Research Article

Global Attractivity for Lasota–Ważewska-Type System with Patch Structure and Multiple Time-Varying Delays

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This paper aims to study the asymptotic behavior of Lasota–Ważewska-type system with patch structure and multiple time-varying delays. Based on the fluctuation lemma and some differential inequality techniques, we prove that the positive equilibrium is a global attractor of the addressed system with small time delay. Finally, we provide an example to illustrate the feasibility of the theoretical results.

1. Introduction

In 1988, in order to describe the survival of red blood cells in animals, Ważewska–Czyżewska and Lasota in [1] presented the following delayed differential equation model

$$x'(t) = -\delta x(t) + \sum_{j=1}^m p_j e^{-a_j x(t-\tau_j(t))}, \quad (1)$$

where $x(t)$ represents the number of red blood cells at time t , δ denotes the death rate of red blood cells, p_j and a_j are related to the production of red blood cells per unit time, $\tau_j(t)$ represents the time required to produce a red blood cell. Since the model was proposed, there have been a large number of results about the dynamical behaviors for (1) and its modifications (see [2–7] and the references therein) due to their comprehensive practical application background.

As pointed out by Yao in [8], populations usually spread between different patches for survival and development. Recently, many scholars have paid attention to the population models with patch structure and time delays (see [9–16]). As far as we know, fewer works have been done concerning with the effect of time delay on dynamical behaviors of Lasota–Ważewska-type model with patch structure. The purpose of the present paper is to establish some sufficient conditions to

guarantee the global attractivity of the following Lasota–Ważewska-type delay system with patch structure

$$x_i'(t) = -\delta_i x_i(t) + \sum_{j=1, j \neq i}^n d_{ij} x_j(t) + \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i(t-\tau_{ij}(t))}, \quad i \in J = \{1, 2, \dots, n\}, \quad (2)$$

where x_i denotes the number of species in the patch i , generation delay function $\tau_{ij} : \mathbb{R} \rightarrow (0, +\infty)$ is bounded and continuous, d_{ij} represents the dispersal coefficient of the species from patch j to patch i , δ_i , p_{ij} and a_{ij} are all positive. In what follows, we always assume that

$$a_i = \max_{1 \leq j \leq m} a_{ij} > 0, \quad r_i = \max_{1 \leq j \leq m} \sup_{t \in \mathbb{R}} \tau_{ij}(t) > 0, \quad \tau = \max_{1 \leq i \leq n} r_i, \quad (3)$$

$$0 < \delta_i - \sum_{j=1, j \neq i}^n d_{ij} < \sum_{j=1}^m p_{ij} \quad \text{for all } i \in J.$$

For convenience, let $C = \prod_{i=1}^n C([-r_i, 0], \mathbb{R})$ and $C_+ = \prod_{i=1}^n C([-r_i, 0], [0, +\infty))$. If $x_i(t)$ is defined on $[t_0 - r_i, \sigma)$ with $t_0, \sigma \in \mathbb{R}$ and $i \in J$, then we write $x_t \in C$ as $x_t = (x_t^1, x_t^2, \dots, x_t^n)$ where $x_t^i(\theta) = x_i(t + \theta)$ for all $\theta \in [-r_i, 0]$ and $t_0 \leq t < \sigma$ and $i \in J$. Denote $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ as an

admissible solution of (2) with the following admissible initial condition:

$$x_{t_0} = \varphi, \quad \varphi \in C_+, \quad \varphi_i(0) > 0, \quad i \in J. \quad (4)$$

Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x_i(t_0, \varphi)$.

In the following, we further assume that there exists at least one positive constant $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ such that x^* is the positive equilibrium point of (2) satisfying

$$-\delta_i x_i^* + \sum_{j=1, j \neq i}^n d_{ij} x_j^* + \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*} = 0, \quad \text{for all } i \in J. \quad (5)$$

2. Global Attractivity of the Positive Equilibrium Point $(x_1^*, x_2^*, \dots, x_n^*)$

First, we will discuss the properties of the solution $x(t; t_0, \varphi)$ of the system (2) with (4).

Lemma 1. $x(t; t_0, \varphi)$ is positive and bounded on $[t_0, \eta(\varphi))$, and $\eta(\varphi) = +\infty$. Moreover, $\liminf_{t \rightarrow +\infty} x_j(t; t_0, \varphi) > 0$, $j \in J$.

