

Research Article

Asymptotic Behavior of a Stochastic Delayed Model for Chronic Hepatitis B Infection

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In this paper, a stochastic delayed model is constructed to describe chronic hepatitis B infection with HBV DNA-containing capsids. At first, the existence and uniqueness of the global positive solution are obtained. Secondly, the sufficient conditions are derived that the solution of the stochastic system fluctuates around the disease-free equilibrium E_0 and the endemic equilibrium E^* . In the end, some numerical simulations are implemented to support our analytical results.

1. Introduction

Hepatitis B virus (HBV) infection, which is a typical liver disease, has raised great attention all over the world [1]. It is generally divided into acute and chronic. In particular, it is likely to suffer from other diseases such as cirrhosis of the liver for those patients who have been sustained infected by Hepatitis B virus [2, 3]. The essence of HBV infection lies in the transformation of the DNA molecule of HBV [2, 4, 5].

The majority of mathematical models whose research objects are classified as common three compartments has been investigated by numerous scholars [6–8]. In order to better explore the mechanism of HBV infection, Manna and Chakrabarty for the first time came up with the model of chronic HBV infection including HBV DNA-containing capsids [9], and their model is given below:

$$\begin{cases} \frac{dH(t)}{dt} = m - \alpha H(t)V(t) - \mu H(t), \\ \frac{dI(t)}{dt} = \alpha H(t)V(t) - \delta I(t), \\ \frac{dD(t)}{dt} = \eta I(t) - \beta D(t) - \delta D(t), \\ \frac{dV(t)}{dt} = \beta D(t) - cV(t), \end{cases} \quad (1)$$

where $H(t)$, $I(t)$, $D(t)$, and $V(t)$ denote the healthy hepatocytes that are not infected by the viruses, the unhealthy hepatocytes which are infected by the viruses, intracellular HBV DNA-containing capsids, and hepatitis B viruses, respectively. Furthermore, the meaning of each parameter is shown as follows:

- (i) m stands for the constant recruitment rate of the uninfected hepatocytes
- (ii) μ is the natural death rate of the uninfected hepatocytes
- (iii) α denotes the rate that these healthy hepatocytes are infected by the viruses and infected hepatocytes come into being
- (iv) δ is the rate of infected hepatocytes that are eliminated and also is the natural death rate for the capsids
- (v) η represents the rate of production of intracellular HBV DNA-containing capsids
- (vi) β is the rate at which the capsids are exported to the blood, producing the virion
- (vii) c is the natural death rate for the viruses

These parameters all are positive constant.

In fact, the process that healthy hepatocytes are infected by the viruses and then transformed into the infected

hepatocyte population is not instantaneous, so the time delay cannot be ignored. Manna and Chakrabarty [10] considered the following model with delay:

$$\begin{cases} \frac{dH(t)}{dt} = m - \alpha H(t)V(t) - \mu H(t), \\ \frac{dI(t)}{dt} = \alpha H(t - \tau)V(t - \tau) - \delta I(t), \\ \frac{dD(t)}{dt} = \eta I(t) - \beta D(t) - \delta D(t), \\ \frac{dV(t)}{dt} = \beta D(t) - cV(t). \end{cases} \quad (2)$$

In system (2) [10], the basic reproduction number is $\mathfrak{R}_0 = (\alpha\beta\eta m)/(c\delta\mu(\beta + \delta))$. If $\mathfrak{R}_0 \leq 1$, then system (2) has only the disease-free equilibrium $E_0(H^0, 0, 0, 0)$ which is globally asymptotically stable, where $H^0 = m/\mu$. If $\mathfrak{R}_0 > 1$, system (2) has two equilibria: $E_0(H^0, 0, 0, 0)$ and $E^*(H^*, I^*, D^*, V^*)$, and E^* is globally asymptotically stable, where $H^* > 0, I^* > 0, D^* > 0$, and $V^* > 0$.

It is worth pointing out that all biological processes are inevitably affected by numerous unpredictable environmental white noise. Hence, the deterministic models have some limitations in predicting the future dynamics of the system accurately; stochastic models produce more valuable real benefits and can predict the future dynamics of the system accurately than deterministic models, and after one studies a deterministic model, extending the results to the stochastic case becomes a hot issue. To understand the impacts due to such randomness and fluctuations, stochastic differential equation (SDE) approach is widely used in many kinds of branches of applied science; many stochastic models have been proposed and studied, such as in the population ecology [11–16] and in the epidemiology [17–27], as well as in other fields [28–30]. Many valuable and interesting results were obtained.

On the basis of the abovementioned works, to make model (2) more reasonable and realistic, including the stochastic perturbation on the natural death rate with white noise, we establish a delayed stochastic model as an extension of system (2) as follows:

$$\begin{cases} dH(t) = [m - \alpha H(t)V(t) - \mu H(t)]dt + \sigma_1 H(t)dB_1(t), \\ dI(t) = [\alpha H(t - \tau)V(t - \tau) - \delta I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dD(t) = [\eta I(t) - \beta D(t) - \delta D(t)]dt + \sigma_3 D(t)dB_3(t), \\ dV(t) = [\beta D(t) - cV(t)]dt + \sigma_4 V(t)dB_4(t), \end{cases} \quad (3)$$

where $B = (B_1(t), B_2(t), B_3(t), B_4(t), t \geq 0)$ is a real-valued standard Brownian motion. It is defined on a complete

probability space $(\Omega, \mathcal{F}, \mathbb{P})$ including a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ according with the general conditions, that is, it is increasing and right continuous nevertheless \mathcal{F}_0 incorporates all \mathbb{P} -null sets. σ_i ($i = 1, 2, 3, 4$) represents the intensities of the white noise, and they are positive. All the other parameters have the same meaning as that of system (2). The initial conditions of system (3) are

$$\begin{cases} H(\theta) = \psi_1(\theta), \\ I(\theta) = \psi_2(\theta), \\ D(\theta) = \psi_3(\theta), \\ V(\theta) = \psi_4(\theta), \\ (\psi_1, \psi_2, \psi_3, \psi_4) \in C, \end{cases} \quad (4)$$

where C means the space in which all functions are continuous, which is expressed as $C([- \tau, 0]; \mathbb{R}_+^4)$, where $\mathbb{R}_+^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4: x_i > 0, i = 1, 2, 3, 4\}$.

This paper is organized as follows: in the Section 2, it is proved that there is a unique global positive solution of system (3) with initial value (4). The asymptotic behavior of the solution of stochastic system (3) around the equilibrium E_0 of deterministic model (2) is discussed in Section 3. In section 4, we show that the solution of the stochastic system (3) oscillates around the infected equilibrium E^* of deterministic model (2) under certain conditions. Numerical simulations are carried out in Section 5 to illustrate the main theoretical results. A brief discussion is given in Section 6 to conclude this work.

2. Existence and Uniqueness of the Global Positive Solution

In this section, we will prove that there is a unique global positive solution of system (3) with initial value (4).

Theorem 1. *If we give any initial value (4), then there is a unique positive solution $(H(t), I(t), D(t), V(t))$ for all $t \geq -\tau$ for system (3). Furthermore, the solution will remain in \mathbb{R}_+^4 with probability one. In brief, $(H(t), I(t), D(t), V(t)) \in \mathbb{R}_+^4$ for all $t \geq -\tau$ almost surely (a.s.).*

Proof. According to the theory of stochastic differential equations, we draw the conclusion that system (3) exists as a unique local solution $(H(t), I(t), D(t), V(t))$ on $t \in [-\tau, \tau_e)$, thereinto τ_e is called as the explosion time [31]. However, we want to illuminate that there is a global solution for system (3). So, it is necessary for us to prove that $\tau_e = \infty$ a.s. For this purpose, we assume $k_0 \geq 1$ to be large enough. Under the circumstances, $H(\theta), I(\theta), D(\theta)$, and $V(\theta)$ ($\theta \in [-\tau, 0]$) all are contained in the interval $[1/k_0, k_0]$. In the following, we introduce the definition of the stopping time:

$$\tau_k = \inf \left\{ t \in [0, \tau_e): H(t) \notin \left(\frac{1}{k}, k \right) \text{ or } I(t) \notin \left(\frac{1}{k}, k \right) \text{ or } D(t) \notin \left(\frac{1}{k}, k \right) \text{ or } V(t) \notin \left(\frac{1}{k}, k \right) \right\}, \quad (5)$$

and we let $\inf \emptyset = \infty$ (in general, \emptyset expresses the empty set). Obviously, τ_k is increasing as $k \rightarrow \infty$. At this time, assume $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$; hence, we can get that $\tau_\infty \leq \tau_e$ a.s. In order to finish the proof, we must prove that $\tau_\infty = \infty$ a.s. If this assertion is false, then there are constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \quad (6)$$

So, there exist an integer $k_0 \geq k_0$ satisfying the following inequality:

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon, \quad \text{for all } k \geq k_1. \quad (7)$$

Define a C^2 -function $W: \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ as follows:

$$\begin{aligned} W(H, I, D, V) = & \left(H - b - b \ln \frac{H}{b} \right) + (I - 1 - \ln I) + fD \\ & + f(V - 1 - \ln V) + \int_{t-\tau}^t \alpha H(\zeta) V(\zeta) d\zeta, \end{aligned} \quad (8)$$

where the positive constants b and f will be determined later.

