Research Article

A New Numerical Approximation Method for Two-Dimensional Wave Equation with Neumann Damped Boundary

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In this paper, a fully discretized finite difference scheme is derived for two-dimensional wave equation with damped Neumann boundary condition. By discrete energy method, the proposed difference scheme is proven to be of second-order convergence and of unconditional stability with respect to both initial conditions and right-hand term in a proper discretized L^2 norm. The theoretical result is verified by a numerical experiment.

1. Introduction

In this paper, we consider the finite difference discretization for the following initial boundary value problem (IBVP) of wave equation in a square $\Omega = (0, 1) \times (0, 1)$:

$$w_{tt}(x, y, t) - \Delta w(x, y, t) = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \quad (1)$$

$$\begin{aligned} w(x, y, 0) &= \varphi(x, y), \\ w_t(x, y, 0) &= \psi(x, y), \\ (x, y) &\in \Omega, \end{aligned} \quad (2)$$

$$\begin{aligned} w(0, y, t) &= 0, \\ w_x(1, y, t) &= -w_t(1, y, t), \\ y &\in [0, 1], t \in [0, T], \end{aligned} \quad (3)$$

$$\begin{aligned} w(x, 0, t) &= 0, \\ w_y(x, 1, t) &= -w_t(x, 1, t), \\ x &\in [0, 1], t \in [0, T], \end{aligned} \quad (4)$$

where $\varphi(x, y)$, $\psi(x, y)$, and $f(x, y, t)$ are given sufficient smooth functions, and $\varphi(0, y) = \varphi(x, 0) = 0$, $\psi(0, y) =$

$\psi(x, 0) = 0$, $\varphi_x(1, y) + \psi(1, y) = \varphi_y(x, 1) + \psi(x, 1) = 0$, which means that the initial conditions are compatible with boundary conditions.

This system arises in many important models for distributed parameter control systems. In particular, in the model of a vibrating flexible membrane, the solution w represents the transverse displacement of the membrane, and in models for acoustic pressure fields, the solution w represents the fluid pressure (see, [1, 2] for more examples). The boundary conditions on the right and top sides of Ω in (3) and (4) are called Neumann actuations with damped controller from the viewpoint of control theory. Here, this kind of boundary condition is regarded as damped Neumann boundary. For this special boundary condition, it is more complicated to construct a proper energy function to prove a priori estimate of proposed finite difference scheme than that of classical boundary conditions (Dirichlet, Neumann, or Robin).

The numerical analysis of second-order hyperbolic equations has been extensively studied for decades. Many numerical methods have been used to approximate this kind of problem, including Finite Element Methods [3, 4], Finite Volume Methods [5, 6], Finite Difference Methods [7, 8], Discontinuous Galerkin methods [9–11], Mixed Finite Element Methods [12] and so on. In [4], wave equation with

homogeneous boundary condition is considered, in which a posteriori error bounds in the $L^\infty(L^2)$ norm for finite element methods is derived under minimal regularity assumptions. A finite volume scheme on the non-conforming meshes for multidimensional wave equation is constructed in [6], where the error estimates for the approximations of the exact solution and its first derivatives is derived. An unconditionally stable and second-order convergent finite difference scheme is constructed for one-dimensional linear hyperbolic equation in [8]. As far as hyperbolic conservation law is concerned, there are many distinguished papers focused on this topic, for example, the classical ENO/WENO schemes [13], the improved ENO/WENO schemes [14–17], and so on. All linear numerical schemes are either dispersive or dissipative. The computational dispersion can lead to noise in the numerical solution. Dispersion and dissipation phenomena were investigated in [18–22], where some composite schemes were proposed to reduce the dispersive effect on the numerical solution. High order compact finite difference scheme is an efficient way for solving PDEs. In [23–25], the high order compact finite difference scheme was constructed for hyperbolic equations subject to homogeneous boundary condition.

There are also some investigations of the numerical methods for viscous or strongly damped hyperbolic equation [26–29]. However, most of them just considered the homogeneous boundary problems. In [3], the generalized version of this kind of boundary condition in [3, (2.2b)] was considered in the context of finite element methods. In [30], the mixed finite element formulation for second-order hyperbolic equation with absorbing boundary condition was investigated. Recently, the weak formulation of hyperbolic problems with inhomogeneous Dirichlet and Neumann boundary was considered in [31]. To the best of our knowledge, there is no contribution to the finite difference approximation to the 2D wave equation with damped Neumann boundary condition so far.

The rest of this paper is organized as follows. Some basic notations and lemmas are given in Section 2, which is essential to the analysis of finite difference schemes. Section 3 constructs the finite difference scheme for (1)–(4). A priori estimate of the proposed difference scheme is shown in Section 4. The unique solvability, convergence, and stability of proposed finite difference scheme are proved in Section 5. A numerical experiment is conducted in Section 6, before a conclusion is stated in Section 7.

2. Preliminary

Let m and n be two positive integers and assume that the space step and time step are $h = 1/m$ and $\tau = T/n$, respectively. Define $\Omega_h = \{(x_i, y_j) \mid x_i = ih, y_j = jh, 0 \leq i, j \leq m\}$, $\Omega_\tau = \{t_k \mid t_k = k\tau, 0 \leq k \leq n\}$ and $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Let $w = \{w_{i,j}^k \mid 0 \leq i, j \leq m, 0 \leq k \leq n\}$ be the grid function on $\Omega_{h\tau}$. We introduce the following difference and averaging operators:

$$\begin{aligned} w_{ij}^{k+(1/2)} &= \frac{w_{ij}^k + w_{ij}^{k+1}}{2}, \\ \delta_t w_{ij}^{k+(1/2)} &= \frac{w_{ij}^{k+1} - w_{ij}^k}{\tau}, \\ D_t w_{ij}^k &= \frac{w_{ij}^{k+1} - w_{ij}^{k-1}}{2\tau}, \\ \delta_t^2 w_{ij}^k &= \frac{\delta_t w_{ij}^{k+(1/2)} - \delta_t w_{ij}^{k-(1/2)}}{\tau}, \\ \delta_x w_{i+(1/2),j}^k &= \frac{w_{i+1,j}^k - w_{i,j}^k}{h}, \\ \delta_y w_{i,j+(1/2)}^k &= \frac{w_{i,j+1}^k - w_{i,j}^k}{h}, \\ \delta_x^2 w_{ij}^k &= \frac{\delta_x w_{i+(1/2),j}^k - \delta_x w_{i-(1/2),j}^k}{h}, \\ \delta_y^2 w_{ij}^k &= \frac{\delta_y w_{i,j+(1/2)}^k - \delta_y w_{i,j-(1/2)}^k}{h}, \\ \bar{w}_{ij}^k &= \frac{w_{ij}^{k+1} + w_{ij}^{k-1}}{2}. \end{aligned} \quad (5)$$

The following lemmas (Lemma 1–Lemma 4) are necessary for analyzing the truncation error of the difference scheme and for proving the convergence of the difference scheme, which are analogous to lemmas in [32].

Lemma 1. Let $h > 0$ and c be two constants.

