Hindawi
Complexity
Volume 2020, Article ID 1941636, 12 pages
https://doi.org/10.1155/2020/1941636

Review Article

Finite-Time Stability Analysis: A Tutorial Survey

Honglei Xu

School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, Perth, WA 6845, Australia

Correspondence should be addressed to Honglei Xu; h.xu@curtin.edu.au

Received 16 July 2020; Revised 9 September 2020; Accepted 23 September 2020; Published 19 October 2020

Academic Editor: Jianquan Lu

In the past decades, there has been a growing research interest in the field of finite-time stability and stabilization. This paper aims to provide a self-contained tutorial review in the field. After a brief introduction to notations and two distinct finite-time stability concepts, dynamical system models, particularly in the form of linear time-varying systems and impulsive linear systems, are studied. The finite-time stability analysis in a quantitative sense is reviewed, and a variety of stability results including state transition matrix conditions, the piecewise continuous Lyapunov-like function theory, and the converse Lyapunov-like theorem are investigated. Then, robustness and time delay issues are studied. Finally, fundamental finite-time stability results in a qualitative sense are briefly reviewed.

1. Introduction

Finite-time stability was first introduced in a Russian journal [1] and later appeared in the western literature [2–4]. The term short-time stability is another name for it [5]. In the current literature, there are two different concepts of finite-time stability. The first is the traditional finite-time stability concept which concerns the restrained system behavior during a specified interval of time. The initial and trajectory domains and the time interval need to be specified in advance, so the traditional concept is a quantitative one. We call it finite-time stability in a quantitative sense. The second one characterizes an asymptotically stable system whose state reaches zero in a finite time, called a settling time. Similar to the Lyapunov stability, it is a qualitative concept, and hence, we call the second concept finite-time stability in the qualitative sense. The analysis and synthesis results of both finite-time stability concepts can be applied to many practical applications such as ATM networks [6], car suspension systems [7], and robot manipulators [8].

Finite-time stability in a quantitative sense emphasizes the following characteristics: the system restrains its trajectory to a predefined time-varying domain over a finite time interval for a bounded initial condition. Even though it mimics the Lyapunov stability, it is quite different from the classical one due to its finite time interval and specified domains for initial conditions and system trajectories, i.e., a system is finite-time stable for some chosen initial and trajectory domains and time intervals but not finite-time stable for different ones. In the past few decades, many finite-time stability analysis and control design problems have been investigated and a variety of stability criteria have been obtained, see, for example, [3, 4, 9] and the references therein. Recently, computationally tractable finite-time stability criteria with less conservatism have been established under the help of new tools such as linear matrix inequalities [10], Lyapunov matrix equations [11], and differential linear matrix inequalities [12]. More recently, studies on the finite-time stability and stabilization have been extended from linear time-varying systems to complex dynamical systems such as switched systems [13–15] and stochastic systems [16].

On the other hand, finite-time stability in a qualitative sense has attracted much attention in recent years and become a growing interdisciplinary research area. It focuses on asymptotical stability analysis for dynamical systems whose trajectories reach an equilibrium point in a finite time. It is a stronger concept than asymptotical stability and has the settling-time characteristic. Relevant results on autonomous and nonautonomous nonlinear systems have been

discussed in [17–20]. Later, switched versions and time-delay versions appeared in the literature, e.g., [21–23]. Recently, relevant issues of underactuated systems with disturbance have been considered in [24]. In the context of this paper, the readers should be not hard to distinguish whether the concept of finite-time stability is quantitative or qualitative, so we can use the term "finite-time stability" in most places without causing confusion.

In this paper, we will summarize the results of finite-time stability from both quantitative and qualitative aspects. Several excellent surveys on finite-time stability have been found, see, for example, the review papers [25–27], the books [7, 28, 29], and the references therein. These publications report and survey finite-time stability on one aspect or another. This paper aims to provide a unified self-contained tutorial review of finite-time stability to introduce the recent discoveries in the field.

The remainder of this paper is as follows. Section 2 gives some basic mathematical preliminaries including two finite-time stability concepts. Section 3 reviews finite-time stability results in a quantitative sense, mostly for linear time-varying systems. Results involving time-dependent and state-dependent impulses, time delays, and uncertainty are also investigated. In Section 4, we briefly overview some results on finite-time stability in a qualitative sense. Finally, a conclusion is drawn in Section 5.

2. Mathematical Preliminaries

2.1. Notations and Definitions. Let $\mathbb{R}^+$ denote a set of nonnegative real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space, and consider the time interval $\Omega = [0, T]$, $T > 0$. Let A^T be the transpose of A and I be the identity matrix with an appropriate dimension. For a square matrix A, we denote by $\lambda(A)$, $\lambda_{\max}(A)$, and $\lambda_{\min}(A)$ the set of eigenvalues, the maximum eigenvalues, and the minimum eigenvalues of A, respectively. The symmetric components in a matrix are represented by $*$. $A \geq 0$ ($A > 0$), called to be positive semi-definite (positive definite), means $x^\top Ax \geq 0$ ($x^\top Ax > 0$) for all $x \in R^n$. $A \geq B$ is equivalent to $A - B \geq 0$. Let $C([-h, 0], R^n)$ denote the set of all vector-valued continuous functions on $[-h, 0]$. For $x(t) \in C([-h, 0], \mathbb{R}^n)$, it is represented by $x_t = \{x(t+s)\colon s \in [-h, 0]\}$ with the norm $\|x_t\| = \sup_{s\in[-h,0]}\|x(t+s)\|$. Let $\mathcal{S}_\rho$ and $\overline{\mathcal{S}}_\rho$ denote the open and closed sets of the allowable system states defined as $\mathcal{S}_\rho = \left\{x \in \mathbb{R}^n | \|x(t)\|_Q^2 < \rho\right\}$ and $\overline{\mathcal{S}}_\rho = \left\{x \in \mathbb{R}^n | \|x(t)\|_Q^2 \leq \rho\right\}$, respectively, where $\rho > 0$ and Q is a symmetric positive definite real matrix. We denote $V_{\min}^\rho(t) = \min_{\|x\|=\rho} V(x, t)$ and $V_{\max}^\rho(t) = \max_{\|x\|=\rho} V(x, t)$. For a set $S_p = \left\{x_1, x_2, \ldots, x_p\right\} \subseteq \mathbb{R}^n$, the conical hull of S_p is the set of all conical combinations, i.e., $\operatorname{cone}(S_p) = \{x | x = \sum_{i=1}^p a_i x_i,\ a_i \geq 0\}$. The set of normalized extremal rays generating S_q, denoted by $\operatorname{extr}(S) = \left\{\overline{x}_1, \ldots, \overline{x}_q\right\}$ with $\|\overline{x}_i\|_2 = 1$, $i = 1, \ldots, q \leq p$, is the minimal set of unit vectors such that $S = \operatorname{cone}(\left\{\overline{x}_1, \ldots, \overline{x}_q\right\})$. Given a piecewise continuous matrix-valued (or vector-valued) function $H(\cdot)$ over Ω and a positive real number ε, we denote $H^-(t) = \lim_{\varepsilon \longrightarrow 0} H(t - \varepsilon)$ and $H^+(t) = \lim_{\varepsilon \longrightarrow 0} H(t + \varepsilon)$, i.e., $H^-(t)$ and $H^+(t)$ are the left and right limits, respectively. Let the set $\mathscr{C}$ be an open set having the origin and a boundary $\partial\mathscr{C}$.

We first look at the basic definition of finite-time stability for a dynamical system

$$\dot{x}(t) = f(t, x(t)), \tag{1}$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in \mathbb{R}^n$ is the state variable, and $f(\cdot)$ is a $\mathbb{R}^n$-valued function. Suppose that system (1) has a unique solution. We first introduce the concepts of "finite-time stability" in a quantitative sense and in a qualitative sense, respectively.

Definition 1 (Finite-Time Stability in a Quantitative Sense). Given two sets $\mathcal{X}_0$ and $\mathcal{X}(t)$, $0 \in \mathcal{X}_0$, system (1) is said to be finite-time stable with respect to $(\Omega, \mathcal{X}_0, \mathcal{X}(t))$ if

$$x_0 \in \mathcal{X}_0 \text{ implies } x(t) \in \mathcal{X}(t) \text{ for } t \in \Omega. \tag{2}$$

In Figure 1, we can see a graphical explanation of Definition 1, where the initial set $\mathcal{X}_0$ and the time-varying set $\mathcal{X}_t$ can be in various forms.

