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Research Article

On Structural Properties of ξ-Complex Fuzzy Sets and Their Applications

Aneeza Imtiaz,[1] Umer Shuaib ,[1] Hanan Alolaiyan ,[2] Abdul Razaq,[3] and Muhammad Gulistan [4]

[1]*Department of Mathematics, Government College University, Faisalabad 38000, Pakistan*
[2]*Department of Mathematics, King Saud University, Riyadh, Saudi Arabia*
[3]*Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54000, Pakistan*
[4]*Department of Mathematics and Statistics, Hazara University Mansehra, Mansehra, KPK, Pakistan*

Correspondence should be addressed to Hanan Alolaiyan; holayan@ksu.edu.sa

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Complex fuzzy sets are the novel extension of Zadeh's fuzzy sets. In this paper, we comprise the introduction to the concept of ξ-complex fuzzy sets and proofs of their various set theoretical properties. We define the notion of (α, δ)-cut sets of ξ-complex fuzzy sets and justify the representation of an ξ-complex fuzzy set as a union of nested intervals of these cut sets. We also apply this newly defined concept to a physical situation in which one may judge the performance of the participants in a given task. In addition, we innovate the phenomena of ξ-complex fuzzy subgroups and investigate some of their fundamental algebraic attributes. Moreover, we utilize this notion to define level subgroups of these groups and prove the necessary and sufficient condition under which an ξ-complex fuzzy set is ξ-complex fuzzy subgroup. Furthermore, we extend the idea of ξ-complex fuzzy normal subgroup to define the quotient group of a group G by this particular ξ-complex fuzzy normal subgroup and establish an isomorphism between this quotient group and a quotient group of G by a specific normal subgroup $G_{A^{\xi}}$.

1. Introduction

The competency of fuzzy logic to articulate steady adaptations from membership to nonmembership and the other way around has played an effective role to solve many physical problems. It provides us not only the powerful and meaningful representations of measuring uncertainty but also a useful approach to view the vague concepts expressed in the natural language. Despite all of the advantages of this logic, we still face immense complications to counter various physical situations based on a real-valued membership function. It is therefore quite necessary to propose an additional development of fuzzy set theory on account of set of complex numbers which is indeed an existing augmentation of real numbers. The complex fuzzy logic is linear augmentation of traditional fuzzy logic, which also allows a natural development of the difficulty based on the fuzzy logic that is impracticable to solve with superficial membership function. This particular set has a paramount part in numerous executions, in particular, advanced control systems and predicting of the periodic events, where multiple fuzzy variables are interrelated in a complex manner that cannot be effectively characterized by simple fuzzy operations. In addition, these sets are also used to solve several problems, especially the numerous periodic aspects and forecast problems. One of the far-reaching significances of studying the CFS is that they illustrate the data with uncertainty and periodicity in a much effective way.

