Research Article

Interval-Valued Pythagorean Hesitant Fuzzy Set and Its Application to Multiattribute Group Decision-Making

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Pythagorean hesitant fuzzy sets are widely watched because of their excellent ability to deal with uncertainty, imprecise and vague information. This paper extends Pythagorean hesitant fuzzy environments to interval-valued Pythagorean hesitant fuzzy environments and proposes the concept of interval-valued Pythagorean hesitant fuzzy set (IVPHFS), which allows the membership of each object to be a set of several pairs of possible interval-valued Pythagorean fuzzy elements. Furthermore, we develop a series of aggregation operators for interval-valued Pythagorean hesitant fuzzy information and apply them to multiattribute group decision-making (MAGDM) problems. Then, some desired operational laws and properties of IVPHFSs are studied. Especially, considering an interval-valued Pythagorean fuzzy element (IVPHFE) is formed by several pairs of interval values, this paper proposes the concepts of score function and accuracy function in the form of two interval numbers which can retain interval-valued Pythagorean fuzzy information as much as possible. Then, the relationship among these operators is discussed by comparing the interval numbers. Eventually, an illustrative example fully shows the feasibility, practicality, and effectiveness of the proposed approach.

1. Introduction

There are numerous uncertain, imprecise, and incomplete problems in the real world. Zadeh's fuzzy set theory [1] is a successful and effective tool to solve many similar problems. The theory and its primary extensions, such as intuitionistic fuzzy sets [2], interval-valued intuitionistic fuzzy sets [3], type-2 fuzzy sets [4], interval type-2 fuzzy sets [5], and q -rung orthopair fuzzy sets [6], have been successfully applied to pattern recognition [7], medical diagnosis [8], fuzzy logic [9], decision-making [10–18], and so on.

In many practical cases, it is difficult to define a membership function of a universe because people would face several possible membership degrees of one object to be chosen and hesitate about which one would be the most right one. Hesitant fuzzy set theory, introduced by Torra and Narukawa [19, 20], has provided successful results dealing with hesitant situations, which are not well managed by the

previous tools [21]. It constructs possible membership degrees of an object as a set and keeps more information in real environment. Many scholars have focused on hesitant fuzzy sets and proposed diverse corresponding extensions, such as dual hesitant fuzzy sets [22], interval-valued dual fuzzy sets [23, 24], intuitionistic hesitant fuzzy sets [25], interval-valued hesitant fuzzy sets [26–28], and interval-valued intuitionistic fuzzy sets [29]. These theories have been applied to decision-making [30, 31], clustering analysis [32], and so on.

As another extension of fuzzy set theory, Yager [33, 34] proposed another class of nonstandard fuzzy sets, called Pythagorean fuzzy sets. The sets are represented by pairs of two values $\langle \mu_P(x), \nu_P(x) \rangle$, which satisfies $\mu_P^2(x) + \nu_P^2(x) \leq 1$. Obviously, their application range is broader than that of intuitionistic fuzzy sets. These situations are more common in different real-world problems. So, Pythagorean fuzzy sets have been paid attention in a short period of time. Yager

[35, 36], Zhang and Xu [37], Ren et al. [38], Liu et al. [39], and Teng et al. [39] have studied several kinds of Pythagorean fuzzy aggregation operators and applied them to decision-making problems. Furthermore, Peng and Yang [40] proposed the definition of interval-valued Pythagorean fuzzy set and Rahamn et al. [41, 42] developed group decision-making with interval-valued Pythagorean fuzzy environments. Yi et al. [43] applied it to multicriteria decision-making problems. Also, Liang et al. [44] introduced the interval-valued Pythagorean fuzzy extended Bonferroni mean operators.

Recently, some scholars have tried to combine Pythagorean fuzzy sets and hesitant fuzzy sets, and all called them Pythagorean hesitant fuzzy sets (PHFSs), but their construction methods are controversial. Let $A = \{\langle x, \Gamma_A(x), \Psi_A(x) \rangle \mid x \in U\}$ be a Pythagorean fuzzy set in U , where $\Gamma_A(x) = \{\mu_A(x) \mid \mu_A(x) \in [0, 1]\}$ and $\Psi_A(x) = \{\nu_A(x) \mid \nu_A(x) \in [0, 1]\}$. Liu and He [45] defined that, for any x , any $\mu_A(x) \in \Gamma_A(x)$, and any $\nu_A(x) \in \Psi_A(x)$, $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$ holds. Khan et al. [46] defined that, for any x and for any $\mu_A(x) \in \Gamma_A(x)$, there is $\nu_A(x) \in \Psi_A(x)$, such that $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$. Also, for any $\nu_A(x) \in \Psi_A(x)$, there is $\mu_A(x) \in \Gamma_A(x)$, such that $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$. Wei et al. [47] also defined another Pythagorean fuzzy set $P = \{\langle x, h_P(x) \rangle \mid x \in U\}$, where $h_P(x)$ is a set of some Pythagorean fuzzy elements in U . The three definitions have their own merits. Considering the pairing of possible membership degrees and possible non-membership degrees, it is more conducive to the aggregation of hesitant fuzzy numbers. In this paper, we choose Wei's definition as the definition of PHFS.

From the above analysis, we can see that PHFSs are more convenient to deal with fuzzy information than hesitant fuzzy sets or Pythagorean fuzzy sets. However, for real multiattribute group decision-making (MAGDM) problems, it is difficult for decision makers to provide some exact and crisp fuzzy values to depict uncertain or insufficient alternatives because of the increasing complexity of social and economic life. The aim of this paper is to extend PHFSs to interval-valued Pythagorean hesitant fuzzy sets (IVPHFSs) and develop MAGDM approaches to interval-valued Pythagorean hesitant fuzzy environments based on newly constructed aggregation operators. In particular, since there is no one-to-one correspondence between interval numbers and real numbers, this paper directly uses interval numbers to define score functions and accuracy functions, which can preserve interval-valued Pythagorean fuzzy information as much as possible.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts and results of hesitant fuzzy sets, interval-valued hesitant fuzzy sets, and interval-valued Pythagorean fuzzy sets. Section 3 proposes the definition of IVPHFS and discusses some basic operational laws. In particular, we propose the concepts of the score function and accuracy function which are both appeared as interval values. Some operators for aggregating interval-valued Pythagorean hesitant fuzzy information are studied and developed in section 4. Section 5 shows the application to MAGDMs in interval-valued Pythagorean hesitant fuzzy

environments and illustrates the feasibility and applicability of the proposed method. Concluding remarks are made in Section 6.

2. Preliminaries

This section will briefly review the basic notations and results of hesitant fuzzy sets, interval-valued Pythagorean fuzzy sets.

Hereinafter, without explanation, let U be a nonempty finite set, called the universe of discourse, and $D[x, y]$ be the set of all closed subintervals of the interval $[x, y]$.

2.1. Hesitant Fuzzy Set (HFS). HFS was introduced by Torra and Narukawa [19] and Torra [20], which permits the membership being a set of possible values. It is very suitable for describing problems that are difficult to determine with only one membership.

Definition 1 (see [20]). A hesitant fuzzy set (HFS) E on U is described as $E = \{\langle x, h_E(x) \rangle \mid x \in U\}$, where $h_E(x) = \{\mu_E(x) \mid \mu_E(x) \in [0, 1]\}$ represents the set of possible membership degrees of E at x .

For convenience, Xia and Xu [30] proposed the concept of a hesitant fuzzy element (HFE), denoted by $h = h_E(x)$, where $h = \{\gamma \mid \gamma \in [0, 1]\}$. They gave some operations on HFEs.

Definition 2 (see [19, 20, 30]). Let h, h_1 , and h_2 be three HFEs and $\lambda > 0$, then

- (1) $h^C = \{1 - \gamma \mid \gamma \in h\}$
- (2) $h^\lambda = \{\gamma^\lambda \mid \gamma \in h\}$
- (3) $\lambda h = \{1 - (1 - \gamma)^\lambda \mid \gamma \in h\}$
- (4) $h_1 \oplus h_2 = \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 \mid \gamma_1 \in h_1, \gamma_2 \in h_2\}$
- (5) $h_1 \otimes h_2 = \{\gamma_1\gamma_2 \mid \gamma_1 \in h_1, \gamma_2 \in h_2\}$
- (6) $h_1 \cup h_2 = \{\max\{\gamma_1, \gamma_2\} \mid \gamma_1 \in h_1, \gamma_2 \in h_2\}$
- (7) $h_1 \cap h_2 = \{\min\{\gamma_1, \gamma_2\} \mid \gamma_1 \in h_1, \gamma_2 \in h_2\}$

Furthermore, to compare the HFEs, the following comparison laws are given.

Definition 3 (see [30]). For an HFE h , $S(h) = 1/|h| \sum_{\gamma \in h} \gamma$ is called the score function of h with $|\cdot|$ denoting the cardinality here and below. For two HFEs, h_1 and h_2 , if $S(h_1) > S(h_2)$, then h_1 is superior to h_2 , denoted by $h_1 \succ h_2$; if $S(h_1) = S(h_2)$, then h_1 is equivalent to h_2 , denoted by $h_1 = h_2$.

2.2. Interval-Valued Hesitant Fuzzy Set (IVHFS). Chen et al. [26, 27] generalized HFSs to interval-valued hesitant fuzzy sets (IVPHFSs), in which the membership degree of each object of the universe is denoted by several possible interval values.

Definition 4 (see [26]). An interval-valued hesitant fuzzy set (IVHFS) $\tilde{E}$ on U is described as $\tilde{E} = \{\langle x, \tilde{h}_{\tilde{E}}(x) \rangle \mid x \in U\}$,

where $\tilde{h}_E(x) = \{\tilde{\mu}_E(x) \mid \tilde{\mu}_E(x) \triangleq [\mu_E^-(x), \mu_E^+(x)] \in D[0, 1]\}$ represents the set of possible membership intervals of $\tilde{E}$ at x .

For convenience, $\tilde{h}_E(x)$ is called an interval-valued hesitant fuzzy element (IVHFE), denoted by $\tilde{h} = \{\tilde{\mu} \mid \tilde{\mu} \triangleq [\mu^-, \mu^+]\}$. Especially, if $\mu^- = \mu^+$, then $\tilde{h}$ degenerates to an HFS.

Definition 5 (see [26, 27]). Let $\tilde{h}, \tilde{h}_1$, and $\tilde{h}_2$ be three IVHFEs and $\lambda > 0$, then

- (1) $\tilde{h}^C = \{[1 - \mu^+, 1 - \mu^-] \mid \tilde{\mu} \in \tilde{h}\}$
- (2) $\tilde{h}^\lambda = \{[(\mu^-)^\lambda, (\mu^+)^\lambda] \mid \tilde{\mu} \in \tilde{h}\}$
- (3) $\lambda\tilde{h} = \{[1 - (1 - \mu^-)^\lambda, 1 - (1 - \mu^+)^\lambda] \mid \tilde{\mu} \in \tilde{h}\}$
- (4) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\mu_1^- + \mu_2^- - \mu_1^- \cdot \mu_2^-, \mu_1^+ + \mu_2^+ - \mu_1^+ \cdot \mu_2^+] \mid \tilde{\mu}_1 \in \tilde{h}_1, \tilde{\mu}_2 \in \tilde{h}_2\}$
- (5) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\mu_1^- \cdot \mu_2^-, \mu_1^+ \cdot \mu_2^+] \mid \tilde{\mu}_1 \in \tilde{h}_1, \tilde{\mu}_2 \in \tilde{h}_2\}$
- (6) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max\{\mu_1^-, \mu_2^-\}, \max\{\mu_1^+, \mu_2^+\}] \mid \tilde{\mu}_1 \in \tilde{h}_1, \tilde{\mu}_2 \in \tilde{h}_2\}$
- (7) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min\{\mu_1^-, \mu_2^-\}, \min\{\mu_1^+, \mu_2^+\}] \mid \tilde{\mu}_1 \in \tilde{h}_1, \tilde{\mu}_2 \in \tilde{h}_2\}$

Definition 6 (see [48]). Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two interval numbers, $\lambda > 0$. The interval arithmetic is defined as

- (1) $a + b = [a^- + b^-, a^+ + b^+]$
- (2) $a - b = [a^- - b^+, a^+ - b^-]$
- (3) $\lambda a = [\lambda a^-, \lambda a^+]$
- (4) $ab = [\min\{a^- b^-, a^- b^+, a^+ b^-, a^+ b^+\}, \max\{a^- b^-, a^- b^+, a^+ b^-, a^+ b^+\}]$

Definition 7 (see [49]). Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two interval numbers and $l(a) = a^+ - a^-$ and $l(b) = b^+ - b^-$, then the possibility degree of $a > b$ is defined as follows:

$$P(a \geq b) = \max \left\{ 1 - \max \left\{ \frac{b^+ - a^-}{l(a) + l(b)}, 0 \right\}, 0 \right\}. \quad (1)$$

The equation of the possibility degree is used to compare two interval numbers. If $P(a \geq b) > 0.5$, then a is superior to b , denoted by $a > b$; if $P(a \geq b) = 0.5$, then a is equivalent to b , denoted by $a = b$.

Since an IVHFE is formed by several interval numbers, the comparative analysis of IVHFEs is different from that of HFEs. Based on the possibility degree of interval numbers in [49], Chen et al. [26] gave the following comparison laws.

Definition 8 (see [26]). For an IVHFE $\tilde{h}$, $S(\tilde{h}) = 1/|\tilde{h}| \sum_{\tilde{\mu} \in \tilde{h}} \tilde{\mu}$ is called the score function of $\tilde{h}$ with $|\cdot|$ denoting the cardinality, and $S(\tilde{h})$ is an interval value belonging to $[0, 1]$. For two IVHFEs, $\tilde{h}_1$ and $\tilde{h}_2$, if $P(S(\tilde{h}_1) \geq S(\tilde{h}_2)) > 0.5$, then $\tilde{h}_1$ is superior to $\tilde{h}_2$, denoted by $\tilde{h}_1 > \tilde{h}_2$.

2.3. Interval-Valued Pythagorean Fuzzy Set (IVPFS). A Pythagorean fuzzy set (PFS) is introduced by Yager [33], which

is characterized by a membership function and a non-membership function, where the sum of the square of the membership degree and the nonmembership degree of x is less than or equal to 1, while an intuitionistic fuzzy set is also characterized by them, where the sum is less than or equal to 1. Obviously, PFSs are more general than intuitionistic fuzzy sets. A PFS has emerged as an effective tool to solve multiattribute decision-making problems [37].

Definition 9 (see [33, 35]). A Pythagorean fuzzy set (PFS) P on U is described as

$$P = \{ \langle x, \mu_P(x), \nu_P(x) \rangle \mid \mu_P^2(x) + \nu_P^2(x) \leq 1, \mu_P(x), \nu_P(x) \in [0, 1], x \in U \}, \quad (2)$$

where $\mu_P(x)$ and $\nu_P(x)$ represent the Pythagorean membership degree and the Pythagorean nonmembership degree of P at x , respectively.

Since people often find it difficult to exactly quantify their opinions facing with incomplete fuzzy decision-making problems, interval-valued fuzzy elements can provide a better solving way. Peng et al. [40] focused on interval-valued Pythagorean fuzzy sets (IVPFSs), whose ideas are similar to interval-valued intuitionistic fuzzy sets.

Definition 10 (see [40]). An interval-valued Pythagorean fuzzy set (IVPFS) $\tilde{P}$ on U is described as

$$\tilde{P} = \left\{ \langle x, [\mu_{\tilde{P}}^-(x), \mu_{\tilde{P}}^+(x)], [\nu_{\tilde{P}}^-(x), \nu_{\tilde{P}}^+(x)] \rangle \mid [\mu_{\tilde{P}}^-(x), \mu_{\tilde{P}}^+(x)] \cdot [\nu_{\tilde{P}}^-(x), \nu_{\tilde{P}}^+(x)] \in D[0, 1], (\mu_{\tilde{P}}^+(x))^2 + (\nu_{\tilde{P}}^+(x))^2 \leq 1, x \in U \right\}, \quad (3)$$

where $[\mu_{\tilde{P}}^-(x), \mu_{\tilde{P}}^+(x)]$ and $[\nu_{\tilde{P}}^-(x), \nu_{\tilde{P}}^+(x)]$ are the Pythagorean membership intervals and the Pythagorean nonmembership intervals of $\tilde{P}$ at x , respectively.

Each pair $\langle [\mu_{\tilde{P}}^-(x), \mu_{\tilde{P}}^+(x)], [\nu_{\tilde{P}}^-(x), \nu_{\tilde{P}}^+(x)] \rangle$ is called an interval-valued Pythagorean fuzzy element (IVPFE), denoted by $\tilde{P} = \langle [\mu^-, \mu^+], [\nu^-, \nu^+] \rangle$. Obviously, for any IVPFE $\tilde{P}$ on U , $[\mu^-, \mu^+]$ and $[\nu^-, \nu^+]$ are both singletons, the IVPFS degenerates into a PFS.