(1) When they are ellipsoids, which are the most common forms existing in the literature (see, e.g., [7, 30–32]), they can be formulated as $\mathcal{X}_0 = \left\{x_0 \in \mathbb{R}^n | x_0^\top R x_0 \leq 1\right\}$ and $\mathcal{X}_t = \{x \in \mathbb{R}^n | x^\top Q(t) x \leq 1\}$, where $Q(t)$ is a bounded and piecewise continuous matrix-valued function of time and $Q(0) < R$. In many cases, $\mathcal{X}_0$ and $\mathcal{X}_t$ can also be expressed as $\mathcal{X}_0 = \left\{x_0 \in \mathbb{R}^n | \|x_0\|_Q^2 < c_1\right\}$ and $\mathcal{X}_t = \left\{x \in \mathbb{R}^n | \|x_t\|_Q^2 < c_2\right\}$ for $c_2 > c_1 > 0$ [33].

(2) When they are in the form of polytopes, they can be described by $\mathcal{X}_0 = \operatorname{conv}\left\{x_1^{(0)}, x_2^{(0)}, \ldots, x_p^{(0)}\right\}$ and $\mathcal{X}_t = \operatorname{conv}\left\{x_1^{(t)}, x_2^{(t)}, \ldots, x_q^{(t)}\right\} = \{x \in \mathbb{R}^n | a_i^\top x \leq 1,\ i = 1, 2, \ldots, q\}$, where p and q are the number of vertices of the polytopes $\mathcal{X}_0$ and $\mathcal{X}_t$, and $x_i^{(0)}$ and $x_j^{(t)}$ are the i-th vertex of the polytope $\mathcal{X}_0$ and the j-th vertex of the polytope $\mathcal{X}_t$ [34].

(3) They can be formulated as much generalized piecewise quadratic domains over conical partitions $\mathcal{P}_0 = \{U_1^0, U_2^0, \ldots, U_u^0\}$ and $\mathcal{P}_t = \{U_1^t, U_2^t, \ldots, U_v^t\}$. Then, $\mathcal{X}_0 = \left\{x_0 \in \mathbb{R}^n | x_0^\top R_i x_0 \leq 1,\ x_0 \in U_i^0,\ i = 1, 2, \ldots, u\right\}$ and $\mathcal{X}_t = \{x \in \mathbb{R}^n | x^\top Q_i(t) x \leq 1,\ x \in U_i^t,\ i = 1, 2, \ldots, v\}$. It is obvious to see that the set of piecewise quadratic domains is a generalized set of ellipsoidal domains since an ellipsoidal domain is indeed a piecewise quadratic domain choosing $Q_i = Q$ for all $i = 1, 2, \ldots, v$. It can also represent the set of polytopic domains whose boundary is a polyhedral function's level curve.

Definition 2 (Finite-Time Stability in a Qualitative Sense). System (1) is said to be finite-time stable if for any $x_0 \in \mathscr{C} \subset \mathbb{R}^n$, $t \geq 0$ and $\varepsilon > 0$, there exist $\delta(t, x_0) > 0$ and

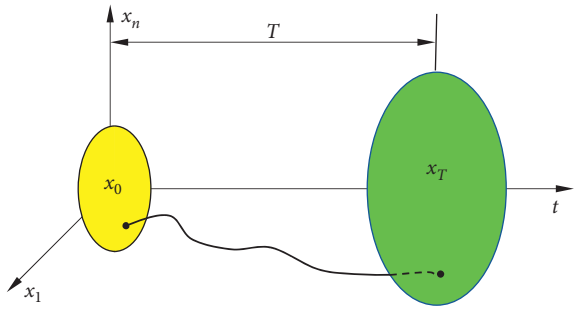

Figure 1: Schematic illustration of finite-time stability in a quantitative sense.

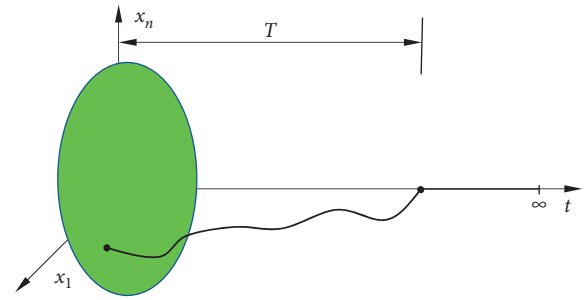

Figure 2: Schematic illustration of finite-time stability in a qualitative sense.

$T(x_0) > 0$ such that $\|x_0\| \le \delta$ implies $\|x(t, x_0)\| \le \varepsilon$, $\lim_{t \longrightarrow \infty} \|x(t, x_0)\| = 0$ and $x(t) = 0$ for all $t > T(x_0)$. Here, $T(x_0) = \inf\{T \ge 0 \mid x(t, x_0) = 0, \forall t \ge T\}$ is called the settling-time function of system (1), and the set $\mathscr{C}$ is called the domain of attraction. Moreover, if $\mathscr{C} = \mathbb{R}^n$, system (1) is globally finite-time stable.

Similarly, a schematic illustration of Definition 2 is given in Figure 2.

2.2. Mathematical Formulations. System (1) is a general model for both linear and nonlinear systems depending on the choice of the function $f(\cdot)$. In the first part of this paper, we will focus on finite-time stability issues in a quantitative sense for continuous-time linear time-varying systems with and without finite jumps. In the second part of this paper, we will analyze finite-time stability in a qualitative sense for continuous-time nonlinear systems.

First, we introduce a linear time-varying system described as

$$\dot{x}(t) = A(t)x(t), \tag{3}$$

for a given initial condition $x(0) = x_0$, which has been considered in many papers, see, e.g., [7, 30]. Here, $A(\cdot)$: $\Omega \mapsto \mathbb{R}^{n \times n}$ is a continuous matrix-valued function.

In many practical scenarios, abrupt state changes and system jumping behaviors are commonly existing, and these finite jumps occur when the time points and/or the system states satisfy a certain triggering condition, say $(t, x(t)) \in \mathcal{S} \subset \Omega \times \mathbb{R}^n$. When the impulses are triggered, the impulsive mappings can be described by

$$x^+(t) = B(t)x^-(t) = B(t)x(t), \quad (t, x(t)) \in \mathcal{S}, \tag{4}$$

where $B(\cdot)$: $\Omega \mapsto \mathbb{R}^{n \times n}$ is a matrix-valued function, which describes the jumping behavior of system (3) with left continuity over the triggering set $\mathcal{S} \subset \Omega \times \mathbb{R}^n$. We call a dynamical system modeled by (3) and (4) to be an impulsive linear system. According to the triggering set $\mathcal{S}$, impulsive linear systems expressed by (3) and (4) can be categorized into two main types: the time-dependent impulsive linear systems and the state-dependent impulsive linear systems, which can be described by

$$\begin{cases} \dot{x}(t) = A(t)x(t), & x(t_0) = x_0, \quad t \notin \mathcal{I} = \{t_1, t_2, \ldots\} \subset \Omega, \\ x(t^+) = B(t)x(t), & t \in \mathcal{I}, \quad k = 1, 2, \ldots, \end{cases} \tag{5}$$

$$\begin{cases} \dot{x}(t) = A(t)x(t), & x(t_0) = x_0, \quad x(t) \in \mathbb{R}^n \setminus \bigcup_{k=1}^{N} \mathcal{S}_k, \\ x(t^+) = B(t)x(t), & x(t) \in \mathcal{S}_k, \quad k = 1, 2, \ldots, \end{cases} \tag{6}$$

respectively [31, 32, 35]. For a time-dependent impulsive linear system, the impulses occur at the given time points, $t \in \mathcal{I} = \{t_1, t_2, \ldots\} \subset \Omega$, so the triggering set $\mathcal{S}$ can be written as $\mathcal{S} = \mathcal{I} \times \mathcal{D}(x_0, \mathcal{I})$, where $\mathcal{D}(x_0, \mathcal{I}) = \{x(t): t \in \mathcal{I}\} \subset \mathbb{R}^n$. For a state-dependent impulsive linear system, the impulses happen when the system state reaches a preassigned set $\mathcal{D} \subset \mathbb{R}^n$, and then, the triggering set $\mathcal{S}$ is written as $\mathcal{I}(x_0, \mathcal{D}) = \{t \in \Omega: x(t) \in \mathcal{D}\} \subset \Omega$. It is worth pointing out that the well-posedness of the triggering times should be guaranteed and the Zeno phenomena need to be avoided in this paper.