Proof. For simplicity, we denote $x(t; t_0, \varphi)$ by $x(t)$. We first prove that

$$x_i(t) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)), \quad i = 1, 2, \dots, n. \quad (6)$$

Assume by contradiction that there exist $t^* \in [t_0, \eta(\varphi))$ and $k \in J$ such that

$$x_k(t^*) = 0, \quad x_i(t) > 0 \quad \text{for all } t \in [t_0, t^*), \quad i = 1, 2, \dots, n. \quad (7)$$

It follows from (2) that

$$\begin{aligned} 0 &\geq D^- x_k(t^*) \\ &= -\delta_k x_k(t^*) + \sum_{j=1, j \neq k}^n d_{kj} x_j(t^*) + \sum_{j=1}^m p_{kj} e^{-a_{kj} x_k(t^* - \tau_{kj}(t^*))} \\ &> 0. \end{aligned} \quad (8)$$

This contradiction implies that (6) holds. Furthermore, define

$$\begin{aligned} M(t) &= \max \left\{ \xi : \xi \leq t \mid \text{there exists } j_0 \in J \text{ such that } x_{j_0}(\xi) \right. \\ &= \max_{t_0 \leq s \leq t} x_{j_0}(s) = \max_{j \in J} \left\{ \max_{t_0 \leq s \leq t} x_j(s) \right\}. \end{aligned} \quad (9)$$

We claim that $x_i(t)$ is bounded on $[t_0, \eta(\varphi))$, $i = 1, 2, \dots, n$. Otherwise, we have $M(t) \rightarrow \eta(\varphi)$ as $t \rightarrow \eta(\varphi)$. Moreover, we can choose $k \in J$ and $\{l_p\}_{p \geq 1}$ with $\lim_{p \rightarrow +\infty} l_p = \eta(\varphi)$ such that

$$x_k(M(l_p)) = \max_{t_0 \leq s \leq l_p} x_k(s) = \max_{j \in J} \left\{ \max_{t_0 \leq s \leq l_p} x_j(s) \right\} \quad (10)$$

and

$$\lim_{p \rightarrow +\infty} x_k(M(l_p)) = +\infty. \quad (11)$$

According to the definition of $M(l_p)$,

$$\begin{aligned} 0 \leq D^- x_k(M(l_p)) &= -\delta_k x_k(M(l_p)) + \sum_{j=1, j \neq k}^n d_{kj} x_j(M(l_p)) \\ &\quad + \sum_{j=1}^m p_{kj} e^{-a_{kj} x_k(M(l_p) - \tau_{kj}(M(l_p)))}. \end{aligned} \quad (12)$$

Furthermore, we have

$$\delta_k \leq \sum_{j=1, j \neq k}^n d_{kj} \frac{x_j(M(l_p))}{x_k(M(l_p))} + \sum_{j=1}^m \frac{p_{kj}}{x_k(M(l_p))} e^{-a_{kj} x_k(M(l_p) - \tau_{kj}(M(l_p)))}. \quad (13)$$

Letting $p \rightarrow +\infty$ gives us that

$$\delta_k \leq \sum_{j=1, j \neq k}^n d_{kj}, \quad (14)$$

which contradicts with the inequality in (3). This shows that $x_i(t)$ is positive and bounded for all $t \in [t_0, \eta(\varphi))$, $i \in J$. From Theorem 2.3.1 in [17], we easily obtain $\eta(\varphi) = +\infty$.

Next we prove that any positive solution of (2) with (4) satisfies

$$\liminf_{t \rightarrow +\infty} x_j(t) > 0, \quad j \in J. \quad (15)$$

Denote $l = \min_{j \in J} \liminf_{t \rightarrow +\infty} x_j(t)$. We claim that $l > 0$. Suppose on the contrary that $l = 0$. Define

$$\begin{aligned} m(t) &= \max \left\{ \xi : \xi \leq t \mid \text{there exists } j_0 \in J \text{ such that } x_{j_0}(\xi) \right. \\ &= \min_{t_0 \leq s \leq t} x_{j_0}(s) = \min_{j \in J} \left\{ \min_{t_0 \leq s \leq t} x_j(s) \right\}. \end{aligned} \quad (16)$$