Considering the inaccuracy in decision-making, Zadeh [1] popularized the concept of fuzzy sets for the first time, in 1965. Roenfeld [2] used Zadeh's work to invent the approach of fuzzy subgroups in 1971. Mukherjee and Bhattacharya [3] initiated the study of the fuzzy cosets along with fuzzy normal subgroups in 1984. Mashour et al. [4] studied many important features of normal fuzzy subgroups. For more details about the recent development of fuzzy subgroups, we refer to [5–7]. In addition, another important aspect of fuzzy sets is the level sets of this notion. These level sets have an incredible significance as they set up a connection between concepts of crisp and fuzzy sets. This phenomenon is utilized to generalize various ideas and techniques for crisp set hypothesis as well as fuzzy sets. In the succession of fuzzification of subgroups, the idea of level subgroup was initiated by Das [8]. Yuan and Li [9] studied the impact of these level sets in the field of intuitionistic fuzzy sets. To see more on the level sets and level subgroups, we refer to [10–12]. Buckley [13] proposed the study of complex fuzzy numbers in 1989. The author [14] applied the theory of complex fuzzy numbers to set up new techniques of differentiation. Moreover, some fundamental properties of fuzzy counter integral in the complex plane were interpreted by the same author in [15]. Zhang [16] established many important properties of complex fuzzy numbers in 1992. Ascia et al. [17] designed a competent specific fuzzy processor which effectively deals with the complex fuzzy inference system. In [18], Ramote et al. launched the study of CFS in 2002. The two novel operations, namely, reflection and rotations, of these sets were introduced in [19]. Zhang et al. [20] formulated the δ-equalities of CFS. In 2011, Chen et al. [21] developed the adaptive neuro complex fuzzy inference system. In [22], Fu and Shen utilized CFS to design a new approach of linguistic evaluation of classifier performance. Tamir and Kandel [23] provided granted bases for first order estimation, complex class theory, and complex fuzzy logic in the same year. Ma et al. [24] defined product sum accruement operator for these particular sets in 2012. Li et al. [25] presented a new self-learning complex neuro fuzzy system using the same idea. In 2014, Alkouri and Salleh [26] illustrated many important multiple distance measures defined on CFS. In 2015, Tamir et al. [27] reintroduced the concept of CFS and complex fuzzy logic in a more logical way. Al-Husban and Salleh [28] interpreted the study of complex fuzzy hyper structure over complex fuzzy space in 2016. The phenomenon of CFSG over complex fuzzy space was discussed in [29]. The operator theory has an important role in modern mathematics because it is extensively used in fuzzy theory for approximate reasoning. The techniques of t-norms and t-co-norms with respect to complex fuzzy sets were presented by Nagarajan et al. [30]. For more on fuzzy set theory, we suggest reading of [31–35]. The competency of the complex fuzzy set has played an effective role to solve many physical problems. It provides us the meaningful representations of measuring uncertainty and periodicity. Despite all of these advantages, we still face vast complications to counter various physical situations based on a complex-valued membership function. This motivates us to define the notion of ξ-complex fuzzy set (ξ-CFS) through which one can have multiple options to investigate a specific real-world situation in much efficient way by choosing appropriate value of the parameter ξ.

In this article, we propose the idea of ξ-CFS as a powerful extension of classical fuzzy set and use this idea to define the notion of the cut sets of these particular sets. We explore the importance of these cut sets by proving the decomposition theorems for a ξ-CFS. Moreover, we introduce the phenomenon of ξ-complex fuzzy subgroup (ξ-CFSG) over an ξ-complex fuzzy set and investigate some of their fundamental algebraic attributes.

After a brief discussion about the historical background and significance of CFS, the rest of the article is organized as follows. Section 2 contains some definitions of the basic introduction of the concept of ξ-CFS and the study of (α, δ)-cut sets and strong (α, δ)-cut sets of this newly defined notion. In addition, we establish the importance of defining these ideas by viewing an ξ-CFS as a union of nested sequence of both (α, δ)-cut sets and strong (α, δ)-cut sets, respectively. Moreover, we apply ξ-CFS to a physical situation in which one may select the most suitable performance of a participant. In Section 4, we utilize the concept of ξ-CFS to propose the study of the idea of ξ-CFSG and prove that each CFSG is ξ-CFSG. Moreover, we extend the importance of these fuzzy subgroups by introducing the notions of ξ-complex fuzzy cosets and ξ-complex fuzzy normal subgroup (ξ-CFNSG) and investigate their many important algebraic aspects.

2. Preliminaries

This section contains a brief review of the notion of CFS and related ideas, which are quite essential to understand the novelty of this article.

Definition 1 (see [19]). A CFS A defined on a universe of discourse U is characterized by a membership function $\mu_A(m)$ that allocates each element of $U$to a unit circle C^* in complex plane and is written as $r_A(m)e^{i\omega_A(m)}$, where $r_A(m)$ denotes the real-valued function from U to the closed unit interval and $e^{i\omega_A(m)}$ is a periodic function whose periodic law and principal period are 2π and $0 < arg_A(m) \leq 2\pi$, respectively.

Note that $\omega_A(m) = \arg_A(m) + 2k\pi$, $k \in Z$ and $arg_A(m)$ is the principal argument.

Definition 2 (see [8]). Let $0 \leq \alpha \leq 1$ and $0 \leq \delta \leq 2\pi$. Then, the (α, δ)-cut set of CFS A is denoted by $A_{(\alpha,\delta)}$ and is defined as $A_{(\alpha,\delta)} = \{m \in U: r_A(m) \geq \alpha, \omega_A(m) \geq \delta\}$.