3. Finite-Time Stability in a Quantitative Sense

In this section, we will provide some results on finite-time stability in a quantitative sense for linear time-varying system (3) and its variants with impulses (4).

3.1. State Transition Matrix. In linear system theory, we know that the solution of (3) can be described as $x(t) = \Phi(t, 0)x(0)$, where the matrix-valued function $\Phi(\cdot, \cdot)$ has the following basic properties:

$$\begin{aligned} &\dot{\Phi}(t, 0) = A(t)\Phi(t, 0), \\ &\Phi(0, 0) = I. \end{aligned} \tag{7}$$

This matrix-valued function $\Phi(\cdot, \cdot)$ is called to be the state transition matrix, and an appropriate assumption on the nature of $A(t)$ can ensure the existence and uniqueness of the state transition matrix. For system (3), it is stable in the sense of Lyapunov if there exists a positive constant M such that $\|\Phi(t, 0)\| < M$ for all $t \ge 0$. Moreover, system (3) is asymptotically stable if it is stable and $\lim_{t \longrightarrow \infty} \Phi(t, 0) = 0$.

As for finite-time stability of the system (3), a necessary and sufficient stability condition will be provided in the following theorem [30].

Theorem 1. *System (3) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where* $\mathcal{X}_0 = \{x \in \mathbb{R}^n \mid x_0^\top R x_0 \le 1\}$ *and* $\mathcal{X}_t = \{x \in \mathbb{R}^n \mid x^\top Q(t) x \le 1\}$ *if and only if the state transition matrix of system (3) satisfies*

$$\Phi(t,0)^\top Q(t) \Phi(t,0) < R. \tag{8}$$

The state transition matrix approach has been extended to impulsive linear system (5) in [31, 32, 36]. Letting $T \in [t_l, t_{l+1}]$, the solution of the impulsive linear system (5) will be $x(t) = \widehat{\Phi}(t,0)x_0$, $t \in \Omega$, where $\widehat{\Phi}(t,0)$, called the state transition matrix of (5), is a piecewise continuous matrix-valued function with discontinuous right-hand sides at the time instants $t_k, k = 1, 2, \ldots, l$. In detail, when $t \in (0, t_1]$, $\widehat{\Phi}(t,0)$ is the solution of the following matrix differential equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\widehat{\Phi}(t,0) = A(t)\Phi(t,0), \widehat{\Phi}(0,0) = I, \\ &\widehat{\Phi}(t_1^+, 0) = B(t_1)\widehat{\Phi}(t_1, 0). \end{aligned} \tag{9}$$

In the sequel intervals for $k = 1, 2, \ldots, l-1$, $\widehat{\Phi}(t, t_k)$ should satisfy

$$\begin{aligned} &\frac{\partial}{\partial t}\widehat{\Phi}(t, t_k^+) = A(t)\widehat{\Phi}(t, t_k^+), \quad t \in (t_k, t_{k+1}], \\ &\widehat{\Phi}(t_{k+1}^+, t_k^+) = B(t_{k+1})\widehat{\Phi}(t_{k+1}, t_k^+). \end{aligned} \tag{10}$$

In the end, when $t \in (t_l, t_{l+1}]$, we have

$$\frac{\partial}{\partial t}\widehat{\Phi}(t, t_l^+) = A(t)\widehat{\Phi}(t, t_l^+), \quad t \in (t_l, T]. \tag{11}$$

Theorem 2. *Impulsive linear system (5) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same with those in Theorem 1, if and only if for all* $t \in [0, T]$, *the following is satisfied:*

$$\widehat{\Phi}(t,0)^\top Q(t) \widehat{\Phi}(t,0) < R. \tag{12}$$

The conditions in the form of state transition matrices (8) and (12) in Theorems 1 and 2 are valuable for theoretical analysis but hard to apply due to the high computational difficulty, particularly for the time-varying case.

To obtain computational conditions for finite-time stability, some Lyapunov-like functions are needed to establish conditions in the form of linear matrix equalities or Lyapunov matrix inequalities. In some early work such as [3], it concludes that system (1) is finite-time stable with respect to $(\Omega, \mathcal{S}_\alpha, \mathcal{S}_\beta)$ if and only if there exists a real-valued Lipschitz function $V(t,x)$, continuous on $\Omega \times \overline{\mathcal{S}}_\beta$, and a real-valued integrable function $\varphi(t)$ such that, for $t \in \Omega$, we have $\dot{V}_0(t,x) \le \varphi(t)$ for all $x \in [\mathcal{S}_\beta - \overline{\mathcal{S}}_\alpha]$ and $\int_{t_1}^{t_2} \varphi(t)\mathrm{d}t < V_{\min}^{\beta}(t_2) - V_{\max}^{\alpha}(t_1)$ for $t_1 < t_2$ and $t_1, t_2 \in \Omega$. A piecewise continuous Lyapunov-like function is the most common one in the literature, and relevant results will be presented in the following subsection.

3.2. Piecewise Continuous Lyapunov-Like Functions. To obtain computationally tractable finite-time stability conditions, we choose a quadratic piecewise continuous Lyapunov-like function $V(t,x) = x^\top P(t) x$ and establish the following conditions containing coupled differential Lyapunov matrix equations and differential linear matrix inequalities.

Theorem 3 (see [7, 30]). *System (3) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same with those in Theorem 1, if and only if for all* $t \in \Omega$, *there exists a symmetric piecewise differentiable matrix-valued function* $P(\cdot)$ *such that the following conditions involving the differential matrix equation with boundary conditions are satisfied:*

$$\begin{aligned} &\dot{P}(\tau) = -A(\tau)^\top P(\tau) - P(\tau)A(\tau) - \varepsilon I, \quad \tau \in [0, t], \ \varepsilon > 0, \\ &P(t) = Q(t), \quad P(0) < R. \end{aligned} \tag{13}$$

We see that Theorem 3 provides a necessary and sufficient condition for finite-stability of system (3). However, it is not practicable to verify the differential matrix equation in (13) for every $\tau \in [0, t]$, and hence, Theorem 3 is not suitable for the computational purpose. Using Theorem 3, we can obtain a sufficient condition with computational tractability for finite-stability of system (3).

Theorem 4 (see [7, 30]). *System (3) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same as those in Theorem 1, if and only if for all* $t \in \Omega$ *there exists a symmetric piecewise differentiable matrix-valued function* $P(\cdot)$ *such that the following conditions involving the differential matrix equation with boundary conditions are satisfied:*

$$\begin{aligned} &\dot{P}(t) = -A(t)^\top P(t) - P(t)A(t) - \varepsilon I, \quad \varepsilon > 0, \\ &P(t) = Q(t), \quad P(0) < R. \end{aligned} \tag{14}$$

Nowadays, computational tools in the convex optimization framework such as linear matrix inequalities or differential linear matrix inequalities are very efficient, and we can obtain the following necessary and sufficient conditions including differential linear matrix inequalities, equivalent to (13):

$$\begin{aligned} &\dot{P}(\tau) < -A(\tau)^\top P(\tau) - P(\tau)A(\tau), \\ &Q(t) \le P(t), \quad P(0) < R. \end{aligned} \tag{15}$$

The piecewise continuous Lyapunov-like function is also applied to the finite-time stability problem of impulsive linear systems. For linear time-varying systems with time-dependent impulses, we have the following finite-time stability results.