Using Itô formula to W , we obtain

$$\begin{aligned} dW = & \left(1 - \frac{b}{H} \right) dH + \left(1 - \frac{1}{I} \right) dI + f dD + f \left(1 - \frac{1}{V} \right) dV + \frac{1}{2} \frac{b}{H^2} (dH)^2 + \frac{1}{2} \frac{1}{I^2} (dI)^2 \\ & + \frac{1}{2} f \frac{1}{V^2} (dV)^2 + [\alpha H(t) V(t) - \alpha H(t-\tau) V(t-\tau)] dt \\ = & \left[\left(1 - \frac{b}{H} \right) (m - \alpha H V - \mu H) + \frac{1}{2} \sigma_1^2 b + \left(1 - \frac{1}{I} \right) [\alpha H(t-\tau) V(t-\tau) - \delta I] + \frac{1}{2} \sigma_2^2 \right. \\ & + f(\eta I - \beta D - \delta D) + f \left(1 - \frac{1}{V} \right) (\beta D - cV) + \frac{1}{2} \sigma_4^2 f + \alpha H V - \alpha H(t-\tau) V(t-\tau) \Big] dt \\ & + \sigma_1 (H - b) dB_1(t) + \sigma_2 (I - 1) dB_2(t) + f \sigma_3 D dB_3(t) + f \sigma_4 (V - 1) dB_4(t) \\ = & LW dt + \sigma_1 (H - b) dB_1(t) + \sigma_2 (I - 1) dB_2(t) + f \sigma_3 D dB_3(t) + f \sigma_4 (V - 1) dB_4(t), \end{aligned} \quad (9)$$

where

$$\begin{aligned} LW = & m - \mu H - \frac{mb}{H} + \alpha b V + \mu b - \delta I - \frac{\alpha H(t-\tau) V(t-\tau)}{I} + \delta + f \eta I \\ & - f(\beta + \delta) D + f \beta D - f c V - \frac{f \beta D}{V} + f c + \frac{1}{2} \sigma_1^2 b + \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_4^2 f \\ \leq & m + \mu b + \delta + f c + (\alpha b - f c) V + (f \eta - \delta) I + f c + \frac{1}{2} \sigma_1^2 b + \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_4^2 f. \end{aligned} \quad (10)$$

Choose the parameters $f = \delta/\eta$ and $b = c\delta/\alpha\eta$ such that $\alpha b - f c = 0$ and $f \eta - \delta = 0$, then

$$LW \leq m + \mu b + \delta + f c + \frac{1}{2} \sigma_1^2 b + \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_4^2 f := K. \quad (11)$$

The following proof is similar to the method in the literature [20], so it is omitted. The proof is completed. \square

3. Asymptotic Behavior around the Disease-free Equilibrium E_0 of Equation (2)

If $\mathfrak{R}_0 \leq 1$, then system (2) has only the disease-free equilibrium $E_0(H^0, 0, 0, 0)$ which is globally asymptotically stable. However, system (3) does not exist in the equilibrium. In the following, we establish the sufficient conditions to ensure that the solution of system (3) oscillates around E_0 of system (2).

Theorem 2. Assume that $(H(t), I(t), D(t), V(t))$ is the solution of system (3) with the initial value (4). If $\mathfrak{R}_0 \leq 1$ and the following conditions are satisfied,

$$\begin{aligned}\sigma_1^2 &< \mu, \\ \sigma_2^2 &< \delta, \\ \sigma_3^2 &< \beta + \delta, \\ \sigma_4^2 &< 2c - \frac{3}{2},\end{aligned}\tag{12}$$

then

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (H(\zeta) - H^0)^2 d\zeta &\leq \frac{\sigma_1^2 H^{02}}{\mu - \sigma_1^2}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t I^2(\zeta) d\zeta &\leq \frac{2\sigma_1^2 H^{02}}{\delta - \sigma_2^2} \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right], \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t D^2(\zeta) d\zeta &\leq \frac{A_1}{\lambda_1}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t V^2(\zeta) d\zeta &\leq \frac{A_2}{\lambda_2},\end{aligned}\tag{13}$$

where $\lambda_1, \lambda_2, A_1$, and A_2 are positive constants, and they are defined in the proof.

Proof. Since E_0 is the disease-free equilibrium of system (2), then $m = \mu H^0$.

On account of system (3), we can obtain

$$\begin{cases} dI(t + \tau) = [\alpha HV - \delta I(t + \tau)]dt + \sigma_2 I(t + \tau)dB_2(t), \\ dD(t + \tau) = [\eta I(t + \tau) - (\beta + \delta)D(t + \tau)]dt + \sigma_3 D(t + \tau)dB_3(t), \\ dV(t + \tau) = [\beta D(t + \tau) - cV(t + \tau)]dt + \sigma_4 V(t + \tau)dB_4(t). \end{cases}\tag{14}$$

Letting

$$S_{11} = \frac{(H - H^0)^2}{2}.\tag{15}$$

Using Itô formula, one can obtain that

$$\begin{aligned}LS_{11} &= -\mu(H - H^0)^2 - \alpha(H - H^0)^2 V - \alpha H^0(H - H^0)V \\ &\quad + \frac{1}{2}\sigma_1^2 H^2 \\ &\leq -\mu(H - H^0)^2 - \alpha H^0(H - H^0)V + \sigma_1^2(H - H^0)^2 \\ &\quad + \sigma_1^2 H^{02} \\ &= -(\mu - \sigma_1^2)(H - H^0)^2 - \alpha H^0(H - H^0)V + \sigma_1^2 H^{02},\end{aligned}\tag{16}$$

where the conclusion that $(a + b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$ is employed.

Similarly, setting

$$S_{12} = I(t) + \frac{\delta}{\eta} D(t) + \frac{\delta(\beta + \delta)}{\beta\eta} V(t) + \alpha \int_{t-\tau}^t H(\zeta)V(\zeta)d\zeta,\tag{17}$$

then

$$\begin{aligned}LS_{12} &= \alpha HV - \frac{c\delta(\beta + \delta)}{\beta\eta} V \\ &= \alpha(H - H^0)V + \alpha H^0 V - \frac{c\delta(\beta + \delta)}{\beta\eta} V \\ &= \alpha(H - H^0)V + \frac{c\delta(\beta + \delta)}{\beta\eta} (\mathfrak{R}_0 - 1)V \\ &\leq \alpha(H - H^0)V.\end{aligned}\tag{18}$$

Now, we define

$$S_1 = S_{11} + H^0 S_{12}.\tag{19}$$

According to (16) and (18), we can calculate that

$$LS_1 \leq -(\mu - \sigma_1^2)(H - H^0)^2 + \sigma_1^2 H^{02}.\tag{20}$$

Integrating (20) from 0 to t and then taking the expectation on both sides, by virtue of Theorem 1.5.8 (ii) [32], it yields

$$0 \leq \mathbb{E}S_1(t) \leq S_1(0) - (\mu - \sigma_1^2) \mathbb{E} \int_0^t (H(\zeta) - H^0)^2 d\zeta + \sigma_1^2 H^{02} t.\tag{21}$$

Thus, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (H(\zeta) - H^0)^2 d\zeta \leq \frac{\sigma_1^2 H^{02}}{\mu - \sigma_1^2}.\tag{22}$$

Similarly, setting

$$S_{13} = \frac{[H - H^0 + I(t + \tau)]^2}{2},\tag{23}$$

we have

$$\begin{aligned}LS_{13} &= -\mu(H - H^0)^2 - \delta I^2(t + \tau) - (\mu + \delta)(H - H^0)I(t + \tau) \\ &\quad + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) \\ &\leq -\mu(H - H^0)^2 - \delta I^2(t + \tau) + \frac{\delta}{2} I^2(t + \tau) \\ &\quad + \frac{(\mu + \delta)^2}{2\delta} (H - H^0)^2 \\ &\quad + \sigma_1^2 (H - H^0)^2 + \sigma_1^2 H^{02} + \frac{1}{2}\sigma_2^2 I^2(t + \tau) \\ &= \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) (H - H^0)^2 - \frac{1}{2}(\delta - \sigma_2^2) I^2(t + \tau) + \sigma_1^2 H^{02},\end{aligned}\tag{24}$$

where we have used the following inequality:

$$-(\mu + \delta)(H - H^0)I(t + \tau) \leq \frac{\delta}{2}I^2(t + \tau) + \frac{(\mu + \delta)^2}{2\delta}(H - H^0)^2. \quad (25)$$

Define

$$S_2 = S_{13} + \frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_1 + \frac{1}{2}(\delta - \sigma_2^2) \int_t^{t+\tau} I^2(\zeta) d\zeta. \quad (26)$$

By means of (20) and (24), we obtain

$$LS_2 \leq -\frac{1}{2}(\delta - \sigma_2^2)I^2(t) + \sigma_1^2 H^{02} \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right]. \quad (27)$$

Let us take the integral of (27) from 0 to t and then take the expectation on both sides, by Theorem 1.5.8 (ii) [32] yield

$$0 \leq \mathbb{E}S_2(t) \leq S_2(0) - \frac{1}{2}(\delta - \sigma_2^2) \mathbb{E} \int_0^t I^2 d\zeta + \sigma_1^2 H^{02} \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right] t. \quad (28)$$

Therefore, we can summarize that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t I^2 d\zeta \leq \frac{2\sigma_1^2 H^{02}}{\delta - \sigma_2^2} \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right]. \quad (29)$$

