


Research Article

Mei Symmetry and New Conserved Quantities of Time-Scale Birkhoff's Equations

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The time-scale dynamic equations play an important role in modeling complex dynamical processes. In this paper, the Mei symmetry and new conserved quantities of time-scale Birkhoff's equations are studied. The definition and criterion of the Mei symmetry of the Birkhoffian system on time scales are given. The conditions and forms of new conserved quantities which are found from the Mei symmetry of the system are derived. As a special case, the Mei symmetry of time-scale Hamilton canonical equations is discussed and new conserved quantities for the Hamiltonian system on time scales are derived. Two examples are given to illustrate the application of results.

1. Introduction

In 1988, Hilger [1] proposed the calculus on time scales to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both. This theory is really important and plays a useful role in modeling complex dynamic processes and has a tremendous potential for applications [2–4].

Symmetry is an important aspect in studying dynamic equations. From a symmetry, we can find a conserved quantity or the first integral of dynamic equations. The famous Noether symmetry theory which reveals the relationships between symmetries and conservation laws has been applied to a time-scale analogue of analytical mechanics, modern theoretical physics, and engineering [5–10]. Except for the Noether symmetry method, the other two popular symmetry methods, the Lie symmetry method [11] and the Mei symmetry method [12], are widely applied in studying dynamic systems. The Mei symmetry means that when the dynamic functions are replaced by the transformed functions under the infinitesimal transformations of group, the forms of the differential equations of motion keep invariant. Here, the dynamic functions are the Lagrangian, the

Hamiltonian, the Birkhoffian, Birkhoff's functions, the generalized forces, the generalized constraint reactions, etc. New kinds of conserved quantity can be led by Mei symmetry. This method has been successfully applied in equations of motion for Lagrangian systems, Hamiltonian systems, Birkhoffian systems, the motion of charged particles in an electromagnetic field, the equation of nonmaterial volumes, the equation of thin elastic rod, etc. [13–21].

The Birkhoffian mechanics [22] represents a new stage in the development of analytical mechanics. It is a natural development of the Hamiltonian mechanics and has valuable applications in hadronic physics, spatial mechanics, statistical mechanics, biophysics, and engineering [23–26]. Over the past two decades, important achievements, including the symmetries and conserved quantities of Birkhoffian systems, the Birkhoffian dynamic inverse problems, the stability of motion of Birkhoff's equations, the Birkhoffian systems with time delay, and the fractional Birkhoffian systems, have been made [27–38].

The Pfaff–Birkhoff principle and Birkhoff's equations were studied in a time-scale analogue, and their corresponding Noether symmetry and Lie symmetry as well as the conserved quantities have been studied [39,40] recently.

However, the Mei symmetry of Birkhoffian systems on time scales has not been done. It is worth going a step further to find more new conserved quantities of Birkhoff's equations by another symmetry method. In this paper, we will study the Mei symmetry and new conserved quantities of time-scale Birkhoff's equations.

This paper is organized as follows: in Section 2, we first give the definition and criterion of Mei symmetry for the Birkhoffian system on time scales. Three kinds of conserved quantities led by Mei symmetry of the system on time scales are given in Section 3. As a special case, Section 4 deals with the Mei symmetry of time-scale Hamilton canonical equations and the corresponding new conserved quantities for the Hamiltonian system on time scales. Two examples are given in Section 5 to show the application of results. In the end, conclusions and future works are given.

2. Mei Symmetry of Time-Scale Birkhoff's Equations

For basic knowledge about the calculus on time scales, the readers can refer to References [2, 3].

The Pfaff–Birkhoff action on time scales is

$$S = \int_{t_1}^{t_2} [R_\omega(t, a_v^\sigma(t)) a_\omega^\Delta - B(t, a_v^\sigma(t))] \Delta t. \quad (1)$$

According to the Pfaff–Birkhoff principle on time scales,

$$\delta S = 0, \quad (2)$$

with boundary conditions

$$\delta a_v|_{t=t_1} = \delta a_v|_{t=t_2} = 0 \quad (3)$$

and exchange relations

$$\begin{aligned} \delta a_v^\Delta &= (\delta a_v)^\Delta, \\ \delta a_v^\sigma &= (\delta a_v)^\sigma, \end{aligned} \quad (4)$$

the time-scale Birkhoff's equations can be derived, i.e.,

$$\begin{aligned} \frac{\partial R_\omega(t, a_v^\sigma(t))}{\partial a_v^\sigma} a_\omega^\Delta - R_v^\Delta(t, a_v^\sigma(t)) - \frac{\partial B(t, a_v^\sigma(t))}{\partial a_v^\sigma} &= 0, \\ (v, \omega, \rho &= 1, 2, \dots, 2n), \end{aligned} \quad (5)$$

where $B: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the Birkhoffian function, $R_\omega: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are Birkhoff's functions, a_ω^Δ are the delta derivatives of the Birkhoffian variables $a_v(t)$ with respect to t , and $a_v^\sigma(t) = (a_v \circ \sigma)(t)$. This result can be found in Ref. [39].

