widetilde{\sigma}_t\mathrm{d}\widetilde{W}(t) + \sigma_t\mathrm{d}W(t), \\ \widehat{L}(t) &= L_0 \exp\left\{\int_0^t\left(\mu_s - \widetilde{\mu}_s + \frac{1}{2}\widetilde{\sigma}_s^2 - \frac{1}{2}\sigma_s^2\right)\mathrm{d}s - \int_0^t \widetilde{\sigma}_s\mathrm{d}\widetilde{W}(s) \right. \\ &\quad \left. + \int_0^t \sigma_s \mathrm{d}W(s)\right\}. \end{aligned} \quad (6)$$

The asset process after eliminating the influence of inflation is defined as $\widehat{X}^\pi(t) = X^\pi(t)/P(t)$; then, we get the following asset process according to Itô's formula:

$$\frac{\mathrm{d}\widehat{X}^{\pi}(t)}{\widehat{X}^{\pi}(t)} = \left[\widetilde{r} - \widetilde{\mu}_t + \widetilde{\sigma}_t^2 + \pi_1(t)\left(\widetilde{\mu}_t + r - \widetilde{r} - \widetilde{\sigma}_t^2\right) + \pi_2(t)\left(\widehat{\mu}_t - \widetilde{r}\right)\right]\mathrm{d}t + \left(\pi_1(t) - 1\right)\widetilde{\sigma}_t \mathrm{d}\widetilde{W}(t) + \pi_2(t)\widehat{\sigma}_t \mathrm{d}\widehat{W}(t). \tag{7}$$

2.3. Asset-Liability Problem with Regulation Constraints. The *asset-liability ratio* is defined as $Z^{\pi}(t) = \widehat{X}^{\pi}(t)/\widehat{L}(t)$, which describes the ratio between the size of asset and the size of liability at time t. According to Itô's formula, the dynamic process of the *asset-liability ratio* can be given by

$$\frac{\mathrm{d}Z^{\pi}(t)}{Z^{\pi}(t)} = \left(\widetilde{r} - \mu_t - \sigma_t^2\right)\mathrm{d}t - \sigma_t \mathrm{d}W(t) + \pi_1(t)\left(\left(\widetilde{\mu}_t + r - \widetilde{r}\right)\mathrm{d}t + \widetilde{\sigma}_t \mathrm{d}\widetilde{W}(t)\right) + \pi_2(t)\left(\left(\widehat{\mu}_t - \widetilde{r}\right)\mathrm{d}t + \widehat{\sigma}_t \mathrm{d}\widehat{W}(t)\right). \tag{8}$$

In practice, the regulation only occurs at certain fixed time which is called regulatory time. The regulatory time is usually selected in advance, which provides a time period of appropriate length, but the length may decrease if the regulatory frequency increased. In accordance with the need of practice, this paper considers the asset-liability problem with the constraints imposed at the regulatory time T to find the optimal investment strategy, in order to reduce the risk of abnormal high-speed growth of asset price within a short period of time or high investment leverage and also to lessen the risk of too low return rate or a sharp fall. Their mathematical descriptions of the *regulation constraints* at the regulatory time T are formulated as the following inequality:

$$\alpha < Z^{\pi}(T) < \beta, \tag{9}$$

where the *bottom regulation bound* $\alpha \geq 1$ and the *top regulation bound* β are two appropriate constants satisfying the requirement $\beta > \alpha$. The bottom bound α should not be too small; otherwise, it cannot cover the liability which means the investment has failed. The top bound β should not be too large; or else, a much larger value of β may lead to excessive leverage multiples and too much investment risks being pulled in the company. In practice, it is highly technical to determine the values of α and β, which requires abundant practical experience and rigorous mathematical calculation, and only can be completed by well-trained and experienced investment managers based on the real market quotation and their practical experiences.

Let Π denote the set of all admissible strategies π. The investor aims to find an optimal portfolio $\pi \in \Pi$ of the following *original problem*:

$$\begin{cases} \min\limits_{\pi \in \prod} & \mathbf{Var}[Z^{\pi}(T)] = \mathbf{E}\left[Z^{\pi}(T) - (\alpha + \rho(\beta - \alpha))\right]^2 \\ \text{s.t.} & \mathbf{E}[Z^{\pi}(T)] = \alpha + \rho(\beta - \alpha) > 0, \\ & Z^{\pi}(t) \text{satisfy}\,(8), \rho \in (0,1), \text{for}\, \pi \in \prod, \end{cases} \tag{10}$$

where ρ is a constant. Only if ρ takes an unequivocal value, then the expectation of $Z^{\pi}(T)$ can be fixed at a certain value between the top regulation bound β and the bottom regulation bound α.

For a determined value of expectation $\alpha + \rho(\beta - \alpha)$, if there exists at least one admissible pair $(Z(\cdot), \pi(\cdot))$ satisfying $\mathbf{E}[Z^{\pi}(T)] = \alpha + \rho(\beta - \alpha)$, then the problem (10) is called *feasible*. Given $\alpha + \rho(\beta - \alpha)$, the optimal strategy π^* of (10) is called an *efficient strategy*. The pair $(Z^{\pi^*}(T), \mathrm{Var}[Z^{\pi^*}(T)])$ under this optimal strategy π^* is called an *efficient point*. The set of all efficient points is called the *efficient frontier*. Obviously, problem (10) is a dynamic quadratic convex optimization problem, and thus, it has an unique solution.

3. Solutions of the Problem

In this section, an unconstrained optimization problem is derived by the Lagrange multiplier method and then, a Hamilton–Jacobi–Bellman equation is deduced to solve the unconstrained problem. Next, solution of the original problem is obtained according to Lagrange duality between the unconstrained problem and the original problem. At the end, solution of a special case is given for comparison.

3.1. Solution of an Unconstrained Problem. For this convex optimization problem (10), the equality constraint $\mathbf{E}[Z^{\pi}(T)] = \alpha + \rho(\beta - \alpha)$ can be eliminated by using the Lagrange multiplier method, so we get a Lagrange dual problem:

$$\begin{cases} \min\limits_{\pi \in \prod} & \mathbf{E}\left[Z^{\pi}(T) - (\alpha + \rho(\beta - \alpha))\right]^2 + 2\lambda \mathbf{E}\left[Z^{\pi}(T) - (\alpha + \rho(\beta - \alpha))\right], \\ \text{s.t.} & Z^{\pi}(T) \text{satisfy}\,(8), \rho \in (0,1), \text{for}\, \pi \in \prod, \end{cases} \tag{11}$$

where $\lambda \in \mathbf{R}$ is a Lagrange multiplier.