In the following, letting

$$S_{14} = \frac{[H - H^0 + I(t + \tau) + (\delta/\eta)D(t + \tau)]^2}{2}, \quad (30)$$

then we use Itô formula and arrive at

$$\begin{aligned} LS_{14} &= -\mu(H - H^0)^2 - \frac{\delta(\beta + \delta + \mu)}{\eta}(H - H^0)D(t + \tau) - \mu(H - H^0)I(t + \tau) \\ &\quad - \frac{\delta(\beta + \delta)}{\eta}I(t + \tau)D(t + \tau) - \left[\frac{\delta^2(\beta + \delta)}{\eta^2} + \frac{\delta^2}{2\eta^2}\sigma_3^2 \right] D^2(t + \tau) + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) \\ &\leq -\mu(H - H^0)^2 + \frac{\delta^2(\beta + \delta)}{4\eta^2} D^2(t + \tau) + \frac{(\beta + \delta + \mu)^2}{\beta + \delta} (H - H^0)^2 + \frac{\mu}{4} (H - H^0)^2 \\ &\quad + \mu I^2(t + \tau) + \frac{\delta^2(\beta + \delta)}{4\eta^2} D^2(t + \tau) + (\beta + \delta)I^2(t + \tau) - \frac{\delta^2(\beta + \delta)}{\eta^2} D^2(t + \tau) \\ &\quad + \sigma_1^2 (H - H^0)^2 + \sigma_1^2 H^{02} + \frac{1}{2}\sigma_2^2 I^2(t + \tau) + \frac{\delta^2}{2\eta^2} \sigma_3^2 D^2(t + \tau) \\ &= \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] (H - H^0)^2 + \left(\mu + \beta + \delta + \frac{1}{2}\sigma_2^2 \right) I^2(t + \tau) \\ &\quad - \frac{\delta^2}{2\eta^2} (\beta + \delta - \sigma_3^2) D^2(t + \tau) + \sigma_1^2 H^{02}, \end{aligned} \quad (31)$$

where we have applied the following inequality:

$$\begin{aligned} -\frac{\delta(\beta + \delta + \mu)}{\eta}(H - H^0)D(t + \tau) &\leq \frac{\delta^2(\beta + \delta)}{4\eta^2} D^2(t + \tau) + \frac{(\beta + \delta + \mu)^2}{\beta + \delta} (H - H^0)^2, \\ -\mu(H - H^0)I(t + \tau) &\leq \frac{\mu}{4} (H - H^0)^2 + \mu I^2(t + \tau), \\ -\frac{\delta(\beta + \delta)}{\eta}I(t + \tau)D(t + \tau) &\leq \frac{\delta^2(\beta + \delta)}{4\eta^2} D^2(t + \tau) + (\beta + \delta)I^2(t + \tau), \end{aligned} \quad (32)$$

$(a + b)^2 \leq 2a^2 + 2b^2$, for any $a, b \in \mathbb{R}$.

We define

$$S_3 = S_{14} + \frac{1}{\mu - \sigma_1^2} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] S_1 + \frac{2}{\delta - \sigma_2^2} \left(\mu + \beta + \delta + \frac{1}{2}\sigma_2^2 \right) \times \left[S_{13} + \frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_1 \right] + \frac{\delta^2}{2\eta^2} (\beta + \delta - \sigma_3^2) \int_t^{t+\tau} D^2(\zeta) d\zeta. \quad (33)$$

From (20), (24), and (31), we have

$$LS_3 \leq -\frac{\delta^2}{2\eta^2} (\beta + \delta - \sigma_3^2) D^2(t) + \frac{\sigma_1^2 H^{02}}{\mu - \sigma_1^2} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] + \frac{2\sigma_1^2 H^{02}}{\delta - \sigma_2^2} \left(\mu + \beta + \delta + \frac{1}{2}\sigma_2^2 \right) \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right] + \sigma_1^2 H^{02} \\ := -\lambda_1 D^2(t) + A_1. \quad (34)$$

Let us take the integral of (34) from 0 to t and then take the expectation on both sides, next relying on Theorem 1.5.8 (ii) [32], we can attain

$$\mathbb{E}S_3(t) - S_3(0) \leq -\lambda_1 \mathbb{E} \int_0^t D^2(\zeta) d\zeta + A_1 t. \quad (35)$$

Therefore, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t D^2(\zeta) d\zeta \leq \frac{A_1}{\lambda_1}. \quad (36)$$

Next choose that

$$S_{15} = \frac{[H - H^0 + I(t + \tau) + (\delta/\eta)D(t + \tau) + (\delta(\beta + \delta)/\beta\eta)V(t + \tau)]^2}{2}, \quad (37)$$

then taking advantage of Itô formula and attaining that

$$LS_{15} = -\mu(H - H^0)^2 - \mu(H - H^0)I(t + \tau) - \frac{\delta\mu}{\eta}(H - H^0)D(t + \tau) \\ - \frac{\delta(\beta + \delta)(\mu + c)}{\beta\eta}(H - H^0)V(t + \tau) - \frac{\delta c(\beta + \delta)}{\beta\eta}I(t + \tau)V(t + \tau) - \frac{\delta^2 c(\beta + \delta)}{\beta\eta^2}D(t + \tau)V(t + \tau) \\ - \frac{\delta^2 c(\beta + \delta)^2}{\beta^2\eta^2}V^2(t + \tau) + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) + \frac{\delta^2}{2\eta^2}\sigma_3^2 D^2(t + \tau) + \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2}\sigma_4^2 V^2(t + \tau) \\ \leq -\mu(H - H^0)^2 + \frac{\mu}{4}(H - H^0)^2 + \mu I^2(t + \tau) + \frac{\delta^2(\beta + \delta)}{4\eta^2}D^2(t + \tau) + \frac{\mu^2}{\beta + \delta}(H - H^0)^2 \\ + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + (\mu + c)^2(H - H^0)^2 + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + c^2 I^2(t + \tau) \\ + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + \frac{\delta^2 c^2}{\eta^2}D^2(t + \tau) - \frac{\delta^2 c(\beta + \delta)^2}{\beta^2\eta^2}V^2(t + \tau) + \sigma_1^2(H - H^0)^2 \\ + \sigma_1^2 H^{02} + \frac{1}{2}\sigma_2^2 I^2(t + \tau) + \frac{\delta^2}{2\eta^2}\sigma_3^2 D^2(t + \tau) + \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2}\sigma_4^2 V^2(t + \tau) \\ = \left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] (H - H^0)^2 + \left(\mu + c^2 + \frac{1}{2}\sigma_2^2 \right) I^2(t + \tau) \\ + \frac{\delta^2}{4\eta^2} (\beta + \delta + 4c^2 + 2\sigma_3^2) D^2(t + \tau) - \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2} \left(2c - \frac{3}{2} - \sigma_4^2 \right) V^2(t + \tau) + \sigma_1^2 H^{02}, \quad (38)$$

in which we have applied the following inequality:

$$\begin{aligned}
-\mu(H - H^0)I(t + \tau) &\leq \frac{\mu}{4}(H - H^0)^2 + \mu I^2(t + \tau), \\
-\frac{\delta\mu}{\eta}(H - H^0)D(t + \tau) &\leq \frac{\delta^2(\beta + \delta)}{4\eta^2}D^2(t + \tau) + \frac{\mu^2}{\beta + \delta}(H - H^0)^2, \\
-\frac{\delta(\beta + \delta)(\mu + c)}{\beta\eta}(H - H^0)V(t + \tau) &\leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + (\mu + c)^2(H - H^0)^2, \\
-\frac{\delta c(\beta + \delta)}{\beta\eta}I(t + \tau)V(t + \tau) &\leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + c^2I^2(t + \tau), \\
-\frac{\delta^2 c(\beta + \delta)}{\beta\eta^2}D(t + \tau)V(t + \tau) &\leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}V^2(t + \tau) + \frac{\delta^2 c^2}{\eta^2}D^2(t + \tau).
\end{aligned} \tag{39}$$

Let

$$\begin{aligned}
S_4 = S_{15} + \frac{1}{\mu - \sigma_1^2} &\left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] S_1 \\
&+ \frac{2}{\delta - \sigma_2^2} \left(\mu + c^2 + \frac{1}{2}\sigma_2^2 \right) \left[S_{13} + \frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_1 \right] \\
&+ \frac{\beta + \delta + 4c^2 + 2\sigma_3^2}{2(\beta + \delta - \sigma_3^2)} \left\{ S_{14} + \frac{1}{\mu - \sigma_1^2} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] S_1 \right. \\
&\quad \left. + \frac{2}{\delta - \sigma_2^2} \left(\mu + \beta + \delta + \frac{1}{2}\sigma_2^2 \right) \right. \\
&\quad \left. \times \left[S_{13} + \frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_1 \right] \right\} + \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2} \left(2c - \frac{3}{2} - \sigma_4^2 \right) \int_t^{t+\tau} V^2(\zeta) d\zeta.
\end{aligned} \tag{40}$$

By means of (20), (24), (31), and (38), we can calculate that

$$\begin{aligned}
LS_4 \leq & -\frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2} \left(2c - \frac{3}{2} - \sigma_4^2 \right) V^2(t) + \frac{\sigma_1^2 H^{02}}{\mu - \sigma_1^2} \left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] \\
&+ \frac{2\sigma_1^2 H^{02}}{\delta - \sigma_2^2} \left(\mu + c^2 + \frac{1}{2}\sigma_2^2 \right) \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) + 1 \right] + \frac{\sigma_1^2 H^{02}(\beta + \delta + 4c^2 + 2\sigma_3^2)}{2(\beta + \delta - 2\sigma_3^2)} \\
&+ \frac{\sigma_1^2 H^{02}(\beta + \delta + 4c^2 + 2\sigma_3^2)}{2(\mu - \sigma_1^2)(\beta + \delta - 2\sigma_3^2)} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] \\
&+ \frac{\sigma_1^2 H^{02}(\mu + \beta + \delta + 1/2\sigma_2^2)(\beta + \delta + 4c^2 + 2\sigma_3^2)}{(\delta - \sigma_2^2)(\beta + \delta - \sigma_3^2)} \left[\frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \right] + \sigma_1^2 H^{02} \\
&:= -\lambda_2 V^2(t) + A_2.
\end{aligned} \tag{41}$$

In a similar way, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t V^2(\zeta) d\zeta \leq \frac{A_2}{\lambda_2}. \quad (42)$$

The proof is completed. \square

Remark 1. For the deterministic systems (1) and (2), when $\mathfrak{R}_0 \leq 1$, the equilibrium E_0 is globally asymptotically stable. This means that the disease will be extinct. However, the stochastic system (3) does not exist in the equilibrium. Therefore, the significance of proving the asymptotic behavior of the solution of the stochastic system (3) around the equilibrium E_0 of system (1) is to show that diseases will be extinct.