Introduce the one-parameter infinitesimal transformations of time and variables a_v as

$$\begin{aligned} t^* &= t + \varepsilon \xi_0(t, a_\omega(t)), \\ a_v^* &= a_v(t) + \varepsilon \xi_v(t, a_\omega(t)), \\ v, \omega &= 1, 2, \dots, 2n, \end{aligned} \quad (6)$$

where ξ_0 and ξ_v are infinitesimals and the infinitesimal parameter $\varepsilon \in \mathbb{R}$. The corresponding infinitesimal generator $X^{(0)}$ is

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_v \frac{\partial}{\partial a_v}, \quad (7)$$

and the first extended infinitesimal generator is

$$X^{(1)} = X^{(0)} + (\xi_v^\Delta - a_v^\Delta \xi_0^\Delta) \frac{\partial}{\partial a_v^\Delta}. \quad (8)$$

The Birkhoffian B and Birkhoff's functions R_ω become the new Birkhoffian B^* and the new Birkhoff's functions R_ω^* under transformations (6), that is,

$$\begin{aligned} B^* &= B(t^*, a_v^{*\sigma^*}) = B(t + \varepsilon \xi_0, a_v^\sigma + \varepsilon \xi_v^\sigma), \\ R_\omega^* &= R_\omega(t^*, a_v^{*\sigma^*}) = R_\omega(t + \varepsilon \xi_0, a_v^\sigma + \varepsilon \xi_v^\sigma). \end{aligned} \quad (9)$$

By performing Taylor series expansion at the point $\varepsilon = 0$, we have

$$B^* = B(t, a_v^\sigma) + \varepsilon X^{(0)}(B) + O(\varepsilon^2), \quad (10)$$

$$R_\omega^* = R_\omega(t, a_v^\sigma) + \varepsilon X^{(0)}(R_\omega) + O(\varepsilon^2). \quad (11)$$

Definition 1. If the form of equation (5) keeps invariant when the Birkhoffian B and Birkhoff's functions R_ω are replaced by B^* and R_ω^* , respectively, i.e.,

$$\frac{\partial R_\omega^*}{\partial a_v^\sigma} a_\omega^\Delta - R_v^{*\Delta} - \frac{\partial B^*}{\partial a_v^\sigma} = 0, \quad (v, \omega = 1, 2, \dots, 2n), \quad (12)$$

then this invariance is called the Mei symmetry of the Birkhoffian system on time scales.

From Definition 1 and formulae (10) and (11), we have the following.

Criterion 1. For the Birkhoffian system (5) on time scales, if the infinitesimals ξ_0 and ξ_v satisfy

$$\frac{\partial X^{(0)}(R_\omega)}{\partial a_v^\sigma} a_\omega^\Delta - (X^{(0)}(R_v))^\Delta - \frac{\partial X^{(0)}(B)}{\partial a_v^\sigma} = 0, \quad (v, \omega = 1, 2, \dots, 2n), \quad (13)$$

then the corresponding invariance is Mei symmetry.

3. New Conserved Quantities of Birkhoffian Systems on Time Scales

For the Birkhoffian system (5) on time scales, new conserved quantities can be found from Mei symmetry.

Theorem 1. *If the infinitesimals ξ_0 and ξ_v of the Mei symmetry of the Birkhoffian system (5) and the gauge function $G_1(t, a_v^\sigma)$ satisfy the following structural equation:*

$$\begin{aligned} [X^{(0)}(R_v) a_v^\Delta - X^{(0)}(B)] \xi_0^\Delta + X^{(1)} [X^{(0)}(R_v) a_v^\Delta - X^{(0)}(B)] \\ + \mu(t) \left(\frac{\partial X^{(0)}(R_\omega)}{\partial a_v^\sigma} - \frac{\partial X^{(0)}(B)}{\partial t} \right) a_v^\Delta \xi_0^\Delta + G_1^\Delta = 0, \end{aligned} \quad (14)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$I_1 = X^{(0)}(R_v)\xi_v - \left[X^{(0)}(B) + \mu(t) \left(\frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial t} \right) \right] \xi_0 + G_1 = \text{const.} \quad (15)$$