(a) If $g(x) \in C^2[c-h, c+h]$, then

$$g(c) = \frac{1}{2} [g(c-h) + g(c+h)] - \frac{h^2}{2} g''(\xi_0), \quad c-h < \xi_0 < c+h. \quad (6)$$

(b) If $g(x) \in C^2[c, c+h]$, then

$$g'(c) = \frac{1}{h} [g(c+h) - g(c)] - \frac{h}{2} g''(\xi_1), \quad c-h < \xi_1 < c+h. \quad (7)$$

(c) If $g(x) \in C^2[c-h, c]$, then

$$g'(c) = \frac{1}{h} [g(c) - g(c-h)] + \frac{h}{2} g''(\xi_2), \quad c-h < \xi_2 < c. \quad (8)$$

(d) If $g(x) \in C^3[c-h, c+h]$, then

$$g'(c) = \frac{1}{2h} [g(c+h) - g(c-h)] - \frac{h^2}{6} g'''(\xi_3), \quad c-h < \xi_3 < c+h. \quad (9)$$

(e) If $g(x) \in C^4[c-h, c+h]$, then

$$g''(c) = \frac{1}{h^2} [g(c+h) - 2g(c) + g(c-h)] - \frac{h^2}{12} g^{(4)}(\xi_4),$$

$$c-h < \xi_4 < c+h.$$
(10)

(f) If $g(x) \in C^3[c-h, c]$, then

$$g''(c) = \frac{2}{h} \left[g'(c) - \frac{g(c) - g(c-h)}{h} \right] + \frac{h}{3} g'''(\xi_5),$$

$$c-h < \xi_5 < c.$$
(11)

Lemma 2 (Gronwall Inequality). Suppose $\{F^k, G^k \mid k \geq 0\}$ are nonnegative sequences such that $F^{k+1} \leq (1 + c\tau)F^k + \tau G^k$, $k = 0, 1, 2, \dots$, then

$$F^k \leq \exp(ck\tau) \left(F^0 + \tau \sum_{l=1}^{k-1} G^l \right), \quad k = 0, 1, 2, \dots, \quad (12)$$

where c is a nonnegative constant.

Lemma 3. Suppose the mesh grids be $I_h = \{x_i \mid 0 \leq i \leq m\}$, where $x_i = ih$, $h = 1/m$. Let $u = \{u_i \mid 0 \leq i \leq m\}$ be mesh function on I_h such that $u_0 = 0$, then

$$\|u\|_\infty \leq |u|_1, \quad \|u\| \leq \frac{\sqrt{2}}{2} |u|_1, \quad (13)$$

where $|u|_1 = \sqrt{h \sum_{i=1}^m (\delta_x u_{i-(1/2)})^2}$, $\|u\| = \sqrt{h((1/2)u_0^2 + \sum_{i=1}^{m-1} u_i^2 + (1/2)u_m^2)}$, $\|u\|_\infty = \max_{0 \leq i \leq m} |u_i|$.

Proof. For $1 \leq i \leq m$, we have

$$u_i = \sum_{j=1}^i (u_j - u_{j-1}) = h \sum_{j=1}^i \delta_x u_{j-(1/2)}. \quad (14)$$

Square both sides of (14) and apply the Cauchy-Schwarz inequality, then

$$(u_i)^2 \leq \left(h \sum_{j=1}^i 1^2 \right) \left(h \sum_{j=1}^i (\delta_x u_{j-(1/2)})^2 \right)$$

$$= x_i h \sum_{j=1}^i (\delta_x u_{j-(1/2)})^2 \leq h \sum_{j=1}^m (\delta_x u_{j-(1/2)})^2, \quad (15)$$

which directly implies

$$\|u\|_\infty \leq |u|_1. \quad (16)$$

The first inequality of (15) implies

$$u_i^2 \leq x_i h \sum_{j=1}^i (\delta_x u_{j-(1/2)})^2 \leq x_i |u|_1^2. \quad (17)$$

Multiply both sides of (17) by h , and sum for i from 1 to $m-1$, then

$$h \sum_{i=1}^{m-1} u_i^2 \leq h \sum_{i=1}^{m-1} x_i |u|_1^2 = \frac{1}{2} m(m-1) h^2 |u|_1^2. \quad (18)$$

Thus, we have

$$\|u\|^2 = h \sum_{i=1}^{m-1} u_i^2 + \frac{h}{2} u_m^2 \leq \left(\frac{h}{2} m(m-1) h + \frac{h}{2} \right) |u|_1^2 = \frac{1}{2} |u|_1^2, \quad (19)$$

which implies

$$\|u\| \leq \frac{\sqrt{2}}{2} |u|_1, \quad (20)$$

and ends the proof. $\square$

Introduce a grid function space on Ω_h :

$$\mathcal{V} = \left\{ v \mid v = (v_{ij})_{i,j=0}^m; v_{i,0} = 0, 0 \leq i \leq m; v_{0,j} = 0, 1 \leq j \leq m \right\}. \quad (21)$$

Then for $v \in \mathcal{V}$, define the norms

$$\|v\| = \sqrt{h^2 \sum_{i,j=1}^{m-1} (v_{ij})^2 + h^2 \frac{1}{2} \sum_{j=1}^{m-1} (v_{mj})^2 + h^2 \frac{1}{2} \sum_{i=1}^{m-1} (v_{im})^2 + h^2 \frac{1}{4} (v_{mm})^2},$$

$$\|\delta_x v\| = \sqrt{h^2 \sum_{i=0}^{m-1} \left(\frac{1}{2} (\delta_x v_{i+(1/2),0})^2 + \sum_{j=1}^{m-1} (\delta_x v_{i+(1/2),j})^2 + \frac{1}{2} (\delta_x v_{i+(1/2),m})^2 \right)}, \quad (22)$$

$$\|\delta_y v\| = \sqrt{h^2 \sum_{j=0}^{m-1} \left(\frac{1}{2} (\delta_y v_{0,j+(1/2)})^2 + \sum_{i=1}^{m-1} (\delta_y v_{i,j+(1/2)})^2 + \frac{1}{2} (\delta_y v_{m,j+(1/2)})^2 \right)}.$$

Lemma 4. Let $v = \{v_{ij} \mid 0 \leq i, j \leq m\}$ be mesh function on Ω_h such that $v_{0j} = v_{i0} = 0$ for $1 \leq i, j \leq m$, then we have

$$\|v\| \leq \frac{1}{2}|v|_1, \quad (23)$$

where $|v|_1 = (\|\delta_x v\|^2 + \|\delta_y v\|^2)^{1/2}$.

Proof. From Lemma 3, we have

$$h \sum_{i=1}^{m-1} u_{ij}^2 + \frac{h}{2} u_{mj}^2 \leq \frac{1}{2} h \sum_{i=0}^{m-1} (\delta_x u_{i+(1/2),j})^2, \quad 1 \leq j \leq m. \quad (24)$$

Multiplying both sides of (24) by h and summing for j from 1 to $m-1$, we get

$$h^2 \sum_{i,j=1}^{m-1} u_{ij}^2 + \frac{h^2}{2} \sum_{j=1}^{m-1} u_{mj}^2 \leq \frac{1}{2} h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} (\delta_x u_{i+(1/2),j})^2. \quad (25)$$

Multiplying both sides of (24) by $h/2$ and letting $j = m$ in (24), then after combining it with (25), we get

$$\|u\|^2 \leq \frac{1}{2} \|\delta_x u\|^2. \quad (26)$$

Similarly, we have

$$\|u\|^2 \leq \frac{1}{2} \|\delta_y u\|^2. \quad (27)$$

Combining (26) and (27), it follows

$$\|u\| \leq \frac{1}{2}|u|_1. \quad (28)$$

This completes the proof. $\square$

3. Derivation of the Finite Difference Scheme

Define the grid function $W = \{W_{ij}^k \mid 0 \leq i, j \leq m, 0 \leq k \leq n\}$ by

$$W_{ij}^k = w(x_i, y_j, t_k), \quad 0 \leq i, j \leq m, 0 \leq k \leq n. \quad (29)$$

Consider equation (1) at the inner points (x_i, y_j, t_k) in $\Omega_{h\tau}$, then it follows

$$\frac{\partial^2 w(x_i, y_j, t_k)}{\partial t^2} - \frac{\partial^2 w(x_i, y_j, t_k)}{\partial x^2} - \frac{\partial^2 w(x_i, y_j, t_k)}{\partial y^2} = f(x_i, y_j, t_k), \quad (30)$$

$$1 \leq i, j \leq m-1, 1 \leq k \leq n-1.$$

According to Lemma 1(e), it follows

$$\delta_t^2 W_{ij}^k - \delta_x^2 W_{ij}^k - \delta_y^2 W_{ij}^k = f(x_i, y_j, t_k) + R_{ij}^k, \quad 1 \leq i, j \leq m-1, 1 \leq k \leq n-1, \quad (31)$$

where $R_{ij}^k = O(\tau^2 + h^2)$. These are discrete equations at inner points.