Definition 3 (see [31]). Let A and B be any two CFS of a universe U. Then,

(1) A is homogeneous CFS if $r_A(m) \leq r_A(n)$ implies $\omega_A(m) \leq \omega_A(n)$ and vice versa $\forall m, n \in U$

(2) A is homogeneous CFS with B if $r_A(m) \leq r_B(n)$ implies $\omega_A(m) \leq \omega_B(n)$ and vice versa $\forall m, n \in U$

Definition 4 (see [31]) A homogeneous CFS A of G is said to be a CFSG if $\mu_A(mn) \geq \min\{\mu_A(m), \mu_A(n)\}$ and $\mu_A(m^{-1}) \geq \mu_A(m)$ $\forall m, n \in G$.

Definition 5 (see [5]). Let A be a CFSG (G) and $m \in G$. Then, the complex fuzzy left coset of A in G is represented by mAand is given by $mA(g) = \{\mu_A(m^{-1}g): g \in G\}$.

Similarly, one can define complex fuzzy right coset of Ain G.

Definition 6 (see [5]). A CFSG A of a group $G$is CFNSG (G) if mA = Am, $\forall m \in G$.

3. Decomposition Theorems of ξ-Complex Fuzzy Sets

In this section, we initiate the idea of ξ-CFS as a powerful extension of classical fuzzy sets. We also define the concepts of (α,δ)-cut sets and strong (α,δ)-cut sets of ξ-CFS and establish fundamental properties of these phenomena. We also prove three decomposition theorems of ξ-CFS.

Definition 7. Let A be a CFS of a universe U and $\xi = \alpha e^{i\delta}$ be an element of a unit circle with $0 \le \alpha \le 1$ and $0 \le \delta \le 2\pi$. Then, the CFS A^{ξ} is called the ξ-complex fuzzy set (ξ-CFS) with respect to CFS A and is expressed as $\mu_{A^{\xi}}(m) = \min\{\mu_A(m), \xi\}$, $\forall m \in U$.

The family of all ξ-CFS defined on the universe U is denoted by $F^{\xi}(U)$.

Definition 8. For any A^{ξ} and $\mathrm{B}^{\xi} \in F^{\xi}(U)$,

(1) The union of ξ-CFS A^{ξ} and B^{ξ} is denoted by $A^{\xi} \cup \mathrm{B}^{\xi}$ and is defined as follows:

$$\begin{aligned}\mu_{A^{\xi}\cup \mathrm{B}^{\xi}}(m) &= r_{A^{\xi}\cup \mathrm{B}^{\xi}}(m)e^{i\omega_{A^{\xi}\cup \mathrm{B}^{\xi}}(m)}\\ &= \max\left(r_{A^{\xi}}(m), r_{\mathrm{B}^{\xi}}(m)\right)e^{i\max\left(\omega_{A^{\xi}}(m),\omega_{B^{\xi}}(m)\right)},\\ &\quad \forall m \in U.\end{aligned} \tag{1}$$

(2) The intersection of ξ-CFS A^{ξ} and B^{ξ} is denoted by $A^{\xi} \cap B^{\xi}$ and is defined as follows:

$$\begin{aligned}\mu_{A^{\xi}\cap B^{\xi}}(m) &= r_{A^{\xi}\cap B^{\xi}}(m)e^{i\omega_{A^{\xi}\cap B^{\xi}}(m)}\\ &= \min\left(r_{A^{\xi}}(m), r_{B^{\xi}}(m)\right)e^{i\min\left(\omega_{A^{\xi}}(m),\omega_{B^{\xi}}(m)\right)},\\ &\quad \forall m \in U.\end{aligned} \tag{2}$$

(3) The complement of ξ-CFS A^{ξ} is denoted by $A^{\xi'}$ and is described as follows:

$$\begin{aligned}\mu_{A^{\xi'}}(m) &= r_{A^{\xi'}}(m)e^{i\omega_{A^{\xi'}}(m)}\\ &= \left(1 - r_{A^{\xi}}(m)\right)e^{i\left(2\pi - \omega_{A^{\xi}}(m)\right)}, \quad \forall m \in U.\end{aligned} \tag{3}$$