Proof. To prove that formula (15) is a conserved quantity, we need to prove

$$\frac{\Delta}{\Delta t} I_1 = 0. \quad (16)$$

Therefore, we have

$$\begin{aligned} \frac{\Delta}{\Delta t} I_1 &= X^{(0)}(R_v)\xi_v^\Delta + \frac{\Delta}{\Delta t} (X^{(0)}(R_v))\xi_v^\sigma - \\ &\cdot \left[X^{(0)}(B) + \mu(t) \left(\frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial t} \right) \right] \xi_0^\Delta \\ &- \frac{\Delta}{\Delta t} \left[X^{(0)}(B) + \mu(t) \left(\frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial t} \right) \right] \xi_0^\sigma + G_1^\Delta. \end{aligned} \quad (17)$$

Here, for system (5), the following energy equation [40]:

$$\frac{\Delta}{\Delta t} \left[B - \mu(t) \left(\frac{\partial B}{\partial t} - \frac{\partial R_\omega}{\partial t} a_\omega^\Delta \right) \right] = \frac{\partial B}{\partial t} - \frac{\partial R_\omega}{\partial t} a_\omega^\Delta, \quad (18)$$

holds. After the infinitesimal transformations, the new energy equation

$$\frac{\Delta}{\Delta t} \left[B^* - \mu(t) \left(\frac{\partial B^*}{\partial t} - \frac{\partial R_\omega^*}{\partial t} a_\omega^\Delta \right) \right] = \frac{\partial B^*}{\partial t} - \frac{\partial R_\omega^*}{\partial t} a_\omega^\Delta, \quad (19)$$

holds. Noting formulae (10) and (11), we get

$$\begin{aligned} \frac{\Delta}{\Delta t} \left[X^{(0)}(B) - \mu(t) \left(\frac{\partial X^{(0)}(B)}{\partial t} - \frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta \right) \right] \\ = \frac{\partial X^{(0)}(B)}{\partial t} - \frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta. \end{aligned} \quad (20)$$

Thus, combining equation (14) with (20) yields

$$\begin{aligned} \frac{\Delta}{\Delta t} I_1 &= [X^{(0)}(R_v)a_v^\Delta - X^{(0)}(B)]\xi_0^\Delta + X^{(1)}[X^{(0)}(R_v)a_v^\Delta \\ &- X^{(0)}(B)] + \mu(t) \left(\frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial t} \right) a_v^\Delta \xi_0^\Delta \\ &+ G_1^\Delta = 0. \end{aligned} \quad (21)$$

The proof is completed.

Theorem 1 gives the Mei conserved quantity (15) on time scales led directly by Mei symmetry of system (5) with considering the structure equation (14). When $\mathbb{T} = \mathbb{R}$, the conserved quantity (15) becomes the classical Mei conserved quantity. \square

Remark 1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$, and the conserved quantity (15) becomes the classical one [14]

$$I_1 = X^{(0)}(R_v)\xi_v - X^{(0)}(B)\xi_0 + G_1 = \text{const.} \quad (22)$$

Remark 2. If $\mathbb{T} = h^{\mathbb{N}_0} = \{h^i: i \in \mathbb{N}_0\}, h > 1$, then $\sigma(t) = ht$ and $\mu(t) = (h-1)t$, and the conserved quantity (15) becomes the quantum one

$$\begin{aligned} I_1 &= X^{(0)}(R_v)\xi_v - \left[X^{(0)}(B) + (h-1)t \left(\frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial t} \right) \right] \xi_0 + G_1 = \text{const}, \end{aligned} \quad (23)$$

where $a_\omega^\Delta = [a_\omega(ht) - a_\omega(t)]/(h-1)t$.

Theorem 2. If the infinitesimals ξ_0 and ξ_v of the Mei symmetry of the Birkhoffian system (5) and the gauge function $G_2(t, a_v^\sigma)$ satisfy the following condition:

$$a_v^\sigma \frac{\Delta}{\Delta t} \left[\frac{\partial X^{(0)}(R_\omega)}{\partial a_v^\sigma} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial a_v^\sigma} \right] - a_v^\Delta \frac{\Delta}{\Delta t} [X^{(0)}(R_v)] + G_2^\Delta = 0, \quad (24)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$I_2 = \left[\frac{\partial X^{(0)}(R_\omega)}{\partial a_v^\sigma} a_\omega^\Delta - \frac{\partial X^{(0)}(B)}{\partial a_v^\sigma} \right] [a_v^\sigma - \mu(t)a_v^\Delta] + G_2 = \text{const.} \quad (25)$$

With taking note of equation (13) and condition (24), we can derive that formula (25) is a conserved quantity of system (5).