Next we will pay our main attention to the derivation of difference equations at the boundary points. Considering the boundary condition 1.3 at (x_m, y_j, t_k) , it follows

$$\frac{\partial w(x_m, y_j, t_k)}{\partial t} + \frac{\partial w(x_m, y_j, t_k)}{\partial x} = 0, \quad 1 \leq j \leq m-1. \quad (32)$$

Similarly, the boundary condition 1.4 at (x_i, y_m, t_k) satisfies

$$\frac{\partial w(x_i, y_m, t_k)}{\partial t} + \frac{\partial w(x_i, y_m, t_k)}{\partial y} = 0, \quad 1 \leq i \leq m-1. \quad (33)$$

By (3) and (4), the differential equation at the corner point (x_m, y_m, t_k) reads

$$\frac{2 \partial w(x_m, y_m, t_k)}{\partial t} + \frac{\partial w(x_m, y_m, t_k)}{\partial x} + \frac{\partial w(x_m, y_m, t_k)}{\partial y} = 0. \quad (34)$$

According to Lemma 1 (a), (c), and (d), we have

$$\begin{aligned} \frac{\partial w(x_m, y_j, t_k)}{\partial t} &= \frac{W(x_m, y_j, t_{k+1}) - W(x_m, y_j, t_{k-1})}{2\tau} \\ &+ O(\tau^2) = D_t W_{mj}^k + O(\tau^2), \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{\partial w(x_i, y_m, t_k)}{\partial t} &= \frac{W(x_i, y_m, t_{k+1}) - W(x_i, y_m, t_{k-1})}{2\tau} + O(\tau^2) \\ &= D_t W_{im}^k + O(\tau^2), \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial w(x_m, y_j, t_k)}{\partial x} &= \frac{1}{2} \left(\frac{\partial w(x_m, y_j, t_{k-1})}{\partial x} + \frac{\partial w(x_m, y_j, t_{k+1})}{\partial x} \right) \\ &+ O(\tau^2) \\ &= D_{x-} W_{mj}^k + \frac{h}{2} \left(\delta_t^2 W_{mj}^k - \delta_y^2 W_{mj}^k \right. \\ &\quad \left. - f(x_m, y_j, t_k) \right) + O(\tau^2 + h^2), \end{aligned} \quad (37)$$

$$\begin{aligned} \frac{\partial w(x_i, y_m, t_k)}{\partial y} &= \frac{1}{2} \left(\frac{\partial w(x_i, y_m, t_{k-1})}{\partial y} + \frac{\partial w(x_i, y_m, t_{k+1})}{\partial y} \right) \\ &+ O(\tau^2) \\ &= D_{y-} W_{im}^k + \frac{h}{2} \left(\delta_t^2 W_{im}^k - \delta_x^2 W_{im}^k \right. \\ &\quad \left. - f(x_i, y_m, t_k) \right) + O(\tau^2 + h^2). \end{aligned} \quad (38)$$

Applying (35) and (37) into (32), we get

$$\begin{aligned} D_{x-} W_{mj}^k + D_t W_{mj}^k + \frac{h}{2} \delta_t^2 W_{mj}^k - \frac{h}{2} \delta_y^2 W_{mj}^k &= \frac{h}{2} f(x_m, y_j, t_k) \\ &+ R_{mj}^k, \quad 1 \leq j \leq m-1, 1 \leq k \leq n-1, \end{aligned} \quad (39)$$

where $R_{mj}^k = O(\tau^2 + h^2)$. Similarly, we have

$$D_{y-}W_{im}^{\bar{k}} + D_t W_{im}^k + \frac{h}{2}\delta_t^2 W_{im}^k - \frac{h}{2}\delta_x^2 W_{im}^{\bar{k}} = \frac{h}{2}f(x_i, y_m, t_k) + R_{im}^k, \quad (40)$$

$$\begin{aligned} D_{x-}W_{mm}^{\bar{k}} + D_{y-}W_{mm}^{\bar{k}} + 2D_t W_{mm}^k + \frac{h}{2}\delta_t^2 W_{mm}^k \\ = hf(x_m, y_m, t_k) + R_{mm}^k, \quad 1 \leq i \leq m-1, 1 \leq k \leq n-1, \end{aligned} \quad (41)$$

where $R_{im}^k = O(\tau^2 + h^2)$ and $R_{mm}^k = O(\tau^2 + h^2)$.

When $k = 0$, equation (1) at initial grids (x_i, y_j, t_0) satisfies

$$\begin{aligned} \frac{\partial^2 w(x_i, y_j, t_0)}{\partial t^2} - \frac{\partial^2 w(x_i, y_j, t_0)}{\partial x^2} - \frac{\partial^2 w(x_i, y_j, t_0)}{\partial y^2} = f(x_i, y_j, t_0), \\ 1 \leq i, j \leq m-1. \end{aligned} \quad (42)$$

From Lemma 1(f) we have,

$$\frac{\partial^2 w(x_i, y_j, t_0)}{\partial t^2} = \frac{2}{\tau}(\delta_t W_{ij}^{1/2} - w_t(x_i, y_j, t_0)) + O(\tau). \quad (43)$$

Putting (43) into (42), then by Lemma 1(a) and (e), the following equations hold:

$$\begin{aligned} \frac{2}{\tau}(\delta_t W_{ij}^{1/2} - w_t(x_i, y_j, t_0)) - \delta_x^2 W_{ij}^{1/2} - \delta_y^2 W_{ij}^{1/2} = f(x_i, y_j, t_0) \\ + r_{ij}, \quad 1 \leq i, j \leq m-1, \end{aligned} \quad (44)$$

where $r_{ij} = O(\tau + h^2)$. At $(x_m, y_j, t_{1/2})$, we construct the following difference scheme:

$$\begin{aligned} \delta_x W_{m-(1/2),j}^{1/2} + \delta_t W_{mj}^{1/2} + \frac{h}{\tau}\delta_t W_{mj}^{1/2} - \frac{h}{2}\delta_y^2 W_{mj}^{1/2} = \frac{h}{\tau}\psi(x_m, y_j) \\ + \frac{h}{2}f(x_m, y_j, t_0) + r_{mj}, \end{aligned} \quad (45)$$

where $r_{mj} = O(\tau^2 + h^2)$. Similarly, at $(x_i, y_m, t_{1/2})$ and $(x_m, y_m, t_{1/2})$, we have

$$\begin{aligned} \delta_y W_{i,m-(1/2)}^{1/2} + \delta_t W_{im}^{1/2} + \frac{h}{\tau}\delta_t W_{im}^{1/2} - \frac{h}{2}\delta_x^2 W_{im}^{1/2} = \frac{h}{\tau}\psi(x_i, y_m) \\ + \frac{h}{2}f(x_i, y_m, t_0) + r_{im}, \end{aligned} \quad (46)$$

$$\begin{aligned} \delta_x W_{m-(1/2),m}^{1/2} + \delta_y W_{m,m-(1/2)}^{1/2} + 2\delta_t W_{mm}^{1/2} + \frac{h}{\tau}\delta_t W_{mm}^{1/2} \\ = \frac{2h}{\tau}\psi(x_m, y_m) + hf(x_m, y_m, t_0) + r_{mm}, \end{aligned} \quad (47)$$

respectively, where $r_{im} = O(\tau^2 + h^2)$ and $r_{mm} = O(\tau^2 + h^2)$. For the initial condition and the left and bottom boundaries, we have

$$W_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leq i, j \leq m, \quad (48)$$

$$W_{0j}^k = W_{i0}^k = 0, \quad 0 \leq i \leq m, 1 \leq j \leq m, 1 \leq k \leq n. \quad (49)$$