Then $m(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Moreover, for a sequence $\{t_p\}_{p \geq 1}$ with $\lim_{p \rightarrow +\infty} t_p = +\infty$, we can choose $i_0 \in J$ and a subsequence $\{t_{p_k}\}_{k \geq 1} \subseteq \{t_p\}_{p \geq 1}$ such that

$$x_{i_0}(m(t_{p_k})) = \min_{t_0 \leq s \leq t_{p_k}} x_{i_0}(s) = \min_{j \in J} \left\{ \min_{t_0 \leq s \leq t_{p_k}} x_j(s) \right\} \quad (17)$$

and

$$\lim_{k \rightarrow +\infty} x_{i_0}(m(t_{p_k})) = 0. \quad (18)$$

By virtue of the definition of $m(t_{p_k})$, we obtain that $D^- x_{i_0}(m(t_{p_k})) \leq 0$ or

$$\begin{aligned} \delta_{i_0} x_{i_0}(m(t_{p_k})) &\geq \sum_{j=1, j \neq i_0}^n d_{i_0 j} x_j(m(t_{p_k})) \\ &\quad + \sum_{j=1}^m p_{i_0 j} \exp(-a_{i_0 j} x_{i_0}(m(t_{p_k}) - \tau_{i_0 j}(m(t_{p_k})))) \\ &\geq \sum_{j=1}^m p_{i_0 j} \exp\left(-a_{i_0 j} \sup_{t \geq t_0 - \tau_{i_0}} x_{i_0}(t)\right), \quad m(t_{p_k}) > t_0 + \tau. \end{aligned} \quad (19)$$

Letting $k \rightarrow +\infty$ leads to

$$0 \geq \sum_{j=1}^m p_{i_0 j} \exp\left(-a_{i_0 j} \sup_{t \geq t_0 - \tau_{i_0}} x_{i_0}(t)\right) > 0. \quad (20)$$

This is a contradiction, and the claim holds. The proof of Lemma 1 is completed. $\square$

Now, we show the global attractivity of $(x_1^*, x_2^*, \dots, x_n^*)$ by the following two propositions:

Proposition 2. If $y_i(t) = x_i(t) - x_i^*$ is eventually nonoscillating about zero, then $\lim_{t \rightarrow +\infty} x_i(t) = x_i^*$, for all $i \in J$.

Proof. We only give the proofs for the case that $y_i(t) = x_i(t) - x_i^*$ is eventually nonnegative for all $i \in J$, since the eventually nonpositive case can be proved by a similar argument. In this case, we can choose $T > t_0$ such that

$$y_i(t) = x_i(t) - x_i^* \geq 0 \quad \text{for all } i \in J, t \geq T. \quad (21)$$

Let $i^* \in J$ such that $\limsup_{t \rightarrow +\infty} y_{i^*}(t) = \max_{j \in J} \limsup_{t \rightarrow +\infty} y_j(t)$. We claim that

$$\limsup_{t \rightarrow +\infty} y_{i^*}(t) = 0. \quad (22)$$

Assume the contrary that $\limsup_{t \rightarrow +\infty} y_{i^*}(t) > 0$. In view of the fluctuation lemma [18, Lemma A.1], there exists a sequence $\{t_k\}_{k \geq 1}$ such that

$$t_k \rightarrow +\infty, y_{i^*}(t_k) \rightarrow \limsup_{t \rightarrow +\infty} y_{i^*}(t), y'_{i^*}(t_k) \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (23)$$

It follows from (2) that

$$\begin{aligned} y'_{i^*}(t_k) &= -\delta_{i^*} x_{i^*}(t_k) + \sum_{j=1, j \neq i^*}^n d_{i^*j} x_j(t_k) \\ &\quad + \sum_{j=1}^m p_{i^*j} e^{-a_{i^*j} x_{i^*}(t_k - \tau_{i^*j}(t_k))}, \quad t_k > T + \tau. \end{aligned} \quad (24)$$

Without loss of generality, we can pick a subsequence of $\{t_k\}$ (not relabelled) such that $\lim_{k \rightarrow +\infty} x_i(t_k)$, $\lim_{k \rightarrow +\infty} x_{i^*}(t_k)$ and $\lim_{k \rightarrow +\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k))$ exist for all $i \in J$, $j \in I = \{1, 2, \dots, m\}$. Then,

$$\begin{aligned} x_{i^*}^* &\leq \lim_{k \rightarrow +\infty} x_{i^*}(t_k - \tau_{i^*j}(t_k)) \\ &\leq x_{i^*}^* + \limsup_{t \rightarrow +\infty} y_{i^*}(t) \\ &= \lim_{k \rightarrow +\infty} x_{i^*}(t_k), \quad j \in I. \end{aligned} \quad (25)$$