Theorem 3. If the infinitesimals ξ_0 and ξ_v of the Mei symmetry of the Birkhoffian system (5) and the gauge function $G_3(t, a_v^\sigma)$ satisfy the following condition:

$$\frac{\partial X^{(0)}(B)}{\partial t} - \frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta + G_3^\Delta = 0, \quad (26)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$\begin{aligned} I_3 &= X^{(0)}(B) - \mu(t) \left(\frac{\partial X^{(0)}(B)}{\partial t} - \frac{\partial X^{(0)}(R_\omega)}{\partial t} a_\omega^\Delta \right) \\ &+ G_3 = \text{const.} \end{aligned} \quad (27)$$

With taking note of formulae (20) and (26), we can derive that formula (27) is a conserved quantity of system (5).

Theorems 2 and 3 give the other two kinds of conserved quantities (25) and (27) on time scales also led by Mei symmetry with considering conditions (24) and (26).

4. Mei Symmetry and Conserved Quantities of Time-Scale Hamilton Canonical Equations

The Birkhoffian mechanics is a natural development of the Hamiltonian mechanics; if we take

$$\begin{aligned} a_v^\sigma &= \begin{cases} q_v^\sigma, & v = 1, 2, \dots, n, \\ p_{v-n}, & v = n+1, n+2, \dots, 2n, \end{cases} \\ R_v &= \begin{cases} p_v, & v = 1, 2, \dots, n, \\ 0, & v = n+1, n+2, \dots, 2n, \end{cases} \\ B &= H, \end{aligned} \quad (28)$$

then we can obtain the time-scale Hamiltonian canonical equations [8]

$$\begin{aligned} q_k^\Delta &= \frac{\partial H}{\partial p_k}, \\ p_k^\Delta &= -\frac{\partial H}{\partial q_k^\sigma}, \\ k &= 1, 2, \dots, n. \end{aligned} \quad (29)$$

Now, we introduce the infinitesimal transformations

$$\begin{cases} t^* = t + \varepsilon \xi_0(t, q_s(t), p_s(t)), \\ q_k^* = q_k(t) + \varepsilon \xi_k(t, q_s(t), p_s(t)), \\ p_k^* = p_k(t) + \varepsilon \eta_k(t, q_s(t), p_s(t)), \end{cases} \quad (30)$$

where ξ_0 , ξ_k , and η_k are infinitesimals and the infinitesimal parameter $\varepsilon \in \mathbb{R}$. Undergoing transformation (30), the Hamiltonian H becomes H^* . Then,

$$H^* = H(t, q_k^\sigma, p_k) + \varepsilon X^{(0)}(H) + O(\varepsilon^2), \quad (31)$$

where

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_k \frac{\partial}{\partial q_k} + \eta_k \frac{\partial}{\partial p_k}. \quad (32)$$

Definition 2. If the form of equations (29) keep invariant when the Hamiltonian H is replaced by H^* , i.e.,

$$\begin{aligned} q_k^\Delta &= \frac{\partial H^*}{\partial p_k}, \\ p_k^\Delta &= -\frac{\partial H^*}{\partial q_k^\sigma}, \\ k &= 1, 2, \dots, n, \end{aligned} \quad (33)$$

then this invariance is called the Mei symmetry of the Hamiltonian system on time scales.

Criterion 2. For the Hamiltonian system (29) on time scales, if the infinitesimals ξ_0 , ξ_k , and η_k satisfy

$$\begin{aligned} \frac{\partial X^{(0)}(H)}{\partial q_k^\sigma} &= 0, \\ \frac{\partial X^{(0)}(H)}{\partial p_k} &= 0, \\ k &= 1, 2, \dots, n, \end{aligned} \quad (34)$$

then the corresponding invariance is Mei symmetry.

Thus, we have the following three kinds of conserved quantities led by Mei symmetry of the Hamiltonian system.