Finally, by dropping the infinitesimals in (31), (39)–(47), with W_{ij}^k replaced by w_{ij}^k , we get the finite difference schemes for (1)–(4) as follows

$$\begin{aligned} \delta_t^2 w_{ij}^k - \delta_x^2 w_{ij}^{\bar{k}} - \delta_y^2 w_{ij}^{\bar{k}} = f(x_i, y_j, t_k), \quad 1 \leq i, j \leq m-1, \\ 1 \leq k \leq n-1, \end{aligned} \quad (50)$$

$$\begin{aligned} D_{x-}w_{mj}^{\bar{k}} + D_t w_{mj}^k + \frac{h}{2}\delta_t^2 w_{mj}^k - \frac{h}{2}\delta_y^2 w_{mj}^{\bar{k}} = \frac{h}{2}f(x_m, y_j, t_k), \\ 1 \leq j \leq m-1, 1 \leq k \leq n-1, \end{aligned} \quad (51)$$

$$\begin{aligned} D_{y-}w_{im}^{\bar{k}} + D_t w_{im}^k + \frac{h}{2}\delta_t^2 w_{im}^k - \frac{h}{2}\delta_x^2 w_{im}^{\bar{k}} = \frac{h}{2}f(x_i, y_m, t_k), \\ 1 \leq i \leq m-1, 1 \leq k \leq n-1, \end{aligned} \quad (52)$$

$$\begin{aligned} D_{x-}w_{mm}^{\bar{k}} + D_{y-}w_{mm}^{\bar{k}} + 2D_t w_{mm}^k + \frac{h}{2}\delta_t^2 w_{mm}^k = \frac{h}{2}f(x_m, y_m, t_k), \\ 1 \leq k \leq n-1, \end{aligned} \quad (53)$$

$$\begin{aligned} \frac{2}{\tau}\delta_t w_{ij}^{1/2} - \delta_x^2 w_{ij}^{1/2} - \delta_y^2 w_{ij}^{1/2} = f(x_i, y_j, t_0) + \frac{2}{\tau}\psi(x_i, y_j), \\ 1 \leq i, j \leq m-1, \end{aligned} \quad (54)$$

$$\begin{aligned} \delta_x w_{m-(1/2),j}^{1/2} + \delta_t w_{mj}^{1/2} + \frac{h}{\tau}\delta_t w_{mj}^{1/2} - \frac{h}{2}\delta_y^2 w_{mj}^{1/2} = \frac{h}{\tau}\psi(x_m, y_j) \\ + \frac{h}{2}f(x_m, y_j, t_0), \quad 1 \leq j \leq m-1, \end{aligned} \quad (55)$$

$$\begin{aligned} \delta_y w_{i,m-(1/2)}^{1/2} + \delta_t w_{im}^{1/2} + \frac{h}{\tau}\delta_t w_{im}^{1/2} - \frac{h}{2}\delta_x^2 w_{im}^{1/2} = \frac{h}{\tau}\psi(x_i, y_m) \\ + \frac{h}{2}f(x_i, y_m, t_0), \quad 1 \leq i \leq m-1, \end{aligned} \quad (56)$$

$$\begin{aligned} \delta_x w_{m-(1/2),m}^{1/2} + \delta_y w_{m,m-(1/2)}^{1/2} + 2\delta_t w_{mm}^{1/2} + \frac{h}{\tau}\delta_t w_{mm}^{1/2} \\ = \frac{2h}{\tau}\psi(x_m, y_m) + \frac{h}{2}f(x_m, y_m, t_0), \end{aligned} \quad (57)$$

$$w_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leq i, j \leq m, \quad (58)$$

$$\begin{aligned} w_{i0}^k &= 0, \\ w_{0j}^k &= 0, \\ 0 \leq i \leq m, 1 \leq j \leq m, 1 \leq k \leq n. \end{aligned} \quad (59)$$

4. A Priori Estimate of the Difference Scheme

In order to prove the convergence and stability of difference schemes, we give a priori estimate of difference schemes (50)–(59).

Theorem 1. Suppose that $\{u_{ij}^k \mid 0 \leq i, j \leq m, 0 \leq k \leq n\}$ solves the following difference scheme:

$$\delta_t^2 u_{ij}^k - \delta_x^2 u_{ij}^k - \delta_y^2 u_{ij}^k = f_{ij}^k, \quad 1 \leq i, j \leq m-1, 1 \leq k \leq n-1, \quad (60)$$

$$\begin{aligned} D_x u_{mj}^k + D_t u_{mj}^k + \frac{h}{2} \delta_t^2 u_{mj}^k - \frac{h}{2} \delta_y^2 u_{mj}^k &= f_{mj}^k, \quad 1 \leq j \leq m-1, \\ 1 \leq k \leq n-1, \end{aligned} \quad (61)$$

$$\begin{aligned} D_y u_{im}^k + D_t u_{im}^k + \frac{h}{2} \delta_t^2 u_{im}^k - \frac{h}{2} \delta_x^2 u_{im}^k &= f_{im}^k, \quad 1 \leq i \leq m-1, \\ 1 \leq k \leq n-1, \end{aligned} \quad (62)$$

$$D_x u_{mm}^k + D_y u_{mm}^k + 2D_t u_{mm}^k + \frac{h}{2} \delta_t^2 u_{mm}^k = f_{mm}^k, \quad 1 \leq k \leq n-1, \quad (63)$$

$$\frac{2}{\tau} \delta_t u_{ij}^{1/2} - \delta_x^2 u_{ij}^{1/2} - \delta_y^2 u_{ij}^{1/2} = f_{ij}^0, \quad 1 \leq i, j \leq m-1, \quad (64)$$

$$\delta_x u_{m-(1/2),j}^{1/2} + \delta_t u_{mj}^{1/2} + \frac{h}{\tau} \delta_t u_{mj}^{1/2} - \frac{h}{2} \delta_y^2 u_{mj}^{1/2} = f_{mj}^0, \quad 1 \leq j \leq m-1, \quad (65)$$

$$\delta_y u_{i,m-(1/2)}^{1/2} + \delta_t u_{im}^{1/2} + \frac{h}{\tau} \delta_t u_{im}^{1/2} - \frac{h}{2} \delta_x^2 u_{im}^{1/2} = f_{im}^0, \quad 1 \leq i \leq m-1, \quad (66)$$

$$\delta_x u_{m-(1/2),m}^{1/2} + \delta_y u_{m,m-(1/2)}^{1/2} + 2\delta_t u_{mm}^{1/2} + \frac{h}{\tau} \delta_t u_{mm}^{1/2} = f_{mm}^0, \quad (67)$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leq i, j \leq m, \quad (68)$$