(4) The product of ξ-CFS A^{ξ} and B^{ξ} is represented by $A^{\xi} \circ B^{\xi}$ and is expressed as follows:

$$\begin{aligned}\mu_{A^{\xi}\circ B^{\xi}}(m) &= r_{A^{\xi}\circ B^{\xi}}(m)e^{i\omega_{A^{\xi}\circ B^{\xi}}(m)}\\ &= \left(r_{A^{\xi}}(m).r_{B^{\xi}}(m)\right)e^{i2\pi\left(\left(\omega_{A^{\xi}}(m)/2\pi\right).\left(\omega_{B^{\xi}}(m)/2\pi\right)\right)},\\ &\quad \forall m \in U.\end{aligned} \tag{4}$$

(5) Let $A_n^{\xi}, n \in N$ be ξ-CFS of a universe U. Then, the Cartesian product of ξ-CFS A_n^{ξ} is represented by $A_1^{\xi} \times A_2^{\xi} \times \cdots \times A_n^{\xi}$ and is defined in the following way:

$$\begin{aligned}&\mu_{A_1^{\xi}\times A_2^{\xi}\times\ldots\times A_n^{\xi}}(m) = r_{A_1^{\xi}\times A_2^{\xi}\times\ldots\times A_n^{\xi}}(m)e^{i\omega_{A_1^{\xi}\times A_2^{\xi}\times\ldots\times A_n^{\xi}}(m)} =\\ &\min\left(r_{A_1^{\xi}}(m_1), r_{A_2^{\xi}}(m_2), \ldots, r_{A_n^{\xi}}(m_n)\right)e^{i\min\left(\omega_{A_1^{\xi}}(m_1),\omega_{A_2^{\xi}}(m_2),\ldots,\omega_{A_n^{\xi}}(m_n)\right)},\end{aligned} \tag{5}$$

where $m = (m_1, m_2, \cdots, m_n)$, $m_i \in U_i$, $i \in N$.

Definition 9. Let A^{ξ} and B^{ξ} be any two ξ-CFS of a universe U. Then,

(1) A^{ξ} is homogeneous ξ-CFS if $r_{A^{\xi}}(m) \le r_{A^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \le \omega_{A^{\xi}}(n)$ and vice versa $\forall\, m, n \in U$

(2) A^{ξ} is homogeneous ξ-CFS with B^{ξ} if $r_{A^{\xi}}(m) \le r_{B^{\xi}}(n)$ implies $\omega_{A^{\xi}}(m) \le \omega_{B^{\xi}}(n)$ and vice versa $\forall m, n \in U$

The next result illustrates that the intersection of any two ξ-CFS is also ξ-CFS.

Proposition 1. *For any two ξ-CFS $A^{\xi}$and B^{ξ}, $(A \cap B)^{\xi} = A^{\xi} \cap B^{\xi}$*

Proof. Consider $\mu_{(A\cap B)^{\xi}}(m) = \min\{\mu_{A\cap B}(m), \xi\}$, $m \in U$.

By applying Definition 8 (2), we have $\mu_{(A\cap B)^{\xi}}(m) = \min(\mu_{A^{\xi}}(m), \mu_{B^{\xi}}(m)) = \mu_{A^{\xi}\cap B^{\xi}}(m)$. □

Remark 1. The union of any two ξ-CFS is also ξ-CFS.

Definition 10. The (α,δ)-cut set of ξ-CFS A^{ξ} is represented by $A^{\xi}_{(\alpha,\delta)}$ and is defined as follows:

$$A^{\xi}_{(\alpha,\delta)} = \{m \in U\colon r_{A^{\xi}}(m) \ge \alpha, \omega_{A^{\xi}}(m) \ge \delta, 0 \le \alpha \le 1, 0 \le \delta \le 2\pi\}. \tag{6}$$

Definition 11. For any $A^{\xi} \in F^{\xi}(U)$ strong (α,δ)-cut set of A^{ξ} is defined as $A^{\xi}_{+(\alpha,\delta)} = \{m \in U\colon r_{A^{\xi}}(m) > \alpha, \omega_{A^{\xi}}(m) > \delta, 0 \le \alpha \le 1, 0 \le \delta \le 2\pi\}$.

Definition 12. The level set Ω_{A^ξ} of A^ξ can be described as $\Omega_{A^\xi} = \{m \in U: r_{A^\xi}(m) = \alpha, \omega_{A^\xi}(m) = \delta\}$, where $\alpha \in [0,1]$ and $\delta \in [0, 2\pi]$.