After a simple calculation, it yields

$$\mathbf{E}\left[Z^{\pi}(T) - (\alpha + \rho(\beta - \alpha))\right]^2 + 2\lambda \mathbf{E}\left[Z^{\pi}(T) - (\alpha + \rho(\beta - \alpha))\right] = \mathbf{E}\left[Z^{\pi}(T) - ((\alpha + \rho(\beta - \alpha)) - \lambda)\right]^2 - \lambda^2. \tag{12}$$

Remark 1. The link between problems (10) and (11) is provided by the Lagrange duality theorem, which can be referred in the study by Luenberger [17] that

$$\min_{\pi\in\prod}\mathbf{Var}\left[Z^{\pi}(T)\right] = \max_{\lambda\in R}\min_{\pi\in\prod}\left\{\mathbf{E}\left[Z^{\pi}(T)-((\alpha+\rho(\beta-\alpha))-\lambda)\right]^{2}-\lambda^{2}\right\}. \tag{13}$$

Thus, for the fixed constants λ and u, problem (11) is equivalent to

$$\begin{cases}\min\limits_{\pi\in\prod} & \mathbf{E}\left[Z^{\pi}(T)-((\alpha+\rho(\beta-\alpha))-\lambda)\right]^{2}\\ \text{s.t.} & Z^{\pi}(t)\text{ satisfy }(8),\rho\in(0,1),\text{ for }\pi\in\prod.\end{cases} \tag{14}$$

Therefore, in order to solve problem (11), we only need to solve problem (14). Fortunately, problem (14) can be solved by using the stochastic optimal control technology.

To solve problem (14), a truncated form beginning at time t is considered here, and the corresponding value function is defined as

$$V(t,z)=\inf_{\pi\in\prod}\mathbf{E}\left[\left(Z^{\pi}(T)-((\alpha+\rho(\beta-\alpha))-\lambda)\right)^{2}\Big|_{Z^{\pi}(t)=z}\right], \tag{15}$$

with the boundary condition $V(T,z)=(z-((\alpha+\rho(\beta-\alpha))-\lambda))^{2}$.

The controlled infinitesimal generator for any test function $\phi(t,x)$ and any admissible control π is deduced as follows:

$$\mathscr{A}^{\pi}\phi(t,x)=\phi_t+x\phi_x\left[\tilde{r}-\mu_t-\sigma_t^2+\pi_1(t)\left(\tilde{\mu}_t+r-\tilde{r}\right)+\pi_2(t)\left(\hat{\mu}_t-\tilde{r}\right)\right]+\frac{x^2}{2}\phi_{xx}\left[\sigma_t^2+\pi_1^2(t)\tilde{\sigma}_t^2+\pi_2^2(t)\hat{\sigma}_t^2\right], \tag{16}$$

for any real valued function $\phi(t,x)\in\mathbf{C}^{1,2}([0,T],\mathbf{R})$ and admissible strategy π, where

$$\mathbf{C}^{1,2}([0,T],\mathbf{R})=\{\psi(t,x)|\psi(t,\cdot)\text{ is once continuously differentiable on }[0,T],\text{ and }\psi(\cdot,x)\text{ is twice continuously differentiable almost surely on }\mathbf{R}\}. \tag{17}$$

If $V(t,z)\in\mathbf{C}^{1,2}([0,T]\times\mathbf{R})$, then $V(t,z)$ satisfies the following HJB equation:

$$\inf_{\pi\in\prod}\mathscr{A}^{\pi}V(t,z)=0, \tag{18}$$

which can be given more specifically as follows:

$$\inf_{\pi\in\prod}\left\{V_t+zV_z\left(\tilde{r}-\mu_t-\sigma_t^2\right)+\frac{z^2}{2}V_{zz}\sigma_t^2+zV_z\left[\pi_1(t)\left(\tilde{\mu}_t+r-\tilde{r}\right)+\pi_2(t)\left(\hat{\mu}_t-\tilde{r}\right)\right]+\frac{z^2}{2}V_{zz}\left(\pi_1^2(t)\tilde{\sigma}_t^2+\pi_2^2(t)\hat{\sigma}_t^2\right)=0\right\}. \tag{19}$$

For convenience, some simple notations are introduced as follows:

$$\begin{aligned}\delta&\triangleq\frac{\left(\tilde{\mu}_t+r-\tilde{r}\right)^2}{\tilde{\sigma}_t^2}+\frac{\left(\hat{\mu}_t-\tilde{r}\right)^2}{\hat{\sigma}_t^2},\\ \varrho(t)&\triangleq\frac{\left(e^{(t-T)\left(\delta+\sigma_t^2\right)}\delta+\sigma_t^2\right)}{\delta+\sigma_t^2},\\ \omega&\triangleq\frac{(\alpha+\rho(\beta-\alpha))}{(1-\hat{\varrho})},\\ \hat{\varrho}&\triangleq\varrho(0)=\frac{\left(e^{-T\left(\delta+\sigma_t^2\right)}\delta+\sigma_t^2\right)}{\delta+\sigma_t^2},\\ \varepsilon&\triangleq\frac{e^{-T\left(-\tilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0}{(1-\hat{\varrho})}.\end{aligned} \tag{20}$$

Solving equation (19) gives the following important theorem.

Theorem 1. *For the asset-liability management with investment regulation and eliminating inflation, the efficient investment strategy $\pi^*=(\pi_0^*,\pi_1^*,\pi_2^*)$ corresponding to problem (11) and problem (14) is given by*

$$\begin{cases}\pi_0^*=1+\delta\left(1-\dfrac{1}{z}\xi_\alpha^\beta(t)\right),\\ \pi_1^*=\dfrac{\left(\tilde{\mu}_t+r-\tilde{r}\right)}{\tilde{\sigma}_t^2}\left(\dfrac{\xi_\alpha^\beta(t)}{z}-1\right),\\ \pi_2^*=\dfrac{\left(\hat{\mu}_t-\tilde{r}\right)}{\hat{\sigma}_t^2}\left(\dfrac{\xi_\alpha^\beta(t)}{z}-1\right),\end{cases} \tag{21}$$

and the value function is

$$V(t,z) = z^2 e^{(t-T)\left(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2\right)} - 2ze^{(t-T)\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)} + ((\alpha+\rho(\beta-\alpha))-\lambda)^2 \varrho(t), \tag{22}$$

where $\xi_\alpha^\beta(t) = e^{(t-T)(\widetilde{r}-\mu_t)}((\alpha+\rho(\beta-\alpha))-\lambda)$.