$$\begin{aligned} u_{i0}^k &= 0, \\ u_{0j}^k &= 0, \\ 0 \leq i \leq m, 1 \leq j \leq m, 1 \leq k \leq n, \end{aligned} \quad (69)$$

then for arbitrary grid ratio $\lambda = \tau/h$, we have

$$\begin{aligned} \|\delta_t u^{k+(1/2)}\|^2 + \frac{1}{2} \left(\|u^{k+1}\|_1^2 + \|u^k\|_1^2 \right) &\leq \exp\left(\frac{3}{2}T\right) \left(\|\varphi\|_1^2 + \frac{\tau^2}{4} \|f^0\|^2 + \frac{h\tau}{4} \sum_{j=1}^{m-1} |f_{mj}^0|^2 + \frac{h\tau}{4} \sum_{i=1}^{m-1} |f_{im}^0|^2 + \frac{h\tau}{16} |f_{mm}^0|^2 + \frac{3\tau}{2} \sum_{l=1}^k G^l \right), \\ 1 \leq k \leq n-1, \end{aligned} \quad (70)$$

where

$$G^l = h^2 \sum_{i,j=1}^{m-1} (f_{ij}^k)^2 + \frac{h}{2} \sum_{j=1}^{m-1} |f_{mj}^l|^2 + \frac{h}{2} \sum_{i=1}^{m-1} |f_{im}^l|^2 + \frac{h}{8} |f_{mm}^l|^2, \quad 1 \leq l \leq n-1. \quad (71)$$

Proof. Define

$$E^k = \|\delta_t u^{k+(1/2)}\|^2 + \frac{1}{2} \left(\|u^{k+1}\|_1^2 + \|u^k\|_1^2 \right). \quad (72)$$

Multiplying both sides of (60) by $2h^2 D_t u_{ij}^k$, and taking summation for i and j from 1 to $m-1$, we get

$$2h^2 \sum_{i,j=1}^{m-1} \delta_t^2 u_{ij}^k D_t u_{ij}^k - 2h^2 \sum_{i,j=1}^{m-1} \delta_x^2 u_{ij}^{\bar{k}} D_t u_{ij}^k - 2h^2 \sum_{i,j=1}^{m-1} \delta_y^2 u_{ij}^{\bar{k}} D_t u_{ij}^k = 2h^2 \sum_{i,j=1}^{m-1} f_{ij}^k D_t u_{ij}^k, \quad 1 \leq k \leq n-1. \quad (73)$$

The first term in (73) implies

$$2h^2 \sum_{i,j=1}^{m-1} \delta_t^2 u_{ij}^k D_t u_{ij}^k = \frac{1}{\tau} \left(h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k+(1/2)})^2 - h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k-(1/2)})^2 \right). \quad (74)$$

The second term and the third term in (73) imply

$$\begin{aligned} & -2h^2 \sum_{i,j=1}^{m-1} \delta_x^2 u_{ij}^{\bar{k}} D_t u_{ij}^k \\ &= -2h \sum_{i,j=1}^{m-1} \left(\delta_x u_{i+(1/2),j}^{\bar{k}} - \delta_x u_{i-(1/2),j}^{\bar{k}} \right) D_t u_{ij}^k \\ &= 2h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} \delta_x u_{i+(1/2),j}^{\bar{k}} D_t \delta_x u_{i+(1/2),j}^k - 2h \sum_{j=1}^{m-1} D_{x-} u_{mj}^{\bar{k}} D_t u_{mj}^k \end{aligned} \quad (75)$$

$$\begin{aligned} &= \frac{1}{2\tau} \left(h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} (\delta_x u_{i+(1/2),j}^{k+1})^2 - h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} (\delta_x u_{i+(1/2),j}^{k-1})^2 \right) - 2h \sum_{j=1}^{m-1} D_{x-} u_{mj}^{\bar{k}} D_t u_{mj}^k, \\ & -2h^2 \sum_{i,j=1}^{m-1} \delta_y^2 u_{ij}^{\bar{k}} D_t u_{ij}^k \\ &= \frac{1}{2\tau} \left(h^2 \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} (\delta_y u_{i,j+(1/2)}^{k+1})^2 - h^2 \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} (\delta_y u_{i,j+(1/2)}^{k-1})^2 \right) - 2h \sum_{i=1}^{m-1} D_{y-} u_{im}^{\bar{k}} D_t u_{im}^k, \end{aligned} \quad (76)$$

respectively. Putting (74), (75) and (76) into (73), we get

$$\begin{aligned} & \frac{1}{\tau} \left(h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k+(1/2)})^2 - h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k-(1/2)})^2 \right) + \frac{1}{2\tau} \left(h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} (\delta_x u_{i+(1/2),j}^{k+1})^2 \right. \\ & \left. - h^2 \sum_{i=0}^{m-1} \sum_{j=1}^{m-1} (\delta_x u_{i+(1/2),j}^{k-1})^2 \right) + \frac{1}{2\tau} \left(h^2 \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} (\delta_y u_{i,j+(1/2)}^{k+1})^2 - h^2 \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} (\delta_y u_{i,j+(1/2)}^{k-1})^2 \right) \\ &= 2h^2 \sum_{i,j=1}^{m-1} f_{ij}^k D_t u_{ij}^k + 2h \sum_{j=1}^{m-1} D_{x-} u_{mj}^{\bar{k}} D_t u_{mj}^k + 2h \sum_{i=1}^{m-1} D_{y-} u_{im}^{\bar{k}} D_t u_{im}^k, \end{aligned} \quad (77)$$

where

$$\begin{aligned}
2h^2 \sum_{i,j=1}^{m-1} f_{ij}^k D_t u_{ij}^k &\leq h^2 \sum_{i,j=1}^{m-1} (D_t u_{ij}^k)^2 + h^2 \sum_{i,j=1}^{m-1} (f_{ij}^k)^2 \\
&\leq \frac{1}{2} \left(h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k+(1/2)})^2 + h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{k-(1/2)})^2 \right) + h^2 \sum_{i,j=1}^{m-1} (f_{ij}^k)^2.
\end{aligned} \tag{78}$$

Considering (61)–(63) and by inequality $2(b-a)a \leq (1/2)b^2$, we get

$$\begin{aligned}
&2h \sum_{j=1}^{m-1} D_{x-} u_{mj}^k D_t u_{mj}^k + 2h \sum_{i=1}^{m-1} D_{y-} u_{im}^k D_t u_{im}^k \\
&= h \sum_{j=1}^{m-1} \left(2(f_{mj}^k - D_t u_{mj}^k) D_t u_{mj}^k - h \delta_t^2 u_{mj}^k D_t u_{mj}^k + h \delta_y^2 u_{mj}^k D_t u_{mj}^k \right) \\
&\quad + h \sum_{i=1}^{m-1} \left(2(f_{im}^k - D_t u_{im}^k) D_t u_{im}^k - h \delta_t^2 u_{im}^k D_t u_{im}^k + h \delta_x^2 u_{im}^k D_t u_{im}^k \right) \\
&\leq -\frac{h^2}{2\tau} \sum_{j=1}^{m-1} \left((\delta_t u_{mj}^{k+(1/2)})^2 - (\delta_t u_{mj}^{k-(1/2)})^2 \right) - \frac{h^2}{4\tau} \sum_{j=0}^{m-1} \left((\delta_y u_{m,j+(1/2)}^{k+1})^2 - (\delta_y u_{m,j+(1/2)}^{k-1})^2 \right) \\
&\quad - \frac{h^2}{2\tau} \sum_{i=1}^{m-1} \left((\delta_t u_{im}^{k+(1/2)})^2 - (\delta_t u_{im}^{k-(1/2)})^2 \right) - \frac{h^2}{4\tau} \sum_{i=0}^{m-1} \left((\delta_x u_{i+(1/2),m}^{k+1})^2 - (\delta_x u_{i+(1/2),m}^{k-1})^2 \right) \\
&\quad + \frac{h}{8} |f_{mm}^k|^2 - \frac{h^2}{4\tau} \left((\delta_t u_{mm}^{k+(1/2)})^2 - (\delta_t u_{mm}^{k-(1/2)})^2 \right) + \frac{h}{2} \sum_{j=1}^{m-1} |f_{mj}^k|^2 + \frac{h}{2} \sum_{i=1}^{m-1} |f_{im}^k|^2.
\end{aligned} \tag{79}$$