4. Asymptotic Behavior Around the Endemic Equilibrium E^* of Equation (2)

In the literature [10], if $\mathfrak{R}_0 > 1$, the endemic equilibrium E^* of system (2) is globally asymptotically stable. However, system (3) does not have the endemic equilibrium E^* . In this section, we show that the solution of system (3) oscillates around E^* of system (2) under certain conditions.

Theorem 3. Assume that $(H(t), I(t), D(t), V(t))$ is the solution of system (3) with the initial value (4). If $\mathfrak{R}_0 > 1$ and the following conditions are satisfied

$$\sigma_1^2 < \mu,$$

$$\sigma_2^2 < \frac{\delta}{2},$$

$$\sigma_3^2 < \frac{\beta + \delta}{2},$$

$$\sigma_4^2 < c - \frac{3}{4},$$

$$\begin{aligned} \max \left\{ \sqrt{\frac{B_1}{m_1}}, \sqrt{\frac{B_2}{m_2}}, \sqrt{\frac{B_3}{m_3}}, \sqrt{\frac{B_4}{m_4}} \right\} &< d(E^*, E_0) \\ &= \sqrt{(H^* - H^0)^2 + I^{*2} + D^{*2} + V^{*2}}, \end{aligned} \quad (43)$$

then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (H(\zeta) - H^*)^2 d\zeta &\leq \frac{B_1}{m_1}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (I(\zeta) - I^*)^2 d\zeta &\leq \frac{B_2}{m_2}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (D(\zeta) - S^*)^2 d\zeta &\leq \frac{B_3}{m_3}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t (V(\zeta) - x^*)^2 d\zeta &\leq \frac{B_4}{m_4}, \end{aligned} \quad (44)$$

where m_i and B_i , $i = 1, 2, 3, 4$, are positive constants, and they are defined in the proof.

Proof. Note that E^* is the endemic equilibrium of system (2), so

$$\begin{aligned} m &= \alpha H^* V^* + \mu H^*, \\ \alpha H^* V^* &= \delta I^*, \\ \eta I^* &= (\beta + \delta) D^*, \\ \beta D^* &= c V^*. \end{aligned} \quad (45)$$

Letting

$$S_{21} = H - H^* - H^* \ln \frac{H}{H^*}, \quad (46)$$

by use of Itô formula, we arrive at

$$\begin{aligned} LS_{21} &= \left(1 - \frac{H^*}{H}\right) (m - \alpha H V - \mu H) + \frac{1}{2} \sigma_1^2 H^* \\ &= m - \alpha H V - \mu H - \frac{m H^*}{H} + \alpha H^* V + \mu H^* + \frac{1}{2} \sigma_1^2 H^* \\ &= \mu H^* \left(2 - \frac{H}{H^*} - \frac{H^*}{H}\right) + \alpha H^* V^* \left(2 - \frac{H}{H^*} - \frac{H^*}{H}\right) \\ &\quad + \alpha H^* V^* \left(\frac{V}{V^*} + \frac{H}{H^*} - \frac{H V}{H^* V^*} - 1\right) + \frac{1}{2} \sigma_1^2 H^* \\ &= -(\mu + \alpha V^*) \frac{(H - H^*)^2}{H} - \alpha (H - H^*) (V - V^*) + \frac{1}{2} \sigma_1^2 H^*. \end{aligned} \quad (47)$$

Taking

$$S_{22} = I - I^* - I^* \ln \frac{I}{I^*}$$

$$+ \alpha H^* V^* \int_{t-\tau}^t \left[\frac{H(\zeta)V(\zeta)}{H^*V^*} - \ln \frac{H(\zeta)V(\zeta)}{H^*V^*} - 1 \right] d\zeta, \quad (48)$$

then by means of Itô formula, we can obtain

$$\begin{aligned} LS_{22} &= \alpha H(t-\tau)V(t-\tau) - \delta I - \frac{\alpha I^* H(t-\tau)V(t-\tau)}{I} + \delta I^* + \frac{1}{2}\sigma_2^2 I^* \\ &\quad + \alpha H^* V^* \left[\frac{HV}{H^*V^*} - \frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \ln \frac{HV}{H^*V^*} + \ln \frac{H(t-\tau)V(t-\tau)}{H^*V^*} \right] \\ &= \alpha H^* V^* \left[\frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \frac{I}{I^*} - \frac{H(t-\tau)I^*}{H^*I} \frac{V(t-\tau)}{V^*} + 1 \right] + \frac{1}{2}\sigma_2^2 I^* \\ &\quad + \alpha H^* V^* \left[\frac{HV}{H^*V^*} - \frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \ln \frac{HV}{H^*V^*} + \ln \frac{H(t-\tau)V(t-\tau)}{H^*V^*} \right] \\ &\leq \alpha H^* V^* \left[\frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \frac{I}{I^*} - \ln \frac{H(t-\tau)V(t-\tau)I^*}{H^*V^*I} \right] + \frac{1}{2}\sigma_2^2 I^* \\ &\quad + \alpha H^* V^* \left[\frac{HV}{H^*V^*} - \frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \ln \frac{HV}{H^*V^*} + \ln \frac{H(t-\tau)V(t-\tau)}{H^*V^*} \right] \\ &= \alpha H^* V^* \left[\frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \frac{I}{I^*} - \ln \frac{I^*}{I} - \ln \frac{H(t-\tau)V(t-\tau)}{H^*V^*} \right] + \frac{1}{2}\sigma_2^2 I^* \\ &\quad + \alpha H^* V^* \left[\frac{HV}{H^*V^*} - \frac{H(t-\tau)V(t-\tau)}{H^*V^*} - \ln \frac{HV}{H^*V^*} + \ln \frac{H(t-\tau)V(t-\tau)}{H^*V^*} \right] \\ &= \alpha HV - \alpha H^* V^* \frac{I}{I^*} - \alpha H^* V^* \ln \frac{HV}{H^*V^*} + \alpha H^* V^* \ln \frac{I}{I^*} + \frac{1}{2}\sigma_2^2 I^* \\ &= \alpha HV - \alpha H^* V^* \frac{I}{I^*} - \alpha H^* V^* \ln \frac{H}{H^*} - \alpha H^* V^* \ln \frac{V}{V^*} + \alpha H^* V^* \ln \frac{I}{I^*} + \frac{1}{2}\sigma_2^2 I^*. \end{aligned} \quad (49)$$

Defining

$$\begin{aligned} S_{23} &= \frac{1}{\eta I^*} \left(D - D^* - D^* \ln \frac{D}{D^*} \right) \\ &\quad + \frac{1}{\beta D^*} \left(V - V^* - V^* \ln \frac{V}{V^*} \right), \end{aligned} \quad (50)$$

we have

$$\begin{aligned} LS_{23} &= \frac{1}{\eta I^*} \left[\eta I - (\beta + \delta)D - \eta D^* \frac{I}{D} + (\beta + \delta)D^* \right] + \frac{1}{2\eta I^*} \sigma_3^2 D^* \\ &\quad + \frac{1}{\beta D^*} \left(\beta D - cV - \beta V^* \frac{D}{V} + cV^* \right) + \frac{1}{2\beta D^*} \sigma_4^2 V^* \\ &= \frac{I}{I^*} - \frac{I}{I^*} \frac{D^*}{D} + 1 - \frac{V}{V^*} - \frac{D}{D^*} \frac{V^*}{V} + 1 + \frac{1}{2\eta I^*} \sigma_3^2 D^* + \frac{1}{2\beta D^*} \sigma_4^2 V^* \\ &\leq \frac{I}{I^*} - \frac{V}{V^*} - \ln \left(\frac{I}{I^*} \frac{D^*}{D} \right) - \ln \left(\frac{D}{D^*} \frac{V^*}{V} \right) + \frac{1}{2\eta I^*} \sigma_3^2 D^* + \frac{1}{2\beta D^*} \sigma_4^2 V^* \\ &= \frac{I}{I^*} - \frac{V}{V^*} - \ln \frac{I}{I^*} + \ln \frac{V}{V^*} + \frac{1}{2\eta I^*} \sigma_3^2 D^* + \frac{1}{2\beta D^*} \sigma_4^2 V^*. \end{aligned} \quad (51)$$