It follows from (24) that (taking limits)

$$\begin{aligned} 0 &\leq -\delta_{i^*} \left(\limsup_{t \rightarrow +\infty} y_{i^*}(t) + x_{i^*}^* \right) + \sum_{j=1, j \neq i^*}^n d_{i^*j} \left(\limsup_{t \rightarrow +\infty} y_i(t) + x_j^* \right) \\ &\quad + \sum_{j=1}^m p_{i^*j} e^{-a_{i^*j} x_{i^*}^*} < \left(-\delta_{i^*} + \sum_{j=1, j \neq i^*}^n d_{i^*j} \right) \limsup_{t \rightarrow +\infty} y_{i^*}(t) < 0, \end{aligned} \quad (26)$$

which leads to a contradiction. Hence, $\limsup_{t \rightarrow +\infty} y_{i^*}(t) = 0$. This completes the proof. $\square$

Remark 3. It is worth noting that, from Proposition 2 the nonoscillating solutions of system (2) converge to the positive equilibrium point which does not depend on the delays.

Inspired by Theorem 4.1 in [19], we can obtain the following more general conclusion.

Lemma 4. Let $v \leq 0 \leq u$ be such that $u \leq e^{-v} - 1$ and $kv \geq e^{-ku} - 1$ for some $k \geq 1$. Then $u = v = 0$.

Proof. Since $u \leq e^{-v} - 1$ and $kv \geq e^{-ku} - 1$, we have

$$1 + u \leq e^{-v} \leq e^{(1/k) - (1/k)e^{-ku}}. \quad (27)$$

Let $f(x) = e^{(1/k) - (1/k)e^{-kx}} - x - 1$. Then,

$$f'(x) = e^{(1/k) - kx - (1/k)e^{-kx}} - 1 \quad (28)$$

and

$$f''(x) = e^{(1/k) - kx - (1/k)e^{-kx}} (-k + e^{-kx}) \leq 0, \quad \text{for all } x > 0. \quad (29)$$

Due to the fact that $f'(0) = 0$, it follows that

$$f'(x) < 0, \quad \text{for all } x > 0. \quad (30)$$

On the other hand, since $f(0) = 0$, we get $f(x) < 0$ for all $x > 0$. This implies that $u = 0$. According to $kv \geq e^{-ku} - 1$, we have $v \geq 0$. As $v \leq 0$, we must have $v = 0$. This finishes the proof. $\square$

Next, we consider the attractivity of (2) on the premise that the conditions in Proposition 5 are not satisfied.

Let

$$y_i(t) = \frac{x_i(t)}{x_i^*} - 1, \quad i \in J. \quad (31)$$

Then, from (2), we get

$$\begin{aligned} y'_i(t) &= -\delta_i(y_i(t) + 1) + \sum_{j=1, j \neq i}^n d_{ij} \frac{x_j^*}{x_i^*} (y_j(t) + 1) \\ &\quad + \sum_{j=1}^m p_{ij} \frac{1}{x_i^*} e^{-a_{ij} x_i^*} (y_i(t - \tau_{ij}(t)) + 1) \\ &= -\delta_i(y_i(t) + 1) + \sum_{j=1, j \neq i}^n d_{ij}^* (y_j(t) + 1) \\ &\quad + \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_i^*} (y_i(t - \tau_{ij}(t)) + 1), \end{aligned} \quad (32)$$

where $d_{ij}^* = d_{ij}(x_j^*/x_i^*)$, $p_{ij}^* = p_{ij}(1/x_i^*)$. It is easy to see that the global attractivity of the equilibrium $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ for (2) is equivalent to the global attractivity of the trivial solution for (35).

Set

$$\lambda = \min_{i \in J} \liminf_{t \rightarrow +\infty} y_i(t), \quad \mu = \max_{i \in J} \limsup_{t \rightarrow +\infty} y_i(t). \quad (33)$$

By the fluctuation lemma [14, Lemma A.1], we can take $i_1^* \in J$ and a strictly monotonically increasing sequence $\{l_q^*\}_{q \geq 1}$ such that

$$l_q^* \rightarrow +\infty, \quad y_{i_1^*}(l_q^*) \rightarrow \mu, \quad \text{as } q \rightarrow +\infty. \quad (34)$$

Furthermore, choose $i_1 \in J$ and a subsequence $\{l_q^{**}\}_{q \geq 1} \subseteq \{l_q^*\}_{q \geq 1}$ satisfying

$$\begin{aligned}
l_q^{**} > t_0, \quad y_{i_1}(\bar{l}_q) &= \max_{t \in [l_q^{**}, l_{q+1}^{**}]} y_{i_1}(t) \\
&= \max_{i \in J} \left\{ \max_{t \in [l_q^{**}, l_{q+1}^{**}]} y_i(t) \right\} > 0, \quad \bar{l}_q \in [l_q^{**}, l_{q+1}^{**}]. \quad (35)
\end{aligned}$$