Theorem 4. If the infinitesimals ξ_0 , ξ_k , and η_k of the Mei symmetry of the Hamiltonian system (29) and the gauge function $G_1(t, q_k^\sigma, p_k)$ satisfy the following structural equation:

$$\begin{aligned} -X^{(0)}(H)\xi_0^\Delta + \frac{\partial X^{(0)}(H)}{\partial p_k} \eta_k + X^{(0)}(p_k)\xi_k^\Delta - X^{(1)} \\ [X^{(0)}(H)] - \mu(t) \frac{\partial X^{(0)}(H)}{\partial q_k^\sigma} q_k^\Delta \xi_0^\Delta + G_1^\Delta = 0, \end{aligned} \quad (35)$$

where

$$X^{(1)} = X^{(0)} + (\xi_k^\Delta - q_k^\Delta \xi_0^\Delta) \frac{\partial}{\partial q_k^\Delta} + (\eta_k^\Delta - p_k^\Delta \xi_0^\Delta) \frac{\partial}{\partial p_k^\Delta}, \quad (36)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$\begin{aligned} I_1 = X^{(0)}(p_k)\xi_k + \left[\mu(t) \frac{\partial X^{(0)}(H)}{\partial t} - X^{(0)}(H) \right] \xi_0 \\ + G_1 = \text{const.} \end{aligned} \quad (37)$$

Remark 3. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$, and the conserved quantity (37) becomes the classical one [16]

$$I_1 = X^{(0)}(p_k)\xi_k - X^{(0)}(H)\xi_0 + G_1 = \text{const.} \quad (38)$$

Remark 4. If $\mathbb{T} = h\mathbb{N}_0 = \{h^i : i \in \mathbb{N}_0\}, h > 1$, then $\sigma(t) = ht$ and $\mu(t) = (h-1)t$, and the conserved quantity (37) becomes the quantum one

$$\begin{aligned} I_1 = X^{(0)}(p_k)\xi_k + \left[(h-1)t \frac{\partial X^{(0)}(H)}{\partial t} - X^{(0)}(H) \right] \xi_0 \\ + G_1 = \text{const.} \end{aligned} \quad (39)$$

Theorem 5. If the infinitesimals ξ_0 , ξ_k , and η_k of the Mei symmetry of the Hamiltonian system (29) and the gauge function $G_2(t, q_k^\sigma, p_k)$ satisfy the following condition:

$$-q_k^\sigma \frac{\Delta}{\Delta t} \left[\frac{\partial X^{(0)}(H)}{\partial q_k^\sigma} \right] - q_k^\Delta \frac{\Delta}{\Delta t} [X^{(0)}(p_k)] + G_2^\Delta = 0, \quad (40)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$I_2 = -\frac{\partial X^{(0)}(H)}{\partial q_k^\sigma} [q_k^\sigma - \mu(t)q_k^\Delta] + G_2 = \text{const.} \quad (41)$$

Theorem 6. If the infinitesimals ξ_0 , ξ_k , and η_k of the Mei symmetry of the Hamiltonian system (29) and the gauge function $G_3(t, q_k^\sigma, p_k)$ satisfy the following condition:

$$\frac{\partial X^{(0)}(H)}{\partial t} + G_3^\Delta = 0, \quad (42)$$

then the Mei symmetry of the system can lead to the new conserved quantity

$$I_3 = X^{(0)}(H) - \mu(t) \frac{\partial X^{(0)}(H)}{\partial t} + G_3 = \text{const.} \quad (43)$$

Theorems 5 and 6 give the other two kinds of conserved quantities (41) and (43) on time scales also led by Mei symmetry with considering conditions (40) and (42).

5. Examples

Example 1. Birkhoff's functions and Birkhoffian of a fourth-order Birkhoffian system on time scales are

$$\begin{aligned} R_1 &= a_3^\sigma, \\ R_2 &= a_4^\sigma, \\ R_3 &= 0, \\ R_4 &= 0, \\ B &= \frac{1}{2} [(a_3^\sigma)^2 + (a_4^\sigma)^2] + a_2^\sigma, \end{aligned} \quad (44)$$

which leads us to study the Mei symmetry and the conserved quantity of the system.

The equations of the system are

$$\begin{aligned} -(a_3^\sigma)^\Delta &= 0, \\ -(a_4^\sigma)^\Delta - 1 &= 0, \\ a_1^\Delta - a_3^\sigma &= 0, \\ a_2^\Delta - a_4^\sigma &= 0. \end{aligned} \quad (46)$$

If we choose the infinitesimals

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= 0, \\ \xi_2 &= a_4^\sigma, \\ \xi_3 &= 0, \\ \xi_4 &= -1, \end{aligned} \quad (47)$$

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= -1, \\ \xi_2 &= \frac{1}{2}(a_3^\sigma)^2, \\ \xi_3 &= a_3^\sigma, \\ \xi_4 &= 0, \end{aligned} \quad (48)$$

we can verify that infinitesimals (47) and (48) satisfy the criterion equation (13); therefore, the corresponding invariance is Mei symmetry.