According to the definition of E^k , (77) implies

$$\begin{aligned}
\frac{1}{\tau} (E^k - E^{k-1}) &\leq \frac{1}{2} (E^k + E^{k-1}) + h^2 \sum_{i,j=1}^{m-1} (f_{ij}^k)^2 + \frac{h}{2} \sum_{j=1}^{m-1} |f_{mj}^k|^2 \\
&\quad + \frac{h}{2} \sum_{i=1}^{m-1} |f_{im}^k|^2 + \frac{h}{8} |f_{mm}^k|^2, \quad 1 \leq k \leq n-1.
\end{aligned} \tag{80}$$

If we confine $\tau \leq (2/3)$, then

$$E^k \leq \left(1 + \frac{3\tau}{2}\right) E^{k-1} + \frac{3\tau}{2} G^k, \quad 1 \leq k \leq n-1. \tag{81}$$

By Gronwall inequality in Lemma 2, we get

$$E^k \leq \exp\left(\frac{3}{2}k\tau\right) \left(E^0 + \frac{3\tau}{2} \sum_{l=1}^k G^l\right), \quad 1 \leq k \leq n-1. \tag{82}$$

Next we estimate E^0 . Multiplying (64) by $h^2 \delta_t u_{ij}^{1/2}$, and summing up for i and j from 1 to $m-1$, we have

$$\begin{aligned}
&\frac{2}{\tau} h^2 \sum_{i,j=1}^{m-1} (\delta_t u_{ij}^{1/2})^2 - h^2 \sum_{i,j=1}^{m-1} (\delta_x^2 u_{ij}^{1/2} + \delta_y^2 u_{ij}^{1/2}) \delta_t u_{ij}^{1/2} \\
&= h^2 \sum_{i,j=1}^{m-1} f_{ij}^0 \delta_t u_{ij}^{1/2},
\end{aligned} \tag{83}$$

which similarly implies

$$\begin{aligned}
&\frac{2}{\tau} \|\delta_t u^{1/2}\|^2 + \frac{1}{2\tau} (|u^1|_1^2 - |u^0|_1^2) \\
&\leq \frac{1}{\tau} \|\delta_t u^{1/2}\|^2 + \frac{\tau}{4} \|f^0\|^2 + \frac{h}{4} \sum_{j=1}^{m-1} |f_{mj}^0|^2 + \frac{h}{4} \sum_{i=1}^{m-1} |f_{im}^0|^2 + \frac{h}{16} |f_{mm}^0|^2.
\end{aligned} \tag{84}$$

According to the definition of E^0 , we obtain

$$\begin{aligned} E^0 &= \|\delta_t u^{1/2}\|^2 + \frac{1}{2} \left(|u^1|_1^2 + |u^0|_1^2 \right) \\ &\leq |u^0|_1^2 + \frac{\tau^2}{4} \|f^0\|^2 + \frac{h\tau}{4} \sum_{j=1}^{m-1} |f_{mj}^0|^2 + \frac{h\tau}{4} \sum_{i=1}^{m-1} |f_{im}^0|^2 + \frac{h\tau}{16} |f_{mm}^0|^2. \end{aligned} \quad (85)$$

Putting (85) into (82) and noticing $k\tau \leq T$, we achieve (70). This completes the proof. $\square$

5. Existence, Convergence, and Stability

In this section, we will discuss the unique solvability, convergence, and stability of the difference scheme (50)–(59).

Theorem 2. *The difference scheme (50)–(59) is uniquely solvable.*

Proof. Notice that w^0 is known by the initial condition. The difference scheme for w^1 is (54)–(59). We consider the following homogeneous equations of (54)–(59):

$$\frac{2}{\tau^2} w_{ij}^1 - \frac{1}{2} \delta_x^2 w_{ij}^1 - \frac{1}{2} \delta_y^2 w_{ij}^1 = 0, \quad 1 \leq i, j \leq m-1, \quad (86)$$

$$\frac{1}{2} \delta_x w_{m-(1/2),j}^1 + \frac{1}{\tau} w_{mj}^1 + \frac{h}{\tau^2} w_{mj}^1 - \frac{h}{4} \delta_y^2 w_{mj}^1 = 0, \quad 1 \leq j \leq m-1, \quad (87)$$

$$\frac{1}{2} \delta_y w_{i,m-(1/2)}^1 + \frac{1}{\tau} w_{im}^1 + \frac{h}{\tau^2} w_{im}^1 - \frac{h}{4} \delta_x^2 w_{im}^1 = 0, \quad 1 \leq i \leq m-1, \quad (88)$$

$$\frac{1}{2} \delta_x w_{m-(1/2),m}^1 + \frac{1}{2} \delta_y w_{m,m-(1/2)}^1 + \frac{2}{\tau} w_{mm}^1 + \frac{h}{\tau^2} w_{mm}^1 = 0, \quad (89)$$

$$\begin{aligned} w_{i0}^1 &= 0, \quad 0 \leq i \leq m, \\ w_{0j}^1 &= 0, \quad 1 \leq j \leq m. \end{aligned} \quad (90)$$

Multiplying (86) by $h^2 w_{ij}^1$, summing up for i and j from 1 to $m-1$, and combining with (87)–(90), we get

$$\frac{2}{\tau^2} \|w^1\|^2 + \frac{1}{2} |w^1|_1^2 + \frac{h}{\tau} \left(\sum_{j=1}^{m-1} (w_{mj}^1)^2 + \sum_{i=1}^{m-1} (w_{im}^1)^2 + (w_{mm}^1)^2 \right) = 0, \quad (91)$$

which implies $w_{ij}^1 = 0$ ($0 \leq i, j \leq m$). Thus, w^1 is uniquely solvable.