By the boundedness of $\{y_{i_1}(\bar{l}_q)\}$, we pick a strictly monotonically increasing sequence $\{l_q\}_{q \geq 1} \subseteq \{\bar{l}_q\}_{q \geq 1}$ such that $\lim_{q \rightarrow +\infty} y_{i_1}'(l_q)$ exists. It follows from (35) that

$$y_{i_1}(l_q) > 0, \quad \lim_{q \rightarrow +\infty} l_q = +\infty, \quad \lim_{q \rightarrow +\infty} y_{i_1}(l_q) = \mu, \quad \lim_{q \rightarrow +\infty} y_{i_1}'(l_q) \geq 0. \quad (36)$$

Adopting the same procedure as in the proof of (12), there exist $i_2 \in J$ and a strictly monotonically increasing sequence $\{s_q\}_{q \geq 1}$ such that

$$y_{i_2}(s_q) < 0, \quad \lim_{q \rightarrow +\infty} s_q = +\infty, \quad \lim_{q \rightarrow +\infty} y_{i_2}(s_q) = \lambda, \quad \lim_{q \rightarrow +\infty} y_{i_2}'(s_q) \leq 0. \quad (37)$$

Subsequently, we prove that there is a positive integer Q_1 such that, for any $q \geq Q_1$, there exists $L_q \in [l_q - r_{i_1}, l_q]$ such that

$$y_{i_1}(L_q) = 0, \quad y_{i_1}(t) > 0, \quad \text{for all } t \in (L_q, l_q). \quad (38)$$

In the contrary case, there exists a subsequence of $\{l_q\}$ (for convenience, we still denote by $\{l_q\}$) such that

$$y_{i_1}(t) > 0, \quad \text{for all } t \in [l_q - r_{i_1}, l_q], \quad q = 1, 2, \dots \quad (39)$$

According to the definition of the l_q , we conclude from (32) that

$$\begin{aligned}
y_{i_1}'(l_q) &= -\delta_{i_1}(y_{i_1}(l_q) + 1) + \sum_{j=1, j \neq i_1}^n d_{i_1 j}^*(y_j(l_q) + 1) \\
&\quad + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{ij} x_{i_1}^*} (y_{i_1}(l_q - \tau_{ij}(l_q)) + 1) \\
&< -\delta_{i_1}(y_{i_1}(l_q) + 1) + \sum_{j=1, j \neq i_1}^n d_{i_1 j}^*(y_j(l_q) + 1) \\
&\quad + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{ij} x_{i_1}^*}, \quad l_q > t_0^* + r_{i_1}. \quad (40)
\end{aligned}$$

Assume that $\lim_{q \rightarrow +\infty} y_j(l_q)$ exists for all $j \in J \setminus \{i_1\}$, from the fact that $\limsup_{t \rightarrow +\infty} y_i(t) \leq \mu (i \in J)$, (40) implies that

$$\begin{aligned}
0 &\leq \lim_{q \rightarrow +\infty} y_{i_1}'(l_q) \\
&\leq -\delta_{i_1} \left(\lim_{q \rightarrow +\infty} y_{i_1}(l_q) + 1 \right) \\
&\quad + \sum_{j=1, j \neq i_1}^m d_{i_1 j}^* \left(\lim_{q \rightarrow +\infty} y_j(l_q) + 1 \right) + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{ij} x_{i_1}^*} \\
&\leq -\delta_{i_1}(\mu + 1) + \sum_{j=1, j \neq i_1}^n d_{i_1 j}^*(\mu + 1) + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{ij} x_{i_1}^*} \\
&< (\mu + 1) \left[-\delta_{i_1} + \sum_{j=1, j \neq i_1}^n d_{i_1 j}^* + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{ij} x_{i_1}^*} \right] = 0, \quad (41)
\end{aligned}$$

which is a contradiction and (38) is true.

Based on the above discussions, the following conclusion can be drawn.

Proposition 5. Suppose that the assumption mentioned in Proposition 2 does not hold, and

$$\frac{a_i(e^{\delta_i r_i} - 1) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}}{\delta_i + ((e^{\delta_i r_i} - 1)/x_i^*) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}} \leq 1, \quad \text{for all } i \in J. \quad (42)$$

$$a_i x_{i_1}^* = \min_{1 \leq i \leq n} \{a_i x_i^*\}. \quad (43)$$

Then $\lim_{t \rightarrow +\infty} x_i(t) = x_i^*, i \in J$.