Substituting infinitesimal (47) into the structure equation (14), we have the gauge function $G_1 = -t$. According to Theorem 1, we have

$$I_1 = -a_4^\sigma - t = \text{const.} \quad (49)$$

Substituting infinitesimal (48) into the structure equation (14), we have the gauge function $G_1 = 0$. From Theorem 1, we have

$$I_1 = -a_3^\sigma = \text{const.} \quad (50)$$

The conserved quantities (49) and (50) show that different infinitesimals correspond to different forms of conserved quantities.

From formulae (47) and (24), we obtain

$$G_2 = 0. \quad (51)$$

According to Theorem 2, we have

$$I_2 = 0. \quad (52)$$

This conserved quantity is a trivial one.

From formulae (48) and (26), we obtain

$$G_3 = 0. \quad (53)$$

According to Theorem 3, we have

$$I_3 = \frac{1}{2}(a_3^\sigma)^2 = \text{const.} \quad (54)$$

Conserved quantity (54) is led by the Mei symmetry of the system.

Example 2. The Hamiltonian of a system on time scales is

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_1^\sigma + q_2^\sigma, \quad (55)$$

which leads us to study the Mei symmetry and the conserved quantity of the system.

The canonical equations of the system are

$$\begin{aligned} q_1^\Delta &= p_1, \\ q_2^\Delta &= p_2, \\ q_3^\Delta &= p_3, \\ p_1^\Delta &= -1, \\ p_2^\Delta &= -1, \\ p_3^\Delta &= 0. \end{aligned} \quad (56)$$

Taking the calculation, we have

$$X^{(0)}(H) = \xi_1 + \xi_2 + \eta_1 p_1 + \eta_2 p_2 + \eta_3 p_3. \quad (57)$$

If we choose the infinitesimals as

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= p_1, \\ \xi_2 &= 0, \\ \xi_3 &= 0, \\ \eta_1 &= -1, \\ \eta_2 &= 0, \\ \eta_3 &= 0, \end{aligned} \quad (58)$$

$$\begin{aligned} \xi_0 &= 0, \\ \xi_1 &= p_2, \\ \xi_2 &= p_1 + p_2, \\ \xi_3 &= 0, \\ \eta_1 &= -1, \\ \eta_2 &= -2, \\ \eta_3 &= 0, \end{aligned} \quad (59)$$

then

$$X^{(0)}(H) = 0. \quad (60)$$

Substituting (58) into (35), we obtain $G_1 = -t$. According to Theorem 4, we have

$$I_1 = -p_1 - t = \text{const}. \quad (61)$$

Substituting (59) into (35), we obtain $G_1 = -5t$. From Theorem 4, we have

$$I_1 = -2p_1 - 3p_2 - 5t = \text{const}. \quad (62)$$

The conserved quantities (61) and (62) show that different infinitesimals correspond to different forms of conserved quantities.

Substituting (58) and (59) into (40), respectively, we obtain $G_2 = 0$. From Theorem 5, we get a trivial conserved quantity $I_2 = 0$.

Taking note of formula (60), we obtain $G_3 = 0$. From Theorem 6, we also get a trivial conserved quantity $I_3 = 0$.

6. Conclusions

In recent years, the theory of time-scale calculus becomes widely available in describing plenty systems that contain both continuous-time and discrete-time domains. Symmetry theory is an important method to find a conserved quantity or the first integral of the equations of dynamical systems. This paper presented and studied the Mei symmetry of Birkhoffian systems on time scales. Three new forms of conserved quantities on time scales were derived from the Mei symmetry. In addition, as a special case, the Mei symmetry and conserved quantities for the Hamiltonian system are studied. The examples given illustrated the effectiveness of the results. The results of this paper show a method to find the conserved quantity or the first integral of dynamical equations on time scales. The method is universal. The continuous results and discrete results are special cases. By choosing different time scales, general results can be obtained directly without repeated proof.

Further works about the Mei symmetry for general holonomic systems, nonholonomic systems, Hamiltonian systems on time scales, and Birkhoffian systems on time scales, as well as its relation to Lie symmetry on time scales, are still worth doing.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this research work.

Authors' Contributions

All authors contributed equally to this manuscript and have read and approved the final version.

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