Theorem 1. *Let A^ξ and B^ξ be any two ξ-CFS. Then, the following attributes hold for any $\alpha, \alpha' \in [0,1]$ and $\beta, \beta' \in [0, 2\pi]$.*

(1) $A^\xi_{+(\alpha,\delta)} \subseteq A^\xi_{(\alpha,\delta)}$
(2) $\alpha \le \alpha', \delta \le \delta\prime$ implies $A^\xi_{(\alpha',\delta')} \subseteq A^\xi_{(\alpha,\delta)}$
(3) $(A^\xi \cap B^\xi)_{(\alpha,\delta)} = A^\xi_{(\alpha,\delta)} \cap B^\xi_{(\alpha,\delta)}$
(4) $(A^\xi \cup B^\xi)_{(\alpha,\delta)} = A^\xi_{(\alpha,\delta)} \cup B^\xi_{(\alpha,\delta)}$
(5) $A^{\xi'}_{(\alpha,\delta)} = (A^{\xi'})_{+(1-\alpha, 2\pi-\delta)}$

Proof

(1) In view of Definition 10, for any element $m \in U$, $r_{A^\xi}(m) > \alpha$ and $\omega_{A^\xi}(m) > \delta$. It means that $r_{A^\xi}(m) \ge \alpha$ and $\omega_{A^\xi}(m) \ge \delta$. Thus, $A^\xi_{+(\alpha,\delta)} \subseteq A^\xi_{(\alpha,\delta)}$.
(2) Let $m \in U$, and by applying Definition 10, we have $r_{A^\xi}(m) \ge \alpha, \omega_{A^\xi}(m) \ge \delta$. Therefore, $A^\xi_{(\alpha',\delta)} \subseteq A^\xi_{(\alpha,\delta)}$.
(3) For any $m \in (A^\xi \cap B^\xi)_{(\alpha,\delta)}$, we have $r_{A^\xi \cap B^\xi}(m) \ge \alpha$ and $\omega_{A^\xi \cap B^\xi}(m) \ge \delta$. This implies that $\min\{r_{A^\xi}(m), r_{B^\xi}(m)\} \ge \alpha$ and $\min\{\omega_{A^\xi}(m), \omega_{B^\xi}(m)\} \ge \delta$. Then, clearly, $r_{A^\xi}(m) \ge \alpha, r_{B^\xi}(m) \ge \alpha$ and $\omega_{A^\xi}(m) \ge \delta, \omega_{B^\xi}(m) \ge \delta$. Therefore, $m \in A^\xi_{(\alpha,\delta)} \cap B^\xi_{(\alpha,\delta)}$, and ultimately, we obtain

$$\left(A^\xi \cap B^\xi\right)_{(\alpha,\delta)} \subseteq A^\xi_{(\alpha,\delta)} \cap B^\xi_{(\alpha,\delta)}. \tag{7}$$

Let $m \in A^\xi_{(\alpha,\delta)} \cap B^\xi_{(\alpha,\delta)}$.

By applying Definition 10, in the above relations, we obtain

$$r_{A^\xi}(m) \ge \alpha, \omega_{A^\xi}(m) \ge \delta, r_{B^\xi}(m) \ge \alpha, \omega_{B^\xi}(m) \ge \delta. \tag{8}$$

This shows that $\min\{r_{A^\xi}(m), r_{B^\xi}(m)\} \ge \alpha$ and $\min\{\omega_{A^\xi}(m), \omega_{B^\xi}(m)\} \ge \delta$.

Consequently,

$$A^\xi_{(\alpha,\delta)} \cap B^\xi_{(\alpha,\delta)} \subseteq \left(A^\xi \cap B^\xi\right)_{(\alpha,\delta)}. \tag{9}$$

From (7) and (9), the required equality holds.

(4) For any $m \in (A^\xi \cup B^\xi)_{(\alpha,\delta)}$, we obtain $r_{A^\xi \cup B^\xi}(m) \ge \alpha$ and $\omega_{A^\xi \cup B^\xi}(m) \ge \delta$. Therefore, $\max\{r_{A^\xi}(m), r_{B^\xi}(m)\} \ge \alpha$ and $\max\{\omega_{A^\xi}(m), \omega_{B^\xi}(m)\} \ge \delta$. This means that $r_{A^\xi}(m) \ge \alpha, r_{B^\xi}(m) \ge \alpha$ and $\omega_{A^\xi}(m) \ge \delta$, $\omega_{B^\xi}(m) \ge \delta$.