Proof. Observe that, from Theorem 6, $-1 < \lambda \leq 0 \leq \mu$. Now, we show that $\lambda = \mu = 0$. Otherwise, either $\mu > 0$ or $\lambda < 0$ holds. We only consider the case that $\mu > 0$ holds (the situation is analogous for $\lambda < 0$).

For any given $\varepsilon > 0$ such that $\lambda - \varepsilon > -1$, by (33) there exists a positive integer $q^* > Q_2 + t_0^*$ such that

$$\lambda - \varepsilon < y_i(t) < \mu + \varepsilon \quad \text{for all } t > \min\{l_{q^*}, s_{q^*}\} - 2\tau, \quad i \in J. \quad (44)$$

Furthermore, from the fact that

$$\begin{aligned}
y_{i_1}'(t) &= -\delta_{i_1}(y_{i_1}(t) + 1) + \sum_{j=1, j \neq i_1}^n d_{ij}^*(y_j(t) + 1) \\
&\quad + \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_{i_1}^*} (y_{i_1}(t - \tau_{ij}(t)) + 1), \quad t \in [L_q, l_q], \quad (45)
\end{aligned}$$

we have

$$\begin{aligned}
&y_{i_1}(l_q) e^{\delta_{i_1} l_q} \\
&\leq -(e^{\delta_{i_1} l_q} - e^{\delta_{i_1} L_q}) + \sum_{j=1, j \neq i_1}^n d_{ij}^* \int_{L_q}^{l_q} (y_j(t) + 1) e^{\delta_{i_1} t} dt \\
&\quad + \sum_{j=1}^m p_{ij}^* \int_{L_q}^{l_q} e^{-a_{ij} x_{i_1}^*} (y_{i_1}(t - \tau_{ij}(t)) + 1) e^{\delta_{i_1} t} dt \\
&\leq \left[-\delta_{i_1} + (\mu + \varepsilon + 1) \left(\sum_{j=1, j \neq i_1}^n d_{ij}^* \right) + \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_{i_1}^*} e^{-a_{ij} x_{i_1}^* (\lambda - \varepsilon)} \right] \\
&\quad \times \frac{e^{\delta_{i_1} l_q} - e^{\delta_{i_1} L_q}}{\delta_{i_1}} \\
&= \left[-\delta_{i_1} + (\mu + \varepsilon + 1) \left(\delta_{i_1} - \sum_{j=1}^m p_{ij}^* e^{-x_{i_1}^*} \right) + \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_{i_1}^*} e^{-a_{ij} x_{i_1}^* (\lambda - \varepsilon)} \right] \\
&\quad \times \frac{e^{\delta_{i_1} l_q} - e^{\delta_{i_1} L_q}}{\delta_{i_1}} \\
&= \left[(\mu + \varepsilon) \delta_{i_1} + \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_{i_1}^*} (e^{-a_{ij} x_{i_1}^* (\lambda - \varepsilon)} - 1) - (\mu + \varepsilon) \sum_{j=1}^m p_{ij}^* e^{-a_{ij} x_{i_1}^*} \right] \\
&\quad \times \frac{e^{\delta_{i_1} l_q} - e^{\delta_{i_1} L_q}}{\delta_{i_1}}, \quad q > q^*, \quad (46)
\end{aligned}$$

that is,

$$y_{i_1}(l_q) \leq \left[(\mu + \varepsilon) \delta_{i_1} + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*} (e^{-a_{i_1 j} x_{i_1}^* (\lambda - \varepsilon)} - 1) - (\mu + \varepsilon) \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*} \right] \frac{1 - e^{-\delta_{i_1} r_{i_1}}}{\delta_{i_1}}, \quad q > q^*, \quad (47)$$

Letting $q \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$, (36) and (47) give us that

$$\mu \leq \mu(1 - e^{-\delta_{i_1} r_{i_1}}) + \left[\sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*} (e^{-a_{i_1 j} x_{i_1}^* \lambda} - 1) - \mu \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*} \right] \frac{1 - e^{-\delta_{i_1} r_{i_1}}}{\delta_{i_1}}, \quad (48)$$

which implies

$$\begin{aligned} \mu &\leq \frac{\sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^* (1 - e^{-\delta_{i_1} r_{i_1}})} / (\delta_{i_1})}{e^{-\delta_{i_1} r_{i_1}} + \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^* (1 - e^{-\delta_{i_1} r_{i_1}})} / (\delta_{i_1})} (e^{-a_{i_1} x_{i_1}^* \lambda} - 1) \\ &= \frac{(e^{\delta_{i_1} r_{i_1}} - 1) \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*}}{\delta_{i_1} + (e^{\delta_{i_1} r_{i_1}} - 1) \sum_{j=1}^m p_{i_1 j}^* e^{-a_{i_1 j} x_{i_1}^*}} (e^{-a_{i_1} x_{i_1}^* \lambda} - 1). \end{aligned} \quad (49)$$