Definition 11 (see [40]). Let $\tilde{P} = \langle [\mu^-, \mu^+], [\nu^-, \nu^+] \rangle$, $\tilde{P}_1 = \langle [\mu_1^-, \mu_1^+], [\nu_1^-, \nu_1^+] \rangle$, and $\tilde{P}_2 = \langle [\mu_2^-, \mu_2^+], [\nu_2^-, \nu_2^+] \rangle$ be three IVPFEs and $\lambda > 0$. The operational laws of IVPFEs are defined as follows:

- (1) $\tilde{P}^C = \langle [\nu^-, \nu^+], [\mu^-, \mu^+] \rangle$
- (2) $\tilde{P}^\lambda = \langle [(\mu^-)^\lambda, (\mu^+)^\lambda], [\sqrt{1 - (1 - (\nu^-)^2)^\lambda}, \sqrt{1 - (1 - (\nu^+)^2)^\lambda}] \rangle$
- (3) $\lambda\tilde{P} = \langle [\sqrt{1 - (1 - (\mu^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu^+)^2)^\lambda}], [(\nu^-)^\lambda, (\nu^+)^\lambda] \rangle$
- (4) $\tilde{P}_1 \oplus \tilde{P}_2 = \langle [\sqrt{(\mu_1^-)^2 + (\mu_2^-)^2 - (\mu_1^-)^2(\mu_2^-)^2}, \sqrt{(\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)^2(\mu_2^+)^2}], [\nu_1^-\nu_2^-, \nu_1^+\nu_2^+] \rangle$

- $$\begin{aligned}
(5) \quad & \tilde{P}_1 \otimes \tilde{P}_2 = \langle [\mu_1^- \mu_2^-, \mu_1^+ \mu_2^+], [\sqrt{(\nu_1^-)^2 + (\nu_2^-)^2 - (\nu_1^+)^2} - (\nu_1^+)^2, \\
& (\nu_2^+)^2, \sqrt{(\nu_1^+)^2 + (\nu_2^+)^2 - (\nu_1^-)^2} - (\nu_1^-)^2] \rangle \\
(6) \quad & \tilde{P}_1 \cup \tilde{P}_2 = \langle [\max\{\mu_1^-, \mu_2^-\}, \max\{\mu_1^+, \mu_2^+\}], \\
& [\min\{\nu_1^-, \nu_2^-\}, \min\{\nu_1^+, \nu_2^+\}] \rangle \\
(7) \quad & \tilde{P}_1 \cap \tilde{P}_2 = \langle [\min\{\mu_1^-, \mu_2^-\}, \min\{\mu_1^+, \mu_2^+\}], [\max\{\nu_1^-, \nu_2^-\}, \\
& \max\{\nu_1^+, \nu_2^+\}] \rangle
\end{aligned}$$

Definition 12 (see [40]). For an IVPFE $\tilde{P} = \langle [\mu^-, \mu^+], [\nu^-, \nu^+] \rangle$, the score function of $\tilde{P}$ is defined as follows:

$$S(\tilde{P}) = \frac{1}{2} [(\mu^-)^2 + (\mu^+)^2 - (\nu^-)^2 - (\nu^+)^2]. \quad (4)$$

The accuracy function of $\tilde{P}$ is defined as follows:

$$H(\tilde{P}) = \frac{1}{2} [(\mu^-)^2 + (\mu^+)^2 + (\nu^-)^2 + (\nu^+)^2]. \quad (5)$$

For two IVPFEs $\tilde{P}_1$ and $\tilde{P}_2$,

- (1) If $S(\tilde{P}_1) > S(\tilde{P}_2)$, then $\tilde{P}_1 > \tilde{P}_2$
- (2) If $S(\tilde{P}_1) = S(\tilde{P}_2)$, then
 - (2.1) If $H(\tilde{P}_1) > H(\tilde{P}_2)$, then $\tilde{P}_1 > \tilde{P}_2$;
 - (2.2) If $H(\tilde{P}_1) = H(\tilde{P}_2)$, then $\tilde{P}_1 = \tilde{P}_2$.

3. Interval-Valued Pythagorean Hesitant Fuzzy Sets (IVPHFSs)

3.1. Interval-Valued Pythagorean Hesitant Fuzzy Elements. As mentioned earlier, in many practical problems, it is difficult for decision makers to determine precise membership degrees or nonmembership degrees, and the evaluation with relatively reasonable interval values often exists in decision-making. In order to better avoid the information loss and enhance the flexibility and applicability of the decision-making models in dealing with qualitative information, we propose the concept of interval-valued Pythagorean hesitant fuzzy set (IVPHFS).

Definition 13. An interval-valued Pythagorean hesitant fuzzy set (IVPHFS) $\mathcal{P}$ on U is described as

$$\mathcal{P} = \{ \langle x, h_{\mathcal{P}}(x) \rangle \mid x \in U \}, \quad (6)$$

where

$$\begin{aligned}
h_{\mathcal{P}}(x) &= \{ \langle \tilde{\mu}_{\mathcal{P}}(x), \tilde{\nu}_{\mathcal{P}}(x) \rangle \mid \tilde{\mu}_{\mathcal{P}}(x) = [\mu_{\mathcal{P}}^-(x), \mu_{\mathcal{P}}^+(x)] \in D[0, 1], \\
& \tilde{\nu}_{\mathcal{P}}(x) = [\nu_{\mathcal{P}}^-(x), \nu_{\mathcal{P}}^+(x)] \in D[0, 1], (\mu_{\mathcal{P}}^+(x))^2 + (\nu_{\mathcal{P}}^+(x))^2 \leq 1 \},
\end{aligned} \quad (7)$$

where $\tilde{\mu}_{\mathcal{P}}(x)$ and $\tilde{\nu}_{\mathcal{P}}(x)$ are the possible Pythagorean membership intervals and the possible Pythagorean nonmembership intervals of $\mathcal{P}$ at x , respectively. The set of all IVPHFSs on U is denoted by Ω .

Obviously, for each $x \in U$, if $h_{\mathcal{P}}(x)$ includes only one pair of intervals, the IVPHFS degenerates into an IVPFS; if both $\tilde{\mu}_{\mathcal{P}}(x)$ and $\tilde{\nu}_{\mathcal{P}}(x)$ degenerate one singleton, the IVPHFS can be seen as a PHFS; if $\tilde{\nu}_{\mathcal{P}}(x) = [0, 0]$, the IVPHFS can be seen as an IVHFS; if $\mu_{\mathcal{P}}^+(x) + \nu_{\mathcal{P}}^+(x) \leq 1$, the

IVPHFS can be seen as an interval-valued intuitionistic hesitant fuzzy set [29].

For convenience, we call each pair $\tilde{\mathcal{P}} = h_{\mathcal{P}}(x)$ as an interval-valued Pythagorean hesitant fuzzy element (IVPHFE), where $\tilde{\mathcal{P}} = \{ \langle \tilde{\mu}, \tilde{\nu} \rangle \mid \tilde{\mu} = [\mu^-, \mu^+], \tilde{\nu} = [\nu^-, \nu^+] \}$.

Based on the operators of IVHFEs [27] and IVPFEs [40], the operational laws of IVPHFEs are defined as follows.

Definition 14. Let $\tilde{\mathcal{P}} = \{ \langle \tilde{\mu}, \tilde{\nu} \rangle \mid \tilde{\mu} = [\mu^-, \mu^+], \tilde{\nu} = [\nu^-, \nu^+] \}$, $\tilde{\mathcal{P}}_1 = \{ \langle \tilde{\mu}_1, \tilde{\nu}_1 \rangle \mid \tilde{\mu}_1 = [\mu_1^-, \mu_1^+], \tilde{\nu}_1 = [\nu_1^-, \nu_1^+] \}$, and $\tilde{\mathcal{P}}_2 = \{ \langle \tilde{\mu}_2, \tilde{\nu}_2 \rangle \mid \tilde{\mu}_2 = [\mu_2^-, \mu_2^+], \tilde{\nu}_2 = [\nu_2^-, \nu_2^+] \}$ be three IVPHFEs and $\lambda > 0$. The operational laws of IVPHFEs are defined as follows:

- (1) $\tilde{\mathcal{P}}^C = \{ \langle [\nu^-, \nu^+], [\mu^-, \mu^+] \rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \}$
- (2) $\tilde{\mathcal{P}}^\lambda = \{ \langle [(\mu^-)^\lambda, (\mu^+)^\lambda], [\sqrt{1 - (1 - (\nu^-)^2)^\lambda}, \sqrt{1 - (1 - (\nu^+)^2)^\lambda}] \rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \}$
- (3) $\lambda \tilde{\mathcal{P}} = \{ \langle [\sqrt{1 - (1 - (\mu^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu^+)^2)^\lambda}], \langle [(\nu^-)^\lambda, (\nu^+)^\lambda] \rangle \rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \}$
- (4) $\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 = \{ \langle [\sqrt{(\mu_1^-)^2 + (\mu_2^-)^2 - (\mu_1^+)(\mu_2^+)}, \sqrt{(\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^-)(\mu_2^-)}], [\nu_1^-, \nu_2^-], [\nu_1^+, \nu_2^+] \rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \}$
- (5) $\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2 = \{ \langle [\mu_1^-, \mu_2^-], \mu_1^+ \mu_2^+ \rangle [\sqrt{(\nu_1^-)^2 + (\nu_2^-)^2 - (\nu_1^+)(\nu_2^+)}, \sqrt{(\nu_1^+)^2 + (\nu_2^+)^2 - (\nu_1^-)(\nu_2^-)}], \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \}$
- (6) $\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2 = \{ \langle [\max\{\mu_1^-, \mu_2^-\}, \max\{\mu_1^+, \mu_2^+\}], [\min\{\nu_1^-, \nu_2^-\}, \min\{\nu_1^+, \nu_2^+\}] \rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \}$
- (7) $\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2 = \{ \langle [\min\{\mu_1^-, \mu_2^-\}, \min\{\mu_1^+, \mu_2^+\}], [\max\{\nu_1^-, \nu_2^-\}, \max\{\nu_1^+, \nu_2^+\}] \rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \}$

Proposition 1. Let $\tilde{\mathcal{P}}, \tilde{\mathcal{P}}_1$, and $\tilde{\mathcal{P}}_2$ be three IVPHFEs and $\lambda > 0$, then $\tilde{\mathcal{P}}^C, \tilde{\mathcal{P}}^\lambda, \lambda \tilde{\mathcal{P}}, \tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2$, and $\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2$ are all IVPHFEs.

Proof. Obviously, $\tilde{\mathcal{P}}^C$ is an IVPHFE.

For any $\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}$, since $(\mu^+)^2 + (\nu^+)^2 \leq 1$,

$$\begin{aligned}
((\mu^+)^2)^\lambda + (\sqrt{1 - (1 - (\nu^+)^2)^\lambda})^2 &= (\mu^+)^{2\lambda} + 1 - (1 - (\nu^+)^2)^\lambda \\
&\leq (\mu^+)^{2\lambda} + 1 - (\mu^+)^{2\lambda} = 1.
\end{aligned} \quad (8)$$

So, $\tilde{\mathcal{P}}^\lambda$ is an IVPHFE. Similarly, $\lambda \tilde{\mathcal{P}}$ is also an IVPHFE. As for $\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2$,

$$\begin{aligned}
& (\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)(\mu_2^+) + (\nu_1^+)^2 + (\nu_2^+)^2 \\
& \leq (\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)(\mu_2^+) + (1 - (\mu_1^+)^2)(1 - (\mu_2^+)^2) \\
& = (\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)(\mu_2^+) + 1 - (\mu_1^+)^2 - (\mu_2^+)^2 \\
& \quad + (\mu_1^+)(\mu_2^+) = 1.
\end{aligned} \quad (9)$$

So, $\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2$ is an IVPHFE. Similarly, $\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2$ is also an IVPHFE.

At last, we prove the last two claims. Assume $\mu_1^+ < \mu_2^+$.

Then, $(\max\{\mu_1^+, \mu_2^+\})^2 = (\mu_2^+)^2 \leq 1 - (\nu_2^+)^2 \leq 1 - (\min\{\nu_1^+, \nu_2^+\})^2$. So, $\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2$ is an IVPHFE. Similarly, $\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2$ is also an IVPHFE. All the claims are proved. $\square$

Proposition 2. Let $\tilde{\mathcal{P}}, \tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2$, and $\tilde{\mathcal{P}}_3$ be four IVPHFEs and $\lambda, \lambda_1, \lambda_2 > 0$, then

- (1) $\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2 \oplus \tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2 \otimes \tilde{\mathcal{P}}_1$
- (2) $\lambda \tilde{\mathcal{P}}_1 \oplus \lambda \tilde{\mathcal{P}}_2 = \lambda (\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2)$ and $(\tilde{\mathcal{P}}_1)^\lambda \otimes (\tilde{\mathcal{P}}_2)^\lambda = (\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2)^\lambda$

- (3) $\lambda_1 \tilde{\mathcal{P}} \oplus \lambda_2 \tilde{\mathcal{P}} = (\lambda_1 + \lambda_2) \tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}^{\lambda_1} \otimes \tilde{\mathcal{P}}^{\lambda_2} = \tilde{\mathcal{P}}^{(\lambda_1 + \lambda_2)}$
- (4) $(\tilde{\mathcal{P}}^C)^\lambda = (\lambda \tilde{\mathcal{P}})^C$ and $\lambda (\tilde{\mathcal{P}}^C) = (\tilde{\mathcal{P}}^\lambda)^C$
- (5) $(\tilde{\mathcal{P}}_1)^C \oplus (\tilde{\mathcal{P}}_2)^C = (\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2)^C$ and $(\tilde{\mathcal{P}}_1)^C \otimes (\tilde{\mathcal{P}}_2)^C = (\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2)^C$
- (6) $(\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2) \oplus \tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_1 \oplus (\tilde{\mathcal{P}}_2 \oplus \tilde{\mathcal{P}}_3)$ and $(\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2) \otimes \tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_1 \otimes (\tilde{\mathcal{P}}_2 \otimes \tilde{\mathcal{P}}_3)$

Proof. (1) Based on Definition 14, claim (1) is obvious, so here the proof process is overleaped.

$$\begin{aligned}
 \lambda \tilde{\mathcal{P}}_1 \oplus \lambda \tilde{\mathcal{P}}_2 &= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu_1^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu_1^+)^2)^\lambda} \right], [(\nu_1^-)^\lambda, (\nu_1^+)^\lambda] \right\rangle \mid \langle \tilde{\mu}_1, \tilde{\nu}_1 \rangle \in \tilde{\mathcal{P}}_1 \right\} \\
 &\quad \oplus \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu_2^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu_2^+)^2)^\lambda} \right], [(\nu_2^-)^\lambda, (\nu_2^+)^\lambda] \right\rangle \mid \langle \tilde{\mu}_2, \tilde{\nu}_2 \rangle \in \tilde{\mathcal{P}}_2 \right\} \\
 &= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu_1^-)^2)^\lambda (1 - (\mu_2^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu_1^+)^2)^\lambda (1 - (\mu_2^+)^2)^\lambda} \right], [(\nu_1^- \nu_2^-)^\lambda, (\nu_1^+ \nu_2^+)^\lambda] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\}, \\
 &\quad \lambda (\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2) \\
 &= \lambda \left\{ \left\langle \left[\sqrt{(\mu_1^-)^2 + (\mu_2^-)^2 - (\mu_1^-)^2 (\mu_2^-)^2}, \sqrt{(\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)^2 (\mu_2^+)^2} \right], [\nu_1^- \nu_2^-, \nu_1^+ \nu_2^+] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\} \\
 &= \lambda \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu_1^-)^2)(1 - (\mu_2^-)^2)}, \sqrt{1 - (1 - (\mu_1^+)^2)(1 - (\mu_2^+)^2)} \right], [\nu_1^- \nu_2^-, \nu_1^+ \nu_2^+] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\} \\
 &= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu_1^-)^2)^\lambda (1 - (\mu_2^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu_1^+)^2)^\lambda (1 - (\mu_2^+)^2)^\lambda} \right], [(\nu_1^- \nu_2^-)^\lambda, (\nu_1^+ \nu_2^+)^\lambda] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\}.
 \end{aligned} \tag{10}$$

So, $\lambda \tilde{\mathcal{P}}_1 \oplus \lambda \tilde{\mathcal{P}}_2 = \lambda (\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2)$ holds. Similarly, we have $(\tilde{\mathcal{P}}_1)^\lambda \otimes (\tilde{\mathcal{P}}_2)^\lambda = (\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2)^\lambda$.

$$\begin{aligned}
 \lambda_1 \tilde{\mathcal{P}} \oplus \lambda_2 \tilde{\mathcal{P}} &= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu^-)^2)^{\lambda_1}}, \sqrt{1 - (1 - (\mu^+)^2)^{\lambda_1}} \right], [(\nu^-)^{\lambda_1}, (\nu^+)^{\lambda_1}] \right\rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \right\} \\
 &\quad \oplus \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu^-)^2)^{\lambda_2}}, \sqrt{1 - (1 - (\mu^+)^2)^{\lambda_2}} \right], [(\nu^-)^{\lambda_2}, (\nu^+)^{\lambda_2}] \right\rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \right\} \\
 &= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu^-)^2)^{(\lambda_1 + \lambda_2)}}, \sqrt{1 - (1 - (\mu^+)^2)^{(\lambda_1 + \lambda_2)}} \right], [(\nu^-)^{(\lambda_1 + \lambda_2)}, (\nu^+)^{(\lambda_1 + \lambda_2)}] \right\rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \right\} \\
 &= (\lambda_1 + \lambda_2) \tilde{\mathcal{P}}.
 \end{aligned} \tag{11}$$