Set

$$S_1 = S_{22} + \alpha H^* V^* S_{23}. \quad (52)$$

According to (49) and (51), we can calculate that

$$\begin{aligned} LS_1 &\leq \alpha H V - \alpha H^* V^* \ln \frac{H}{H^*} - \alpha H^* V^* \frac{V}{V^*} + \frac{1}{2} \sigma_2^2 I^* \\ &\quad + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \\ &= \alpha H^* V^* \left(\frac{H}{H^*} - 1 \right) \left(\frac{V}{V^*} - 1 \right) + \alpha H^* V^* \left(\frac{H}{H^*} - 1 - \ln \frac{H}{H^*} \right) \\ &\quad + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \\ &\leq \alpha H^* V^* \left(\frac{H}{H^*} - 1 \right) \left(\frac{V}{V^*} - 1 \right) + \alpha H^* V^* \left(\frac{H}{H^*} + \frac{H^*}{H} - 2 \right) \\ &\quad + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \\ &= \alpha (H - H^*) (V - V^*) + \alpha V^* \frac{(H - H^*)^2}{H} + \frac{1}{2} \sigma_2^2 I^* \\ &\quad + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^*, \end{aligned} \quad (53)$$

in the inequality above, we make use of the equality

$$\begin{aligned} \frac{H}{H^*} - 1 - \ln \frac{H}{H^*} &= \frac{H}{H^*} - 1 + \ln \frac{H^*}{H} \leq \frac{H}{H^*} + \frac{H^*}{H} - 2 \\ &= \frac{(H - H^*)^2}{HH^*}. \end{aligned} \quad (54)$$

Choosing

$$S_{24} = \frac{(H - H^*)^2}{2}, \quad (55)$$

we can obtain

$$\begin{aligned} LS_{24} &= -\mu (H - H^*)^2 - \alpha (H - H^*)^2 V \\ &\quad - \alpha H^* (H - H^*) (V - V^*) + \frac{1}{2} \sigma_1^2 H^2 \\ &\leq -(\mu - \sigma_1^2) (H - H^*)^2 - \alpha H^* (H - H^*) (V - V^*) + \sigma_1^2 H^{*2}. \end{aligned} \quad (56)$$

Taking

$$S_2 = \frac{\alpha V^*}{\mu + \alpha V^*} S_{21} + S_1 + \frac{\mu}{\mu + \alpha V^*} S_{24}. \quad (57)$$

By virtue of (47), (53), and (56), we obtain

$$\begin{aligned} LS_2 &\leq -\frac{\mu(\mu - \sigma_1^2)}{\mu + \alpha V^*} [H(t) - H^*]^2 + \frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^{*2} \\ &\quad + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \\ &:= -m_1 [H(t) - H^*]^2 + B_1. \end{aligned} \quad (58)$$

Let us take the integral of (58) from 0 to t and then take the expectation on both sides, and we derive

$$\mathbb{E} S_2(t) - S_2(0) \leq -m_1 \mathbb{E} \int_0^t [H(\zeta) - H^*]^2 d\zeta + B_1 t. \quad (59)$$

Therefore, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t [H(\zeta) - H^*]^2 d\zeta \leq \frac{B_1}{m_1}. \quad (60)$$

Set

$$S_{31} = \frac{[H - H^* + I(t + \tau) - I^*]^2}{2}. \quad (61)$$

At this point, one obtains

$$\begin{aligned}
LS_{31} &= -\mu(H - H^*)^2 - \delta[I(t + \tau) - I^*]^2 - (\mu + \delta)(H - H^*)[I(t + \tau) - I^*] \\
&\quad + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) \\
&\leq -\mu(H - H^*)^2 - \delta[I(t + \tau) - I^*]^2 + \frac{\delta}{2}[I(t + \tau) - I^*]^2 + \frac{(\mu + \delta)^2}{2\delta}(H - H^*)^2 \\
&\quad + \sigma_1^2(H - H^*)^2 + \sigma_1^2 H^{*2} + \sigma_2^2[I(t + \tau) - I^*]^2 + \sigma_2^2 I^{*2} \\
&= -\frac{1}{2}(\delta - 2\sigma_2^2)[I(t + \tau) - I^*]^2 + \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2\right)(H - H^*)^2 + \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2},
\end{aligned} \tag{62}$$

where we have applied the following inequality to the abovementioned inequality, that is,

$$\begin{aligned}
-(\mu + \delta)(H - H^*)[I(t + \tau) - I^*] &\leq \frac{\delta}{2}[I(t + \tau) - I^*]^2 \\
&\quad + \frac{(\mu + \delta)^2}{2\delta}(H - H^*)^2.
\end{aligned} \tag{63}$$

Define

$$\begin{aligned}
S_3 &= S_{31} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_2 \\
&\quad + \frac{1}{2}(\delta - 2\sigma_2^2) \int_t^{t+\tau} [I(\zeta) - I^*]^2 d\zeta.
\end{aligned} \tag{64}$$

By means of (58) and (62), we derive

$$\begin{aligned}
LS_3 &\leq -\frac{1}{2}(\delta - 2\sigma_2^2)[I(t) - I^*]^2 + \left[\sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{1}{\mu - \sigma_1^2} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \sigma_1^2 H^* \right. \\
&\quad \left. + \frac{\mu + \alpha V^*}{2\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \sigma_2^2 I^* + \frac{\alpha H^* V^* (\mu + \alpha V^*)}{2\eta\mu I^* (\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \sigma_3^2 D^* + \frac{\alpha H^* V^* (\mu + \alpha V^*)}{2\beta\mu D^* (\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \sigma_4^2 V^* \right] \\
&:= -m_2[I(t) - I^*]^2 + B_2.
\end{aligned} \tag{65}$$

Let us take the integral of (65) from 0 to t and then take the expectation on both sides; next according to Theorem 1.5.8 (ii) [32], we have

$$\mathbb{E}S_3(t) - S_3(0) \leq -m_2 \mathbb{E} \int_0^t [I(\zeta) - I^*]^2 d\zeta + B_2 t. \tag{66}$$

Therefore, we can summarize that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t [I(\zeta) - I^*]^2 d\zeta \leq \frac{B_2}{m_2}. \tag{67}$$

Writing

$$S_{32} = \frac{[H - H^* + I(t + \tau) - I^* + \delta/\eta(D(t + \tau) - D^*)]^2}{2}, \tag{68}$$

we arrive at

$$\begin{aligned}
LS_{32} &= -\mu(H - H^*)^2 - \frac{\delta(\beta + \delta + \mu)}{\eta} (H - H^*)[D(t + \tau) - D^*] - \mu(H - H^*)[I(t + \tau) - I^*] \\
&\quad - \frac{\delta(\beta + \delta)}{\eta} [I(t + \tau) - I^*][D(t + \tau) - D^*] - \frac{\delta^2(\beta + \delta)}{\eta^2} [D(t + \tau) - D^*]^2 \\
&\quad + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) + \frac{\delta^2}{2\eta^2}\sigma_3^2 D^2(t + \tau) \\
&\leq -\mu(H - H^*)^2 + \frac{\delta^2(\beta + \delta)}{4\eta^2} [D(t + \tau) - D^*]^2 + \frac{(\beta + \delta + \mu)^2}{\beta + \delta} (H - H^*)^2 + \frac{\mu}{4} (H - H^*)^2 \\
&\quad + \mu[I(t + \tau) - I^*]^2 + \frac{\delta^2(\beta + \delta)}{4\eta^2} [D(t + \tau) - D^*]^2 + (\beta + \delta)[I(t + \tau) - I^*]^2 \\
&\quad - \frac{\delta^2(\beta + \delta)}{\eta^2} [D(t + \tau) - D^*]^2 + \sigma_1^2 (H - H^*)^2 + \sigma_1^2 H^{*2} + \sigma_2^2 [I(t + \tau) - I^*]^2 \\
&\quad + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2}\sigma_3^2 [D(t + \tau) - D^*]^2 + \frac{\delta^2}{\eta^2}\sigma_3^2 D^{*2} \\
&= \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] (H - H^*)^2 + (\mu + \beta + \delta + \sigma_2^2) [I(t + \tau) - I^*]^2 \\
&\quad - \frac{\delta^2}{2\eta^2} (\beta + \delta - 2\sigma_3^2) [D(t + \tau) - D^*]^2 + \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2}\sigma_3^2 D^{*2},
\end{aligned} \tag{69}$$

where we have applied the following inequality:

$$\begin{aligned}
-\frac{\delta(\beta + \delta + \mu)}{\eta} (H - H^*)[D(t + \tau) - D^*] &\leq \frac{\delta^2(\beta + \delta)}{4\eta^2} [D(t + \tau) - D^*]^2 + \frac{(\beta + \delta + \mu)^2}{\beta + \delta} (H - H^*)^2, \\
-\mu(H - H^*)[I(t + \tau) - I^*] &\leq \frac{\mu}{4} (H - H^*)^2 + \mu[I(t + \tau) - I^*]^2, \\
-\frac{\delta(\beta + \delta)}{\eta} [I(t + \tau) - I^*][D(t + \tau) - D^*] &\leq \frac{\delta^2(\beta + \delta)}{4\eta^2} [D(t + \tau) - D^*]^2 + (\beta + \delta)[I(t + \tau) - I^*]^2.
\end{aligned} \tag{70}$$