Consequently,

$$\left(A^\xi \cup B^\xi\right)_{(\alpha,\delta)} \subseteq A^\xi_{(\alpha,\delta)} \cup B^\xi_{(\alpha,\delta)}. \tag{10}$$

Now, suppose that $m \in A^\xi_{(\alpha,\delta)} \cup B^\xi_{(\alpha,\delta)}$; then, $m \in A^\xi_{(\alpha,\delta)}$ or $m \in B^\xi_{(\alpha,\delta)}$. By applying Definition 10 in the above relations, we get $r_{A^\xi}(m) \ge \alpha, \omega_{A^\xi}(m) \ge \delta$ or $r_{B^\xi}(m) \ge \alpha, \omega_{B^\xi}(m) \ge \delta$. It further shows that $\max\{r_{A^\xi}(m), r_{B^\xi}(m)\} \ge \alpha$ and $\max\{\omega_{A^\xi}(m), \omega_{B^\xi}(m)\} \ge \delta$.

Consequently,

$$A^\xi_{(\alpha,\delta)} \cup B^\xi_{(\alpha,\delta)} \subseteq \left(A^\xi \cup B^\xi\right)_{(\alpha,\delta)}. \tag{11}$$

From (10) and (11), the required equality is satisfied.

(5) Let $m \in A^{\xi'}_{(\alpha,\delta)}$, then $\mu_{A^{\xi'}}(m) = 1 - r_{A^\xi}(m) e^{i(2\pi - \omega_{A^\xi}(m))}$ implying that $1 - r_{A^\xi}(m) \ge \alpha$, $2\pi - \omega_{A^\xi}(m) \ge \delta$.

It follows that $r_{A^\xi}(m) \le 1 - \alpha$, $\omega_{A^\xi}(m) \le 2\pi - \delta$ which shows that $m \notin (A^\xi)_{+(1-\alpha, 2\pi-\delta)}$.

Therefore, $m \in (A^{\xi'})_{+(1-\alpha,2\pi-\delta)}$ and hence

$$A^{\xi'}_{(\alpha,\delta)} \subseteq \left(A^{\xi'}\right)_{+(1-\alpha,2\pi-\delta)}. \tag{12}$$

Now, suppose $m \in (A^{\xi'})_{+(1-\alpha,2\pi-\delta)}$; then, $m \notin (A^\xi)_+ (1-\alpha, 2\pi-\delta)$.

This implies that $1 - \alpha \ge r_{A^\xi}(m)$, $\omega_{A^\xi}(m) \le 2\pi - \delta$, $\alpha \le 1 - r_{A^\xi}(m)$, and $\delta \le 2\pi - \omega_{A^\xi}(m)$.

It means that $m \in A^{\xi'}_{(\alpha,\delta)}$; therefore,

$$\left(A^{\xi'}\right)_{+(1-\alpha,2\pi-\delta)} \subseteq A^{\xi'}_{(\alpha,\delta)}. \tag{13}$$

From (12) and (13), the required result is satisfied. □

Theorem 2. *Let A^ξ and B^ξ be any two ξ-CFS. Then, the following attributes hold for all $\alpha, \alpha' \in [0,1]$ and $\delta, \delta' \in [0, 2\pi]$*

(1) $\alpha \le \alpha', \delta \le \delta'$ implies $A^\xi_{+(\alpha',\delta)} \subseteq A^\xi_{+(\alpha,\delta)}$
(2) $(A^\xi \cap B^\xi)_{+(\alpha,\delta)} = A^\xi_{+(\alpha,\delta)} \cap B^\xi_{+(\alpha,\delta)}$
(3) $(A^\xi \cup B^\xi)_{(\alpha,\delta)} = A^\xi_{(\alpha,\delta)} \cup B^\xi_{(\alpha,\delta)}$

Theorem 3. *The following properties hold for any family of ξ-CFS A_i^ξ, $i \in$ I.*