Similarly, we have $\tilde{\mathcal{P}}^{\lambda_1} \otimes \tilde{\mathcal{P}}^{\lambda_2} = \tilde{\mathcal{P}}^{(\lambda_1 + \lambda_2)}$.

$$\begin{aligned}
(\tilde{\mathcal{P}}^C)^\lambda &= \left\{ \left\langle \left[(v^-)^\lambda, (v^+)^\lambda \right], \left[\sqrt{1 - (1 - (\mu^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu^+)^2)^\lambda} \right] \right\rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \right\} \\
&= \left\{ \left\langle \left[\sqrt{1 - (1 - (\mu^-)^2)^\lambda}, \sqrt{1 - (1 - (\mu^+)^2)^\lambda} \right], \left[(v^-)^\lambda, (v^+)^\lambda \right] \right\rangle \mid \langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}} \right\} \\
&= (\lambda \tilde{\mathcal{P}})^C.
\end{aligned} \tag{12}$$

Similarly, we have $\lambda(\tilde{\mathcal{P}}^C) = (\tilde{\mathcal{P}}^\lambda)^C$.

$$\begin{aligned}
\tilde{\mathcal{P}}_1^C \oplus \tilde{\mathcal{P}}_2^C &= \left\{ \langle [v_1^-, v_1^+], [\mu_1^-, \mu_1^+] \rangle \mid \langle \tilde{\mu}_1, \tilde{\nu}_1 \rangle \in \tilde{\mathcal{P}}_1 \right\} \oplus \left\{ \langle [v_2^-, v_2^+], [\mu_2^-, \mu_2^+] \rangle \mid \langle \tilde{\mu}_2, \tilde{\nu}_2 \rangle \in \tilde{\mathcal{P}}_2 \right\} \\
&= \left\{ \left\langle \left[\sqrt{(v_1^-)^2 + (v_2^-)^2 - (v_1^-)^2 (v_2^-)^2}, \sqrt{(v_1^+)^2 + (v_2^+)^2 - (v_1^+)^2 (v_2^+)^2} \right], [\mu_1^-, \mu_2^-, \mu_1^+, \mu_2^+] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\} \\
&= \left\{ \left\langle [\mu_1^-, \mu_2^-, \mu_1^+, \mu_2^+], \left[\sqrt{(v_1^-)^2 + (v_2^-)^2 - (v_1^-)^2 (v_2^-)^2}, \sqrt{(v_1^+)^2 + (v_2^+)^2 - (v_1^+)^2 (v_2^+)^2} \right] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2 \right\} \\
&= (\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2)^C.
\end{aligned} \tag{13}$$

Similarly, $\tilde{\mathcal{P}}_1^C \otimes \tilde{\mathcal{P}}_2^C = (\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2)^C$ holds.

$$\begin{aligned}
(\tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2) \oplus \tilde{\mathcal{P}}_3 &= \left\{ \left\langle \left[\sqrt{(\mu_1^-)^2 + (\mu_2^-)^2 - (\mu_1^-)^2 (\mu_2^-)^2}, \sqrt{(\mu_1^+)^2 + (\mu_2^+)^2 - (\mu_1^+)^2 (\mu_2^+)^2} \right], [v_1^-, v_2^-, v_1^+, v_2^+] \right\rangle \mid \langle \tilde{\mu}_1, \tilde{\nu}_1 \rangle \in \tilde{\mathcal{P}}_1, \langle \tilde{\mu}_2, \tilde{\nu}_2 \rangle \in \tilde{\mathcal{P}}_2 \right\} \\
&\quad \oplus \left\{ \langle [\mu_3^-, \mu_3^+], [v_3^-, v_3^+] \rangle \mid \langle \tilde{\mu}_3, \tilde{\nu}_3 \rangle \in \tilde{\mathcal{P}}_3 \right\} \\
&= \left\{ \left\langle \left[\sqrt{(\mu_1^-)^2 + (\mu_2^-)^2 + (\mu_3^-)^2 - (\mu_1^-)^2 (\mu_2^-)^2 - (\mu_1^-)^2 (\mu_3^-)^2 - (\mu_2^-)^2 (\mu_3^-)^2 + (\mu_1^-)^2 (\mu_2^-)^2 (\mu_3^-)^2}, \right. \right. \\
&\quad \left. \left. \sqrt{(\mu_1^+)^2 + (\mu_2^+)^2 + (\mu_3^+)^2 - (\mu_1^+)^2 (\mu_2^+)^2 - (\mu_1^+)^2 (\mu_3^+)^2 - (\mu_2^+)^2 (\mu_3^+)^2 + (\mu_1^+)^2 (\mu_2^+)^2 (\mu_3^+)^2} \right], \right. \\
&\quad \left. [v_1^-, v_2^-, v_3^-, v_1^+, v_2^+, v_3^+] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2, 3 \right\} \\
&= \left\{ \langle [\mu_1^-, \mu_1^+], [v_1^-, v_1^+] \rangle \mid \langle \tilde{\mu}_1, \tilde{\nu}_1 \rangle \in \tilde{\mathcal{P}}_1 \right\} \oplus \left\{ \left\langle \left[\sqrt{(\mu_2^-)^2 + (\mu_3^-)^2 - (\mu_2^-)^2 (\mu_3^-)^2}, \sqrt{(\mu_2^+)^2 + (\mu_3^+)^2 - (\mu_2^+)^2 (\mu_3^+)^2} \right], \right. \right. \\
&\quad \left. \left. [v_2^-, v_3^-, v_2^+, v_3^+] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 2, 3 \right\} \\
&= \tilde{\mathcal{P}}_1 \oplus (\tilde{\mathcal{P}}_2 \oplus \tilde{\mathcal{P}}_3).
\end{aligned} \tag{14}$$

Similarly, we have $(\tilde{\mathcal{P}}_1 \otimes \tilde{\mathcal{P}}_2) \otimes \tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_1 \otimes (\tilde{\mathcal{P}}_2 \otimes \tilde{\mathcal{P}}_3)$.
All the claims of the proposition are proved. $\square$

Proposition 3. Let $\tilde{\mathcal{P}}, \tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2$, and $\tilde{\mathcal{P}}_3$ be four IVPFHEs and $\lambda > 0$, then

- (1) $\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2 \cup \tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2 \cap \tilde{\mathcal{P}}_1$
- (2) $\lambda \tilde{\mathcal{P}}_1 \cup \lambda \tilde{\mathcal{P}}_2 = \lambda(\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2)$ and $(\tilde{\mathcal{P}}_1)^\lambda \cap (\tilde{\mathcal{P}}_2)^\lambda = (\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2)^\lambda$

- (3) $(\tilde{\mathcal{P}}_1)^C \cup (\tilde{\mathcal{P}}_2)^C = (\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2)^C$ and $(\tilde{\mathcal{P}}_1)^C \cap (\tilde{\mathcal{P}}_2)^C = (\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2)^C$
- (4) $(\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2) \cup \tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_1 \cup (\tilde{\mathcal{P}}_2 \cap \tilde{\mathcal{P}}_3)$ and $(\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2) \cap \tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_1 \cap (\tilde{\mathcal{P}}_2 \cap \tilde{\mathcal{P}}_3)$
- (5) $(\tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2) \cap \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2$ and $(\tilde{\mathcal{P}}_1 \cap \tilde{\mathcal{P}}_2) \cup \tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_2$

Proof. They are trial. We omit them. $\square$

3.2. Score Function of IVPHFEs. To determine the priorities of the alternatives of an interval-valued Pythagorean hesitant fuzzy group decision-making problem, we need the concept of score functions for IVPHFEs. Since an IVPHFE includes several pairs formed by possible Pythagorean membership intervals and possible Pythagorean nonmembership intervals, if we use a method similar to Definition 12, the intervals are represented by the average values of the intervals, which must lose some information because there is no one-to-one correspondence between an interval number and a value.

To facilitate comparison of IVPHFEs, we shall give the following comparison laws.

Definition 15. Let $\tilde{\mathcal{P}} = \{\langle \tilde{\mu}, \tilde{\nu} \rangle \mid \tilde{\mu} = [\mu^-, \mu^+], \tilde{\nu} = [\nu^-, \nu^+]\}$ be an IVPHFE. The score function $S(\tilde{\mathcal{P}})$ is described as

$$S(\tilde{\mathcal{P}}) = \frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} (\tilde{\mu}^2 - \tilde{\nu}^2) = \left[\frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} ((\mu^-)^2 - (\nu^+)^2), \frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} ((\mu^+)^2 - (\nu^-)^2) \right]. \quad (15)$$

The accuracy function $H(\tilde{\mathcal{P}})$ is described as

$$H(\tilde{\mathcal{P}}) = \frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} (\tilde{\mu}^2 + \tilde{\nu}^2) = \left[\frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} ((\mu^-)^2 + (\nu^-)^2), \frac{1}{2|\tilde{\mathcal{P}}|} \sum_{\langle \tilde{\mu}, \tilde{\nu} \rangle \in \tilde{\mathcal{P}}} ((\mu^+)^2 + (\nu^+)^2) \right]. \quad (16)$$

The idea of the above concepts originated from Definition 8 and used the definition of interval arithmetic in Definition 6. The score function of an IVPHFE is the mean of the difference between possible Pythagorean membership intervals and possible Pythagorean nonmembership intervals; also, the accuracy function reflects the overall accuracy degree of an IVPHFE. For keeping fuzzy information as much as possible, both the two functions are represented by interval values. For an IVPHFE $\tilde{\mathcal{P}}$, the score function $S(\tilde{\mathcal{P}}) \in [-1, 1]$ and the accuracy function $H(\tilde{\mathcal{P}}) \in [0, 1]$ hold obviously. Based on Definition 7, we have the following definition.

Definition 16. Let $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$ be two IVPHFEs.

- (1) If $P(S(\tilde{\mathcal{P}}_1) > S(\tilde{\mathcal{P}}_2)) < 0.5$, then we say $\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2$
- (2) If $P(S(\tilde{\mathcal{P}}_1) > S(\tilde{\mathcal{P}}_2)) = 0.5$, then
 - (2.1) If $P(H(\tilde{\mathcal{P}}_1) > H(\tilde{\mathcal{P}}_2)) < 0.5$, we say $\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2$
 - (2.2) If $P(H(\tilde{\mathcal{P}}_1) > H(\tilde{\mathcal{P}}_2)) = 0.5$, we say $\tilde{\mathcal{P}}_1 = \tilde{\mathcal{P}}_2$

Proposition 4. Let $\tilde{\mathcal{P}}_1 = \{\langle \tilde{\mu}_{1j}, \tilde{\nu}_{1j} \rangle \mid \tilde{\mu}_{1j} = [\mu_{1j}^-, \mu_{1j}^+], \tilde{\nu}_{1j} = [\nu_{1j}^-, \nu_{1j}^+], j = 1, 2, \dots, m\}$ and $\tilde{\mathcal{P}}_2 = \{\langle \tilde{\mu}_{2j}, \tilde{\nu}_{2j} \rangle \mid \tilde{\mu}_{2j} = [\mu_{2j}^-, \mu_{2j}^+], \tilde{\nu}_{2j} = [\nu_{2j}^-, \nu_{2j}^+], j = 1, 2, \dots, m\}$ be two IVPHFEs. If for any $j = 1, \dots, m$, $\mu_{1j}^- \leq \mu_{2j}^-$, $\mu_{1j}^+ \leq \mu_{2j}^+$, $\nu_{1j}^- \geq \nu_{2j}^-$, and $\nu_{1j}^+ \geq \nu_{2j}^+$, then $\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2$.

Proof. Based on Definition 15, we have

$$S(\tilde{\mathcal{P}}_1) = \frac{1}{2m} \sum_{j=1}^m [(\mu_{1j}^-)^2 - (\nu_{1j}^+)^2, (\mu_{1j}^+)^2 - (\nu_{1j}^-)^2], \quad (17)$$

$$S(\tilde{\mathcal{P}}_2) = \frac{1}{2m} \sum_{j=1}^m [(\mu_{2j}^-)^2 - (\nu_{2j}^+)^2, (\mu_{2j}^+)^2 - (\nu_{2j}^-)^2].$$

Suppose

$$L \triangleq \frac{\sum_{j=1}^m ((\mu_{2j}^+)^2 - (\nu_{2j}^-)^2 + (\nu_{1j}^+)^2 - (\mu_{1j}^-)^2)}{\sum_{j=1}^m ((\mu_{1j}^+)^2 - (\nu_{1j}^-)^2 + (\nu_{1j}^+)^2 - (\mu_{1j}^-)^2 + (\mu_{2j}^+)^2 - (\nu_{2j}^-)^2 + (\nu_{2j}^+)^2 - (\mu_{2j}^-)^2)}. \quad (18)$$

Hence, we obtain $P(S(\tilde{\mathcal{P}}_1) > S(\tilde{\mathcal{P}}_2)) = \max\{1 - \max\{L, 0\}, 0\}$ by Definition 7.

For any $j = 1, \dots, m$, $\mu_{1j}^- \leq \mu_{2j}^-$, $\mu_{1j}^+ \leq \mu_{2j}^+$, $\nu_{1j}^- \geq \nu_{2j}^-$, and $\nu_{1j}^+ \geq \nu_{2j}^+$, $(\mu_{2j}^+)^2 - (\mu_{1j}^-)^2 + (\nu_{1j}^+)^2 - (\nu_{2j}^-)^2 \geq (\mu_{1j}^+)^2 - (\mu_{2j}^-)^2 + (\nu_{2j}^+)^2 - (\nu_{1j}^-)^2 \geq 0$; then, $0.5 \leq L \leq 1$. So, $0 \leq 1 - \max\{L, 0\} = 1 - L \leq 0.5$. Hence, $P(S(\tilde{\mathcal{P}}_1) > S(\tilde{\mathcal{P}}_2)) = 1 - L \leq 0.5$. That means $\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2$ holds. $\square$

4. Aggregation Operators for Interval-Valued Pythagorean Hesitant Fuzzy Information

In multiattribute decision-making problems, the selection of aggregation operators is a basis problem, which is also important in the interval-valued Pythagorean hesitant

fuzzy environment. Considering an IVPHFE is regarded as the extension of an IVHFE, IVPFE, or PHFE, we propose a series of aggregation operators for interval-valued Pythagorean hesitant fuzzy information in this section based on the discussion of aggregation operators in [27, 45, 50, 51] and deduce some desirable properties.

4.1. The IVPHFWA, IVPHFWG, GIVPHFWA, and GIVPHFWG Operators

Definition 17. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of some IVPHFEs and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$), with $\omega_i \in [0, 1]$, $\sum_{i=1}^n \omega_i = 1$, and $\lambda > 0$.

- (1) An interval-valued Pythagorean hesitant fuzzy weighted averaging (IVPHFWA) operator can be seen a map IVPHFWA: $\Omega^n \longrightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \omega_1 \tilde{\mathcal{P}}_1 \oplus \dots \oplus \omega_n \tilde{\mathcal{P}}_n \\ &= \left\{ \left\langle \left[\sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^-)^2)^{\omega_i}}, \right. \right. \right. \\ &\quad \left. \sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^+)^2)^{\omega_i}} \right], \\ &\quad \left. \left[\prod_{i=1}^n (\nu_i^-)^{\omega_i}, \prod_{i=1}^n (\nu_i^+)^{\omega_i} \right] \right\rangle \\ &\quad \left. \mid \langle \tilde{\mu}_i, \tilde{\nu} \rangle_i \in \tilde{\mathcal{P}}_i, i = 1, \dots, n \right\}. \end{aligned} \quad (19)$$

- (2) An interval-valued Pythagorean hesitant fuzzy weighted geometric (IVPHFWG) operator can be seen a map IVPHFWG: $\Omega^n \longrightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFWG}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \tilde{\mathcal{P}}_1^{\omega_1} \otimes \dots \otimes \tilde{\mathcal{P}}_n^{\omega_n} \\ &= \left\{ \left\langle \left[\prod_{i=1}^n (\mu_i^-)^{\omega_i}, \prod_{i=1}^n (\mu_i^+)^{\omega_i} \right], \right. \right. \\ &\quad \cdot \left[1 - \prod_{i=1}^n (1 - (\nu_i^-)^2)^{\omega_i}, \right. \\ &\quad \cdot \left. \left. \left. \left. \sqrt{1 - \prod_{i=1}^n (1 - (\nu_i^+)^2)^{\omega_i}} \right] \right] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \right. \\ &\quad \left. \in \tilde{\mathcal{P}}_i, i = 1, \dots, n \right\}. \end{aligned} \quad (20)$$

- (3) A generalized interval-valued Pythagorean hesitant fuzzy weighted averaging (GIVPHFWA) operator can be seen a map GIVPHFWA: $\Omega^n \longrightarrow \Omega$, which satisfies

$$\begin{aligned} \text{GIVPHFWA}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \left(\omega_1 \tilde{\mathcal{P}}_1^\lambda \oplus \dots \oplus \omega_n \tilde{\mathcal{P}}_n^\lambda \right)^{1/\lambda} \\ &= \left\{ \left\langle \left[\sqrt{\left(1 - \prod_{i=1}^n (1 - (\mu_i^-)^{2\lambda})^{\omega_i} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{i=1}^n (1 - (\mu_i^+)^{2\lambda})^{\omega_i} \right)^{1/\lambda}} \right], \right. \right. \\ &\quad \cdot \left[\sqrt{1 - \left(1 - \prod_{i=1}^n (1 - (1 - (\nu_i^-)^2)^\lambda)^{\omega_i} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{i=1}^n (1 - (1 - (\nu_i^+)^2)^\lambda)^{\omega_i} \right)^{1/\lambda}} \right] \right\rangle \\ &\quad \left. \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, \dots, n \right\}, \end{aligned} \quad (21)$$

where $\lambda > 0$.