Now suppose that w^{k-1} and w^k are uniquely solvable. The difference schemes for w^{k+1} satisfy (50)–(53) and (59). The homogeneous equations read:

$$\frac{1}{\tau^2} w_{ij}^{k+1} - \frac{1}{2} \delta_x^2 w_{ij}^{k+1} - \frac{1}{2} \delta_y^2 w_{ij}^{k+1} = 0, \quad 1 \leq i, j \leq m-1, 1 \leq k \leq n-1, \quad (92)$$

$$\frac{1}{2} D_{x-} w_{mj}^{k+1} + \frac{1}{2\tau} w_{mj}^{k+1} + \frac{h}{2\tau^2} w_{mj}^{k+1} - \frac{h}{4} \delta_y^2 w_{mj}^{k+1} = 0, \quad 1 \leq j \leq m-1, 1 \leq k \leq n-1, \quad (93)$$

$$\frac{1}{2} D_{y-} w_{im}^{k+1} + \frac{1}{2\tau} w_{im}^{k+1} + \frac{h}{2\tau^2} w_{im}^{k+1} - \frac{h}{4} \delta_x^2 w_{im}^{k+1} = 0, \quad 1 \leq i \leq m-1, 1 \leq k \leq n-1, \quad (94)$$

$$\frac{1}{2} D_{x-} w_{mm}^{k+1} + \frac{1}{2} D_{y-} w_{mm}^{k+1} + \frac{1}{\tau} w_{mm}^{k+1} + \frac{h}{2\tau^2} w_{mm}^{k+1} = 0, \quad 1 \leq k \leq n-1, \quad (95)$$

$$\begin{aligned} w_{i0}^k &= 0, \\ w_{0j}^k &= 0, \\ 0 &\leq i \leq m, 0 \leq j \leq m, 1 \leq k \leq n. \end{aligned} \quad (96)$$

Multiplying (92) by $h^2 w_{ij}^{k+1}$, and summing up for i and j from 1 to $m-1$, and applying equations (93)–(96), we have

$$\frac{1}{\tau^2} \|w^{k+1}\|^2 + \frac{1}{2} |w^{k+1}|_1^2 + \frac{h}{2\tau} \left(\sum_{j=1}^{m-1} (w_{mj}^{k+1})^2 + \sum_{i=1}^{m-1} (w_{im}^{k+1})^2 + (w_{mm}^{k+1})^2 \right) = 0, \quad (97)$$

which means that $w_{ij}^{k+1} = 0$ ($0 \leq i, j \leq m$). Thus w^{k+1} is uniquely solvable.

According to the principle of induction, the difference scheme of (50)–(59) is uniquely solvable for all w^k ($1 \leq k \leq n$). This completes the proof. $\square$

Define the pointwise error by

$$e_{ij}^k = W_{ij}^k - w_{ij}^k, \quad 0 \leq i, j \leq m, 0 \leq k \leq n. \quad (98)$$

The following theorem states the convergence of difference scheme (50)–(59).

Theorem 3. *Suppose that $w(x, y, t) \in C_{x,y,t}^{4,4,3}(\overline{\Omega} \times [0, T])$ is the solution of (1)–(4) and $\{w_{ij}^k | 0 \leq i, j \leq m, 0 \leq k \leq n\}$ is the solution of difference scheme (50)–(59). Then there exists a constant C_T such that*

$$\|\delta_t e^{k+(1/2)}\|^2 + \frac{1}{2} \left(|e^{k+1}|_1^2 + |e^k|_1^2 \right) \leq C_T (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1, \quad (99)$$

where $C_T = \exp((3/2)T)((5/4) + (3/2)T)c^2$.

Proof. Subtracting (50)–(59) from (31), (39)–(41) and (44)–(49) respectively, we have the following error equation

$$\begin{aligned}
& \delta_t^2 e_{ij}^k - \delta_x^2 e_{ij}^{\bar{k}} - \delta_y^2 e_{ij}^{\bar{k}} = R_{ij}^k, \quad 1 \leq i, j \leq m-1, 1 \leq k \leq n-1, \\
& D_{x-} e_{mj}^{\bar{k}} + D_t e_{mj}^k + \frac{h}{2} \delta_t^2 e_{mj}^k - \frac{h}{2} \delta_y^2 e_{mj}^{\bar{k}} = R_{mj}^k, \quad 1 \leq j \leq m-1, 1 \leq k \leq n-1, \\
& D_{y-} e_{im}^{\bar{k}} + D_t e_{im}^k + \frac{h}{2} \delta_t^2 e_{im}^k - \frac{h}{2} \delta_x^2 e_{im}^{\bar{k}} = R_{im}^k, \quad 1 \leq i \leq m-1, 1 \leq k \leq n-1, \\
& D_{x-} e_{mm}^{\bar{k}} + D_{y-} e_{mm}^{\bar{k}} + 2D_t e_{mm}^k + \frac{h}{2} \delta_t^2 e_{mm}^k = R_{mm}^k, \quad 1 \leq k \leq n-1, \\
& \frac{2}{\tau} \delta_t e_{ij}^{1/2} - \delta_x^2 e_{ij}^{1/2} - \delta_y^2 e_{ij}^{1/2} = r_{ij}, \quad 1 \leq i, j \leq m-1, \\
& \delta_x e_{m-(1/2),j}^{1/2} + \delta_t e_{mj}^{1/2} + \frac{h}{\tau} \delta_t e_{mj}^{1/2} - \frac{h}{2} \delta_y^2 e_{mj}^{1/2} = r_{mj}, \quad 1 \leq j \leq m-1, \\
& \delta_y e_{i,m-(1/2)}^{1/2} + \delta_t e_{im}^{1/2} + \frac{h}{\tau} \delta_t e_{im}^{1/2} - \frac{h}{2} \delta_x^2 e_{im}^{1/2} = r_{im}, \quad 1 \leq i \leq m-1, \\
& \delta_x e_{m-(1/2),m}^{1/2} + \delta_y e_{m,m-(1/2)}^{1/2} + 2\delta_t e_{mm}^{1/2} + \frac{h}{\tau} \delta_t e_{mm}^{1/2} = r_{mm}, \\
& e_{ij}^0 = 0, \quad 0 \leq i, j \leq m, \\
& e_{0j}^k = e_{i0}^k = 0, \quad 0 \leq i \leq m, 1 \leq j \leq m, 0 \leq k \leq n.
\end{aligned} \tag{100}$$

According to Theorem 1, we get

$$\begin{aligned}
& \left\| \delta_t e^{k+(1/2)} \right\|^2 + \frac{1}{2} \left(|e^{k+1}|_1^2 + |e^k|_1^2 \right) \leq \exp\left(\frac{3}{2}k\tau\right) \left(|e^0|_1^2 + \frac{\tau^2}{4} \|r\|^2 + \frac{h\tau}{4} \sum_{j=1}^{m-1} |r_{mj}|^2 \right. \\
& \quad \left. + \frac{h\tau}{4} \sum_{i=1}^{m-1} |r_{im}|^2 + \frac{h\tau}{16} |r_{mm}|^2 + \frac{3\tau}{2} \sum_{l=1}^k \widehat{G}^l \right), \quad 1 \leq k \leq n-1,
\end{aligned} \tag{101}$$

where

$$\widehat{G}^l = \|R^l\|^2 + \frac{h}{2} \sum_{j=1}^{m-1} |R_{mj}^l|^2 + \frac{h}{2} \sum_{i=1}^{m-1} |R_{im}^l|^2 + \frac{h}{8} |R_{mm}^l|^2, \quad 1 \leq l \leq n-1. \tag{102}$$

Taking into account the order of r_{ij} and R_{ij}^k ($0 \leq i, j \leq m, 1 \leq k \leq n$), and the fact that $e_{ij}^0 = 0$, there exists a generic constant c such that

$$\begin{aligned}
& \left\| \delta_t e^{k+(1/2)} \right\|^2 + \frac{1}{2} \left(|e^{k+1}|_1^2 + |e^k|_1^2 \right) \\
& \leq \exp\left(\frac{3}{2}k\tau\right) \left(\frac{\tau^2}{4} c^2 (\tau + h^2)^2 + \frac{\tau h}{4} c^2 (\tau^2 + h^2)^2 + \frac{\tau h}{4} c^2 (\tau^2 + h^2)^2 + \frac{\tau h}{16} c^2 (\tau^2 + h^2)^2 + \frac{3}{2} k \tau c^2 (\tau^2 + h^2)^2 \right) \\
& \leq \exp\left(\frac{3}{2}T\right) \left(\frac{5}{4} + \frac{3}{2}T \right) c^2 (\tau^2 + h^2)^2 = C_T (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1.
\end{aligned} \tag{103}$$

FIGURE 1: The evolution mesh (a) and contour map (b) of numerical solution.