Theorem 5 (see [36]). *System (5) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same with those in Theorem 1, if and only if for all* $t \in \Omega$ *there exists a piecewise differentiable positive definite matrix-valued*

function $P(\cdot)$ such that the following conditions involving differential linear matrix inequalities are satisfied:

$$\begin{aligned}&\dot{P}(t)+A(t)^{\top}P(t)+P(t)A(t)<0, \quad t\notin\mathscr{J},\\&P(t_k)>B(t_k)^{\top}P(t_k^{+})B(t_k), \quad k=1,2,\ldots,\\&P(t)\geq Q(t), \quad P(0)<R.\end{aligned}\tag{16}$$

Moreover, it is finite-time stable with respect to $(\Omega,\mathscr{X}_0,\mathscr{X}_t)$ if and only if there exists a piecewise continuous positive definite matrix-valued solution $Z(\cdot)$: $\Omega\mapsto\mathbb{R}^{n\times n}$ such that the following conditions involving differential/difference Lyapunov equations are satisfied:

$$\begin{aligned}&\dot{Z}(t)-A(t)Z(t)-Z(t)A(t)^{\top}=0, \quad t\notin\mathscr{J},\\&Z^{+}(t_k)=B(t_k)Z^{-}(t_k)B(t_k)^{\top},\\&t\in\mathscr{J},\ Z(0)=R^{-1},\\&G(t)W(t)G^{\top}(t)<I, \quad \forall t\in\Omega,\end{aligned}\tag{17}$$

where $G(\cdot)$ is a nonsingular matrix-valued function satisfying $Q(t)=G^{\top}(t)G(t)$ in Ω.

As for a linear time-varying system with state-dependent impulses (6), we have the following theorem to provide a sufficient condition for its finite-time stability.

Theorem 6 (see [35]). *System (6) is finite-time stable with respect to $(\Omega,\mathscr{X}_0,\mathscr{X}_t)$, where the sets $\mathscr{X}_0$ and $\mathscr{X}_t$ are the same with those in Theorem 1, if and only if for all $t\in\Omega$ there exists a piecewise differentiable positive definite matrix-valued function $P(\cdot)$ such that the following conditions involving linear differential/difference matrix inequalities are satisfied:*

$$\begin{aligned}&\dot{P}(t)+A(t)^{\top}P(t)+P(t)A(t)<0, \quad x(t)\in\mathbb{R}^n\setminus\bigcup_{k=1}^{N}\mathscr{S}_k,\\&x^{\top}\left(B(t)^{\top}P(t)B(t)-P(t)\right)x<0, \quad x(t)\in\mathscr{S}_k,\ k=1,\ldots,N,\\&P(t)\geq G(t), \quad P(0)<R.\end{aligned}\tag{18}$$

When the initial set $\mathscr{X}_0$ and the time-varying set $\mathscr{X}_t$ are two given polytopes and a quadratic function $V(x)=x^{\top}Px$ is chosen, we have following sufficient conditions of finite-time stability for a quadratic system:

$$\dot{x}=Ax+\left[x^{\top}C_1^{\top}x,x^{\top}C_2^{\top}x,\ldots,x^{\top}C_n^{\top}x\right]^{\top},\tag{19}$$

where $C_i\in\mathbb{R}^{n\times n}$, $i=1,\ldots,n$.

Theorem 7 (see [34]). *System (19) is finite-time stable with respect to $(\Omega,\mathscr{X}_0,\mathscr{X}_t)$, where the sets $\mathscr{X}_0$ and $\mathscr{X}_t$ are two given polytopes, if there exists a positive definite symmetric matrix $P\in\mathbb{R}^{n\times n}$ such that*

$$\begin{aligned}&\lambda_{\max}(P)\max_i\left\|x_{\mathscr{X}_0}^{(i)}\right\|^2e^{\alpha T}\\&\begin{pmatrix}1 & a_k^{\top}\\ a_k & P\end{pmatrix}\\&\left[A^{\top}+\left(B_1^{\top}x_{\mathscr{X}_t}^{(i)}B_2^{\top}x_{\mathscr{X}_t}^{(i)}\ldots B_n^{\top}x_{\mathscr{X}_t}^{(i)}\right)\right]P+P\left[A+\left(B_1^{\top}x_{\mathscr{X}_t}^{(i)}B_2^{\top}x_{\mathscr{X}_t}^{(i)}\ldots B_n^{\top}x_{\mathscr{X}_t}^{(i)}\right)^{\top}\right]-\alpha P\leq 0.\end{aligned}\tag{20}$$

The initial set $\mathscr{X}_0$ and the time-varying set $\mathscr{X}_t$ can also be piecewise quadratic domains over conical partitions $\mathscr{P}_0=\{U_1^0,U_2^0,\ldots,U_u^0\}$ and $\mathscr{P}_t=\{U_1^t,U_2^t,\ldots,U_v^t\}$, say $\mathscr{X}_0=\{x\in\mathbb{R}^n|x_0^{\top}R_ix_0\leq 1,\ x_0\in U_i^0,\ i=1,2,\ldots,u\}$ and $\mathscr{X}_t=\{x^{\top}Q_i(t)x\leq 1,\ x\in U_i^t,\ i=1,2,\ldots,v\}$ [37, 38]. In such cases, we choose a time-varying piecewise quadratic Lyapunov-like function defined over the abovementioned conical partition as

$$V_{\mathscr{X}_t}(t,x)=x^{\top}P_i(t)x, \quad \forall x\in S_i \text{ with } i=1,\ldots,v,\tag{21}$$

where $P_i\in\mathbb{R}^{n\times n}$, $i=1,\ldots,v$, are symmetric matrices. Then, a sufficient condition for the finite-time stability of linear-varying system (1) can be presented as follows.

Theorem 8 (see [37]). *System (3) is finite-time stable with respect to $(\Omega,\mathscr{X}_0,\mathscr{X}_t)$, where the sets $\mathscr{X}_0$ and $\mathscr{X}_t$ are the given piecewise quadratic domains, if there exist positive definite symmetric matrices $P_i\in\mathbb{R}^{n\times n}$ such that*

$$\begin{aligned}&x^{\top}\left(\dot{P}_i(t)+A(t)^{\top}P_i(t)+P_i(t)A(t)\right)x<0,\\&x^{\top}\left(P_i(t)-Q_i(t)\right)x\geq 0,\\&x^{\top}\left(P_i(0)-R_i\right)x<0.\end{aligned}\tag{22}$$

For $t\in\Omega$ and $x\in S_i$ with $i=1,\ldots,v$.

The sufficient conditions in (22) are not applicable due to the infinite number of matrix inequalities. Applying S-procedure arguments and considering the conical partition, a computationally tractable sufficient condition is also obtained in [37].

Theorem 9 (see [37]). *System (3) is finite-time stable with respect to $(\Omega,\mathscr{X}_0,\mathscr{X}_t)$, where the sets $\mathscr{X}_0$ and $\mathscr{X}_t$ are the given piecewise quadratic domains, if there exist positive numbers $b_{i,k}$, positive real-valued functions $c_{i,k}(t)$, $z_{i,k}(t)$ and matrices $H_{i,k}$, $i=1,\ldots,v$ and $k=1,\ldots,s$, and positive definite symmetric matrices $P_i\in\mathbb{R}^{n\times n}$ such that $x^{\top}H_{i,k}x\leq 0$, $\forall x\in S_i$ and there exist positive piecewise continuously differentiable matrix-valued functions $P_i(t)\in\mathbb{R}^{n\times n}$, such that the following*

conditions involving differential linear matrix inequalities and linear matrix inequalities are satisfied:

$$\begin{aligned}
&\dot{P}_i(t) + A(t)^\top P_i(t) + P_i(t)A(t) - \sum_{k=1}^{s} c_{i,k}(t)H_{i,k} < 0,\\
&P_i(t) - Q_i(t) + \sum_{k=1}^{s} z_{i,k}(t)H_{i,k} \geq 0,\\
&P_i(0) - R_i - \sum_{k=1}^{s} b_{i,k}H_{i,k} < 0,\\
&\widehat{x}_k^\top P_i(t)\widehat{x}_k = \widehat{x}_k^\top P_j(t)\widehat{x}_k, \forall \widehat{x}_k \in \operatorname{extr}\{S_i \cap S_j\},\\
&\widehat{x}_h^\top P_i(t)\widehat{x}_k = \widehat{x}_h^\top P_j(t)\widehat{x}_k, \forall \widehat{x}_h, \widehat{x}_k \in \operatorname{extr}\{S_i \cap S_j\}.
\end{aligned} \tag{23}$$

3.3. Converse Lyapunov-Like Theorem. In [9], a converse Lyapunov-like theorem is established for finite-time uniformly stable continuous-time nonautonomous system (1). It provides the characterization of finite-time stability with regards to the existence of Lyapunov-like functions.