Proof. According to the first-order necessary condition of extremum, equation (19) yields

$$\begin{cases} \pi_1^*(t) = -\dfrac{(\widetilde{\mu}_t + r - \widetilde{r})}{z\widetilde{\sigma}_t^2}\dfrac{V_z}{V_{zz}}, \\ \pi_2^*(t) = -\dfrac{(\widehat{\mu}_t - \widetilde{r})}{z\widehat{\sigma}_t^2}\dfrac{V_z}{V_{zz}}. \end{cases} \tag{23}$$

Plugging the above expressions of $\pi_1^*(t)$ and $\pi_2^*(t)$ into (19), the HJB equation becomes

$$V_t + zV_z\left(\widetilde{r} - \mu_t - \sigma_t^2\right) + \frac{z^2}{2}V_{zz}\sigma_t^2 - \delta\frac{V_z^2}{2V_{zz}} = 0. \tag{24}$$

Based on the boundary condition $V(T,z) = (z - ((\alpha+\rho(\beta-\alpha))-\lambda))^2$, we try a conjecture

$$W(t,z) = f(t)z^2 + g(t)z + h(t), \tag{25}$$

for equation (24), which satisfies

$$\begin{aligned} f(T) &= 1, \\ g(T) &= -2((\alpha+\rho(\beta-\alpha))-\lambda), \\ h(T) &= ((\alpha+\rho(\beta-\alpha))-\lambda)^2. \end{aligned} \tag{26}$$

The derivatives of this conjecture $W(t, z)$ are given as follows:

$$\begin{aligned} W_t &= f'(t)z^2 + g'(t)z + h'(t), \\ W_z &= 2f(t)z + g(t), \\ W_{zz} &= 2f(t). \end{aligned} \tag{27}$$

Plugging the expressions of W_z and W_{zz} into (23), the optimal strategy becomes

$$\begin{cases} \pi_1^*(t) = -\dfrac{(\widetilde{\mu}_t + r - \widetilde{r})}{z\widetilde{\sigma}_t^2}\left(z + \dfrac{g(t)}{2f(t)}\right), \\ \pi_2^*(t) = -\dfrac{(\widehat{\mu}_t - \widetilde{r})}{z\widehat{\sigma}_t^2}\left(z + \dfrac{g(t)}{2f(t)}\right). \end{cases} \tag{28}$$

Substituting these derivatives of $W(t, z)$ into (24), the HJB equation becomes

$$f'(t)z^2 + g'(t)z + h'(t) + \left(2f(t)z^2 + g(t)z\right)\left(\widetilde{r} - \mu_t - \sigma_t^2\right) + z^2 f(t)\sigma_t^2 - \delta\left(f(t)z^2 + g(t)z + \frac{g^2(t)}{4f(t)}\right) = 0. \tag{29}$$

Equation (29) can be split into three ordinary differential equations as follows:

$$\begin{cases} f'(t) + 2f(t)(\widetilde{r} - \mu_t) - f(t)\sigma_t^2 - \delta f(t) = 0, \\ f(T) = 1, \\ g'(t) + g(t)(\widetilde{r} - \mu_t - \sigma_t^2) - \delta g(t) = 0, \\ g(T) = -2((\alpha+\rho(\beta-\alpha))-\lambda), \\ h'(t) - \dfrac{\delta}{4}\dfrac{g(t)^2}{f(t)} = 0, \\ h(T) = ((\alpha+\rho(\beta-\alpha))-\lambda)^2. \end{cases} \tag{30}$$

Solving these ordinary differential equations in (30) yields

$$\begin{cases} f(t) = e^{(t-T)\left(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2\right)}, \\ g(t) = -2e^{(t-T)\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}((\alpha+\rho(\beta-\alpha))-\lambda), \\ h(t) = ((\alpha+\rho(\beta-\alpha))-\lambda)^2\varrho(t). \end{cases} \tag{31}$$

Plugging the explicit expression of $f(t)$ and $g(t)$ into (28), and using the equality $\pi_0^*(t) + \pi_1^*(t) + \pi_2^*(t) = 1$, the optimal strategy is obtained as follows:

$$\begin{cases} \pi_0^*(t) = 1 + \delta\left(1 - e^{(t-T)(\widetilde{r}-\mu_t)}\dfrac{((\alpha+\rho(\beta-\alpha))-\lambda)}{z}\right), \\ \pi_1^*(t) = -\dfrac{(\widetilde{\mu}_t + r - \widetilde{r})}{\widetilde{\sigma}_t^2} + \dfrac{(\widetilde{\mu}_t + r - \widetilde{r})}{z\widetilde{\sigma}_t^2}e^{(t-T)(\widetilde{r}-\mu_t)}((\alpha+\rho(\beta-\alpha))-\lambda), \\ \pi_2^*(t) = -\dfrac{(\widehat{\mu}_t - \widetilde{r})}{\widehat{\sigma}_t^2} + \dfrac{(\widehat{\mu}_t - \widetilde{r})}{z\widehat{\sigma}_t^2}e^{(t-T)(\widetilde{r}-\mu_t)}((\alpha+\rho(\beta-\alpha))-\lambda). \end{cases} \tag{32}$$

Denoting

$$\xi_\alpha^\beta(t) \triangleq e^{(t-T)(\widetilde{r}-\mu_t)}((\alpha+\rho(\beta-\alpha))-\lambda), \tag{33}$$

the optimal strategy can be simplified into

$$\begin{cases} \pi_0^*(t) = 1+\delta\left(1-\dfrac{1}{z}\xi_\alpha^\beta(t)\right), \\ \pi_1^*(t) = -\dfrac{(\widetilde{\mu}_t+r-\widetilde{r})}{\widetilde{\sigma}_t^2}+\dfrac{(\widetilde{\mu}_t+r-\widetilde{r})}{z\widetilde{\sigma}_t^2}\xi_\alpha^\beta(t), \\ \pi_2^*(t) = -\dfrac{(\widehat{\mu}_t-\widetilde{r})}{\widehat{\sigma}_t^2}+\dfrac{(\widehat{\mu}_t-\widetilde{r})}{z\widehat{\sigma}_t^2}\xi_\alpha^\beta(t). \end{cases} \tag{34}$$

Meanwhile, substituting the expressions of $f(t)$, $g(t)$, and $h(t)$ into $W(t, z)$, the explicit solution of HJB equation is obtained as follows:

$$\begin{aligned} W(t,z) = {} & z^2 e^{(t-T)(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2)}+((\alpha+\rho(\beta-\alpha))-\lambda)^2 \\ & \varrho(t)-2ze^{(t-T)(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}((\alpha+\rho(\beta-\alpha))-\lambda). \end{aligned} \tag{35}$$

□

The following verification theorem shows the obtained results are exactly the optimal strategy and the optimal value function as required.