- (4) A generalized interval-valued Pythagorean hesitant fuzzy weighted geometric (GIVPHFWG) operator

can be seen a map GIVPHFWG: $\Omega^n \longrightarrow \Omega$, which satisfies

$$\begin{aligned} \text{GIVPHFWG}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \frac{1}{\lambda} \left((\lambda \tilde{\mathcal{P}}_1)^{\omega_1} \otimes \dots \otimes (\lambda \tilde{\mathcal{P}}_n)^{\omega_n} \right) \\ &= \left\{ \left\langle \left[\sqrt{1 - \left(1 - \prod_{i=1}^n (1 - (1 - (\mu_i^-)^2)^\lambda)^{\omega_i} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{i=1}^n (1 - (1 - (\mu_i^+)^2)^\lambda)^{\omega_i} \right)^{1/\lambda}} \right], \right. \right. \\ &\quad \cdot \left[\sqrt{\left(1 - \prod_{i=1}^n (1 - (\nu_i^-)^{2\lambda})^{\omega_i} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{i=1}^n (1 - (\nu_i^+)^{2\lambda})^{\omega_i} \right)^{1/\lambda}} \right] \right\rangle, \\ &\quad \left. \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, \dots, n \right\}, \end{aligned} \quad (22)$$

where $\lambda > 0$. If $\lambda = 1$, then the GIVPHFWA and GIVPHFWG operators reduce to the IVPHFWA and IVPHFWG operators, respectively.

Example 1. Suppose $\tilde{\mathcal{P}}_1 = \{ \langle [0.3, 0.5], [0.2, 0.4] \rangle \}$,

$$\begin{aligned}\tilde{\mathcal{P}}_2 &= \{ \langle [0.1, 0.4], [0.3, 0.6] \rangle, \langle [0.1, 0.4], [0.5, 0.7] \rangle \}, \\ \tilde{\mathcal{P}}_3 &= \{ \langle [0.3, 0.4], [0.2, 0.6] \rangle, \langle [0.1, 0.3], [0.2, 0.6] \rangle, \langle [0.4, 0.7], [0.2, 0.6] \rangle \},\end{aligned}\tag{23}$$

are three IVPHFEs, and their weight vector is $\omega = (0.15, 0.30, 0.55)^T$.

Then, by Definition 17, we have

$$\begin{aligned}\text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \omega_1 \tilde{\mathcal{P}}_1 \oplus \omega_2 \tilde{\mathcal{P}}_2 \oplus \omega_3 \tilde{\mathcal{P}}_3 \\ &= \left\{ \left\langle \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_i^-)^2)^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_i^+)^2)^{\omega_i}} \right], \right. \right. \\ &\quad \left. \cdot \left[\prod_{i=1}^3 (\nu_i^-)^{\omega_i}, \prod_{i=1}^3 (\nu_i^+)^{\omega_i} \right] \right\rangle \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2, 3 \right\} \\ &= \{ \langle [0.2583, 0.4173], [0.2259, 0.5646] \rangle, \langle [0.1498, 0.3701], [0.2259, 0.5646] \rangle, \\ &\quad \cdot \langle [0.3270, 0.6102], [0.2259, 0.5646] \rangle, \langle [0.2583, 0.4173], [0.2633, 0.5913] \rangle, \\ &\quad \cdot \langle [0.1498, 0.3701], [0.2633, 0.5913] \rangle, \langle [0.3270, 0.6102], [0.2633, 0.5913] \rangle \}; \\ \text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \tilde{\mathcal{P}}_1^{\omega_1} \otimes \tilde{\mathcal{P}}_2^{\omega_2} \otimes \tilde{\mathcal{P}}_3^{\omega_3} \\ &= \left\{ \left\langle \left[\prod_{i=1}^3 (\mu_i^-)^{\omega_i}, \prod_{i=1}^3 (\mu_i^+)^{\omega_i} \right], \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_i^-)^2)^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_i^+)^2)^{\omega_i}} \right] \right\rangle \right. \\ &\quad \left. \cdot \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2, 3 \right\} \\ &= \{ \langle [0.2158, 0.4136], [0.2351, 0.5774] \rangle, \langle [0.1179, 0.3531], [0.2351, 0.5774] \rangle, \\ &\quad \cdot \langle [0.2528, 0.5627], [0.2351, 0.5774] \rangle, \langle [0.2158, 0.4137], [0.3294, 0.6142] \rangle, \langle [0.1179, 0.3531], \\ &\quad \cdot [0.3294, 0.6142] \rangle, \langle [0.2528, 0.5627], [0.3294, 0.6142] \rangle \}; \\ \text{GIVPHFWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \left(\omega_1 \tilde{\mathcal{P}}_1^3 \oplus \omega_2 \tilde{\mathcal{P}}_2^3 \oplus \omega_3 \tilde{\mathcal{P}}_3^3 \right)^{1/3} \\ &= \left\{ \left\langle \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_i^-)^{2 \times 3})^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_i^+)^{2 \times 3})^{\omega_i}} \right], \right. \right. \\ &\quad \left. \cdot \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\nu_i^-)^2)^3 \right)^{\omega_i}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\nu_i^+)^2)^3 \right)^{\omega_i}} \right] \right\rangle, \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2, 3 \right\} \\ &= \{ \langle [0.2827, 0.4243], [0.2254, 0.5589] \rangle, \langle [0.2190, 0.3982], [0.2254, 0.5589] \rangle, \langle [0.3650, 0.6418], \\ &\quad \cdot [0.2254, 0.5589] \rangle, \langle [0.2827, 0.4243], [0.2582, 0.5805] \rangle, \langle [0.2190, 0.3982], [0.2582, 0.5805] \rangle, \langle [0.3650, 0.6418], \\ &\quad \cdot [0.2582, 0.5805] \rangle \}; \\ \text{GIVPHFWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \frac{1}{3} \left((3\tilde{\mathcal{P}}_1)^{\omega_1} \otimes (3\tilde{\mathcal{P}}_2)^{\omega_2} \otimes (3\tilde{\mathcal{P}}_3)^{\omega_3} \right) \\ &= \left\{ \left\langle \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\mu_i^-)^2)^3 \right)^{\omega_i}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\mu_i^+)^2)^3 \right)^{\omega_i}} \right] \right], \right. \right. \\ &\quad \left. \cdot \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_i^-)^{2 \times 3})^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_i^+)^{2 \times 3})^{\omega_i}} \right] \right\rangle, \mid \langle \tilde{\mu}_i, \tilde{\nu}_i \rangle \in \tilde{\mathcal{P}}_i, i = 1, 2, 3 \right\} \\ &= \{ \langle [0.2135, 0.4129], [0.2532, 0.5858] \rangle, \langle [0.1174, 0.3508], [0.2532, 0.5858] \rangle, \langle [0.2471, 0.5437], [0.2532, 0.5858] \rangle, \\ &\quad \cdot \langle [0.2135, 0.4129], [0.4101, 0.6298] \rangle, \langle [0.1174, 0.3508], [0.4101, 0.6298] \rangle, \langle [0.2471, 0.5437], [0.4101, 0.6298] \rangle \}.\end{aligned}\tag{24}$$

Based on Proposition 1, we know the above four aggregation operators are all IVPHFEs. Also, we can obtain that the score values are

$$\begin{aligned}
S(\text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1344, 0.0838]; \\
S(\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1569, 0.0611]; \\
S(\text{GIVPHFWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1188, 0.0957]; \\
S(\text{GIVPHFWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1649, 0.0401]; \\
\text{So, } P(\text{SIVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) \leq S(\text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.5518; \\
P(\text{SIVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) \leq S(\text{GIVPHFWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.5840; \\
P(\text{SGIVPHFWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) \leq S(\text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.5876; \\
\text{Hence, } \text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3); \\
\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{GIVPHFWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3); \\
\text{GIVPHFWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3).
\end{aligned} \tag{25}$$

In fact, we have the following.

Lemma 1 (see [52]). Let $x_i > 0$, $\omega_i > 0$ ($i = 1, 2, \dots, n$), and $\sum_{i=1}^n \omega_i = 1$, then

$$\prod_{i=1}^n x_i^{\omega_i} \leq \sum_{i=1}^n \omega_i x_i, \tag{26}$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Proposition 5. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of IVPHFEs, $\omega_i > 0$, and $\sum_{i=1}^n \omega_i = 1$, then

$$\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) \leq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \tag{27}$$

with equality if and only if $\tilde{\mathcal{P}}_1 = \tilde{\mathcal{P}}_2 = \dots = \tilde{\mathcal{P}}_n$.

Proof. Suppose $\langle [\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+] \rangle \in \tilde{\mathcal{P}}_i$, for any i . Based on Lemma 1, we have

$$\prod_{i=1}^n ((\mu_i^-)^2)^{\omega_i} \leq \sum_{i=1}^n \omega_i (\mu_i^-)^2 = 1 - \sum_{i=1}^n \omega_i (1 - (\mu_i^-)^2) \leq 1 - \prod_{i=1}^n (1 - (\mu_i^-)^2)^{\omega_i}. \tag{28}$$

So,

$$\prod_{i=1}^n (\mu_i^-)^{\omega_i} = \sqrt{\prod_{i=1}^n ((\mu_i^-)^2)^{\omega_i}} \leq$$

$\sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^-)^2)^{\omega_i}}$. Similarly, we have

$$\begin{aligned}
\prod_{i=1}^n (\mu_i^+)^{\omega_i} &\leq \sqrt{1 - \prod_{i=1}^n (1 - (\mu_i^+)^2)^{\omega_i}}, \\
\prod_{i=1}^n (\nu_i^-)^{\omega_i} &\leq \sqrt{1 - \prod_{i=1}^n (1 - (\nu_i^-)^2)^{\omega_i}}, \\
\prod_{i=1}^n (\nu_i^+)^{\omega_i} &\leq \sqrt{1 - \prod_{i=1}^n (1 - (\nu_i^+)^2)^{\omega_i}}.
\end{aligned} \tag{29}$$

By Definition 17 and Proposition 4, $P(S(\tilde{\mathcal{P}}_1^{\omega_1} \otimes \dots \otimes \tilde{\mathcal{P}}_n^{\omega_n}) \geq S(\omega_1 \tilde{\mathcal{P}}_1 \oplus \dots \oplus \omega_n \tilde{\mathcal{P}}_n)) \leq 0.5$ holds, which means

$$\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) \leq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n). \tag{30}$$

□

Proposition 6. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of IVPHFEs, $\lambda > 0$, $\omega_i > 0$, and $\sum_{i=1}^n \omega_i = 1$, then

$$\begin{aligned}
\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{GIVPHFWA}_\lambda(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\
\text{GIVPHFWG}_\lambda(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n).
\end{aligned} \tag{31}$$

Proof. Suppose $\langle [\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+] \rangle \in \tilde{\mathcal{P}}_i$, for any i . Since $\prod_{i=1}^n ((\mu_i^-)^2)^{\omega_i} = (\prod_{i=1}^n ((\mu_i^-)^{2\lambda})^{\omega_i})^{1/\lambda} \leq (\sum_{i=1}^n \omega_i (\mu_i^-)^{2\lambda})^{1/\lambda} = (1 - \sum_{i=1}^n \omega_i (1 - (\mu_i^-)^{2\lambda}))^{1/\lambda} \leq 1 - (\prod_{i=1}^n (1 - (\mu_i^-)^{2\lambda})^{\omega_i})^{1/\lambda}$, we have $\prod_{i=1}^n (\mu_i^-)^{\omega_i} = \sqrt[\lambda]{\prod_{i=1}^n ((\mu_i^-)^2)^{\omega_i}} \leq$

$\sqrt[\lambda]{1 - \prod_{i=1}^n (1 - (\mu_i^-)^{2\lambda})^{\omega_i}}$. Similarly, $\prod_{i=1}^n (\mu_i^+)^{\omega_i} = \sqrt[\lambda]{\prod_{i=1}^n ((\mu_i^+)^2)^{\omega_i}} \leq \sqrt[\lambda]{1 - \prod_{i=1}^n (1 - (\mu_i^+)^{2\lambda})^{\omega_i}}$ holds.

Also,

$$\begin{aligned}
 \sqrt[1/\lambda]{1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - (v_i^-)^2)^\lambda\right)^{\omega_i}\right)} &\leq \sqrt[1/\lambda]{1 - \left(1 - \sum_{i=1}^n \omega_i \left(1 - (1 - (v_i^-)^2)^\lambda\right)\right)} \\
 &= \sqrt[1/\lambda]{1 - \left(\sum_{i=1}^n \omega_i \left(1 - (v_i^-)^2\right)^\lambda\right)} \leq \sqrt[1/\lambda]{1 - \left(\prod_{i=1}^n \left((1 - (v_i^-)^2)^\lambda\right)^{\omega_i}\right)} \\
 &= \sqrt[1/\lambda]{1 - \prod_{i=1}^n (1 - (v_i^-)^2)^{\omega_i}}.
 \end{aligned} \tag{32}$$

Similarly, $\sqrt[1/\lambda]{1 - (1 - \prod_{i=1}^n (1 - (1 - (v_i^+)^2)^\lambda)^{\omega_i})} \leq$

$\sqrt[1/\lambda]{1 - \prod_{i=1}^n (1 - (v_i^+)^2)^{\omega_i}}$ holds.

All imply that $\text{IVPHFWG}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) \preceq \text{GIVPHFWA}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n)$.

In the same way, we have $\text{GIVPHFWG}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) \preceq \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n)$ and complete the proof of Proposition 6. $\square$

Proposition 4.2 shows no matter how the parameter λ ($\lambda > 0$) changes, the values obtained by IVPHFWG operators are not bigger than the ones obtained by GIVPHFWA operators and the values obtained by GIVPHFWG operators are not bigger than the ones obtained by IVPHFWA operators.