Theorem 10 (see [9]). *If system (1) is uniformly stable with respect to* $(\Omega, \mathcal{S}_\alpha, \mathcal{S}_\beta)$, $\alpha < \beta$, *then there exists a real-valued Lyapunov-like function* $V(t,x)$ *satisfying*

$$\dot{V}_0(t,x) \leq \varphi(t), \tag{24}$$

for all $x \in [\mathcal{S}_\beta - \overline{\mathcal{S}}_\alpha]$, *and*

$$\int_{t_1}^{t_2} \varphi(t)\mathrm{d}t < V_{\min}^{\beta}(t_2) - V_{\max}^{\alpha}(t_1), \tag{25}$$

for $t_1 < t_2$ *and* $t_1, t_2 \in \Omega$.

Moreover, two necessary and sufficient conditions for the existence of a Lyapunov-like function $V(t,x)$ for finite-time stability are given as follows.

Theorem 11 (see [9]). *System (1) is finite-time stable with respect to* $(\Omega, \mathcal{S}_\alpha, \mathcal{S}_\beta)$, $\alpha < \beta$, *if and only if there exist a Lipschitz continuous real-valued Lyapunov-like function* $V(t,x)$ *and a continuous real-valued function* $\mu(\cdot)$ *such that the following conditions are satisfied: (i)* $V(t,x) \geq \mu(\|x\|)$ *for all, (ii)* $\dot{V}(t,x) \leq 0$ *for all* $x \in [\mathcal{S}_\beta - \overline{\mathcal{S}}_\alpha]$, *and (iii)* $V_{\max}^{\alpha}(t) < \mu(\beta)$.

Theorem 12 (see [39]). *System (1) is finite-time stable with respect to* $(\Omega, \mathcal{S}_\alpha, \mathcal{S}_\beta)$, $\alpha < \beta$, *if and only if there exist a continuous real-valued Lyapunov-like function* $V(t,x)$ *such that*

$$\dot{V}(t,x) \leq 0, \tag{26}$$

for all $x \in \overline{\mathcal{S}}_\beta$, *and*

$$V_{\min}^{\beta}(t_2) < V_{\max}^{\alpha}(t_1), \tag{27}$$

for all $t_2 > t_1$ *and* $\delta < \alpha$ *with* $t_1, t_2 \in \Omega$.

3.4. Miscellaneous Issues. Time delays are often encountered in many practical systems such as chemical processes, electric circuits, and networked systems, leading to unsatisfactory system behaviours and even instability. So, various stability problems for delayed systems have attracted much attention to lots of researchers. Among them, finite-time stability analysis has been of particular interest bringing forth many papers such as [40–42].

A linear time-invariant delayed system can be represented by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad h > 0, \tag{28}$$

with an associated initial state function

$$x(t) = \psi(t), \quad -h \leq t \leq 0. \tag{29}$$

To proceed, we need the following definitions.

Definition 3. System (28) associated with initial condition (29) is said to be finite-time stable with respect to $(\zeta(\cdot), \beta, h)$ if $\psi(t)^\top \psi(t) < \zeta(t)$, $\forall t \in [-\tau, 0]$, which implies $x(t)^\top x(t) < \beta$, $\forall t \in \Omega$, where $\zeta(\cdot)$ is a positive scalar-valued function satisfying $\zeta(t) \leq \alpha$ for $-h \leq t \leq 0$, and $\beta > \alpha > 0$.

Definition 4. System (28) associated with initial condition (29) is said to be finite-time stable with respect to (Ω, α, β) if $(\sup_{-h \leq \theta \leq 0} \|\psi(\theta)\|)^2 < \alpha$ implies $\|x(t)\|^2 < \beta$, $\forall t \in \Omega$.

Next, we introduce two sufficient conditions for finite-time stability of linear delayed system (28) in the following two theorems.

Theorem 13 (see [41]). *System (28) associated with the initial condition (29) is finite-time stable with respect to* (α, β, h) *if for all* $t \in \Omega$, *we have*

$$\|\Phi(t)\| < \frac{\sqrt{(\beta/\alpha)}}{1 + \|A_1\|}, \tag{30}$$

where $\Phi(t)$ *is the fundamental matrix of linear delayed system (28).*

Theorem 14 (see [41]). *System (28) associated with initial condition (29) is finite-time stable with respect to* (Ω, α, β) *if*

$$(1 + \sigma_m t)^2 e^{2\sigma_m t} < \frac{\beta}{\alpha}, \tag{31}$$

where

$$\sigma_m = \sigma_{\max}(A_0) + \sigma_{\max}(A_1), \tag{32}$$

with $\sigma_m(\cdot)$ *being the largest singular value of the corresponding matrix.*

In [40], the authors construct a delay-dependent Lyapunov-like function

$$V(t) = \left(x(t) + \int_0^h J(\theta)x(t-\theta)\mathrm{d}\theta\right)^\top \left(x(t) + \int_0^h J(\theta)x(t-\theta)\mathrm{d}\theta\right), \tag{33}$$

where $J(t) \in \mathbb{R}^{n \times n}$ is a differentiable matrix-valued function on $[0, h]$ such that the following differential matrix equation is satisfied:

$$\dot{J}(\theta) = (A_0 + J(0))J(\theta), \quad \theta \in [0, h], \tag{34}$$

with the initial condition $J(h) = A_1$. Then, the following result based on Lyapunov function (33) can be given.

Theorem 15 (see [40]). *System (28) associated with initial condition (29) is finite-time stable with respect to (α, β, h) if there exists a positive real number γ such that the following conditions are satisfied:*

$$\begin{aligned}
&x(t-\theta)^{\top} x(t-\theta) \\
&\quad \cdot (1+h)(1+\psi)\left(1 - \gamma\psi - \frac{ah}{\gamma}\right)^{-1} e^{\lambda_{\max}(R^{\top}+R)T} < \frac{\beta}{\alpha}, \\
&\gamma \in (\max\{\gamma_1, 0\}, \gamma_2), \gamma_{1,2} = \frac{1 \pm \sqrt{1 - 4\psi h a}}{2\psi}, \quad 4\psi h a < 1, \\
&\psi = \lambda_{\max}\left(J(0)J^{\top}(0)\right) \frac{e^{2\mu_1(\overline{A}_0)h} - 1}{2\mu_1(\overline{A}_0)}, \overline{A}_0 = A_0 + J(0),
\end{aligned} \tag{35}$$

where $\mu_1(\cdot)$ is a matrix measure of the given matrix and $J(0)$ is the solution of the following transcendental matrix equation:

$$e^{A_0 + J(0)h} J(0) = A_1. \tag{36}$$

Next, consider a singular linear delayed system

$$E\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad h > 0, \tag{37}$$

where $E \in \mathbb{R}^{n \times n}$ is a singular matrix with rank $r < n$. There exist two nonsingular matrices $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and G such that G such that $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ = MEG. Let

$$\begin{aligned}
&MA_0G = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} 0 \\ M_2 \end{pmatrix} A_0 G = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}, MA_2G = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \\
&W_1 = PA_0 + A_0^{\top}P^{\top} + Q_1 + Q_2\overline{M}A_0 + A_0^{\top}\overline{M}^{\top}Q_2^{\top} - \eta PE, W_2 = PA_1 + Q_2\overline{M}A_1, \\
&\overline{M} = \begin{pmatrix} 0 \\ M_2 \end{pmatrix}, G^{\top}PM^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, M^{-1^{\top}}M^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ * & R_{22} \end{pmatrix}, \\
&\alpha_1 = \frac{\lambda_{\min}(P_{11})}{\lambda_{\max}(R_{11})}, \alpha_2 = \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^{\top}G)} + h\lambda_{\max}(Q_1), \alpha_3 = \frac{\alpha_2\alpha}{\alpha_1}, \\
&\alpha_4 = \sum_{i=0}^{[\top/h]-1} \left\| A_{22}^{-1}D_{22} \right\|^i, \eta = \left\| A_{22}^{-1}D_{21} + A_{22}^{-1}D_{22} \right\|, \alpha_6 = \lambda_{\max}(G^{\top}G).
\end{aligned} \tag{38}$$

Then, a sufficient condition for the finite-time stability of singular linear delayed system (37) was established in [42].