Let's assume that $\lambda < 0$ (if $\lambda = 0$, from (49) and $\mu \geq 0$ we have $\mu = 0$). Using the same arguments in the proof of (38), we can obtain that there is a positive integer $Q_2 > Q_1$ such that, for any $q \geq Q_2$, there exists $S_q \in [s_q - r_{i_2}, s_q]$ such that

$$y_{i_2}(S_q) = 0, \text{ and } y_{i_2}(t) < 0, \quad \text{for all } t \in (S_q, s_q). \quad (50)$$

Again from (32), we have

$$\begin{aligned} y_{i_2}(s_q) e^{\delta_{i_2} s_q} &\geq -(e^{\delta_{i_2} s_q} - e^{\delta_{i_2} S_q}) + \sum_{j=1, j \neq i_2}^m d_{i_2 j}^* \int_{S_q}^{s_q} (y_j(t) + 1) e^{\delta_{i_2} t} dt \\ &\quad + \sum_{j=1}^m p_{i_2 j}^* \int_{S_q}^{s_q} e^{-a_{i_2 j} x_{i_2}^*} (y_j(s_q - \tau_{i_2 j}(s_q)) + 1) e^{\delta_{i_2} t} dt \\ &\geq \left[-\delta_{i_2} + (\lambda - \varepsilon + 1) \left(\sum_{j=1, j \neq i_2}^m d_{i_2 j}^* \right) + \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} e^{-a_{i_2 j} x_{i_2}^* (\mu + \varepsilon)} \right] \\ &\quad \times \frac{e^{\delta_{i_2} s_q} - e^{\delta_{i_2} S_q}}{\delta_{i_2}} \\ &= \left[-\delta_{i_2} + (\lambda - \varepsilon + 1) \left(\delta_{i_2} - \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} \right) + \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} e^{-a_{i_2 j} x_{i_2}^* (\mu + \varepsilon)} \right] \\ &\quad \times \frac{e^{\delta_{i_2} s_q} - e^{\delta_{i_2} S_q}}{\delta_{i_2}} \\ &\geq \left[(\lambda - \varepsilon) \delta_{i_2} + \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} (e^{-a_{i_2 j} x_{i_2}^* (\mu + \varepsilon)} - 1) - (\lambda - \varepsilon) \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} \right] \\ &\quad \times \frac{e^{\delta_{i_2} s_q} - e^{\delta_{i_2} S_q}}{\delta_{i_2}}, \quad q > q^*, \end{aligned} \quad (51)$$

that is,

$$y_{i_2}(s_q) \geq \left[(\lambda - \varepsilon) \delta_{i_2} + \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} (e^{-a_{i_2 j} x_{i_2}^* (\mu + \varepsilon)} - 1) - (\lambda - \varepsilon) \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} \right] \frac{1 - e^{-\delta_{i_2} r_{i_2}}}{\delta_{i_2}}, \quad q > q^*. \quad (52)$$

Letting $q \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$, (37) and (52) give us that

$$\begin{aligned} \lambda &\geq \lambda(1 - e^{-\delta_{i_2} r_{i_2}}) \\ &\quad + \left[\sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} (e^{-a_{i_2 j} x_{i_2}^* \mu} - 1) - \lambda \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*} \right] \\ &\quad \times \frac{1 - e^{-\delta_{i_2} r_{i_2}}}{\delta_{i_2}}, \end{aligned} \quad (53)$$

which implies

$$\begin{aligned} \lambda &\geq \frac{\sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^* (1 - e^{-\delta_{i_2} r_{i_2}})} / (\delta_{i_2})}{e^{-\delta_{i_2} r_{i_2}} + \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^* (1 - e^{-\delta_{i_2} r_{i_2}})} / (\delta_{i_2})} (e^{-a_{i_2} x_{i_2}^* \mu} - 1) \\ &= \frac{(e^{\delta_{i_2} r_{i_2}} - 1) \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*}}{\delta_{i_2} + (e^{\delta_{i_2} r_{i_2}} - 1) \sum_{j=1}^m p_{i_2 j}^* e^{-a_{i_2 j} x_{i_2}^*}} (e^{-a_{i_2} x_{i_2}^* \mu} - 1). \end{aligned} \quad (54)$$