Proposition 7. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of IVPHFEs, $\lambda_2 \geq \lambda_1 > 0$, $\omega_i > 0$, and $\sum_{i=1}^n \omega_i = 1$, then

$$\begin{aligned}
 \text{GIVPHFWA}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\preceq \text{GIVPHFWA}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\
 \text{GIVPHFWG}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\succeq \text{GIVPHFWG}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n).
 \end{aligned} \tag{33}$$

Proof. According to Theorem 3.8 in [31], we have

$$\begin{aligned}
 \sqrt[1/\lambda_1]{1 - \prod_{i=1}^n (1 - (\mu_i^-)^{2\lambda_1})^{\omega_i}} &\leq \sqrt[1/\lambda_2]{1 - \prod_{i=1}^n (1 - (\mu_i^-)^{2\lambda_2})^{\omega_i}}, \\
 \sqrt[1/\lambda_1]{1 - \prod_{i=1}^n (1 - (\mu_i^+)^{2\lambda_1})^{\omega_i}} &\leq \sqrt[1/\lambda_2]{1 - \prod_{i=1}^n (1 - (\mu_i^+)^{2\lambda_2})^{\omega_i}}.
 \end{aligned} \tag{34}$$

Also, we can conclude that

$$\sqrt[1/\lambda_1]{1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - (v_i^-)^2)^{\lambda_1}\right)^{\omega_i}\right)} \geq \sqrt[1/\lambda_2]{1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - (v_i^-)^2)^{\lambda_2}\right)^{\omega_i}\right)}. \tag{35}$$

Similarly,

$$\sqrt[1/\lambda_1]{1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - (v_i^+)^2)^{\lambda_1}\right)^{\omega_i}\right)} \geq \sqrt[1/\lambda_2]{1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - (v_i^+)^2)^{\lambda_2}\right)^{\omega_i}\right)}. \tag{36}$$

Therefore, by Proposition 4, we have

$$\begin{aligned} \text{GIVPHFWA}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\preceq \text{GIVPHFWA}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\ \text{GIVPHFWG}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\succeq \text{GIVPHFWG}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n). \end{aligned} \quad (37)$$

□

4.2. The IVPHFOWA, IVPHFOWG, GIVPHFWA, and GIVPHFWG Operators

Definition 18. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of some IVPHFEs, $\tilde{\mathcal{P}}_{\sigma(i)}$ be the i^{th} largest of them, and

$\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)^T$ be the associated vector such that $\kappa_i \in [0, 1]$, $\sum_{i=1}^n \kappa_i = 1$, and $\lambda > 0$. Then,

- (1) An interval-valued Pythagorean hesitant fuzzy ordered weighted averaging (IVPHFWA) operator can be seen a map IVPHFOWA: $\Omega^n \longrightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFWA}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \kappa_1 \tilde{\mathcal{P}}_{\sigma(1)} \oplus \dots \oplus \kappa_n \tilde{\mathcal{P}}_{\sigma(n)} \\ &= \left\{ \left\langle \left[\sqrt{1 - \prod_{i=1}^n \left(1 - (\mu_{\sigma(i)}^-)^2 \right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^n \left(1 - (\mu_{\sigma(i)}^+)^2 \right)^{\kappa_i}} \right], \left[\prod_{i=1}^n (\nu_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\nu_{\sigma(i)}^+)^{\kappa_i} \right] \right\rangle \right. \\ &\quad \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \Big\}. \end{aligned} \quad (38)$$

- (2) An interval-valued Pythagorean hesitant fuzzy ordered weighted geometric (IVPHFWG) operator

can be seen a map IVPHFOWG: $\Omega^n \longrightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFWG}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \tilde{\mathcal{P}}_{\sigma(1)}^{\kappa_1} \otimes \dots \otimes \tilde{\mathcal{P}}_{\sigma(n)}^{\kappa_n} \\ &= \left\{ \left\langle \left[\prod_{i=1}^n (\mu_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\mu_{\sigma(i)}^+)^{\kappa_i} \right], \left[\sqrt{1 - \prod_{i=1}^n \left(1 - (\nu_{\sigma(i)}^-)^2 \right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^n \left(1 - (\nu_{\sigma(i)}^+)^2 \right)^{\kappa_i}} \right] \right\rangle \right. \\ &\quad \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \Big\}. \end{aligned} \quad (39)$$

- (3) A generalized interval-valued Pythagorean hesitant fuzzy ordered weighted averaging (GIVPHFWA)

operator can be seen a map GIVPHFWA: $\Omega^n \longrightarrow \Omega$, which satisfies

$$\begin{aligned} \text{GIVPHFWA}_{\lambda}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \left(\kappa_1 \dot{\tilde{\mathcal{P}}}_{\sigma(1)}^{\lambda} \oplus \dots \oplus \kappa_n \dot{\tilde{\mathcal{P}}}_{\sigma(n)}^{\lambda} \right)^{1/\lambda} \\ &= \left\{ \left\langle \left[\sqrt{\left(1 - \prod_{i=1}^n \left(1 - (\mu_{\sigma(i)}^-)^{2\lambda} \right)^{\kappa_i} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{i=1}^n \left(1 - (\mu_{\sigma(i)}^+)^{2\lambda} \right)^{\kappa_i} \right)^{1/\lambda}} \right], \right. \right. \\ &\quad \cdot \left[\sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - (\nu_{\sigma(i)}^-)^{\lambda} \right)^{\kappa_i} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - (\nu_{\sigma(i)}^+)^{\lambda} \right)^{\kappa_i} \right)^{1/\lambda}} \right] \Big\rangle, \\ &\quad \cdot \mid \langle \dot{\tilde{\mu}}_{\sigma(i)}, \dot{\tilde{\nu}}_{\sigma(i)} \rangle \in \dot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, \dots, n \Big\}, \end{aligned} \quad (40)$$

where $\lambda > 0$.

- (4) A generalized interval-valued Pythagorean hesitant fuzzy ordered weighted geometric (GIVPHFOWG)

operator can be seen a map
GIVPHFOWG: $\Omega^n \longrightarrow \Omega$, which satisfies

$$\begin{aligned} \text{GIVPHFOWG}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \frac{1}{\lambda} \left(\left(\lambda \ddot{\mathcal{P}}_{\sigma(1)} \right)^{\kappa_1} \otimes \dots \otimes \left(\lambda \ddot{\mathcal{P}}_{\sigma(n)} \right)^{\kappa_n} \right) \\ &= \left\{ \left\langle \left[\sqrt[1/\lambda]{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - (\ddot{\mu}_{\sigma(i)}^-)^2 \right)^\lambda \right)^{\kappa_i} \right)} \right]^{1/\lambda}, \sqrt[1/\lambda]{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - (\ddot{\mu}_{\sigma(i)}^+)^2 \right)^\lambda \right)^{\kappa_i} \right)} \right]^{1/\lambda} \right. \\ &\quad \cdot \left[\sqrt[1/\lambda]{\left(1 - \prod_{i=1}^n \left(1 - (\ddot{\nu}_{\sigma(i)}^-)^{2\lambda} \right)^{\kappa_i} \right)}^{1/\lambda}, \sqrt[1/\lambda]{\left(1 - \prod_{i=1}^n \left(1 - (\ddot{\nu}_{\sigma(i)}^+)^{2\lambda} \right)^{\kappa_i} \right)}^{1/\lambda} \right] \rangle, \\ &\quad \left. \cdot \mid \langle \ddot{\mu}_{\sigma(i)}, \ddot{\nu}_{\sigma(i)} \rangle \in \ddot{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \right\}, \end{aligned} \quad (41)$$

where $\lambda > 0$. If $\lambda = 1$, then the GIVPHFOWA and GIVPHFOWG operators reduce to the IVPHFOWA and IVPHFOWG operators, respectively.

By comparing Definition 17 with Definition 18, we can find that the IVPHFOWA, IVPHFOWG, GIVPHFOWA, and GIVPHFOWG operators weight the ordered positions of the IVPHFEs instead of weighting the IVPHFEs themselves. Also, their characteristics lie in reordering original IVPHFEs according to a decreasing order and in aggregating through the associated weights of positions where the weight κ_i is associated with the i^{th} position in the collection of the IVPHFEs during aggregation processes.

Example 2. Suppose

$$\begin{aligned} \tilde{\mathcal{P}}_1 &= \{ \langle [0.3, 0.5], [0.2, 0.4] \rangle \}, \\ \tilde{\mathcal{P}}_2 &= \{ \langle [0.1, 0.4], [0.3, 0.6] \rangle, \langle [0.1, 0.4], [0.5, 0.7] \rangle \}, \\ \tilde{\mathcal{P}}_3 &= \{ \langle [0.3, 0.4], [0.2, 0.6] \rangle, \langle [0.1, 0.3], [0.2, 0.6] \rangle, \\ &\quad \langle [0.4, 0.7], [0.2, 0.6] \rangle \}, \end{aligned} \quad (42)$$

are three IVPHFEs, and the aggregation-associated vector is $\kappa = (0.2, 0.45, 0.35)^T$.

Then, by Definition 15, we have the score values of $\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2$, and $\tilde{\mathcal{P}}_3$ as follows:

$$\begin{aligned} S(\tilde{\mathcal{P}}_1) &= \left[\frac{1}{2} \left((0.3)^2 - (0.4)^2 \right), \frac{1}{2} \left((0.5)^2 - (0.2)^2 \right) \right] = [-0.035, 0.045], \\ S(\tilde{\mathcal{P}}_2) &= \left[\frac{1}{4} \left((0.1)^2 - (0.6)^2 + (0.1)^2 - (0.7)^2 \right), \frac{1}{4} \left((0.4)^2 - (0.3)^2 + (0.4)^2 - (0.5)^2 \right) \right] \\ &= [-0.2075, -0.005], \\ S(\tilde{\mathcal{P}}_3) &= \left[\frac{1}{6} \left((0.3)^2 - (0.6)^2 + (0.1)^2 - (0.6)^2 + (0.4)^2 - (0.6)^2 \right), \frac{1}{6} \left((0.4)^2 - (0.2)^2 + (0.3)^2 - (0.2)^2 + (0.7)^2 - (0.2)^2 \right) \right] \\ &= [-0.1367, 0.1033]. \end{aligned} \quad (43)$$

Based on Definition 16, we can obtain that $P(S(\tilde{\mathcal{P}}_1) > S(\tilde{\mathcal{P}}_3)) = 0.5678$ and $P(S(\tilde{\mathcal{P}}_3) > S(\tilde{\mathcal{P}}_2)) = 0.7024$ and then $\tilde{\mathcal{P}}_{\sigma(1)} = \tilde{\mathcal{P}}_1$, $\tilde{\mathcal{P}}_{\sigma(2)} = \tilde{\mathcal{P}}_3$, and $\tilde{\mathcal{P}}_{\sigma(3)} = \tilde{\mathcal{P}}_2$.

According to Definition 18, we have

$$\begin{aligned}
\text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \kappa_1 \tilde{\mathcal{P}}_{\sigma(1)} \oplus \kappa_1 \tilde{\mathcal{P}}_{\sigma(2)} \oplus \kappa_n \tilde{\mathcal{P}}_{\sigma(3)} \\
&= \left\{ \left\langle \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_{\sigma(i)}^-)^2)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\mu_{\sigma(i)}^+)^2)^{\kappa_i}} \right], \left[\prod_{i=1}^3 (\nu_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^3 (\nu_{\sigma(i)}^+)^{\kappa_i} \right] \right\rangle \right. \\
&\quad \left. \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, 2, 3 \right\}. \\
&= \{ \langle [0.2505, 0.4229], [0.2305, 0.5533] \rangle, \langle [0.2505, 0.4229], [0.2756, 0.5839] \rangle, \langle [0.1629, 0.3856], [0.2305, 0.5533] \rangle, \\
&\quad \langle [0.1629, 0.3856], [0.2756, 0.5839] \rangle, \langle [0.3097, 0.5865], [0.2305, 0.5930] \rangle, \\
&\quad \langle [0.3097, 0.5865], [0.2756, 0.6259] \rangle \}; \\
\text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \tilde{\mathcal{P}}_{\sigma(1)}^{\kappa_1} \otimes \tilde{\mathcal{P}}_{\sigma(2)}^{\kappa_2} \otimes \tilde{\mathcal{P}}_{\sigma(3)}^{\kappa_3} \\
&= \left\{ \left\langle \left[\prod_{i=1}^3 (\mu_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^3 (\mu_{\sigma(i)}^+)^{\kappa_i} \right], \left[\sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_{\sigma(i)}^-)^2)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^3 (1 - (\nu_{\sigma(i)}^+)^2)^{\kappa_i}} \right] \right\rangle \right. \\
&\quad \left. \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, 2, 3 \right\}. \\
&= \{ \langle [0.2042, 0.4183], [0.2404, 0.5694] \rangle, \langle [0.2042, 0.4183], [0.3456, 0.6130] \rangle, \langle [0.1246, 0.3675], [0.2404, 0.5694] \rangle, \\
&\quad \langle [0.1246, 0.3675], [0.3456, 0.6131] \rangle, \langle [0.2325, 0.5380], [0.2404, 0.6244] \rangle, \\
&\quad \langle [0.2325, 0.5380], [0.3456, 0.6607] \rangle \}; \\
\text{GIVPHFOWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \left(\kappa_1 \tilde{\mathcal{P}}_{\sigma(1)}^3 \oplus \kappa_2 \tilde{\mathcal{P}}_{\sigma(2)}^3 \oplus \kappa_3 \tilde{\mathcal{P}}_{\sigma(3)}^3 \right)^{1/3} \\
&= \left\{ \left\langle \left[\sqrt[3]{\left(1 - \prod_{i=1}^3 (1 - (\mu_{\sigma(i)}^-)^{2 \times 3})^{\kappa_i} \right)^{1/3}}, \sqrt[3]{\left(1 - \prod_{i=1}^3 (1 - (\mu_{\sigma(i)}^+)^{2 \times 3})^{\kappa_i} \right)^{1/3}} \right], \right. \right. \\
&\quad \left. \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\nu_{\sigma(i)}^-)^2)^3)^{\kappa_i} \right)^{1/3}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\nu_{\sigma(i)}^+)^2)^3)^{\kappa_i} \right)^{1/3}} \right] \right\rangle, \\
&\quad \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, 2, 3 \}, \\
&= \{ \langle [0.2793, 0.4310], [0.2299, 0.5466] \rangle, \langle [0.2793, 0.4310], [0.2697, 0.5706] \rangle, \\
&\quad \langle [0.2296, 0.4122], [0.2300, 0.5466] \rangle, \langle [0.2296, 0.4122], [0.2697, 0.5706] \rangle, \\
&\quad \langle [0.3547, 0.6241], [0.2299, 0.5780] \rangle, \langle [0.3547, 0.6241], [0.2697, 0.6053] \rangle \}; \\
\text{GIVPHFOWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \frac{1}{3} \left((3 \tilde{\mathcal{P}}_{\sigma(1)})^{\kappa_1} \otimes (3 \tilde{\mathcal{P}}_{\sigma(2)})^{\kappa_2} \otimes (3 \tilde{\mathcal{P}}_{\sigma(3)})^{\kappa_3} \right) \\
&= \left\{ \left\langle \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\mu_{\sigma(i)}^-)^2)^3)^{\kappa_i} \right)^{1/3}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^3 (1 - (1 - (\mu_{\sigma(i)}^+)^2)^3)^{\kappa_i} \right)^{1/3}} \right], \right. \right. \\
&\quad \left. \left[\sqrt[3]{\left(1 - \prod_{i=1}^3 (1 - (\nu_{\sigma(i)}^-)^{2 \times 3})^{\kappa_i} \right)^{1/3}}, \sqrt[3]{\left(1 - \prod_{i=1}^3 (1 - (\nu_{\sigma(i)}^+)^{2 \times 3})^{\kappa_i} \right)^{1/3}} \right] \right\rangle, \\
&\quad \cdot \mid \langle \tilde{\mu}_{\sigma(i)}, \tilde{\nu}_{\sigma(i)} \rangle \in \tilde{\mathcal{P}}_{\sigma(i)}, i = 1, 2, 3 \}, \\
&= \{ \langle [0.2020, 0.4174], [0.2583, 0.5806] \rangle, \langle [0.2020, 0.4174], [0.4206, 0.6324] \rangle, \\
&\quad \langle [0.1239, 0.3648], [0.2583, 0.5806] \rangle, \langle [0.1239, 0.3648], [0.4206, 0.6324] \rangle, \\
&\quad \langle [0.2275, 0.5213], [0.2583, 0.6438] \rangle, \langle [0.2275, 0.5213], [0.4206, 0.6767] \rangle \}.
\end{aligned}$$

Based on Definition 14, we can obtain that

$$\begin{aligned} S(\text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1389, 0.0796], \\ S(\text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1670, 0.0556], \\ S(\text{GIVPHFOWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1197, 0.1001], \\ S(\text{GIVPHFOWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) &= [-0.1776, 0.0356]. \end{aligned} \quad (45)$$

Then,

$$\begin{aligned} P(S(\text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) \leq S(\text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.5591, \\ P(S(\text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) \leq S(\text{GIVPHFOWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.6038, \\ P(S(\text{GIVPHFOWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)) \leq S(\text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3))) &= 0.5958. \end{aligned} \quad (46)$$

So,

$$\begin{aligned} \text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3), \\ \text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{GIVPHFOWA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3), \\ \text{GIVPHFOWG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &\leq \text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3). \end{aligned} \quad (47)$$

Similar to Proposition 5–7, we easily obtain the following properties of ordered aggregation operators of IVPHFes.

Proposition 8. Let $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$) be a collection of IVPHFes, $\lambda > 0$, and $\lambda_2 > \lambda_1 > 0$, then

$$\begin{aligned} \text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\ \text{IVPHFOWG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{GIVPHFOWA}_{\lambda}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\ \text{GIVPHFOWG}_{\lambda}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{IVPHFOWA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\ \text{GIVPHFOWA}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\leq \text{GIVPHFOWA}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n), \\ \text{GIVPHFOWG}_{\lambda_1}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n) &\geq \text{GIVPHFOWG}_{\lambda_2}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots, \tilde{\mathcal{P}}_n). \end{aligned} \quad (48)$$

From Definition 17 and Definition 18, we know that the IVPHFWA, IVPHFOWG, GIVPHFWA, and GIVPHFOWG operators weight the interval-valued hesitant fuzzy argument itself and there is no consideration of the impact of the order of parameters, while the IVPHFOWA, IVPHFOWG, GIVPHFOWA, and GIVPHFOWG operators only weight the ordered position of each given IVPHFE and there is no consideration of the impact of the argument itself. Hence, we propose a new hybrid aggregation operator which can weight both all the given arguments and their ordered positions.

4.3. The IVPHFHA, IVPHFHG, GIVPHFHA, and GIVPHFHG Operators

Definition 19. For a collection of some IVPHFes $\tilde{\mathcal{P}}_i$ ($i = 1, 2, \dots, n$), $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of them with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Suppose $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)^T$ is the associated vector such that $\kappa_i \in [0, 1]$, $\sum_{i=1}^n \kappa_i = 1$, and $\lambda > 0$. Then,

- (1) An interval-valued Pythagorean hesitant fuzzy hybrid averaging (IVPHFHA) operator can be seen a map IVPHFHA: $\Omega^n \rightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFHA}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \kappa_1 \dot{\tilde{\mathcal{P}}}_{\sigma(1)} \oplus \dots \oplus \kappa_n \dot{\tilde{\mathcal{P}}}_{\sigma(n)} = \left\{ \left\langle \left[\sqrt{1 - \prod_{i=1}^n \left(1 - (\dot{\mu}_{\sigma(i)}^-)^2 \right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^n \left(1 - (\dot{\mu}_{\sigma(i)}^+)^2 \right)^{\kappa_i}} \right], \right. \right. \\ &\quad \left. \left. \left[\prod_{i=1}^n (\dot{\nu}_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\dot{\nu}_{\sigma(i)}^+)^{\kappa_i} \right] \right\rangle \mid \langle \dot{\mu}_{\sigma(i)}, \dot{\nu}_{\sigma(i)} \rangle \in \dot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, \dots, n \right\}, \end{aligned} \quad (49)$$

where $\ddot{\mathcal{P}}_{\sigma(i)}$ is the largest i^{th} of $\ddot{\mathcal{P}}_k = n\omega_k \tilde{\mathcal{P}}_k$ ($k = 1, 2, \dots, n$).