Theorem 16 (see [42]). *System (37) associated with initial condition (29) is finite-time stable with respect to (α, β, h) if there exists a positive number γ, a symmetric positive definite matrix $Q_1 \in \mathbb{R}^{n \times n}$, a nonsingular matrix $P \in \mathbb{R}^{n \times n}$, and a matrix $Q_2 \in \mathbb{R}^{n \times n}$ such that*

$$\begin{aligned}
&PE = E^{\top}P \geq 0, \begin{pmatrix} W_1 & W_2 \\ * & -Q_1 \end{pmatrix} < 0, \\
&e^{\gamma T}\alpha_3 + \left(\alpha_5 + \eta\alpha_4\sqrt{e^{\gamma T}\alpha_3}\right)^2 \leq \frac{\beta}{\alpha_6}.
\end{aligned} \tag{39}$$

More recently, the Lyapunov–Razumikhin approach is extended to finite-time stability for a nonlinear delayed system

$$\dot{x}(t) = f(t, x(t-h)), \tag{40}$$

in [43]. Sufficient conditions can be illustrated through the following theorem.

Theorem 17 (see [43]). *System (40) associated with initial condition (29) is finite-time stable with respect to (α, β, h) if there exists positive scalars $\alpha, \beta, \eta, \sigma, T$ with $\eta < \alpha < \beta$ and $\sigma \in (0, T)$, integrable real-valued function $c(\cdot)$: $\mathbb{R}^+ \mapsto \mathbb{R}$, class $\mathcal{K}$ functions γ_1, γ_2, and a differentiable function V: $[-h, T] \times \mathbb{R}^n \mapsto \mathbb{R}^+$ such that (i) $\gamma_1(|x|) \leq V(t, x) \leq \gamma_2(|x|), \forall (t, x) \in [-h, T] \times \mathbb{R}^n$, (ii) $\dot{V}(t, \psi(0)) \leq c(t)V(t,$*

$\psi(0))$, *whenever* $V(t+s,\psi(s)) \le \Theta(t,s)V(t,\psi(0))$ *for all* $t \in [0,T]$, $s \in [-h,0]$, *where*

$$\Theta(t,s) = \exp\left(-\int_{\max\{t+s,0\}}^{t} c(u)\mathrm{d}u\right), \tag{41}$$

and (iii)

$$\int_0^t c(u)\mathrm{d}u \le \ln\frac{\gamma_1(\eta)}{\gamma_2(\sqrt{\alpha})}, \quad \forall t \in [0,T]. \tag{42}$$

Besides time delays, uncertainty is another important phenomenon commonly encountered in practical systems. The existence of uncertainty causes the poor performance and even instability. The finite-stability concept has been extended to uncertain linear systems [44–46]. The uncertainty can be expressed as norm-bounded uncertainty and structured uncertainty. Consider an uncertain linear system

$$\dot{x} = \left(A(t) + F_1(t)(I - \Delta(t)H(t))^{-1}\Delta(t)F_2(t)\right)x(t), \tag{43}$$

where $x(t) \in \mathbb{R}^n$, $\Delta(\cdot)$ is a norm-bounded uncertainty function such as $\|\Delta(t)\| \le 1$ and $F_1(\cdot), F_2(\cdot)$ are known matrices of appropriate dimensions. The following theorem will present a necessary and sufficient condition for finite-time stability of system (43).

Theorem 18 (see [46]). *System (43) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same as those in Theorem 1, if and only if for all* $t \in \Omega$, *there exist a piecewise continuous function* γ *and a symmetric piecewise differentiable matrix-valued function* $P(\cdot)$ *such that the following conditions involving differential linear matrix inequalities are satisfied:*

$$\begin{pmatrix} \dot{P}(\tau) + A^\top(\tau)P(\tau) + P(\tau)A(\tau) + \gamma(\tau)F_2^\top(\tau)F_2(\tau) & P(\tau)F_1(\tau) + \gamma(\tau)F_2^\top(\tau)H(\tau) \\ F_1^\top(\tau)P(\tau) + \gamma(\tau)H(\tau)F_2(\tau) & -\gamma(\tau)\left(I - H^\top(\tau)H(\tau)\right) \end{pmatrix} < 0,$$
$$P(t) \ge Q(t), P(0) < R, \gamma(t) > 0, \quad \tau \in [0,t]. \tag{44}$$

Similar to Theorem 3, necessary and sufficient conditions in Theorem 18 are not computationally tractable. For the computational purposes, we can see the following sufficient condition for finite-time stability of system (43).

Theorem 19 (see [46]). *System (43) is finite-time stable with respect to* $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, *where the sets* $\mathcal{X}_0$ *and* $\mathcal{X}_t$ *are the same as those in Theorem 1, if and only if for all* $t \in \Omega$, *there exist a piecewise continuous function* γ *and a symmetric piecewise differentiable matrix-valued function* $P(\cdot)$ *such that the following conditions involving differential linear matrix inequalities are satisfied:*

$$\begin{pmatrix} \dot{P}(t) + A^\top(t)P(t) + P(t)A(t) + \gamma(t)F_2^\top(t)F_2(t) & P(t)F_1(t) + \gamma(t)F_2^\top(t)H(t) \\ F_1^\top(t)P(t) + \gamma(t)H(t)F_2(t) & -\gamma(t)\left(I - H^\top(t)H(t)\right) \end{pmatrix} < 0,$$
$$P(t) \ge Q(t), P(0) < R, \gamma(t) > 0, \quad t \in \Omega. \tag{45}$$

Robustness analysis for a linear delayed system with structured uncertainty was conducted in [45], where the uncertain system is described by

$$\dot{x}(t) = \left(A_0 + D_0F(t)E_0\right)x(t) + \left(A_1 + D_1F(t)E_1\right)x(t-h), \tag{46}$$

where D_0, E_0, D_1, E_1 are known matrices of appropriate dimension, and $F(t)$ is the uncertain time-varying matrix with $F(t)F(t)^\top \le I$, $\forall t \in \Omega$.

Theorem 20. (see [45]). *System (46) associated with initial condition (29) is finite-time stable with respect to* (α, β, h) *if*

there exists positive real numbers $\gamma, \delta, \beta_0, \beta_1, \beta_2, \beta_3$ and symmetric positive definite matrices P_1 and P_2 such that the following conditions are satisfied:

$$\begin{bmatrix} \Omega_{11} & P_1A_1 & P_1D_0 & P_1D_1 \\ * & -P_2 + \delta E_1^\top E_1 & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\delta I \end{bmatrix} < 0,$$
$$\Omega_{11} = A_0^\top P_1 + P_1A_0 + P_2 - \beta_0 P_1 + \gamma E_0^\top E_0, \quad \beta_1 I < P_1 < \beta_2 I,\ 0 < P_2 < \beta_3 I, \tag{47}$$
$$\begin{bmatrix} -\beta e^{-\beta_0 T}\beta_1 & \sqrt{\alpha}\beta_2 & \sqrt{\alpha h}\beta_3 \\ * & -\beta_2 & 0 \\ * & * & -\beta_3 \end{bmatrix} < 0.$$

If uncertain linear system (43) is affected by finite impulses (4), the system state will undergo abrupt changes at discrete time instants, which leads to more difficulty to analyze its stability performance. The following theorem from [46] gives a sufficient condition for finite-time stability of the linear time-varying system with both time-dependent impulses and uncertainty. More complex cases with a norm-bounded uncertainty on the impulsive matrix-valued function and state-dependent impulses were also provided in [46] as well.