Theorem 2 (Verification Theorem). *If $W(t, z)$ is a solution of HJB equation (19) and satisfies the boundary condition $W(T,z) = (z-((\alpha+\rho(\beta-\alpha))-\lambda))^2$, then for all admissible strategies $\pi \in \Pi$, $V(t, z) \geq W(t, z)$. If π^* satisfies*

$$\pi^* \in \arg\inf_{\pi\in\prod} \mathbf{E}\left[\left(Z^\pi(T)-((\alpha+\rho(\beta-\alpha))-\lambda)\right)^2 | Z^\pi(t)=z\right], \tag{36}$$

then $V(t, z) = W(t, z)$ and π^* is the optimal investment strategy of problem (14).

Proof. Proof is similar to that given by Pham [18]; Chang [19]; and so on, so the detail is omitted. □

3.2. Solution of the Original Problem. The optimal value function of problem (11) is defined as

$$\widehat{V}(0,z_0) = \inf_{\pi\in\prod} \mathbf{E}\left[\left(Z^\pi(T)-((\alpha+\rho(\beta-\alpha))-\lambda)\right)^2\right]-\lambda^2, \tag{37}$$

where $z_0 = Z(0)$.

In this section, the Lagrange duality method is used to find a solution of original optimization problem (10) based on the obtained results of the unconstrained problem.

Theorem 3. *For original problem (10), the efficient strategy $\pi^* = (\pi_0^*, \pi_1^*, \pi_2^*)$ is given by*

$$\begin{cases} \pi_0^* = 1-\delta\left(\dfrac{\zeta_\alpha^\beta(t)}{z}-1\right), \\ \pi_1^* = \dfrac{(\widetilde{\mu}_t+r-\widetilde{r})}{\widetilde{\sigma}_t^2}\left(\dfrac{\zeta_\alpha^\beta(t)}{z}-1\right), \\ \pi_2^* = \dfrac{(\widehat{\mu}_t-\widetilde{r})}{\widehat{\sigma}_t^2}\left(\dfrac{\zeta_\alpha^\beta(t)}{z}-1\right), \end{cases} \tag{38}$$

and the optimal value function is given by

$$\begin{aligned} \mathbf{Var}\left[Z^{\pi^*}(T)\right] = {} & e^{-T(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2)}z_0^2+(\omega-\varepsilon)^2\widehat{\varrho} \\ & -2e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}(\omega-\varepsilon)z_0-(\varepsilon-\widehat{\varrho}\omega)^2, \end{aligned} \tag{39}$$

where

$$\zeta_\alpha^\beta(t) = e^{(t-T)(\widetilde{r}-\mu_t)}(\omega-\varepsilon). \tag{40}$$

Proof. The optimal value function $\widehat{V}(0,z_0)$ can be obtained with the equivalence between problem (11) and problem (14). By taking $z_0 = Z(0)$ and setting $t = 0$ in $V(t, z)$, it yields

$$\begin{aligned} \widehat{V}(0,z_0) &= V(0,z_0)-\lambda^2 = f(0)z_0^2+g(0)z_0+h(0)-\lambda^2 \\ &= e^{-T(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2)}z_0^2+((\alpha+\rho(\beta-\alpha))-\lambda)^2 \\ &\quad \widehat{\varrho}-2e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}((\alpha+\rho(\beta-\alpha))-\lambda)z_0-\lambda^2. \end{aligned} \tag{41}$$

Since $(\delta+\sigma_t^2) > 0$, then $e^{-T(\delta+\sigma_t^2)} < 1$, and it yields

$$\left(e^{-T(\delta+\sigma_t^2)}\delta+\sigma_t^2\right) < \delta+\sigma_t^2, \tag{42}$$

and then

$$\widehat{\varrho} = \frac{\left(e^{-T(\delta+\sigma_t^2)}\delta+\sigma_t^2\right)}{\delta+\sigma_t^2} < 1. \tag{43}$$

Note that the optimal value function $\widehat{V}(0,z_0)$ is a quadratic function with respect to λ and $\widehat{\varrho}-1$ is the coefficient of quadratic term:

$$\widehat{\varrho}-1 = \frac{\left(e^{-T(\delta+\sigma_t^2)}\delta+\sigma_t^2\right)}{\delta+\sigma_t^2}-1 < 0. \tag{44}$$

Thus, the finite maximum value of $\widehat{V}(0,z_0)$ can be obtained at a specific value λ^*, and

$$\lambda^* = \frac{e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}z_0}{(1-\widehat{\varrho})}-\frac{\widehat{\varrho}(\alpha+\rho(\beta-\alpha))}{(1-\widehat{\varrho})}. \tag{45}$$

Applying the Lagrangian duality, substituting λ^* into $\widehat{V}(0,z_0)$, the minimum variance corresponding to an arbitrarily given expectation $(\alpha+\rho(\beta-\alpha))$ is obtained as follows:

$$\mathbf{Var}\left[Z^{\pi^*}(T)\right] = e^{-T\left(-2\widetilde{r}+\delta+2\mu_t+\sigma_t^2\right)}z_0^2 + \left(\frac{(\alpha+\rho(\beta-\alpha))}{(1-\widehat{\varrho})} - \frac{e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0}{(1-\widehat{\varrho})}\right)^2$$
$$\cdot\,\widehat{\varrho} - 2e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}\left(\frac{(\alpha+\rho(\beta-\alpha))}{(1-\widehat{\varrho})} - \frac{e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0}{(1-\widehat{\varrho})}\right)z_0 - \left(\frac{e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0}{(1-\widehat{\varrho})} - \frac{\widehat{\varrho}(\alpha+\rho(\beta-\alpha))}{(1-\widehat{\varrho})}\right)^2. \tag{46}$$

Substitute (45) into (21), the optimal investment strategy of problem (10) is given by

$$\begin{cases} \pi_0^* = 1-\delta\left(\dfrac{1}{z}\zeta_\alpha^\beta(t)-1\right), \\ \pi_1^* = -\dfrac{(\widetilde{\mu}_t+r-\widetilde{r})}{\widetilde{\sigma}_t^2} + \dfrac{(\widetilde{\mu}_t+r-\widetilde{r})}{z\widetilde{\sigma}_t^2}\zeta_\alpha^\beta(t), \\ \pi_2^* = -\dfrac{(\widehat{\mu}_t-\widetilde{r})}{\widehat{\sigma}_t^2} + \dfrac{(\widehat{\mu}_t-\widetilde{r})}{z\widehat{\sigma}_t^2}\zeta_\alpha^\beta(t), \end{cases} \tag{47}$$

where

$$\zeta_\alpha^\beta(t) = e^{(t-T)(\widetilde{r}-\mu_t)}\left(\frac{(\alpha+\rho(\beta-\alpha))}{1-\widehat{\varrho}} - \frac{e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0}{1-\widehat{\varrho}}\right). \tag{48}$$