- (2) An interval-valued Pythagorean hesitant fuzzy hybrid geometric (IVPHFHG) operator can be seen a map IVPHFHG: $\Omega^n \longrightarrow \Omega$, such that

$$\begin{aligned} \text{IVPHFHG}(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \left(\ddot{\mathcal{P}}_{\sigma(1)} \right)^{\kappa_1} \otimes \dots \otimes \left(\ddot{\mathcal{P}}_{\sigma(n)} \right)^{\kappa_n} \\ &= \left\{ \left\langle \left[\prod_{i=1}^n \left(\ddot{\mu}_{\sigma(i)}^- \right)^{\kappa_i}, \prod_{i=1}^n \left(\ddot{\mu}_{\sigma(i)}^+ \right)^{\kappa_i} \right], \left[\sqrt{1 - \prod_{i=1}^n \left(1 - \left(\ddot{\nu}_{\sigma(i)}^- \right)^2 \right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^n \left(1 - \left(\ddot{\nu}_{\sigma(i)}^+ \right)^2 \right)^{\kappa_i}} \right] \right\rangle \right. \\ &\quad \left. | \langle \ddot{\mu}_{\sigma(i)}, \ddot{\nu}_{\sigma(i)} \rangle \in \ddot{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \right\}. \end{aligned} \quad (50)$$

where $\ddot{\mathcal{P}}_{\sigma(i)}$ is the largest i^{th} of $\ddot{\mathcal{P}}_k = (\tilde{\mathcal{P}}_k)^{n\omega_k}$ ($k = 1, 2, \dots, n$).

be seen a map GIVPHFHA: $\Omega^n \longrightarrow \Omega$, which satisfies

- (3) A generalized interval-valued Pythagorean hesitant fuzzy hybrid averaging (GIVPHFHA) operator can

$$\begin{aligned} \text{GIVPHFHA}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \left(\kappa_1 \left(\dot{\mathcal{P}}_{\sigma(1)} \right)^\lambda \oplus \dots \oplus \kappa_n \left(\dot{\mathcal{P}}_{\sigma(n)} \right)^\lambda \right)^{1/\lambda} \\ &= \left\{ \left\langle \left[\sqrt{\left(1 - \prod_{i=1}^n \left(1 - \left(\dot{\mu}_{\sigma(i)}^- \right)^{2\lambda} \right)^{\kappa_i}} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{i=1}^n \left(1 - \left(\dot{\mu}_{\sigma(i)}^+ \right)^{2\lambda} \right)^{\kappa_i}} \right)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - \left(\dot{\nu}_{\sigma(i)}^- \right)^2 \right)^\lambda \right)^{\kappa_i}} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - \left(\dot{\nu}_{\sigma(i)}^+ \right)^2 \right)^\lambda \right)^{\kappa_i}} \right)^{1/\lambda}} \right] \right\rangle, \\ &\quad | \langle \dot{\mu}_{\sigma(i)}, \dot{\nu}_{\sigma(i)} \rangle \in \dot{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \right\}, \end{aligned} \quad (51)$$

where $\dot{\mathcal{P}}_{\sigma(i)}$ is the largest i^{th} of $\dot{\mathcal{P}}_k = n\omega_k \tilde{\mathcal{P}}_k$ ($k = 1, 2, \dots, n$).

be seen a map GIVPHFHG: $\Omega^n \longrightarrow \Omega$, which satisfies

- (4) A generalized interval-valued Pythagorean hesitant fuzzy hybrid geometric (GIVPHFHG) operator can

$$\begin{aligned} \text{GIVPHFHG}_\lambda(\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_n) &= \frac{1}{\lambda} \left(\left(\lambda \ddot{\mathcal{P}}_{\sigma(1)} \right)^{\kappa_1} \otimes \dots \otimes \left(\lambda \ddot{\mathcal{P}}_{\sigma(n)} \right)^{\kappa_n} \right) \\ &= \left\{ \left\langle \left[\sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - \left(\ddot{\mu}_{\sigma(i)}^- \right)^2 \right)^\lambda \right)^{\kappa_i}} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{i=1}^n \left(1 - \left(1 - \left(\ddot{\mu}_{\sigma(i)}^+ \right)^2 \right)^\lambda \right)^{\kappa_i}} \right)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{\left(1 - \prod_{i=1}^n \left(1 - \left(\ddot{\nu}_{\sigma(i)}^- \right)^{2\lambda} \right)^{\kappa_i}} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{i=1}^n \left(1 - \left(\ddot{\nu}_{\sigma(i)}^+ \right)^{2\lambda} \right)^{\kappa_i}} \right)^{1/\lambda}} \right] \right\rangle, \\ &\quad | \langle \ddot{\mu}_{\sigma(i)}, \ddot{\nu}_{\sigma(i)} \rangle \in \ddot{\mathcal{P}}_{\sigma(i)}, i = 1, \dots, n \right\}, \end{aligned} \quad (52)$$

where $\ddot{\mathcal{P}}_{\sigma(i)}$ is the largest i^{th} of $\ddot{\mathcal{P}}_k = (\tilde{\mathcal{P}}_k)^{n\omega_k}$ ($k = 1, 2, \dots, n$).

If $\lambda = 1$, then the GIVPHFHA and GIVPHFHG operators reduce to the IVPHFHA and IVPHFHG operators, respectively.

Example 3. Suppose $\tilde{\mathcal{P}}_1 = \{\langle [0.3, 0.5], [0.2, 0.4] \rangle\}$,

$$\begin{aligned}\tilde{\mathcal{P}}_2 &= \{\langle [0.1, 0.4], [0.3, 0.6] \rangle, \langle [0.1, 0.4], [0.5, 0.7] \rangle\}, \\ \tilde{\mathcal{P}}_3 &= \{\langle [0.3, 0.4], [0.2, 0.6] \rangle, \langle [0.1, 0.3], [0.2, 0.6] \rangle, \langle [0.4, 0.7], [0.2, 0.6] \rangle\},\end{aligned}\tag{53}$$

are three IVPHFes, the weight vector of the attributes is $\omega = (0.15, 0.30, 0.55)^T$, and the aggregation-associated vector is $\kappa = (0.2, 0.45, 0.35)^T$. Then, we can obtain

$$\begin{aligned}\dot{\tilde{\mathcal{P}}}_1 &= (3 \times 0.15) \otimes \tilde{\mathcal{P}}_1 = \{\langle [0.2038, 0.3485], [0.4847, 0.6621] \rangle\}, \\ \dot{\tilde{\mathcal{P}}}_2 &= (3 \times 0.3) \otimes \tilde{\mathcal{P}}_2 = \{\langle [0.0949, 0.3811], [0.3384, 0.6314] \rangle, \langle [0.0949, 0.3811], [0.5359, 0.7254] \rangle\}, \\ \dot{\tilde{\mathcal{P}}}_3 &= (3 \times 0.3) \otimes \tilde{\mathcal{P}}_3 = \{\langle [0.3796, 0.5000], [0.0703, 0.4305] \rangle, \langle [0.1282, 0.3796], [0.0703, 0.4305] \rangle, \\ &\quad \langle [0.5000, 0.8190], [0.0703, 0.4305] \rangle\}, \\ \ddot{\tilde{\mathcal{P}}}_1 &= (\tilde{\mathcal{P}}_1)^{3 \times 0.15} = \{\langle [0.5817, 0.7320], [0.1349, 0.2747] \rangle\}, \\ \ddot{\tilde{\mathcal{P}}}_2 &= (\tilde{\mathcal{P}}_2)^{3 \times 0.3} = \{\langle [0.1259, 0.4384], [0.2853, 0.5751] \rangle, \langle [0.1259, 0.4384], [0.4776, 0.6741] \rangle\}, \\ \ddot{\tilde{\mathcal{P}}}_3 &= (\tilde{\mathcal{P}}_3)^{3 \times 0.3} = \{\langle [0.1372, 0.2205], [0.2552, 0.7219] \rangle, \langle [0.0224, 0.1372], [0.2552, 0.7219] \rangle, \\ &\quad \langle [0.2205, 0.5552], [0.2552, 0.7219] \rangle\}.\end{aligned}\tag{54}$$

$$\begin{aligned}S(\dot{\tilde{\mathcal{P}}}_1) &= [-0.1984, -0.0567], S(\dot{\tilde{\mathcal{P}}}_2) = [-0.2267, -0.0278], \\ S(\dot{\tilde{\mathcal{P}}}_3) &= [-0.0242, 0.1750], S(\ddot{\tilde{\mathcal{P}}}_1) = [0.1315, 0.2588], \\ S(\ddot{\tilde{\mathcal{P}}}_2) &= [-0.1884, 0.0187], S(\ddot{\tilde{\mathcal{P}}}_3) = [-0.2493, 0.0300].\end{aligned}$$

By Definition 16, we have $P(S(\dot{\tilde{\mathcal{P}}}_1) \leq S(\dot{\tilde{\mathcal{P}}}_2)) = 1$, $P(S(\dot{\tilde{\mathcal{P}}}_2) \leq S(\dot{\tilde{\mathcal{P}}}_3)) = 0.5009$, and $P(S(\dot{\tilde{\mathcal{P}}}_3) \leq S(\ddot{\tilde{\mathcal{P}}}_2)) = 0.5509$, and $P(S(\ddot{\tilde{\mathcal{P}}}_2) \leq S(\ddot{\tilde{\mathcal{P}}}_1)) = 1$. So, we can get

$$\begin{aligned}\dot{\tilde{\mathcal{P}}}_{\sigma(1)} &= \dot{\tilde{\mathcal{P}}}_3, \\ \dot{\tilde{\mathcal{P}}}_{\sigma(2)} &= \dot{\tilde{\mathcal{P}}}_2, \\ \dot{\tilde{\mathcal{P}}}_{\sigma(3)} &= \dot{\tilde{\mathcal{P}}}_1, \\ \ddot{\tilde{\mathcal{P}}}_{\sigma(1)} &= \ddot{\tilde{\mathcal{P}}}_1, \\ \ddot{\tilde{\mathcal{P}}}_{\sigma(2)} &= \ddot{\tilde{\mathcal{P}}}_2, \\ \ddot{\tilde{\mathcal{P}}}_{\sigma(3)} &= \ddot{\tilde{\mathcal{P}}}_3.\end{aligned}\tag{55}$$

According to Definition 19, we can obtain

$$\begin{aligned}
\text{IVPHFHA}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \kappa_1 \dot{\tilde{\mathcal{P}}}_{\sigma(1)} \oplus \kappa_1 \dot{\tilde{\mathcal{P}}}_{\sigma(2)} \oplus \kappa_n \dot{\tilde{\mathcal{P}}}_{\sigma(3)} \\
&= \left\{ \left\langle \left[\sqrt{1 - \prod_{i=1}^3 \left(1 - (\dot{\mu}_{\sigma(i)}^-)^2\right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^3 \left(1 - (\dot{\mu}_{\sigma(i)}^+)^2\right)^{\kappa_i}} \right], \left[\prod_{i=1}^3 (\dot{\nu}_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^3 (\dot{\nu}_{\sigma(i)}^+)^{\kappa_i} \right] \right\rangle \right. \\
&\quad \left. \mid \langle \dot{\mu}_{\sigma(i)}, \dot{\nu}_{\sigma(i)} \rangle \in \dot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, 2, 3 \right\} \\
&= \{ \langle [0.2209, 0.3991], [0.2802, 0.5947] \rangle, \langle [0.2209, 0.3991], [0.3446, 0.6330] \rangle, \\
&\quad \langle [0.1483, 0.3698], [0.2802, 0.5947] \rangle, \langle [0.1483, 0.3698], [0.3446, 0.6330] \rangle, \\
&\quad \langle [0.2713, 0.5356], [0.2802, 0.5947] \rangle, \langle [0.2713, 0.5356], [0.3446, 0.6330] \rangle \}; \\
\text{IVPHFHG}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \left(\ddot{\tilde{\mathcal{P}}}_{\sigma(1)} \right)^{\kappa_1} \otimes \left(\ddot{\tilde{\mathcal{P}}}_{\sigma(2)} \right)^{\kappa_2} \otimes \left(\ddot{\tilde{\mathcal{P}}}_{\sigma(3)} \right)^{\kappa_3} \\
&= \left\{ \left\langle \left[\prod_{i=1}^3 (\ddot{\mu}_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^3 (\ddot{\mu}_{\sigma(i)}^+)^{\kappa_i} \right], \left[\sqrt{1 - \prod_{i=1}^3 \left(1 - (\ddot{\nu}_{\sigma(i)}^-)^2\right)^{\kappa_i}}, \sqrt{1 - \prod_{i=1}^3 \left(1 - (\ddot{\nu}_{\sigma(i)}^+)^2\right)^{\kappa_i}} \right] \right\rangle \right. \\
&\quad \left. \mid \langle \ddot{\mu}_{\sigma(i)}, \ddot{\nu}_{\sigma(i)} \rangle \in \ddot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, 2, 3 \right\} \\
&= \{ \langle [0.1762, 0.3819], [0.2517, 0.6042] \rangle, \langle [0.0934, 0.3235], [0.2517, 0.6042] \rangle, \\
&\quad \langle [0.2080, 0.5276], [0.2517, 0.6042] \rangle, \langle [0.1762, 0.3819], [0.3659, 0.6487] \rangle, \\
&\quad \langle [0.0934, 0.3234], [0.3659, 0.6487] \rangle, \langle [0.2080, 0.5276], [0.3659, 0.6487] \rangle \}; \\
\text{GIVPHFHA}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \left(\kappa_1 \left(\dot{\tilde{\mathcal{P}}}_{\sigma(1)} \right)^3 \oplus \kappa_2 \left(\dot{\tilde{\mathcal{P}}}_{\sigma(2)} \right)^3 \oplus \kappa_3 \left(\dot{\tilde{\mathcal{P}}}_{\sigma(3)} \right)^3 \right)^{1/3} \\
&= \left\{ \left\langle \left[\sqrt{\left(1 - \prod_{i=1}^3 \left(1 - (\dot{\mu}_{\sigma(i)}^-)^{2 \times 3}\right)^{\kappa_i}\right)^{1/3}}, \sqrt{\left(1 - \prod_{i=1}^3 \left(1 - (\dot{\mu}_{\sigma(i)}^+)^{2 \times 3}\right)^{\kappa_i}\right)^{1/3}} \right], \right. \right. \\
&\quad \left. \left[\sqrt{1 - \left(1 - \prod_{i=1}^3 \left(1 - (\dot{\nu}_{\sigma(i)}^-)^2\right)^3\right)^{\kappa_i}}^{1/3}, \sqrt{1 - \left(1 - \prod_{i=1}^3 \left(1 - (\dot{\nu}_{\sigma(i)}^+)^2\right)^3\right)^{\kappa_i}}^{1/3} \right] \right\rangle, \right. \\
&\quad \left. \mid \langle \dot{\mu}_{\sigma(i)}, \dot{\nu}_{\sigma(i)} \rangle \in \dot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, 2, 3 \right\} \\
&= \{ \langle [0.2924, 0.4155], [0.2713, 0.5854] \rangle, \langle [0.2924, 0.4155], [0.3255, 0.6153] \rangle, \\
&\quad \langle [0.1725, 0.3709], [0.2713, 0.5854] \rangle, \langle [0.1725, 0.3709], [0.3256, 0.6153] \rangle, \\
&\quad \langle [0.3833, 0.6438], [0.2713, 0.5854] \rangle, \langle [0.3833, 0.6438], [0.3256, 0.6153] \rangle \}; \\
\text{GIVPHFHG}_3(\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3) &= \frac{1}{3} \left(\left(3 \ddot{\tilde{\mathcal{P}}}_{\sigma(1)} \right)^{\kappa_1} \otimes \left(3 \ddot{\tilde{\mathcal{P}}}_{\sigma(2)} \right)^{\kappa_2} \otimes \left(3 \ddot{\tilde{\mathcal{P}}}_{\sigma(3)} \right)^{\kappa_3} \right) \\
&= \left\{ \left\langle \left[\sqrt{1 - \left(1 - \prod_{i=1}^3 \left(1 - (\ddot{\mu}_{\sigma(i)}^-)^2\right)^3\right)^{\kappa_i}}^{1/3}, \sqrt{1 - \left(1 - \prod_{i=1}^3 \left(1 - (\ddot{\mu}_{\sigma(i)}^+)^2\right)^3\right)^{\kappa_i}}^{1/3} \right], \right. \right. \\
&\quad \left. \left[\sqrt{\left(1 - \prod_{i=1}^3 \left(1 - (\ddot{\nu}_{\sigma(i)}^-)^{2 \times 3}\right)^{\kappa_i}\right)^{1/3}}, \sqrt{\left(1 - \prod_{i=1}^3 \left(1 - (\ddot{\nu}_{\sigma(i)}^+)^{2 \times 3}\right)^{\kappa_i}\right)^{1/3}} \right] \right\rangle, \right. \\
&\quad \left. \mid \langle \ddot{\mu}_{\sigma(i)}, \ddot{\nu}_{\sigma(i)} \rangle \in \ddot{\tilde{\mathcal{P}}}_{\sigma(i)}, i = 1, 2, 3 \right\} \\
&= \{ \langle [0.1714, 0.3659], [0.2642, 0.6384] \rangle, \langle [0.0902, 0.3050], [0.2642, 0.6384] \rangle, \\
&\quad \langle [0.2024, 0.5174], [0.2642, 0.6384] \rangle, \langle [0.1714, 0.3659], [0.4196, 0.6734] \rangle, \\
&\quad \langle [0.0902, 0.3050], [0.4196, 0.6734] \rangle, \langle [0.2024, 0.5174], [0.4196, 0.6734] \rangle \}.
\end{aligned} \tag{56}$$

5. The Application of the Aggregation Operators of IVPHFEs for MAGDM

In multiattribute group decision-making (MAGDM) problems, because the different decision makers usually come from different research directions and backgrounds, they usually have diverging opinions. Thus, how to construct an optimal model that can be obtained with the maximum degree of consensus or agreement of these experts for the given alternatives is an important and interesting research content in MAGDM. In this section, we propose a new MAGDM method based on the interval-valued Pythagorean hesitant fuzzy integration operators. Furthermore, the ranking of integrated interval-valued Pythagorean hesitant fuzzy numbers is accomplished by adopting the score function. The specific details of the MAGDM method is shown in following.