This, combined with (42) and (49) imply that

$$a_{i_1} x_{i_1}^* \mu \leq e^{-a_{i_1} x_{i_1}^* \lambda} - 1, \quad a_{i_2} x_{i_2}^* \lambda \geq e^{-a_{i_2} x_{i_2}^* \mu} - 1. \quad (55)$$

Denote

$$\mu^* = a_{i_1} x_{i_1}^* \mu, \quad \lambda^* = a_{i_2} x_{i_2}^* \lambda, \quad k = \frac{a_{i_2} x_{i_2}^*}{a_{i_1} x_{i_1}^*}. \quad (56)$$

From (43), one can find that $k \geq 1$. By virtue of (55) and (56), we have

$$\mu^* \leq e^{-\lambda^*} - 1, \quad k \lambda^* \geq e^{-k \mu^*} - 1. \quad (57)$$

According to Lemma 4, it is easy to see that $\lambda^* = \mu^* = 0$. Furthermore, we get $\lambda = \mu = 0$, which is a contradiction, and $\mu > 0$ is not true. This completes the proof. $\square$

Combining Propositions 2 and 5, we have the following delay-dependent criterion of global attraction.

Theorem 6. Assume that conditions (3), (42) and (43) are satisfied. Then the positive equilibrium $(x_1^*, x_2^*, \dots, x_n^*)$ is a global attractor of (2).

Remark 7. Let us note that, from

$$\lim_{r_i \rightarrow 0^+} \frac{a_i (e^{\delta_i r_i} - 1) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}}{\delta_i + ((e^{\delta_i r_i} - 1) / x_i^*) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}} = 0 \quad \text{for all } i \in J, \quad (58)$$

one can find that condition (42) naturally holds under the sufficiently small delay, and the positive equilibrium point $(x_1^*, x_2^*, \dots, x_n^*)$ is a global attractor of (2) with small delays. Moreover,

$$\lim_{r_i \rightarrow +\infty} \frac{a_i (e^{\delta_i r_i} - 1) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}}{\delta_i + ((e^{\delta_i r_i} - 1) / x_i^*) \sum_{j=1}^m p_{ij} e^{-a_{ij} x_i^*}} = a_i x_i^* \quad \text{for all } i \in J, \quad (59)$$

implies that condition (42) is not satisfied when the delays in (2) is sufficiently large and $a_i x_i^* > 1$ for $i \in J$.

FIGURE 1: Numerical solution $(x_1(t), x_2(t))$ of Eq. (60) with different initial values $(x_1(s), x_2(s)) \equiv (4, 9), (6, 10), (7, 11), s \in [-0.05, 0]$.

3. A Numerical Example

Example 1. Consider the following Lasota–Ważewska-type delay system with two groups and patch structures:

$$\begin{cases} x_1'(t) = -8x_1(t) + 3x_2(t) + 4e^5 e^{-x_1(t-\tau_{11}(t))} + 6e e^{-(1/5)-x_1(t-\tau_{12}(t))}, \\ x_2'(t) = -5x_2(t) + 4x_1(t) + 10e^5 e^{-(1/2)x_2(t-\tau_{21}(t))} + 20e^{(5/2)} e^{-(1/4)-x_2(t-\tau_{22}(t))}. \end{cases} \quad (60)$$

It is easy to check that

$$a_1 = 1, a_2 = \frac{1}{2}, 0 < 8 - 3 < 4e^5 + 6e, 0 < 5 - 4 < 10e^5 + 20e^{5/2}. \quad (61)$$

Therefore, (3) is true. Obviously, $(x_1^*, x_2^*) = (5, 10)$ is the positive equilibrium point of (60). In addition, $a_1 x_1^* = a_2 x_2^* = 5$ implies (43) holds. Now, we choose

$$\tau_{ij}(t) = 0.05 |\sin(i + j)t|, \quad i, j = 1, 2, \quad (62)$$

such that (42) holds. By Theorem 6, we conclude that the positive equilibrium point $(5, 10)$ is a global attractor of (60) with delays (62). This implies that small delays are harmless on the asymptotic behavior of system (60). Numerical runs with Matlab illustrate convergence of positive solutions to (x_1^*, x_2^*) (see Figure 1).

Remark 8. Observe that the methods used in [13, 14] are not suitable for (60) with (62) since the system (60) with time-varying delays (62) does not generate a semiflow. In addition, it is also worth pointing out that the components of the positive equilibrium point in this paper are not required to be equal, which is also different from the literature [12].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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