□

Remark 2. However, in the obtained results (47) and (46) above, the expected value $(\alpha+\rho(\beta-\alpha))$ of $Z^\pi(t)$ is not fixed at a determined position but can take any value between α and β depending on the value of $\rho\in(0,1)$. Therefore, the obtained minimum variance depends on the parameter value $\rho\in(0,1)$ in the expression of expectation $(\alpha+\rho(\beta-\alpha))$. Next, let us turn our attention to the parameter ρ and determine the optimal parameter value ρ^* for minimizing the variance with respect to ρ as well. The optimal ρ^* to achieve the minimum of $\mathbf{Var}[Z^{\pi^*}(T)]$ is given as follows:

$$\rho^* = \frac{\left(\left(e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}z_0\right)/\widehat{\varrho}\right)-\alpha}{(\beta-\alpha)}, \tag{49}$$

and the expected value of $Z^\pi(t)$ corresponding to the optimal parameter value ρ^* is given by

$$\alpha+\rho^*(\beta-\alpha) = \frac{z_0}{\widehat{\varrho}}e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}. \tag{50}$$

Substituting ρ^* into (47), we can get the optimal strategy corresponding to the optimal value $\alpha+\rho^*-\alpha)$ of expectation. Moreover, plugging the expression of ρ^* into the formula of $\mathbf{Var}[Z^{\pi^*}(T)]$, the optimal minimum variance $\mathbf{Var}[Z^{\pi^*,\rho^*}(T)]$ can also be determined corresponding to the uniquely optimal value $(z_0/\widehat{\varrho})e^{-T\left(-\widetilde{r}+\delta+\mu_t+\sigma_t^2\right)}$ of the expectation.

3.3. Special Case. The investment regulation is also important for an investment process without liabilities. Let $\mu_t=\sigma_t=0$, the original problem degenerates to the case of no liability, and the process $Z^\pi(t)$ becomes the dynamics process $\widehat{X}^\pi(t)$, as follows:

$$\begin{aligned} \mathrm{d}\widehat{X}^\pi(t) &= \left[\widetilde{r}+\pi_1(t)(\widetilde{\mu}_t+r-\widetilde{r})+\pi_2(t)(\widehat{\mu}_t-\widetilde{r})-\pi_1(t)\widetilde{\sigma}_t^2-\widetilde{\mu}_t\right] \\ &\quad\cdot\,\widehat{X}^\pi(t)\mathrm{d}t + (\pi_1(t)-1)\widetilde{\sigma}_t\widehat{X}^\pi(t)\mathrm{d}\widetilde{W}(t)+\pi_2(t)\widehat{\sigma}_t \\ &\quad\cdot\,\widehat{X}^\pi(t)\mathrm{d}\widehat{W}(t). \end{aligned} \tag{51}$$

Meanwhile, the regulation bound constraints also degenerate into a restriction on the investment dynamics process after eliminating inflation:

$$\alpha < \widehat{X}^\pi(T) < \beta. \tag{52}$$

Proposition 1. *For the special case $\mu_t=\sigma_t=0$, the minimum variance of $Z^\pi(t)$ is*

$$\begin{aligned} \mathbf{Var}\left[Z^*(T)\right] &= e^{-T\widetilde{\delta}}\left(\frac{(\alpha+\rho(\beta-\alpha))}{\left(1-e^{-T\widetilde{\delta}}\right)} - \frac{e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0}{\left(1-e^{-T\widetilde{\delta}}\right)}\right)^2 - 2e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\left(\frac{(\alpha+\rho(\beta-\alpha))}{\left(1-e^{-T\widetilde{\delta}}\right)} - \frac{e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0}{\left(1-e^{-T\widetilde{\delta}}\right)}\right)\widehat{x}_0 \\ &\quad - \left(\frac{e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0 - e^{-T\widetilde{\delta}}(\alpha+\rho(\beta-\alpha))}{\left(1-e^{-T\widetilde{\delta}}\right)}\right)^2 + e^{-T\left(-2r+\widetilde{\delta}+2\widetilde{\sigma}_t^2\right)}\widehat{x}_0^2, \end{aligned} \tag{53}$$

and the optimal strategy of the special case is given as follows:

$$\pi_1^*(t) = 1 - \frac{\left(\widetilde{\mu}_t + r - \widetilde{r} - \widetilde{\sigma}_t^2\right)}{\widetilde{\sigma}_t^2}\left(1 - \frac{e^{(t-T)\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}(\alpha + \rho(\beta-\alpha)) - e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0}{\widehat{x}e^{(t-T)\left(-2r+\widetilde{\delta}+2\widetilde{\sigma}_t^2\right)}\left(1-e^{-T\widetilde{\delta}}\right)}\right),$$

$$\pi_2^*(t) = -\frac{\left(\widehat{\mu}_t - \widetilde{r}\right)}{\widehat{\sigma}_t^2}\left(1 - \frac{e^{(t-T)\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}(\alpha + \rho(\beta-\alpha)) - e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0}{\widehat{x}e^{(t-T)\left(-2r+\widetilde{\delta}+2\widetilde{\sigma}_t^2\right)}\left(1-e^{-T\widetilde{\delta}}\right)}\right), \tag{54}$$

$$\pi_0^*(t) = \widetilde{\delta}\left(1 - \frac{e^{(t-T)\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}(\alpha + \rho(\beta-\alpha)) - e^{-T\left(-r+\widetilde{\delta}+\widetilde{\sigma}_t^2\right)}\widehat{x}_0}{\widehat{x}e^{(t-T)\left(-2r+\widetilde{\delta}+2\widetilde{\sigma}_t^2\right)}\left(1-e^{-T\widetilde{\delta}}\right)}\right),$$

where

$$\widetilde{\delta} = \frac{\left(\widetilde{\mu}_t + r - \widetilde{r} - \widetilde{\sigma}_t^2\right)^2}{\widetilde{\sigma}_t^2} + \frac{\left(\widehat{\mu}_t - \widetilde{r}\right)^2}{\widehat{\sigma}_t^2}. \tag{55}$$

Proof. The results can be obtained by the same calculation process as the original problem, so the details are omitted here. □