Let $Y = \{Y_i \mid i = 1, 2, \dots, m\}$ be a finite set of alternatives, $C = \{C_j \mid j = 1, 2, \dots, n\}$ be a set of attributes, and $D = \{D_k \mid k = 1, 2, \dots, l\}$ be a set of l decision makers. Then, let $M^{(k)} = (\tilde{\mathcal{P}}_{ij}^{(k)})_{m \times n}$ is an interval-valued Pythagorean hesitant fuzzy decision matrix (IVPHFDM) of the k^{th} decision maker, where $\tilde{\mathcal{P}}_{ij}^{(k)} = \{\langle \tilde{\mu}_{ij}^{(k)}, \tilde{\nu}_{ij}^{(k)} \rangle\}$, ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$) is an IVPHFE given by the decision maker D_k , in which $\tilde{\mu}_{ij}^{(k)}$ indicates the possible membership intervals that the alternative Y_i satisfies the attribute C_j and $\tilde{\nu}_{ij}^{(k)}$ indicates the possible nonmembership intervals that the alternative Y_i nonmembership intervals not satisfy the attribute C_j .

In the actual MAGDM problem, we will encounter some attributes are benefit (i.e., the bigger the attribute values, the better) and the other attributes are cost (i.e., the smaller the attribute values, the better). In such cases, transform the cost attribute values into the benefit attribute values and normalize the IVPHFD $M^{(k)} = (\tilde{\mathcal{P}}_{ij}^{(k)})_{m \times n}$ into the corresponding IVPHFD $N^{(k)} = (\tilde{P}_{ij}^{(k)})_{m \times n}$ by the method in [53] where

$$\tilde{P}_{ij}^{(k)} = \begin{cases} \tilde{\mathcal{P}}_{ij}^{(k)}, & \text{for bene fit attribute } C_j, \\ (\tilde{\mathcal{P}}_{ij}^{(k)})^C, & \text{for cost attribute } C_j, \end{cases} \quad (57)$$

where $(\tilde{\mathcal{P}}_{ij}^{(k)})^C$ is the complement of $\tilde{\mathcal{P}}_{ij}^{(k)}$.

Based on the above discussion and analysis, we develop an approach for multiattribute decision-making in interval-valued Pythagorean hesitant fuzzy environments. The algorithm involves the following steps:

Step 1: construct the IVPHFD $M^{(k)} = (\tilde{\mathcal{P}}_{ij}^{(k)})_{m \times n}$ and transform $M^{(k)}$ into the corresponding normalized matrix $N^{(k)} = (\tilde{P}_{ij}^{(k)})_{m \times n}$.

Step 2: utilize the GIVPHFWA operator (or the GIVPHFWG operator) to aggregate all the IVPHFDs $N^{(k)} = (\tilde{P}_{ij}^{(k)})_{m \times n}$ ($k = 1, 2, \dots, l$) into the collective IVPHFD $N = (\tilde{P}_{ij})_{m \times n}$, where $\xi = (\xi_1, \xi_2, \dots, \xi_l)^T$ is the weight vector of the decision makers D_k ($k = 1, 2, \dots, l$), and the specific integration operation is as follows:

$$\begin{aligned} \tilde{P}_{ij} &= \text{GIVPHFWA}_\lambda(\tilde{P}_{ij}^{(1)}, \dots, \tilde{P}_{ij}^{(l)}) \\ &= \left\langle \left[\sqrt{\left(1 - \prod_{k=1}^l \left(1 - \left(\mu_{\tilde{P}_{ij}^{(k)}}^-\right)^{2\lambda}\right)^{\xi_k}\right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{k=1}^l \left(1 - \left(\mu_{\tilde{P}_{ij}^{(k)}}^+\right)^{2\lambda}\right)^{\xi_k}\right)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{1 - \left(1 - \prod_{k=1}^l \left(1 - \left(1 - \left(\nu_{\tilde{P}_{ij}^{(k)}}^-\right)^2\right)^{\lambda}\right)^{\xi_k}\right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{k=1}^l \left(1 - \left(1 - \left(\nu_{\tilde{P}_{ij}^{(k)}}^+\right)^2\right)^{\lambda}\right)^{\xi_k}\right)^{1/\lambda}} \right] \right\rangle, \\ &\quad | \langle \tilde{\mu}_{\tilde{P}_{ij}}^{(k)}, \tilde{\nu}_{\tilde{P}_{ij}}^{(k)} \rangle \in \tilde{P}_{ij}^{(k)}, i = 1, \dots, m, j = 1, \dots, n \}. \end{aligned} \quad (58)$$

or

$$\begin{aligned} \tilde{P}_{ij} &= \text{GIVPHFWG}_\lambda(\tilde{P}_{ij}^{(1)}, \dots, \tilde{P}_{ij}^{(l)}) \\ &= \left\langle \left[\sqrt{1 - \left(1 - \prod_{k=1}^l \left(1 - \left(1 - \left(\mu_{\tilde{P}_{ij}^{(k)}}^-\right)^2\right)^{\lambda}\right)^{\xi_k}\right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{k=1}^l \left(1 - \left(1 - \left(\mu_{\tilde{P}_{ij}^{(k)}}^+\right)^2\right)^{\lambda}\right)^{\xi_k}\right)^{1/\lambda}} \right], \right. \\ &\quad \left[\sqrt{\left(1 - \prod_{k=1}^l \left(1 - \left(\nu_{\tilde{P}_{ij}^{(k)}}^-\right)^{2\lambda}\right)^{\xi_k}\right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{k=1}^l \left(1 - \left(\nu_{\tilde{P}_{ij}^{(k)}}^+\right)^{2\lambda}\right)^{\xi_k}\right)^{1/\lambda}} \right] \right\rangle, | \langle \tilde{\mu}_{\tilde{P}_{ij}}^{(k)}, \tilde{\nu}_{\tilde{P}_{ij}}^{(k)} \rangle \in \tilde{P}_{ij}^{(k)}, i = 1, \dots, m, j = 1, \dots, n \}. \end{aligned} \quad (59)$$

Step 3: utilize the GIVPHFHA operator (or the GIVPHFHG operator) to aggregate all the preference values $\tilde{P}_i$ again, where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)^T$ is the

associated weight vector. The specific integration operation is as follows:

$$\begin{aligned} \tilde{P}_i &= \text{GIVPHFHA}_\lambda(\tilde{P}_{i1}, \dots, \tilde{P}_{in}) \\ &= \left\{ \left\langle \left[\sqrt{\left(1 - \prod_{j=1}^n \left(1 - \left(\dot{\mu}_{\tilde{P}_{i\sigma(j)}}^- \right)^{2\lambda} \right)^{\kappa_j}} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{j=1}^n \left(1 - \left(\dot{\mu}_{\tilde{P}_{i\sigma(j)}}^+ \right)^{2\lambda} \right)^{\kappa_j} \right)^{1/\lambda}} \right], \right. \\ &\quad \left. \left[\sqrt{1 - \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \left(\dot{\nu}_{\tilde{P}_{i\sigma(j)}}^- \right)^2 \right)^{\lambda} \right)^{\kappa_j}} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \left(\dot{\nu}_{\tilde{P}_{i\sigma(j)}}^+ \right)^2 \right)^{\lambda} \right)^{\kappa_j} \right)^{1/\lambda}} \right] \right\rangle, \\ &\quad \left. \left| \left\langle \dot{\mu}_{\tilde{P}_{i\sigma(j)}}^-, \dot{\nu}_{\tilde{P}_{i\sigma(j)}}^- \right\rangle \in \dot{\tilde{P}}_{i\sigma(j)}, i = 1, \dots, m, j = 1, \dots, n \right\} \right\}. \end{aligned} \quad (60)$$

or $\tilde{P}_i = \text{GIVPHFHG}_\lambda(\tilde{P}_{i1}, \dots, \tilde{P}_{in})$

$$\begin{aligned} &= \left\{ \left\langle \left[\sqrt{1 - \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \left(\ddot{\mu}_{\tilde{P}_{i\sigma(j)}}^- \right)^2 \right)^{\lambda} \right)^{\kappa_j}} \right)^{1/\lambda}}, \sqrt{1 - \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \left(\ddot{\mu}_{\tilde{P}_{i\sigma(j)}}^+ \right)^2 \right)^{\lambda} \right)^{\kappa_j} \right)^{1/\lambda}} \right], \right. \\ &\quad \left. \left[\sqrt{\left(1 - \prod_{j=1}^n \left(1 - \left(\ddot{\nu}_{\tilde{P}_{i\sigma(j)}}^- \right)^{2\lambda} \right)^{\kappa_j}} \right)^{1/\lambda}}, \sqrt{\left(1 - \prod_{j=1}^n \left(1 - \left(\ddot{\nu}_{\tilde{P}_{i\sigma(j)}}^+ \right)^{2\lambda} \right)^{\kappa_j} \right)^{1/\lambda}} \right] \right\rangle, \\ &\quad \left. \left| \left\langle \ddot{\nu}_{\tilde{P}_{i\sigma(j)}}^-, \ddot{\nu}_{\tilde{P}_{i\sigma(j)}}^- \right\rangle \in \ddot{\tilde{P}}_{i\sigma(j)}, i = 1, \dots, m, j = 1, \dots, n \right\} \right\}. \end{aligned} \quad (61)$$

Here, let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of the attributes C_j ($j = 1, 2, \dots, n$), where $\omega_j \in [0, 1]$

($j = 1, 2, \dots, n$) and $\sum_{j=1}^n \omega_j = 1$. So, we can get $\dot{\tilde{P}}_{i\sigma(j)}$ (or $\ddot{\tilde{P}}_{i\sigma(j)}$) defined by

$$\begin{aligned} \dot{\tilde{P}}_{ij} &= (n \times \omega_j) \otimes \tilde{P}_{ij} \\ &= \left\{ \left\langle \left[\sqrt{1 - \left(1 - \left(\mu_{\tilde{P}_{ij}}^- \right)^2 \right)^{(n \times \omega_j)}}, \sqrt{1 - \left(1 - \left(\mu_{\tilde{P}_{ij}}^+ \right)^2 \right)^{(n \times \omega_j)}} \right], \left[\left(\nu_{\tilde{P}_{ij}}^- \right)^{(n \times \omega_j)}, \left(\nu_{\tilde{P}_{ij}}^+ \right)^{(n \times \omega_j)} \right] \right\rangle \right. \\ &\quad \left. \left| \left\langle \tilde{\mu}_{\tilde{P}_{ij}}^-, \tilde{\nu}_{\tilde{P}_{ij}}^- \right\rangle \in \tilde{P}_{ij}, i = 1, \dots, m, j = 1, \dots, n \right\} \right\}, \\ \ddot{\tilde{P}}_{ij} &= (\tilde{P}_{ij})^{(n \times \omega_j)} \\ &= \left\{ \left\langle \left[\left(\mu_{\tilde{P}_{ij}}^- \right)^{(n \times \omega_j)}, \left(\mu_{\tilde{P}_{ij}}^+ \right)^{(n \times \omega_j)} \right], \left[\sqrt{1 - \left(1 - \left(\nu_{\tilde{P}_{ij}}^- \right)^2 \right)^{(n \times \omega_j)}}, \sqrt{1 - \left(1 - \left(\nu_{\tilde{P}_{ij}}^+ \right)^2 \right)^{(n \times \omega_j)}} \right] \right\rangle \right. \\ &\quad \left. \left| \left\langle \tilde{\mu}_{\tilde{P}_{ij}}^-, \tilde{\nu}_{\tilde{P}_{ij}}^- \right\rangle \in \tilde{P}_{ij}, i = 1, \dots, m, j = 1, \dots, n \right\} \right\}. \end{aligned} \quad (62)$$

TABLE 1: IVPHFDM $M^{(1)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.4, 0.5], [0.3, 0.5] \rangle\}$	$\{\langle [0.2, 0.3], [0.6, 0.8] \rangle\}$	$\{\langle [0.6, 0.8], [0.1, 0.3] \rangle, \langle [0.6, 0.7], [0.3, 0.3] \rangle, \langle [0.7, 0.9], [0.1, 0.1] \rangle\}$
Y_2	$\{\langle [0.1, 0.1], [0.8, 0.9] \rangle, \langle [0.2, 0.4], [0.5, 0.6] \rangle\}$	$\{\langle [0.4, 0.5], [0.5, 0.5] \rangle, \langle [0.6, 0.7], [0.1, 0.2] \rangle, \langle [0.6, 0.7], [0.1, 0.2] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle\}$
Y_3	$\{\langle [0.2, 0.3], [0.6, 0.6] \rangle, \langle [0.1, 0.4], [0.6, 0.7] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle\}$	$\{\langle [0.3, 0.5], [0.4, 0.4] \rangle\}$

TABLE 2: IVPHFDM $M^{(2)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.7, 0.9], [0.1, 0.2] \rangle, \langle [0.7, 0.8], [0.1, 0.2] \rangle, \langle [0.6, 0.8], [0.1, 0.1] \rangle\}$	$\{\langle [0.4, 0.6], [0.3, 0.5] \rangle, \langle [0.5, 0.7], [0.2, 0.3] \rangle\}$	$\{\langle [0.9, 0.9], [0.1, 0.2] \rangle\}$
Y_2	$\{\langle [0.7, 0.8], [0.1, 0.2] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.3] \rangle\}$	$\{\langle [0.5, 0.8], [0.2, 0.3] \rangle, \langle [0.6, 0.8], [0.1, 0.2] \rangle\}$
Y_3	$\{\langle [0.2, 0.3], [0.5, 0.6] \rangle\}$	$\{\langle [0.6, 0.8], [0.2, 0.2] \rangle, \langle [0.7, 0.9], [0.1, 0.1] \rangle\}$	$\{\langle [0.1, 0.3], [0.6, 0.7] \rangle, \langle [0.2, 0.2], [0.7, 0.8] \rangle, \langle [0.3, 0.4], [0.6, 0.6] \rangle\}$

TABLE 3: IVPHFDM $M^{(3)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.3, 0.5], [0.4, 0.5] \rangle\}$	$\{\langle [0.2, 0.3], [0.5, 0.6] \rangle, \langle [0.1, 0.2], [0.8, 0.9] \rangle\}$	$\{\langle [0.7, 0.8], [0.2, 0.2] \rangle\}$
Y_2	$\{\langle [0.4, 0.6], [0.3, 0.4] \rangle, \langle [0.4, 0.5], [0.5, 0.5] \rangle\}$	$\{\langle [0.1, 0.3], [0.6, 0.9] \rangle\}$	$\{\langle [0.3, 0.5], [0.5, 0.5] \rangle, \langle [0.1, 0.2], [0.7, 0.7] \rangle, \langle [0.3, 0.4], [0.5, 0.6] \rangle\}$
Y_3	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle, \langle [0.5, 0.7], [0.1, 0.2] \rangle\}$	$\{\langle [0.3, 0.6], [0.4, 0.4] \rangle, \langle [0.4, 0.5], [0.4, 0.5] \rangle, \langle [0.3, 0.5], [0.4, 0.5] \rangle\}$	$\{\langle [0.9, 0.9], [0.1, 0.1] \rangle\}$

Step 4: compute the score values $S(\tilde{P}_i)$ and the accuracy values $H(\tilde{P}_i)$ of $\tilde{P}_i$ ($i = 1, 2, \dots, m$) by Definition 15.