Define

$$\begin{aligned}
S_4 &= S_{32} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] S_2 + \frac{2}{\delta - 2\sigma_2^2} (\mu + \beta + \delta + \sigma_2^2) \\
&\quad \times \left[S_{31} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_2 \right] + \frac{\delta^2}{2\eta^2} (\beta + \delta - 2\sigma_3^2) \int_t^{t+\tau} [D(\zeta) - D^*]^2 d\zeta.
\end{aligned} \tag{71}$$

According to (58), (62), and (69), we obtain

$$\begin{aligned}
LS_4 \leq & -\frac{\delta^2}{2\eta^2}(\beta + \delta - 2\sigma_3^2)[D(t) - D^*]^2 + \left\{ \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2} \sigma_3^2 D^{*2} \right. \\
& + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] \left[\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right] \\
& + \frac{2(\mu + \beta + \delta + \sigma_2^2)}{\delta - 2\sigma_2^2} \left[\sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \left(\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} \right. \right. \\
& + \left. \left. \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right) \right] \Big\} \\
& := -m_3 [D(t) - D^*]^2 + B_3.
\end{aligned} \tag{72}$$

Integrating (72) from 0 to t and then taking the expectation on both sides, we have

$$\mathbb{E}S_4(t) - S_4(0) \leq -m_3 \mathbb{E} \int_0^t [D(\zeta) - D^*]^2 d\zeta + B_3 t. \tag{73}$$

Therefore, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t [D(\zeta) - D^*]^2 d\zeta \leq \frac{B_3}{m_3}. \tag{74}$$

Define

$$S_{33} = \frac{[H - H^* + I(t + \tau) - I^* + (\delta/\eta)(D(t + \tau) - D^*) + (\delta(\beta + \delta)/\beta\eta)(V(t + \tau) - V^*)]^2}{2}, \tag{75}$$

and we can derive

$$\begin{aligned}
LS_{33} = & -\mu(H - H^*)^2 - \mu(H - H^*)[I(t + \tau) - I^*] - \frac{\delta\mu}{\eta}(H - H^*)[D(t + \tau) - D^*] \\
& - \frac{\delta(\beta + \delta)(\mu + c)}{\beta\eta}(H - H^*)[V(t + \tau) - V^*] - \frac{\delta c(\beta + \delta)}{\beta\eta}[I(t + \tau) - I^*][V(t + \tau) - V^*] \\
& - \frac{\delta^2 c(\beta + \delta)}{\beta\eta^2}[D(t + \tau) - D^*][V(t + \tau) - V^*] - \frac{\delta^2 c(\beta + \delta)^2}{\beta^2\eta^2}[V(t + \tau) - V^*]^2 \\
& + \frac{1}{2}\sigma_1^2 H^2 + \frac{1}{2}\sigma_2^2 I^2(t + \tau) + \frac{\delta^2}{2\eta^2}\sigma_3^2 D^2(t + \tau) + \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2}\sigma_4^2 V^2(t + \tau) \\
\leq & -\mu(H - H^*)^2 + \frac{\mu}{4}(H - H^*)^2 + \mu[I(t + \tau) - I^*]^2 + \frac{\delta^2(\beta + \delta)}{4\eta^2}[D(t + \tau) - D^*]^2 \\
& + \frac{\mu^2}{\beta + \delta}(H - H^*)^2 + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + c^2[I(t + \tau) - I^*]^2 + (\mu + c)^2(H - H^*)^2 \\
& + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + \frac{\delta^2 c^2}{\eta^2}[D(t + \tau) - D^*]^2 \\
& - \frac{\delta^2 c(\beta + \delta)^2}{\beta^2\eta^2}[V(t + \tau) - V^*]^2 + \sigma_1^2(H - H^*)^2 + \sigma_1^2 H^{*2} + \sigma_2^2[I(t + \tau) - I^*]^2 + \sigma_2^2 I^{*2} \\
& + \frac{\delta^2}{\eta^2}\sigma_3^2[D(t + \tau) - D^*]^2 + \frac{\delta^2}{\eta^2}\sigma_3^2 D^{*2} + \frac{\delta^2(\beta + \delta)^2}{\beta^2\eta^2}\sigma_4^2[V(t + \tau) - V^*]^2 + \frac{\delta^2(\beta + \delta)^2}{\beta^2\eta^2}\sigma_4^2 V^{*2} \\
& \cdot \left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] (H - H^*)^2 + (\mu + c^2 + \sigma_2^2)[I(t + \tau) - I^*]^2 \\
& + \frac{\delta^2}{4\eta^2}(\beta + \delta + 4c^2 + \sigma_3^2)[D(t + \tau) - D^*]^2 - \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2}\left(2c - \frac{3}{2} - 2\sigma_4^2\right)[V(t + \tau) - V^*]^2 \\
& + \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2}\sigma_3^2 D^{*2} + \frac{\delta^2(\beta + \delta)^2}{\beta^2\eta^2}\sigma_4^2 V^{*2},
\end{aligned} \tag{76}$$

in which we have applied the following inequality:

$$\begin{aligned}
& -\mu(H - H^*)[I(t + \tau) - I^*] \leq \frac{\mu}{4}(H - H^*)^2 + \mu[I(t + \tau) - I^*]^2, \\
& -\frac{\delta\mu}{\eta}(H - H^*)[D(t + \tau) - D^*] \leq \frac{\delta^2(\beta + \delta)}{4\eta^2}[D(t + \tau) - D^*]^2 + \frac{\mu^2}{\beta + \delta}(H - H^*)^2, \\
& -\frac{\delta(\beta + \delta)(\mu + c)}{\beta\eta}(H - H^*)[V(t + \tau) - V^*] \leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + (\mu + c)^2(H - H^*)^2, \\
& -\frac{\delta c(\beta + \delta)}{\beta\eta}[I(t + \tau) - I^*][V(t + \tau) - V^*] \leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + c^2[I(t + \tau) - I^*]^2, \\
& -\frac{\delta^2 c(\beta + \delta)}{\beta\eta^2}[D(t + \tau) - D^*][V(t + \tau) - V^*] \leq \frac{\delta^2(\beta + \delta)^2}{4\beta^2\eta^2}[V(t + \tau) - V^*]^2 + \frac{\delta^2 c^2}{\eta^2}[D(t + \tau) - D^*]^2.
\end{aligned} \tag{77}$$

Let

$$\begin{aligned}
S_5 = & S_{33} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] S_2 \\
& + \frac{2}{\delta - 2\sigma_2^2} (\mu + c^2 + \sigma_2^2) \left[S_{31} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_2 \right] \\
& + \frac{\beta + \delta + 4c^2 + \sigma_3^2}{2(\beta + \delta - 2\sigma_3^2)} \left\{ S_{32} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] S_2 + \frac{2}{\delta - 2\sigma_2^2} (\mu + \beta + \delta + \sigma_2^2) \right. \\
& \times \left[S_{31} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) S_2 \right] \left. \right\} + \frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2} \left(2c - \frac{3}{2} - 2\sigma_4^2 \right) \int_t^{t+\tau} [V(\zeta) - V^*]^2 d\zeta.
\end{aligned} \tag{78}$$

By means of (58), (62), (69), and (76), we obtain

$$\begin{aligned}
LS_5 \leq & -\frac{\delta^2(\beta + \delta)^2}{2\beta^2\eta^2} \left(2c - \frac{3}{2} - 2\sigma_4^2 \right) [V(t) - V^*]^2 + \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2} \sigma_3^2 D^{*2} \\
& + \frac{\delta^2(\beta + \delta)^2}{\beta^2\eta^2} \sigma_4^2 V^{*2} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left[\frac{4(\beta + \delta)(\mu + c)^2 + 4\mu^2 - 3\mu(\beta + \delta)}{4(\beta + \delta)} + \sigma_1^2 \right] \\
& \times \left[\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* + \frac{1}{2} \sigma_2^2 I^{*2} + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right] \\
& + \frac{2}{\delta - 2\sigma_2^2} (\mu + c^2 + \sigma_2^2) \left\{ \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \right. \\
& \times \left[\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* + \frac{1}{2} \sigma_2^2 I^{*2} + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\beta + \delta + 4c^2 + \sigma_3^2}{2(\beta + \delta - 2\sigma_3^2)} \left\{ \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\delta^2}{\eta^2} \sigma_3^2 D^{*2} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \right. \\
& \times \left[\frac{4(\beta + \delta)^2 + 5\mu(\beta + \delta) + 4\mu^2}{4(\beta + \delta)} + \sigma_1^2 \right] \left[\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* \right. \\
& \left. \left. + \frac{1}{2} \sigma_2^2 I^* + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right] \right\} \frac{(\beta + \delta + 4c^2 + \sigma_3^2)(\mu + \beta + \delta + \sigma_2^2)}{(\delta - 2\sigma_2^2)(\beta + \delta - 2\sigma_3^2)} \\
& \times \left\{ \sigma_1^2 H^{*2} + \sigma_2^2 I^{*2} + \frac{\mu + \alpha V^*}{\mu(\mu - \sigma_1^2)} \left(\frac{\mu^2 + \delta^2}{2\delta} + \sigma_1^2 \right) \left[\frac{\mu}{\mu + \alpha V^*} \sigma_1^2 H^{*2} + \frac{\alpha V^*}{2(\mu + \alpha V^*)} \sigma_1^2 H^* + \frac{1}{2} \sigma_2^2 I^* \right. \right. \\
& \left. \left. + \frac{\alpha H^* V^*}{2\eta I^*} \sigma_3^2 D^* + \frac{\alpha H^* V^*}{2\beta D^*} \sigma_4^2 V^* \right] \right\} \\
& := -m_4 [V(t) - V^*]^2 + B_4.
\end{aligned} \tag{79}$$

In a similar way, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t [V(\zeta) - V^*]^2 d\zeta \leq \frac{B_4}{m_4}. \tag{80}$$

The proof is completed. \square

Remark 2. For the deterministic systems (1) and (2), E^* is globally asymptotically stable when $\mathfrak{R}_0 > 1$; this means that the disease will be persistent. However, the stochastic system (3) does not have equilibrium E^* . So, the sense of proving the asymptotic behavior of the solution of the stochastic system (3) around the equilibrium E^* of system (2) is to illustrate that the disease will be persistent.