Remark 3. The expected value $(\alpha + \rho(\beta - \alpha))$ is not fixed at a determined position in the above proposition, but it can take any value between α and β depending on the value of $\rho \in (0, 1)$. Therefore, the obtained minimum variance depends on the parameter value $\rho \in (0, 1)$. Using the first-order necessary condition of extremum value for $\mathbf{Var}[Z^*(T)]$ with respect to the parameter ρ, the optimal value of parameter ρ is given by

$$\rho^* = \frac{e^{-T\left(-r+\widetilde{\sigma}_t^2\right)}\widehat{x}_0 - \alpha}{(\beta - \alpha)}, \tag{56}$$

and the corresponding optimal expectation

$$\left(\alpha + \rho^*(\beta - \alpha)\right) = e^{-T\left(-r+\widetilde{\sigma}_t^2\right)}\widehat{x}_0. \tag{57}$$

Substituting ρ^* into the optimal strategy $(\pi_0^*(t), \pi_1^*(t), \pi_2^*(t))$, the optimal strategy corresponding to the optimal value $\alpha + \rho^*(\beta - \alpha)$ of expectation can be obtained. Meanwhile, plugging the expression of ρ^* into $\mathbf{Var}[Z^{\pi^*}(t)]$, the optimal minimum variance $\mathbf{Var}[Z^{\pi^*,\rho^*}(T)]$ can also be determined. It is worth mentioning that the optimal minimum variance for this special case with no liability is zero under the uniquely optimal expectation $(z_0/\widehat{\varrho})e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}$.

4. Interpretations of the Main Results

The expressions of the obtained results are so abstract that it is necessary to give some interpretations for the main results to clarify their specific meaning, especially for the investment share $\pi_1(t)$ of the inflation-linked index bond and the investment share $\pi_2(t)$ of the risky stock.

Scrutinizing features of the expressions $\pi_1(t)$ and $\pi_2(t)$,

$$\pi_1^* = \frac{\left(\widetilde{\mu}_t + r - \widetilde{r}\right)}{\widetilde{\sigma}_t^2}\left(\frac{\zeta_\alpha^\beta(t)}{z} - 1\right),$$
$$\pi_2^* = \frac{\left(\widehat{\mu}_t - \widetilde{r}\right)}{\widehat{\sigma}_t^2}\left(\frac{\zeta_\alpha^\beta(t)}{z} - 1\right), \tag{58}$$

where $\zeta_\alpha^\beta(t) = e^{(t-T)(\widetilde{r}-\mu_t)}(\omega - \varepsilon)$, $\omega = (\alpha + \rho(\beta - \alpha))/(1 - \widehat{\varrho})$, and $\varepsilon = (e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}z_0)/(1 - \widehat{\varrho})$, it is easy to find that understanding implications of these simplified notations $\delta, \widehat{\varrho}, \omega, \varepsilon$, and $\zeta_\alpha^\beta(t)$ is the key point. Subsequently, the interpretations will start from descriptions of these simplified notations.

In notation $\delta = ((\widetilde{\mu}_t + r - \widetilde{r})^2/\widetilde{\sigma}_t^2) + ((\widehat{\mu}_t - \widetilde{r})^2/\widehat{\sigma}_t^2)$, the one part $(\widetilde{\mu}_t + r - \widetilde{r})^2/\widetilde{\sigma}_t^2$ is the risk compensation rate of inflation-linked index bond and the other part $(\widehat{\mu}_t - \widetilde{r})^2/\widehat{\sigma}_t^2$ is the risk compensation rate of risky stock. Thus, δ synthetically reflects the risk compensation of two risky assets.

Obviously, $e^{-T(\delta+\sigma_t^2)} < 1$, so the notation $\widehat{\varrho} = (e^{-T(\delta+\sigma_t^2)}\delta + \sigma_t^2)/(\delta + \sigma_t^2)$ is less than 1. Notation $\widehat{\varrho}$ gives a ratio of one sum with the risk compensation rate δ being converted by $e^{-T(\delta+\sigma_t^2)}$ divided by the other sum in which the risk compensation rate δ has no conversion.

The notation $\omega = (\alpha + \rho(\beta - \alpha))/(1 - \widehat{\varrho})$ is an expression of top regulation bound β and bottom regulation bound α, where $\alpha + \rho(\beta - \alpha)$ is a value of the expectation $\mathbf{E}[Z^\pi(T)]$. Because ρ takes a determined value, the changes of regulation bounds directly result in the changes of ω. Therefore, ω is the key point reflecting the influences of both regulation bounds on investment strategy.

In notation ε, the parameter z_0 has the most obvious economic connotation, since it represents the initial value of asset-liability ratio at time 0. The expression $(e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}z_0)/(1 - \widehat{\varrho})$ gives a value resulted from the initial asset-liability ratio z_0 influenced by several different factors (return rates and volatilities) and then converted at the constant relative rate $\widehat{\varrho}$.

The following expression

$$\frac{\zeta_\alpha^\beta(t)}{z} = \frac{e^{(t-T)(\widetilde{r}-\mu_t)}}{(1 - \widehat{\varrho})} \cdot \frac{\left((\alpha + \rho(\beta - \alpha)) - e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}z_0\right)}{z} \tag{59}$$

is key to understand the optimal strategy $(\pi_0^*, \pi_1^*, \pi_2^*)$. The denominator z of this fraction $\zeta_\alpha^\beta(t)/z$ represents the current level of asset-liability ratio at time t.

But the numerator $\zeta_\alpha^\beta(t)$ of this fraction is more complicated. The one multiplier $((\alpha+\rho(\beta-\alpha))-z_0e^{-T(-\widetilde{r}+\delta+\mu+\widetilde{\sigma}_t^2)})$ is a difference between the expected value $(\alpha+\rho(\beta-\alpha))$ of $Z(T)$ and the converted value $e^{-T(-\widetilde{r}+\delta+\mu_t+\sigma_t^2)}z_0$ of the initial value z_0, which characterizes the distance between initial value affected by various economic factors and the expectation of $Z(T)$. The other multiplier $e^{(t-T)(\widetilde{r}-\mu_t)}/(1-\widehat{\varrho})$ is a converted value comprehensively influenced by several parameters embodying the states of financial market.

In addition, it is easy to see that if the relative rate $\zeta_\alpha^\beta(t)/z$ is larger than 1, $\pi_1^*(t)=((\widetilde{\mu}_t+r-\widetilde{r})/\widetilde{\sigma}_t^2)((\zeta_\alpha^\beta(t)/z)-1)$ and $\pi_2^*(t)=((\widehat{\mu}_t-\widetilde{r})/\widehat{\sigma}_t^2)(\zeta_\alpha^\beta(t)/z-1)$ are positive; otherwise, both will be negative. Since $\pi_0^*(t)=1-\pi_1^*(t)-\pi_2^*(t)$, it indicates that $\pi_0^*(t)=1+\delta(1-(1/z)\zeta_\alpha^\beta(t))$ can be determined as long as the investment shares of risky assets have been determined, so the adjustment of risk-free asset is usually passive. If the relative rate $\zeta_\alpha^\beta(t)/z$ is greater than 1, the investment shares on risk-free asset can be ensured within a much better range between 0 and 1. These observations can serve as an important reference for determining both the top regulation bound β and the bottom regulation bound α.