TABLE 1: Numerical convergence of scheme (50)–(59) in time ($h = 1/100$, $T = 1$).

τ	$EW(\tau, h)$	Order
1/20	$1.2948e-002$	—
1/40	$2.9438e-003$	2.1370
1/50	$1.8099e-003$	2.1799
1/60	$1.2057e-003$	2.2280
1/70	$8.4660e-004$	2.2938

TABLE 2: Numerical convergence of scheme (50)–(59) in space ($\tau = 1/200$, $T = 1$).

h	$EW(\tau, h)$	Order
1/20	$2.4487e-003$	—
1/30	$1.0292e-003$	2.1377
1/40	$5.4095e-004$	2.2358
1/50	$3.1889e-004$	2.3683
1/60	$2.0234e-004$	2.4950

This completes the proof. $\square$

According to Lemma 4, we can conclude that $\|e^k\| = O(\tau^2 + h^2)$ ($1 \leq k \leq n$), namely, the difference solution is of second-order convergence in L^2 norm.

TABLE 3: Numerical convergence of scheme (50)–(59) with $\lambda = 1$ ($T = 1$).

h	$EW(\tau, h)$	Order
1/20	$1.1049e-002$	—
1/30	$4.8033e-003$	2.0545
1/40	$2.6446e-003$	2.0745
1/50	$1.6725e-003$	2.0462
1/60	$1.1536e-003$	2.0461

TABLE 4: Numerical convergence of scheme (50)–(59) with $\lambda = 2$ ($T = 1$).

h	$EW(\tau, h)$	Order
1/30	$2.1489e-002$	—
1/40	$1.2564e-002$	1.8656
1/50	$7.9166e-003$	2.0699
1/60	$5.4036e-003$	2.0946
1/70	$3.9183e-003$	2.0850

Theorem 4. For any fixed grid ratio $\lambda = \tau/h$, the difference scheme (50)–(59) is stable with respect to the initial value φ , ψ , and the source term f . More precisely, we have

$$\|\delta_t w^{k+(1/2)}\|^2 + \frac{1}{2} \left(\|w^{k+1}\|_1^2 + \|w^k\|_1^2 \right) \leq \exp\left(\frac{3}{2}T\right) \left(\|\varphi\|_1^2 + c\|\psi\|^2 + \frac{c}{9}\|f^0\|^2 + \frac{3\tau}{2} \sum_{l=1}^k \|f^l\|^2 \right), \quad 0 \leq k \leq n-1, \quad (104)$$

FIGURE 2: The curve of convergence order in Tables 1–4.

where $c = \max\{2, (1/\lambda)\}$.

Proof. Suppose that the solution of difference scheme (50)–(59) is $\{w_{ij}^k \mid 0 \leq i, j \leq m, 0 \leq k \leq n\}$, then for any λ we can get similarly to Theorem 1

$$\|\delta_t w^{k+(1/2)}\|^2 + \frac{1}{2}(|w^{k+1}|_1^2 + |w^k|_1^2) \leq \exp\left(\frac{3}{2}T\right) \left(E^0 + \frac{3\tau}{2} \sum_{l=1}^k \|f^l\|^2\right), \quad 0 \leq k \leq n-1, \quad (105)$$

where

$$\begin{aligned} E^0 &= \|\delta_t w^{1/2}\|^2 + \frac{1}{2}(|w^1|_1^2 + |w^0|_1^2) \\ &\leq |\varphi|_1^2 + 2h^2 \sum_{i,j=1}^{m-1} \psi_{ij}^2 + \frac{h^3}{2\tau} \sum_{j=1}^{m-1} \psi_{mj}^2 + \frac{h^3}{2\tau} \sum_{i=1}^{m-1} \psi_{im}^2 + \frac{h^3}{8\tau} \psi_{mm}^2 \\ &\quad + \frac{\tau^2 h^2}{2} \sum_{i,j=1}^{m-1} (f_{ij}^0)^2 + \frac{\tau h^3}{8} \sum_{j=1}^{m-1} (f_{mj}^0)^2 + \frac{\tau h^3}{8} \sum_{i=1}^{m-1} (f_{im}^0)^2 + \frac{\tau h^3}{32} (f_{mm}^0)^2. \end{aligned} \quad (106)$$

Taking account into the fact that $\tau \leq (2/3)$, $h = (\tau/\lambda) \leq (2/3\lambda)$, there exists constant $c = \max\{2, (1/\lambda)\}$ such that

$$E^0 \leq |\varphi|_1^2 + c\|\psi\|^2 + \frac{c}{9}\|f^0\|^2, \quad (107)$$

which ends the proof. $\square$

6. Numerical Result

In order to test the convergence order of proposed difference schemes (50)–(59), we consider a two-dimensional damped wave equation, in which the source function $f(x, y, t)$ is defined by

$$f(x, y, t) = \frac{243\pi^2}{16} \exp\left(-\frac{9}{4}\pi t\right) \sin\left(\frac{9}{4}\pi x\right) \sin\left(\frac{9}{4}\pi y\right), \quad (108)$$

with initial conditions $\varphi(x, y)$ and $\psi(x, y)$ designed as

$$\begin{aligned} \varphi(x, y) &= \sin\left(\frac{9}{4}\pi x\right) \sin\left(\frac{9}{4}\pi y\right), \\ \psi(x, y) &= -\frac{9}{4}\pi \sin\left(\frac{9}{4}\pi x\right) \sin\left(\frac{9}{4}\pi y\right). \end{aligned} \quad (109)$$

For this specified problem, the exact solution is

$$w(x, y, t) = \exp\left(-\frac{9}{4}\pi t\right) \sin\left(\frac{9}{4}\pi x\right) \sin\left(\frac{9}{4}\pi y\right). \quad (110)$$

For the source term and initial conditions defined above, the evolution mesh and contour map of the numerical solution of difference schemes (50)–(59) with $N=40$ and $T=0.5$ are shown in Figure 1.

Let us denote the error by

$$EW(\tau, h) = \|w(\tau, h) - W(\tau, h)\|. \quad (111)$$

Table 1 provides the computed results with a fixed $h = 1/100$. The data demonstrate that the temporal convergence order of difference scheme (50)–(59) is second order in discrete L^2 -norm, which is in accord with

theoretical analysis of Theorem 3. When the temporal size is fixed with $\tau = 1/200$, Table 2 gives the discrete L^2 -errors and numerical convergence orders in space direction. From these data, we can see that the spatial convergence order of difference scheme (50)–(59) is also second order, which is in agreement with theoretical results in Theorem 3.

The convergence results with fixed grid ratio $\lambda = 1$ and $\lambda = 2$ are shown in Tables 3 and 4, respectively. For these two cases, the proposed difference scheme (50)–(59) can preserve both the convergence and stability. From the numerical results, we can verify the unconditional stability of difference scheme (50)–(59). Also for other different values of λ , the same stability results can be drawn, which are accordant with Theorem 4. The curve of convergence order in Tables 1–4 is shown in Figure 2.

7. Conclusion

In this paper, a finite difference scheme is constructed for the two-dimensional wave equation with a special boundary condition, damped boundary condition. By introducing a proper discrete L^2 -norm, the proposed finite difference scheme is proved to be second-order convergent in both time and space and unconditionally stable with respect to both initial conditions and source term. The theoretical analysis method can be applied to three-dimensional problems accordingly.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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