Theorem 21 (see [44]). *System (3) is finite-time stable with respect to $(\Omega, \mathcal{X}_0, \mathcal{X}_t)$, where the sets $\mathcal{X}_0$ and $\mathcal{X}_t$ are the same as those in Theorem 1, if and only if for all $t \in \Omega$, there exists positive real number γ and a symmetric piecewise differentiable matrix-valued function $P(\cdot)$ such that the following conditions are satisfied:*

$$\begin{pmatrix} \dot{P}(t) + A^\top(t)P(t) + P(t)A(t) + \gamma F_2^\top F_2 & P(t)F_1 + \gamma F_2^\top H \\ F_1^\top P(t) + \gamma H^\top F_2 & -\gamma(I - H^\top H) \end{pmatrix} < 0, \quad t \notin \mathcal{J},$$
$$B(t)^\top P^+(t)B(t) - P(t) \le 0, \quad t \in \mathcal{J}, \tag{48}$$

All results mentioned above have illustrated the finite-time stability conditions in a quantitative sense, and we continue to introduce more results in a qualitative sense.

4. Finite-Time Stability in a Qualitative Sense

It is well known that a radially unbounded positive definite function $V: \mathcal{C} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^+$ with the property $\dot{V}(x) < 0$ is a Lyapunov function. Lyapunov's second method demonstrates that the existence of the Lyapunov function is also equivalent to the asymptotical stability of system (1), which provides the foundation for the following necessary and sufficient conditions for finite-time stability results.

Theorem 22 (see [27, 47]). *Consider an autonomous nonlinear system*

$$\dot{V}(x) \le -cV(x)^\eta, \quad \text{for all } x \in \mathcal{C}. \tag{49}$$

System (49) is finite-time stable if and only if there exists a smooth Lyapunov function (equivalently, all smooth Lyapunov functions) $V: \mathcal{C} \longrightarrow \mathbb{R}^+$ such that for all $x \in \mathcal{C}$,

$$T(x) = \int_{V(x)}^0 \frac{ds}{\dot{V}(x(\theta_x(s), x))} < +\infty, \tag{50}$$

where the map θ_x is the inverse of $t \mapsto V(x(t, x))$.

Theorem 23 (see [27, 47]). *System (49) is finite-time stable with a continuous settling-time function at the origin if and only if there exist a scalar $\eta \in (0, 1)$, a positive scalar c, and a smooth Lyapunov function $V: \mathcal{C} \longrightarrow \mathbb{R}^+$ such that*

$$\dot{V}(x) \le -cV(x)^\eta, \quad \text{for all } x \in \mathcal{C}. \tag{51}$$

Moreover, the settling-time function $T(x)$ should satisfy

$$T(x) \le \frac{V(x)^{1-\eta}}{c(1-\eta)}. \tag{52}$$

A converse Lyapunov theorem was obtained for finite-time stability of nonlinear system (49) in [18]. As for the case that settling-time is continuous, we have the following converse theorem.

Theorem 24 (see [18]). *If system (49) is finite-time stable with a continuous settling-time function at the origin and $\eta \in (0,1)$, then there exists a continuous function $V: \mathscr{C} \longrightarrow \mathbb{R}^+$ such that*

$$\dot{V}(x) \leq -cV(x)^{\eta}, \quad \text{for all } x \in \mathscr{C}. \tag{53}$$

The abovementioned Lyapunov-based methods for analysis of finite-time stability may not be suitable for constructive design. Recently, an implicit Lyapunov function method to solve an algebraic equation was derived in [48], which provides a design method for a robust controller for the closed-loop systems to handle exogenous disturbances. The implicit Lyapunov function theorem only verifies stability conditions in an implicit way and does not need to solve the equation.

Theorem 25 (see [48]). *System (1) is finite-time stable with a settling-time function $T(x_0) \leq (V_0^{\mu}/(c\mu))$, where $Q(V_0, x_0) = 0$ if there exists a continuously differentiable function $Z: \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that for any $x \in \mathscr{C}$, there exists a radially unbounded function $V \in \mathbb{R}^+$ such that*

$$\begin{aligned} &Z(V,x) = 0, \frac{\partial Z(V,x)}{\partial V} < 0, \\ &\sup_{t \in \mathbb{R}^+} \frac{\partial Z(V,x)}{\partial x} f < 0, \end{aligned} \tag{54}$$

for all $(V,x) \in \Omega$, where $\Omega = \{(V,x) \in \mathbb{R}^+ \times \mathbb{R}^n : Z(V,x) = 0\}$ and $\lim_{x \longrightarrow 0} V = 0$.

Recently, the notion of finite-time stability for nonlinear autonomous system (49) was extended to nonautonomous nonlinear system (1). These Lyapunov and converse Lyapunov results are derived and introduced in the following theorems.

Theorem 26. (see [17]). *System (1) is finite-time stable if there exist a scalar $\eta \in (0,1)$, a positive function c(t), a class $\mathscr{K}$ function $\gamma_1(\cdot)$, and a continuously differentiable function $V: \mathbb{R}^+ \times \mathscr{C} \longrightarrow \mathbb{R}^+$ such that $V(t,0) = 0$, $V(t,x) \geq \gamma_1(\cdot)$, and*

$$\dot{V}(x) \leq -c(t)V(x)^{\eta}, \quad \text{for all } x \in \mathscr{C}. \tag{55}$$

Moreover, for the case $\mathscr{C} = \mathbb{R}^n$, then system (1) is be globally finite-time stable. If there exists a class $\mathscr{K}$ function $\gamma_2(\cdot)$ such that $V(t,x) \leq \gamma_2(\cdot)$, then system (1) is uniformly finite-time stable.

Theorem 27 (see [17]). *Let $\eta \in (0,1)$ and there exists a class $\mathscr{K}$ function $\varphi: [0,r] \longrightarrow \mathbb{R}^+$, where $r > 0$ such that $\mathscr{B}_r(0) \subseteq \mathscr{C}$ and*

$$\|f(t,x)\| \leq \varphi(\|x\|), \quad t \in [0,\infty),\ x \in \mathscr{B}_r(0). \tag{56}$$

If system (1) is uniformly finite-time stable and the settling-time function $T(\cdot,\cdot)$ is jointly continuous at $(t,0)$, $t \geq 0$, then there exist a positive scalar c, a class $\mathscr{K}$ function $\gamma(\cdot)$, and a continuously differentiable function $V: \mathbb{R}^+ \times \mathscr{C} \longrightarrow \mathbb{R}^+$ such that $V(t,0) = 0$, $V(t,x) \geq \gamma(\cdot)$, and

$$\dot{V}(x) \leq -cV(x)^{\eta}, \quad \text{for all } x \in \mathscr{C}. \tag{57}$$

5. Conclusions

This paper has overviewed the fundamental results of the finite-time stability analysis of dynamical systems. The concepts of finite-time stability are classified into those in the quantitative and qualitative senses. Finite-time stability in a quantitative sense is firstly investigated. Then, finite-time stability results in a qualitative sense are outlined. This review paper is far from complete due to our limitations and nonawareness. We hope that this paper can be a useful resource for practitioners, researchers, and graduate students working in this field.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported in part by the Australian Research Council under Grant DP160102819.

References

[1] G. Kamenkov, "On stability of motion over a finite interval of time," *Journal of Applied Mathematics and Mechanics*, vol. 17, pp. 529–540, 2019, in Russian.

[2] L. Weiss and E. Infante, "Finite time stability under perturbing forces and on product spaces," *IEEE Transactions on Automatic Control*, vol. 12, no. 1, pp. 54–59, 1967.

[3] L. Weiss, "On uniform and nonuniform finite-time stability," *IEEE Transactions on Automatic Control*, vol. 14, no. 3, pp. 313-314, 1969.

[4] A. A. Kayande and J. S W. Wong, "Finite time stability and comparison principles," in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 64, pp. 749–756, Cambridge University Press, Cambridge, England, 1968.

[5] P. Dorato, "Short-time stability in linear time-varying systems," Technical Report, Polytechnic Inst Of Brooklyn Ny Microwave Research Inst, New York City, NY, USA, 1961.

[6] F. Amato, M. Ariola, C. Abdallah, and C. Cosentino, "Application of finite-time stability concepts to the control of ATM networks," in *Proceedings of the Annual Allerton Conference on Communication Control and Computing*, vol. 40, pp. 1071–1079, 2002.