5. Numerical Examples

This section discusses the variation tendency of optimal strategies π_1^* and π_2^* varying with some important parameters, mainly focusing on the current value z of asset-liability ratio, the top regulation bound β, the bottom regulation bound α, and the parameter ρ. The values of parameters are given in Table 1; unless otherwise specified, other values for the same variable can be consulted from the corresponding graphic legend.

First, let top regulation bound β be fixed; if the bottom regulation bound takes much higher values, then the investment shares of both inflation-linked index bond and risky stock increase significantly. These variety trends can be observed intuitively from Figures 1 and 2, respectively. Meanwhile, Figures 1 and 2 also show that the appropriate bottom regulation bound should be taken at one value between 1 and 2 for an economic environment same as this example. Otherwise, the investment shares on risky assets may increase too much, but the investment share on risk-free asset becomes negative, which may result in leverage risk increases rapidly.

Next, let bottom regulation bound α be fixed; if the top regulation bound β takes much higher value, then the investment shares of both inflation-linked index bond and risky stock should also increase significantly. The variety trends can be perceived from Figures 3 and 4, respectively. Meanwhile, Figures 3 and 4 also show that the appropriate top regulation bound should be taken at one value between 6 and 7 for an economic environment same as this example. Otherwise, the investment shares on risky assets increase too much, but the investment share on risk-free asset becomes negative, which also result in leverage risk increases rapidly.

As can be seen from Figures 3–6, taking the larger value of either the top regulation bound or the bottom regulation bound means allowing a wider volatility range of investment returns, which may lead to the result that assets with greater investment risk are allowed to invest, or greater market risks are incorporated into the wealth process. In addition, the above numerical analysis roughly gives a valuable reference for selecting an appropriate range of top regulation bound and bottom regulation bound according to the relationship between the reasonable values of investment strategy and regulation bounds for an economic environment set by values of parameters.

Table 1: Values of the parameters.

$\widetilde{\mu}=0.08$	$r=0.01$	$\widetilde{\sigma}=0.2$	$\alpha=1$	$\beta=7$	$z_0=3$	$z=5$
$\widetilde{r}=0.05$	$T=6$	$\widehat{\mu}=0.15$	$\widehat{\sigma}=0.6$	$\mu=0.03$	$\sigma=0.1$	$\rho=0.5$

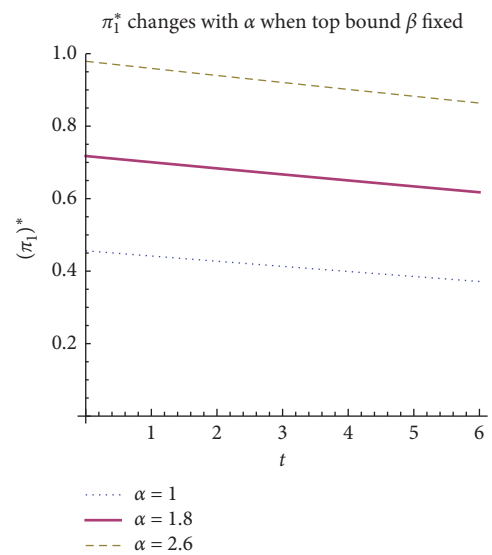

Figure 1: Values of π_1^* change with t for different bottom bound α.

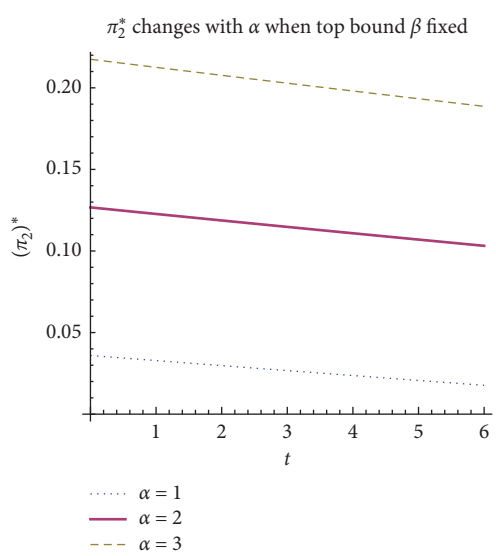

Figure 2: Values of π_2^* change with t for different bottom bound α.

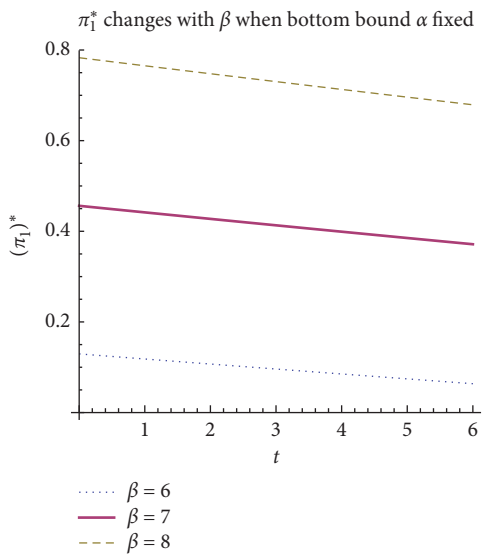

Figure 3: Values of π_1^* change with t for different top bound β.

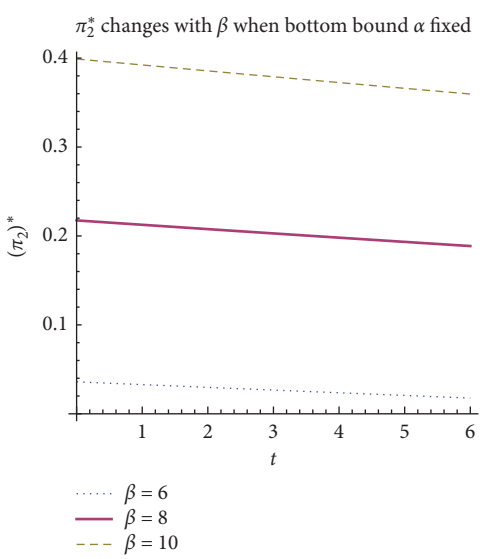

Figure 4: Values of π_2^* change with t for different top bound β.