Step 5: get the priority of the alternatives Y_i by ranking $S(\tilde{P}_i)$ ($i = 1, 2, \dots, m$) based on Definition 16, End.

Example 4. A company needs to select a new site for buildings. Three alternatives Y_i ($i = 1, 2, 3$) are available, and the three decision makers D_k ($k = 1, 2, 3$) consider three attributes to decide which project to choose: (1) C_1 (price), (2) C_2 (environment), and (3) C_3 (location). Among the considered attributes, C_1 is of cost type and C_2 and C_3 are of benefit type. The weight vector of the decision makers D_k ($k = 1, 2, 3$) is $\xi = (0.1, 0.6, 0.3)^T$. The weight vector of the attributes C_j ($j = 1, 2, 3$) is $\omega = (0.5, 0.2, 0.3)^T$. Suppose that the decision makers provide their own IVPHFDM $M^{(k)} = (\tilde{\mathcal{P}}_{ij}^{(k)})_{3 \times 3}$ as listed in Tables 1–3, respectively, where $\tilde{\mathcal{P}}_{ij}^{(k)}$ is an IVPHFE given by the decision maker D_k (Tables 4–7).

Step 1: transform the decision matrix $M^{(k)}$ into the corresponding normalized matrix $N^{(k)} = (\tilde{P}_{ij}^{(k)})_{3 \times 3}$ ($k = 1, 2, 3$) (in Tables 4–6).

Step 2: suppose $\lambda = 5$, and utilize the GIVPHFWA operator to aggregate the three IVPHFDMs $N^{(k)} = (\tilde{P}_{ij}^{(k)})_{3 \times 3}$ ($k = 1, 2, 3$) into the collective IVPHFDM $N = (\tilde{P}_{ij})_{3 \times 3}$ (in Table 7).

Step 3: aggregate all the preference values $\tilde{P}_{ij}$ ($j = 1, 2, 3$) in the i^{th} line of N based on the GIVPHFHA operators, whose associated weighting vector is $\kappa = (0.25, 0.65, 0.1)^T$.

Step 4: compute the three score values as follows:

$$\begin{aligned} S(\tilde{P}_1) &= [0.2736, 0.2833], \\ S(\tilde{P}_2) &= [0.2142, 0.2988], \\ S(\tilde{P}_3) &= [0.2398, 0.2741]. \end{aligned} \quad (63)$$

Step 5: get the priority of the alternatives by ranking the score functions. We can get the ranking order of all alternatives: $Y_1 \succ Y_3 \succ Y_2$. So, the optimal scheme is Y_1 .

Furthermore, we study the change of the GIVPHFHA and GIVPHFHG operators with the parameter λ .

In Table 8, we can find that the score values obtained by the GIVPHFHA operators become bigger as the parameter λ

TABLE 4: IVPHFDM $N^{(1)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.3, 0.5], [0.4, 0.5] \rangle\}$	$\{\langle [0.2, 0.3], [0.6, 0.8] \rangle\}$	$\{\langle [0.6, 0.8], [0.1, 0.3] \rangle, \langle [0.6, 0.7], [0.3, 0.3] \rangle, \langle [0.7, 0.9], [0.1, 0.1] \rangle\}$
Y_2	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle, \langle [0.5, 0.6], [0.2, 0.4] \rangle\}$	$\{\langle [0.4, 0.5], [0.5, 0.5] \rangle, \langle [0.6, 0.7], [0.1, 0.2] \rangle, \langle [0.6, 0.7], [0.1, 0.2] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle\}$
Y_3	$\{\langle [0.6, 0.6], [0.2, 0.3] \rangle, \langle [0.6, 0.7], [0.1, 0.4] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.1] \rangle\}$	$\{\langle [0.3, 0.5], [0.4, 0.4] \rangle\}$

TABLE 5: IVPHFDM $N^{(2)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.1, 0.2], [0.7, 0.9] \rangle, \langle [0.1, 0.2], [0.7, 0.8] \rangle, \langle [0.1, 0.1], [0.6, 0.8] \rangle\}$	$\{\langle [0.4, 0.6], [0.3, 0.5] \rangle, \langle [0.5, 0.7], [0.2, 0.3] \rangle\}$	$\{\langle [0.9, 0.9], [0.1, 0.2] \rangle\}$
Y_2	$\{\langle [0.1, 0.2], [0.7, 0.8] \rangle\}$	$\{\langle [0.8, 0.9], [0.1, 0.3] \rangle\}$	$\{\langle [0.5, 0.8], [0.2, 0.3] \rangle, \langle [0.6, 0.8], [0.1, 0.2] \rangle\}$
Y_3	$\{\langle [0.5, 0.6], [0.2, 0.3] \rangle\}$	$\{\langle [0.6, 0.8], [0.2, 0.2] \rangle, \langle [0.7, 0.9], [0.1, 0.1] \rangle\}$	$\{\langle [0.1, 0.3], [0.6, 0.7] \rangle, \langle [0.2, 0.2], [0.7, 0.8] \rangle, \langle [0.3, 0.4], [0.6, 0.6] \rangle\}$

TABLE 6: IVPHFDM $N^{(3)}$.

	C_1	C_2	C_3
Y_1	$\{\langle [0.4, 0.5], [0.3, 0.5] \rangle\}$	$\{\langle [0.2, 0.3], [0.5, 0.6] \rangle, \langle [0.1, 0.2], [0.8, 0.9] \rangle\}$	$\{\langle [0.7, 0.8], [0.2, 0.2] \rangle\}$
Y_2	$\{\langle [0.3, 0.4], [0.4, 0.6] \rangle, \langle [0.5, 0.5], [0.4, 0.5] \rangle\}$	$\{\langle [0.1, 0.3], [0.6, 0.9] \rangle\}$	$\{\langle [0.3, 0.5], [0.5, 0.5] \rangle, \langle [0.1, 0.2], [0.7, 0.7] \rangle, \langle [0.3, 0.4], [0.5, 0.6] \rangle\}$
Y_3	$\{\langle [0.1, 0.1], [0.8, 0.9] \rangle, \langle [0.1, 0.2], [0.5, 0.7] \rangle\}$	$\{\langle [0.3, 0.6], [0.4, 0.4] \rangle, \langle [0.4, 0.5], [0.4, 0.5] \rangle, \langle [0.3, 0.5], [0.4, 0.5] \rangle\}$	$\{\langle [0.9, 0.9], [0.1, 0.1] \rangle\}$

TABLE 7: IVPHFDM N .

	C_1	C_2	C_3
Y_1	$\{\langle [0.3553, 0.4562], [0.4579, 0.6046] \rangle, \langle [0.3559, 0.4562], [0.4579, 0.6014] \rangle, \langle [0.3553, 0.4562], [0.4404, 0.6014] \rangle\}$	$\{\langle [0.3801, 0.5702], [0.3631, 0.5396] \rangle, \langle [0.4751, 0.6655], [0.2778, 0.3778] \rangle, \langle [0.3801, 0.5702], [0.3836, 0.5614] \rangle, \langle [0.4751, 0.6655], [0.2913, 0.3870] \rangle\}$	$\{\langle [0.8648, 0.8744], [0.1226, 0.2079] \rangle, \langle [0.8648, 0.8722], [0.1363, 0.2079] \rangle, \langle [0.8654, 0.8818], [0.1226, 0.1862] \rangle\}$
Y_2	$\{\langle [0.6387, 0.7283], [0.4172, 0.4696] \rangle, \langle [0.3979, 0.4791], [0.4643, 0.6207] \rangle, \langle [0.6403, 0.7288], [0.4172, 0.4511] \rangle, \langle [0.4562, 0.4959], [0.4643, 0.5790] \rangle\}$	$\{\langle [0.7619, 0.8622], [0.1814, 0.3798] \rangle, \langle [0.7625, 0.8630], [0.1568, 0.3451] \rangle, \langle [0.7620, 0.8630], [0.1568, 0.3451] \rangle\}$	$\{\langle [0.6419, 0.7985], [0.2351, 0.3016] \rangle, \langle [0.6566, 0.7988], [0.1530, 0.2351] \rangle, \langle [0.6419, 0.7983], [0.2447, 0.3150] \rangle, \langle [0.6566, 0.7983], [0.1588, 0.2447] \rangle, \langle [0.6419, 0.7983], [0.2351, 0.3104] \rangle, \langle [0.6566, 0.7983], [0.1530, 0.2414] \rangle\}$
Y_3	$\{\langle [0.4767, 0.4767], [0.5428, 0.6111] \rangle, \langle [0.4767, 0.5567], [0.4758, 0.6616] \rangle, \langle [0.4767, 0.4767], [0.4998, 0.6014] \rangle, \langle [0.4767, 0.5567], [0.4447, 0.6475] \rangle\}$	$\{\langle [0.6566, 0.7995], [0.2247, 0.2247] \rangle, \langle [0.6998, 0.8742], [0.1466, 0.1466] \rangle, \langle [0.6567, 0.7985], [0.2247, 0.2351] \rangle, \langle [0.6998, 0.8738], [0.1466, 0.1530] \rangle, \langle [0.6566, 0.7985], [0.2247, 0.2351] \rangle, \langle [0.6998, 0.8738], [0.1466, 0.1530] \rangle\}$	$\{\langle [0.8094, 0.8095], [0.2913, 0.2997] \rangle, \langle [0.8094, 0.8095], [0.2997, 0.3029] \rangle, \langle [0.8094, 0.8095], [0.2913, 0.2913] \rangle\}$

increases for the same aggregation arguments, and the decision makers can choose the values of λ according to their preferences, see that, as the parameter λ changes, we have different results. Suppose $\lambda = 0.1, 0.2, \dots, 50$, then we analyze the impact of the role of λ on the aggregation results, and see the trend in Figure 1. Because of space limitations, we gave the image within the range of $\lambda \in [0.7, 11.9]$, the same hereinafter.

From Figure 1, we can see that all of $S(\tilde{P}_i) (i = 1, 2, 3)$ increase with the increase of λ . When $\lambda = 0.7$, the ranking

order of the three alternatives is $Y_2 > Y_3 > Y_1$ and the best choice is Y_2 ; when $\lambda \in [0.8, 1.5]$, the ranking order of the three alternatives is $Y_3 > Y_2 > Y_1$ and the best choice is Y_3 ; when $\lambda \in [1.6, 11.4]$, the ranking order of the three alternatives is $Y_1 > Y_3 > Y_2$ and the best choice is Y_1 ; when $\lambda \in [11.5, 50]$, the ranking order of the three alternatives is $Y_2 > Y_1 > Y_3$ and the best choice is Y_2 .

In Table 9, we use the GIVPHFHG operators to aggregate the values of the alternatives. We find the score

TABLE 8: Score values based on the GIVPHFHA operators.

	$\lambda = 0.1$	$\lambda = 10$	$\lambda = 20$
Y_1	$[-0.4036, -0.3452]$	$[0.3207, 0.3240]$	$[0.3519, 0.3530]$
Y_2	$[0.0264, 0.0736]$	$[0.2865, 0.3506]$	$[0.3352, 0.3988]$
Y_3	$[-0.0184, 0.0027]$	$[0.2979, 0.3073]$	$[0.3388, 0.3396]$
Ranking	$Y_2 > Y_3 > Y_1$	$Y_1 > Y_2 > Y_3$	$Y_2 > Y_1 > Y_3$
	$\lambda = 30$	$\lambda = 40$	$\lambda = 50$
Y_1	$[0.3634, 0.3641]$	$[0.3694, 0.3699]$	$[0.3730, 0.3734]$
Y_2	$[0.3533, 0.4179]$	$[0.3627, 0.4279]$	$[0.3684, 0.4339]$
Y_3	$[0.3543, 0.3544]$	$[0.3624, 0.3624]$	$[0.3673, 0.3673]$
Ranking	$Y_2 > Y_1 > Y_3$	$Y_2 > Y_1 > Y_3$	$Y_2 > Y_1 > Y_3$

FIGURE 1: Variation of the score functions of the GIVPHFHA operators by the parameter λ .

TABLE 9: Score values based on the GIVPHFHG operators.

	$\lambda = 0.1$	$\lambda = 10$	$\lambda = 20$
Y_1	$[0.392, 0.4493]$	$[-0.3225, -0.2264]$	$[-0.3633, -0.2567]$
Y_2	$[-0.0184, 0.0047]$	$[-0.2448, -0.1454]$	$[-0.2774, -0.1621]$
Y_3	$[-0.0148, 0.0178]$	$[-0.3424, -0.2848]$	$[-0.3950, -0.3340]$
Ranking	$Y_1 > Y_2 > Y_3$	$Y_2 > Y_1 > Y_3$	$Y_2 > Y_1 > Y_3$
	$\lambda = 30$	$\lambda = 40$	$\lambda = 50$
Y_1	$[-0.3793, -0.2681]$	$[-0.3877, -0.2344]$	$[-0.3928, -0.2011]$
Y_2	$[-0.2899, -0.0968]$	$[-0.2217, -0.0990]$	$[-0.2244, -0.1006]$
Y_3	$[-0.4149, -0.3524]$	$[-0.4254, -0.3619]$	$[-0.4318, -0.3677]$
Ranking	$Y_2 > Y_1 > Y_3$	$Y_2 > Y_1 > Y_3$	$Y_2 > Y_1 > Y_3$

FIGURE 2: Variation of the score functions of the GIVPHFHG operators by the parameter λ .

values obtained by the GIVPHFHG operators become smaller as the parameter λ increases for the same aggregation arguments, and the decision makers can choose the values of λ according to their preferences. See the details in Figure 2.

From Figure 2, we can see that all of $S(\tilde{P}_i)$ ($i = 1, 2, 3$) decrease with the increase of λ . When $\lambda \in [0.7, 1.2]$, the ranking order of the three alternatives is $Y_1 > Y_2 > Y_3$ and the best choice is Y_1 ; when $\lambda \in [1.3, 50]$, the ranking order of the three alternatives is $Y_2 > Y_1 > Y_3$ and the best choice is Y_2 .

6. Conclusions

This paper extends IVHFs and IVPFs to IVPHFSs. Firstly, the new score functions and accuracy functions of IVPHFSs are introduced to compare the size of IVPHFEs based on the comparison of interval numbers. Then, we focus on a new series of interval-valued Pythagorean hesitant fuzzy aggregation operators, including the IVPHFWA, IVPHFWG, GIVPHFWA, GIVPHFWG, IVPHFOWA, IVPHFOWG, GIVPHFOWA, GIVPHFOWG, IVPHFHA, IVPHFHG, GIVPHFHA, and GIVPHFHG operators. The relations of these operators are developed. Furthermore, a new approach has been developed based on the proposed operators to solve the MAGDM problems under the IVPHF environment. Finally, a numerical example is used to illustrate the effectiveness and feasibility of our proposed method.

The following work is to enhance the study of the aggregation operators of weighted IVPHFSs; we shall develop the new potential application of the proposed operators in another field such as clustering analysis, image processing, pattern recognition, and so on. We hope that these will enrich and provide more new ideas and new methods for these fields under the interval-valued Pythagorean hesitant fuzzy environment.

Abbreviations

GIVPHFHA:	Generalized interval-valued Pythagorean hesitant fuzzy hybrid averaging
GIVPHFHG:	Generalized interval-valued Pythagorean hesitant fuzzy hybrid geometric
GIVPHFOWA:	Generalized interval-valued Pythagorean hesitant fuzzy ordered weighted averaging
GIVPHFOWG:	Generalized interval-valued Pythagorean hesitant fuzzy ordered weighted geometric
GIVPHFWA:	Generalized interval-valued Pythagorean hesitant fuzzy weighted averaging
GIVPHFWG:	Generalized interval-valued Pythagorean hesitant fuzzy weighted geometric
HFE:	Hesitant fuzzy element
HFS:	Hesitant fuzzy set
IVHFS:	Interval-valued hesitant fuzzy set
IVHFE:	Interval-valued hesitant fuzzy element
IVPHFDM:	Interval-valued Pythagorean hesitant fuzzy decision matrix
IVPHFE:	Interval-valued Pythagorean hesitant fuzzy element

IVPHFHA:	Interval-valued Pythagorean hesitant fuzzy hybrid averaging
IVPHFHG:	Interval-valued Pythagorean hesitant fuzzy hybrid geometric
IVPHFOWA:	Interval-valued Pythagorean hesitant fuzzy ordered weighted averaging
IVPHFOWG:	Interval-valued Pythagorean hesitant fuzzy ordered weighted geometric
IVPHFWA:	Interval-valued Pythagorean hesitant fuzzy weighted averaging
IVPHFWG:	Interval-valued Pythagorean hesitant fuzzy weighted geometric
IVPHFS:	Interval-valued Pythagorean hesitant fuzzy set
IVPFE:	Interval-valued Pythagorean fuzzy element
IVPFS:	Interval-valued Pythagorean fuzzy set
MAGDM:	Multiattribute group decision-making
PHFE:	Pythagorean hesitant fuzzy element
PHFS:	Pythagorean hesitant fuzzy set.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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