[7] F. Amato, R. Ambrosino, M. Ariola, C. Cosentino, G. De Tommasi et al., *Finite-time Stability and Control*, Springer, Berlin, Germany, 2014.

[8] Y. Orlov, "Finite time stability and quasihomogeneous control synthesis of uncertain switched systems with application to underactuated manipulators," in *Proceedings of the 44th IEEE*

Conference on Decision and Control, pp. 4566–4571, Seville, Spain, December 2005.

[9] L. Weiss, "Converse theorems for finite time stability," *SIAM Journal on Applied Mathematics*, vol. 16, no. 6, pp. 1319–1324, 1968.

[10] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Lodhi, New Delhi, 1994.

[11] Z. Gajic and M. T. J. Qureshi, *Lyapunov Matrix Equation in System Stability and Control*, Courier Corporation, Chelmsford, MA, USA, 2008.

[12] A. Ohara and Y. Sasaki, "On solvability and numerical solutions of parameter-dependent differential matrix inequality,"vol. 4, pp. 3593-3594, in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 4, IEEE, Orlando, FL, USA, December 2001.

[13] H. Du, X. Lin, and S. Li, "Finite-time stability and stabilization of switched linear systems," in *Proceedings of the 48th IEEE Conference on Decision and Control (CDC)*, pp. 1938–1943, Shanghai, China, 2009.

[14] X. Li, X. Lin, S. Li, and Y. Zou, "Finite-time stability of switched nonlinear systems with finite-time unstable subsystems," *Journal of the Franklin Institute*, vol. 352, no. 3, pp. 1192–1214, 2015.

[15] R. Kuiava, B. K. Matos, and G. R. Razente, "Finite-time stability of a class of continuous-time non-homogeneous switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 26, pp. 101–114, 2017.

[16] Z. Yan, W. Zhang, and G. Zhang, "Finite-time stability and stabilization of Ito stochastic systems with Markovian switching: mode-dependent parameter approach," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2428–2433, 2014.

[17] W. M. Haddad, S. G. Nersesov, and L. Du, "Finite-time stability for time-varying nonlinear dynamical systems," in *Proceedings of the American Control Conference*, pp. 4135–4139, Seattle, WA, USA, 2008.

[18] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[19] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals, and Systems*, vol. 17, no. 2, pp. 101–127, 2005.

[20] E. Moulay and W. Perruquetti, "Finite time stability of nonlinear systems," in *Proceedings of the 42nd IEEE International Conference on Decision and Control*, pp. 3641–3646, Lahaina, HI, USA, December 2003.

[21] D. Liu and Q. Huang, "Research on finite-time stability of nonlinear switched systems," in *Proceedings of the 25th Chinese Control and Decision Conference (CCDC)*, pp. 1314–1319, Guiyang, China, May 2013.

[22] R. Yang and Y. Wang, "Finite-time stability and stabilization of a class of nonlinear time-delay systems," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 3113–3131, 2012.

[23] H. Yang, B. Jiang, and J. Zhao, "On finite-time stability of cyclic switched nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2201–2206, 2014.

[24] J. Wu, D. Yang, X. He, and X. Li, "Finite-time stability for a class of underactuated systems subject to time-varying disturbance," *Complexity*, vol. 2020, pp. 1–7, Article ID 8704505, 2020.

[25] F. Amato, M. Ariola, M. Carbone, and C. Cosentino, "Finite-time control of linear systems: A survey," in *Current Trends in Nonlinear Systems and Control*, pp. 195–213, Springer, Berlin, Germany, 2006.

[26] S.-H. Ding and S.-H. Li, "A survey for finite-time control problems," *Control and Decision*, vol. 26, no. 2, pp. 161–169, 2011.

[27] E. Moulay and W. Perruquetti, "Finite-time stability and stabilization: state of the art," in *Advances in Variable Structure and Sliding Mode Control*, pp. 23–41, Springer, Berlin, Germany, 2006.

[28] F. Amato, G. De Tommasi, and A. Pironti, *Finite-time Stability: An Input-Output Approach*, John Wiley & Sons, Hoboken, NJ, USA, 2018.

[29] Y. Xia, J. Zhang, K. Lu, and N. Zhou, *Finite Time and Cooperative Control of Flight Vehicles*, Springer, Berlin, Germany, 2019.

[30] F. Amato, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems: Analysis and controller design," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1003–1008, 2010.

[31] F. Amato, R. Ambrosino, M. Ariola, F. Calabrese, and C. Cosentino, "Finite-time stability of linear time-varying systems with jumps: Analysis and controller design," in *Proceedings of the American Control Conference*, pp. 1638–1643, Seattle, WA, USA, September 2008.

[32] F. Amato, R. Ambrosino, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems with jumps," *Automatica*, vol. 45, no. 5, pp. 1354–1358, 2009.

[33] S. Zhao, J. Sun, and L. Liu, "Finite-time stability of linear time-varying singular systems with impulsive effects," *International Journal of Control*, vol. 81, no. 11, pp. 1824–1829, 2008.

[34] F. Amato, C. Cosentino, and A. Merola, "Sufficient conditions for finite-time stability and stabilization of nonlinear quadratic systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 430–434, 2010.

[35] R. Ambrosino, F. Calabrese, C. Cosentino, and G. De Tommasi, "Sufficient conditions for finite-time stability of impulsive dynamical systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 861–865, 2009.

[36] F. Amato, G. De Tommasi, and A. Pironti, "Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems," *Automatica*, vol. 49, no. 8, pp. 2546–2550, 2013.

[37] R. Ambrosino, E. Garone, M. Ariola, and F. Amato, "Piecewise quadratic functions for finite-time stability analysis," in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, pp. 6535–6540, Wailea, HI, USA, December 2012.

[38] J. Huang, J. Li, and H. Xu, "Feedback finite-time stabilization of impulsive linear systems over piecewise quadratic domains," *Complexity*, vol. 2020, 2020.

[39] J. Kaplan, "Converse theorems for finite-time stability and practical stability," *IEEE Transactions on Circuit Theory*, vol. 20, no. 1, pp. 66-67, 1973.

[40] D. L. Debeljkovic, S. B. Stojanovic, and A. M. Jovanovic, "Finite-time stability of continuous time delay systems: Lyapunov-like approach with Jensen's and Coppel's inequality," *Acta Polytechnica Hungarica*, vol. 10, no. 7, pp. 135–150, 2013.

[41] M. P. Lazarevic, D. L. Debeljkovic, Z. L. Nenadic, and S. A. Milinkovic, "Finite-time stability of delayed systems," *IMA Journal of Mathematical Control and Information*, vol. 17, no. 2, pp. 101–109, 2000.

[42] V. N. Phat, N. H. Muoi, and M. V. Bulatov, "Robust finite-time stability of linear differential-algebraic delay equations,"

Linear Algebra and Its Applications, vol. 487, pp. 146–157, 2015.

[43] X. Li, X. Yang, and S. Song, "Lyapunov conditions for finite-time stability of time-varying time-delay systems," *Automatica*, vol. 103, pp. 135–140, 2019.

[44] F. Amato, R. Ambrosino, M. Ariola, and G. De Tommasi, "Robust finite-time stability of impulsive dynamical linear systems subject to norm-bounded uncertainties," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 10, pp. 1080–1092, 2011.

[45] S. B. Stojanovic, D. L. Debeljkovic, and D. S. Antic, "Robust finite-time stability and stabilization of linear uncertain time-delay systems," *Asian Journal of Control*, vol. 15, no. 5, pp. 1548–1554, 2013.

[46] F. Amato, M. Ariola, and C. Cosentino, "Robust finite-time stabilisation of uncertain linear systems," *International Journal of Control*, vol. 84, no. 12, pp. 2117–2127, 2011.

[47] E. Moulay and W. Perruquetti, "Lyapunov-based approach for finite time stability and stabilization," in *Proceedings of the 44th IEEE Conference on Decision and Control*, pp. 4742–4747, Seville, Spain, December 2005.

[48] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: implicit Lyapunov function approach," *Automatica*, vol. 51, pp. 332–340, 2015.