5. Numerical Simulations

In this section, we will carry out some numerical simulations to demonstrate the theoretical results obtained in this paper.

Example 1. We choose the parameters as follows:

$$\begin{aligned}
m &= 2.6 \times 10^7, \\
\alpha &= 3 \times 10^{-13}, \\
\mu &= 0.01, \\
\delta &= 0.053, \\
\eta &= 150, \\
\tau &= 5, \\
\beta &= 0.87, \\
c &= 3.8, \\
\sigma_1 &= 0.03, \\
\sigma_2 &= 0.1, \\
\sigma_3 &= 0.2, \\
\sigma_4 &= 0.1,
\end{aligned} \tag{81}$$

where the value of parameters $m, \alpha, \mu, \delta, \eta, \beta$, and c are from the literature [9] and the rest of the parameters are assumed. In addition, we assume that the initial values of system (3) are

$$\begin{aligned}
H(\theta) &= 3.412 \times 10^8, \\
I(\theta) &= 1.32 \times 10^8, \\
D(\theta) &= 2.144 \times 10^{10}, \\
V(\theta) &= 4.92 \times 10^9, \\
\theta &\in [-\tau, 0].
\end{aligned} \tag{82}$$

For the deterministic model (2), by calculating, we obtain $\mathfrak{R}_0 = 0.5476 < 1$; therefore, it shows that the infection-free equilibrium $E_0 = (2.6 \times 10^9, 0, 0, 0)$ is globally asymptotically stable (see Figure 1).

For the stochastic model (3), we have

$$\begin{aligned}
\sigma_1^2 &= 0.0009 < \mu = 0.01, \\
\sigma_2^2 &= 0.01 < \delta = 0.053, \\
\sigma_3^2 &= 0.04 < \beta + \delta = 0.923, \\
\sigma_4^2 &= 0.01 < 2c - \frac{3}{2} = 6.1.
\end{aligned} \tag{83}$$

So, the conditions of Theorem 2 are satisfied, one can see the asymptotic behavior around the infection-free equilibrium E_0 of system (2), that is, the infected hepatocytes I , intracellular HBV DNA-containing capsids D , and hepatitis B viruses V will become extinct almost surely. The results are supported in Figure 1.

Example 2. We choose the parameters as follows:

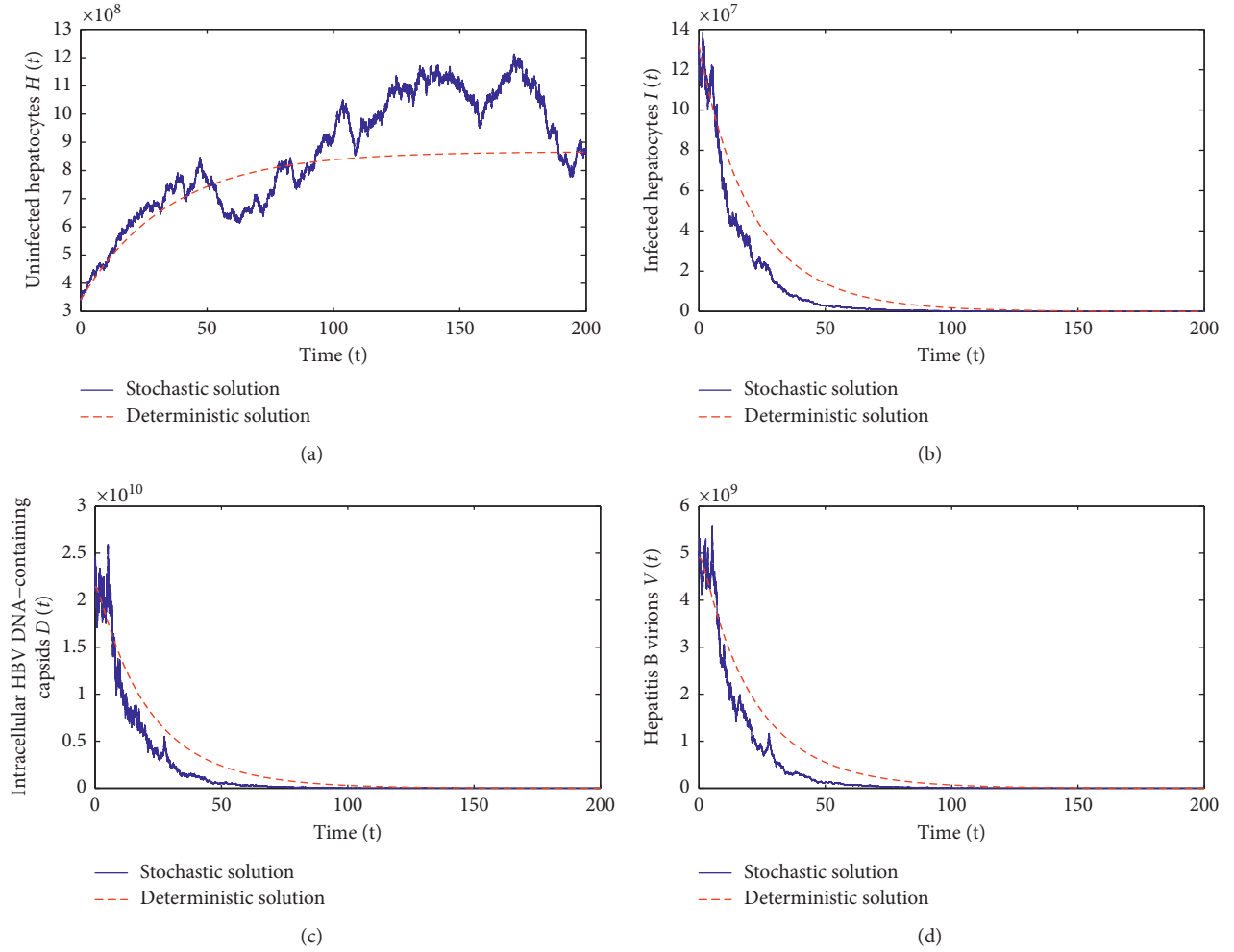


FIGURE 1: Numerical simulations of the solutions of systems (2) and (3). It is shown that the solution of (3) fluctuates around E_0 of (2).

$$\begin{aligned}
 m &= 0.8, \\
 \alpha &= 0.8, \\
 \mu &= 0.6, \\
 \delta &= 0.4, \\
 \eta &= 1.5, \\
 \beta &= 0.87, \\
 c &= 1.0, \\
 \sigma_1 &= 0.03, \\
 \sigma_2 &= 0.03, \\
 \sigma_3 &= 0.03, \\
 \sigma_4 &= 0.03,
 \end{aligned} \tag{84}$$

where the value of parameters m, α, μ, δ , and η are from the literature [20] and β is from the literature [9] and the rest of

the parameters are assumed. Assume that the initial values of system (3) are

$$\begin{aligned}
 H(\theta) &= 0.1, \\
 I(\theta) &= 1.2, \\
 D(\theta) &= 1.0, \\
 V(\theta) &= 0.8, \\
 \theta &\in [-\tau, 0].
 \end{aligned} \tag{85}$$

For the deterministic model (2), by calculating, we obtain $\mathfrak{R}_0 = 2.74 > 1$; therefore, the endemic equilibrium $E^* = (0.4866, 1.2701, 1.5001, 1.3051)$ is globally asymptotically stable (see Figure 2).