In fact, the regulatory bound cannot be determined arbitrarily. Both the top regulatory bound and bottom regulatory bound determine the fluctuation range that decision-making needs to control. Too narrow fluctuation range of investment returns results in loss of great investment opportunities and makes investment in the financial market meaningless. If the regulatory bound is too large, the potential huge risks cannot be dealt with effectively when the investment return soars fast, which is almost equivalent to situations without regulatory bounds and also lose the significance of using regulatory bounds. It is hard to determine the regulatory bounds only by quantitative calculation. Once there is a model error, it may result in serious deviation between theory and practice in actual management; thus, the determination of upper and lower limits should not be purely theoretical. Nevertheless, it should be

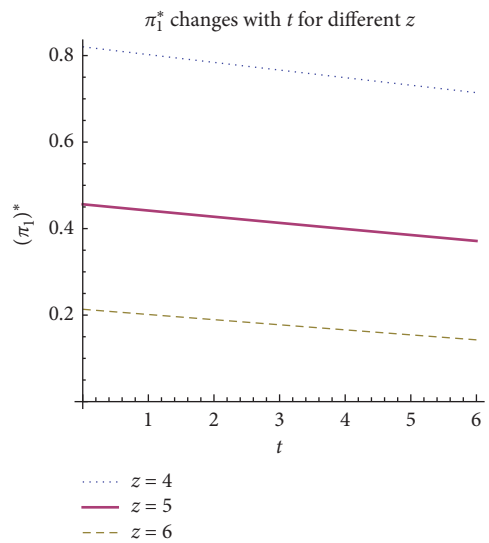

Figure 5: Values of π_1^* change with t for different current asset-liability ratio z.

noted that advanced statistical analysis and maturely practical experience of personnel are essential and of great advantage for determining an exact value of α or β.

Assuming that both top regulation bound β and bottom regulation bound α are determined, the impact of asset-liability ratio z at time t on investment strategy can be observed in Figures 5 and 6. If the current asset-liability ratio z at time t takes much greater value, then the investment shares of both inflation-linked index bond and risky stock should be much lower, which shows that increasing shares of investment on risk-free asset is an important measure to reduce investment risk. For the numerical examples shown in Figures 5 and 6, the regulation bounds being fixed at $\alpha = 1$ and $\beta = 7$, the most appropriate asset-liability ratio should take values around 5 which is much more advantageous for the company.

The impact of parameter ρ on investment strategy can be observed in Figures 7 and 8. If the parameter ρ takes a much greater value, then the investment shares of risky assets must increase correspondingly. Since the top regulation bound β and the bottom regulation bound α are predetermined, the expected value $\alpha + \rho(\beta - \alpha)$ of asset-liability ratio entirely depends on the value of parameter ρ. Therefore, the variation tendency of investment shares $\pi_1^*(t)$ and $\pi_2^*(t)$ with parameter ρ actually illustrates the trend of investment shares changing with the expectation of asset-liability ratio $Z(T)$. In one word, the greater the expectation of $Z(T)$, the much higher the shares of the company's wealth to be invested on risky assets.

6. Conclusion

Some empirical studies in the banking literature deal with the effects of prolonged periods of low interest rates in the economy on risk-taking, real effects on the real sector and also on financial stability and so on as given by Chaudron [20]; Bikker; and Vervliet [21]. However, this paper uses

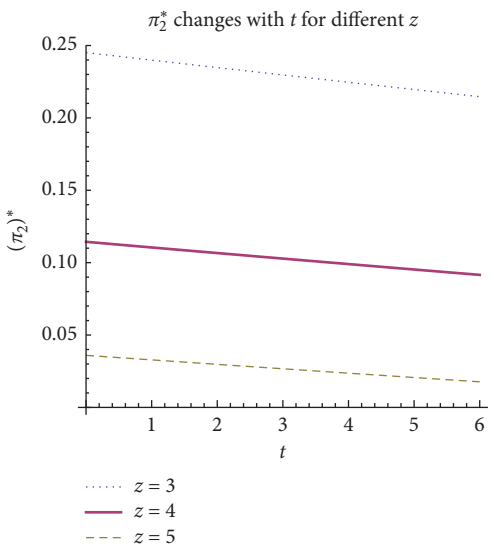

Figure 6: Values of π_2^* change with t for different current asset-liability ratio z.

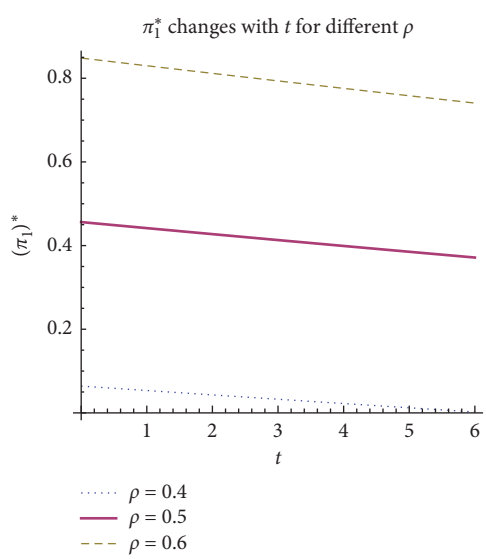

Figure 7: Values of π_1^* change with t for different ρ.

mathematical models and methods to study a problem of asset-liability management with financial market risks: risks of too low return rate, risks of the abnormal rapid growth of return rate, and risks of investment leverage. A model of quantitative regulation (both the top regulation bound and the bottom regulation bound being imposed on the asset-liability ratio) is put forward. The efficient strategy and efficient frontier are obtained under the objective of variance minimization with regulation constraints at the regulatory time T. But the most important value of a theoretical research lies in its guidance and reference for practice. Through numerical examples, it is found that the obtained explicit optimal strategy can also provide reference for determination of regulation bounds, which reinforces the theoretical significance of this research work.

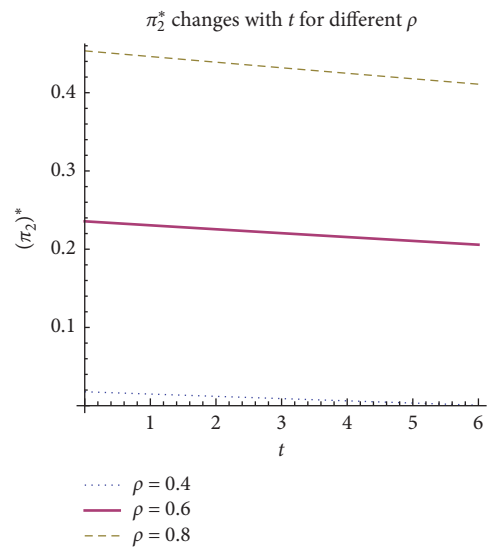

Figure 8: Values of π_2^* change with t for different ρ.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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