For the stochastic model (3), by a simple computation, we have

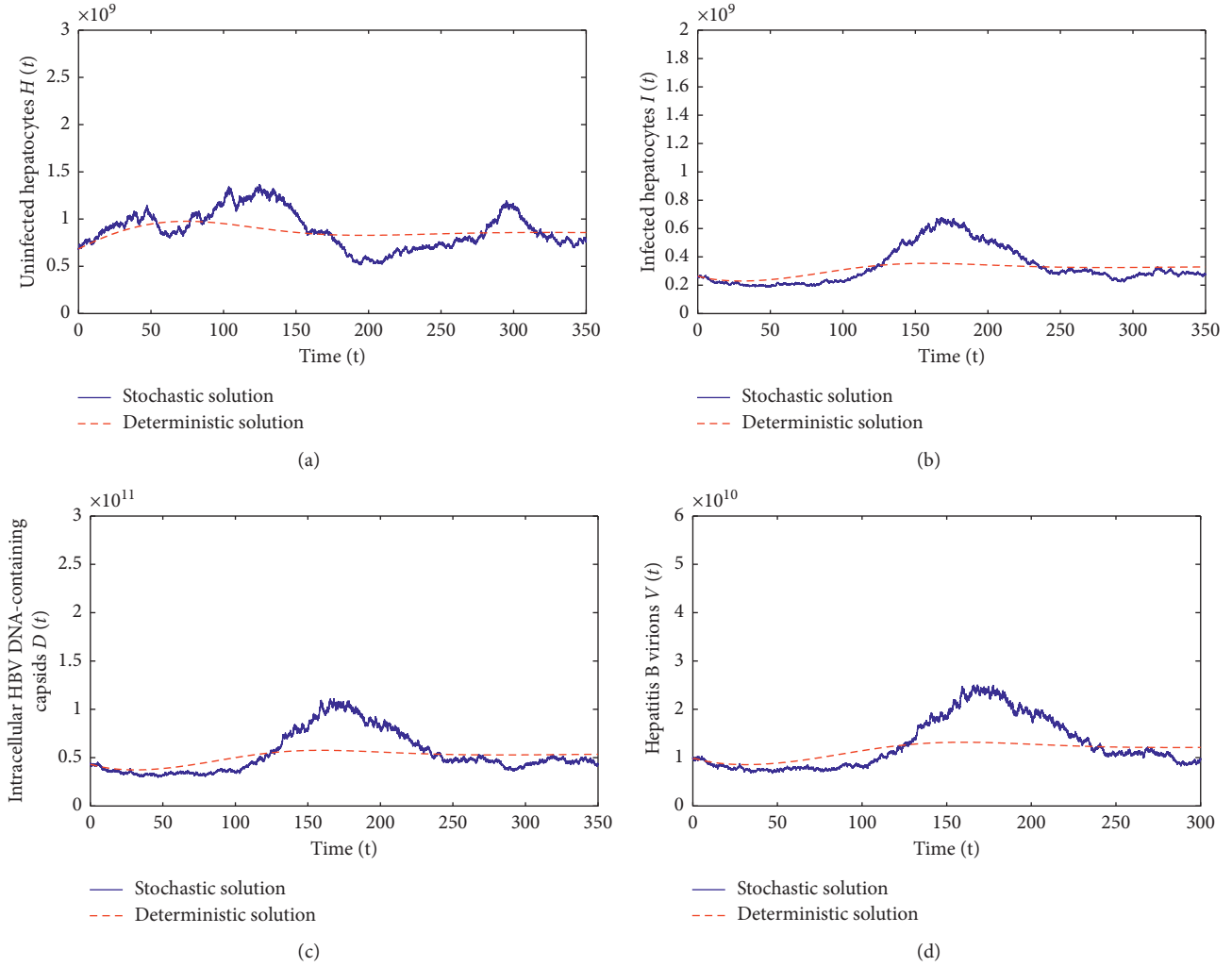


FIGURE 2: Numerical simulations of the solutions of systems (2) and (3). It is shown that the solution of the stochastic system oscillates around the endemic equilibrium E^* .

$$\begin{aligned}
 \sigma_1^2 &= 0.0009 < \mu = 0.8, \\
 \sigma_2^2 &= 0.0009 < \frac{\delta}{2} = 0.2, \\
 \sigma_3^2 &= 0.0009 < \frac{\beta + \delta}{2} = 0.635, \\
 \sigma_4^2 &= 0.0009 < c - \frac{3}{4} = 0.25, \\
 \max\{\sqrt{A}, \sqrt{B}, \sqrt{M}, \sqrt{N}\} &= \max\{0.0740, 0.1621, 1.1872, 2.2392\} \\
 &< d(E^*, E_0) \\
 &= \sqrt{(H^* - H^0)^2 + I^{*2} + D^{*2} + V^{*2}} \\
 &= 2.5067.
 \end{aligned} \tag{86}$$

So, the conditions of Theorem 3 are satisfied. In Figure 2, one can see that the asymptotic behavior around the endemic equilibrium E^* of system (2), that is, the infected hepatocytes I , intracellular HBV DNA-containing capsids D , and hepatitis B viruses V will become persistent almost surely (see Figures 2(b)–2(d)).

Example 3. Based on Example 2, we choose $\sigma_1 = 1.0$, $\sigma_2 = 1.0$, $\sigma_3 = 1.0$, and $\sigma_4 = 1.0$ and other parameters do not change. Here, we have $\Re_0 = 2.74 > 1$, $\sigma_1^2 > \mu$, $\sigma_2^2 > \delta/2$, $\sigma_3^2 > \beta + \delta/2$, and $\sigma_4^2 > c - 3/4$. From the numerical simulations given in Figure 3, we can see that large noise may result in infected hepatocytes, intracellular HBV DNA-containing capsids, and hepatitis B viruses of (3) become extinct almost surely, although the endemic equilibrium E^* of system (2) is globally asymptotically stable.

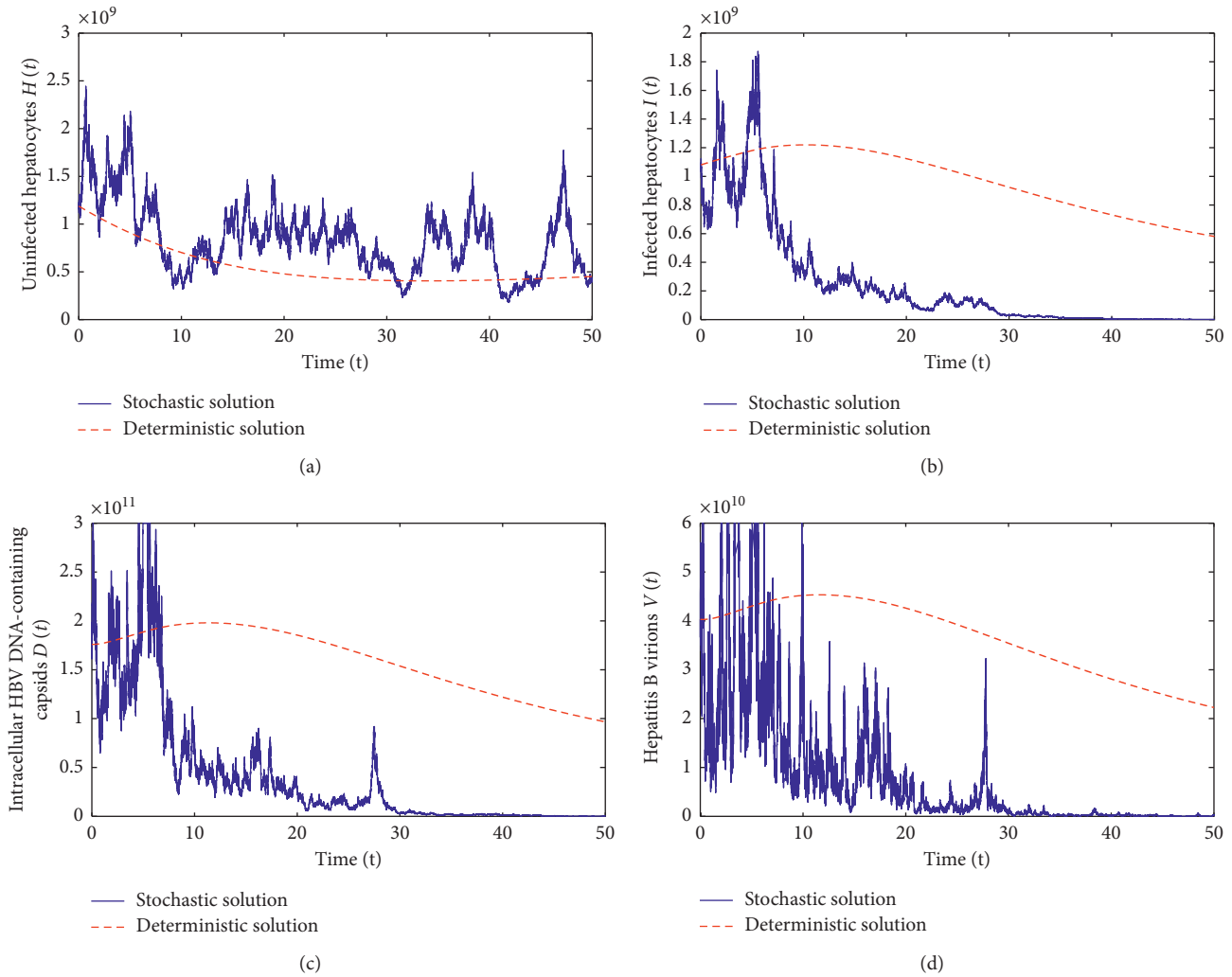


FIGURE 3: Numerical simulations of the solutions of system (3). It is shown that the larger intensity of white noise set may help to eliminate $I(t)$, $D(t)$, and $V(t)$.

6. Conclusions

This paper discusses a stochastic delayed model for chronic hepatitis B infection with HBV DNA-containing capsids. At first, we illustrate that there exists a unique global positive solution for system (3) with the initial value (4). Then, we obtain sufficient conditions to guarantee that the solution of the stochastic system fluctuates around the disease-free equilibrium E_0 and the endemic equilibrium E^* . At last, we carry out the numerical simulation in order to confirm the analytical results. Numerical simulations further reveal that the larger intensity of white noise may help to eliminate the infected hepatocytes, intracellular HBV DNA-containing capsids, and hepatitis B viruses (see Figure 3), and we leave these cases as our future work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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