(1) $\cup_{i\in I}(A_i^\xi)_{(\alpha,\delta)} \subseteq \cup_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$
(2) $\cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)} = \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$

Proof

(1) Let $m \in \cup_{i\in \mathrm{I}}(A_i^\xi)_{(\alpha,\delta)}$; in view of Definition 10, we get $r_{A^\xi_{i_o}}(m) \ge \alpha$ and $\omega_{A^\xi_{i_o}}(m) \ge \delta$.
The above relation holds only if $\mathrm{Sup} r_{A_i^\xi}(m) \ge \alpha$ and $\mathrm{Sup} \omega_{A_i^\xi}(m) \ge \delta$, that is, $\cup_{i\in\mathrm{I}}(r_i^\xi)(m) \ge \alpha$ and $\cup_{i\in\mathrm{I}}(\omega_i^\xi)(m) \ge \delta$.
It follows that $m \in \cup_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$.
(2) Let $m \in \cap_{i\in\mathrm{I}}(A_i^\xi)_{(\alpha,\delta)}$; by using Definition 10, we obtain $r_{A^\xi_{i_o}}(m) \ge \alpha$, $\omega_{A^\xi_{i_o}}(m) \ge \delta$.

The above inequality holds only if $\inf r_{A_i^\xi}(m) \ge \alpha$, $\inf \omega_{A_i^\xi}(m) \ge \delta$, that is, $\cap_{i\in I}(r_i^\xi)(m) \ge \alpha$ and $\cap_{i\in I}(\omega_i^\xi)(m) \ge \delta$. This implies that $m \in \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$. Hence, $\cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)} \subseteq \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$. Now, suppose that $m \in \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$. Again, by applying Definition 10, we get

$r_{A_i^\xi}(m) \geq \alpha$, $\omega_{A_i^\xi}(m) \geq \delta$. Then, obviously $m \in \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$. Consequently, $\cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)} \subseteq \cap_{i\in I}(A_i^\xi)_{(\alpha,\delta)}$. □

Theorem 4. *Any family of ξ-CFS A_i^ξ: $i \in$ I admits the following properties:*

(1) $\cup_{i\in I}(A_i^\xi)_{+(\alpha,\delta)} = \cup_{i\in I}(A_i^\xi)_{+(\alpha,\delta)}$
(2) $\cup_{i\in I}(A_i^\xi)_{+(\alpha,\delta)} \subseteq \cup_{i\in I}(A_i^\xi)_{+(\alpha,\delta)}$

Theorem 5. *Any two ξ-CFS A^ξ and B^ξ satisfy the following relations:*

(1) $A^\xi \subseteq B^\xi$ *if and only if* $A^\xi_{(\alpha,\delta)} \subseteq B^\xi_{(\alpha,\delta)}$
(2) $A^\xi = B^\xi$ *if and only if* $A^\xi_{(\alpha,\delta)} = B^\xi_{(\alpha,\delta)}$

Proof

(1) Suppose $A^\xi \subseteq B^\xi$. Assume that there exist $\alpha_o \in [0,1]$ and $\delta_o \in [0,2\pi]$ such that $A^\xi_{(\alpha_o,\delta_o)} \subseteq B^\xi_{(\alpha_o,\delta_o)}$.
It means that $m_o \in U$ such that $m_o \in A^\xi_{(\alpha_o,\delta_o)}$, but $m_o \notin B^\xi_{(\alpha_o,\delta_o)}$. Then, $r_{A^\xi}(m_o) \geq \alpha_o$, $\omega_{A^\xi}(m_o) \geq \delta_o$, $r_{B^\xi}(m_o) < \alpha_o$, and $\omega_{B^\xi}(m_o) < \delta_o$. Hence, $r_{B^\xi}(m_o) < r_{A^\xi}(m_o)$ and $\omega_{B^\xi}(m_o) < \omega_{A^\xi}(m_o)$, which is contradiction to our supposition.
Conversely, let $A^\xi_{(\alpha,\delta)} \subseteq B^\xi_{(\alpha,\delta)}$. Consider $A^\xi \subseteq B^\xi$, implying that $m_o \in U$, such that$r_{A^\xi}(m_o) > r_{B^\xi}(m_o)$, $\omega_{A^\xi}(m_o) > \omega_{B^\xi}(m_o)$. This further shows that $m_o \in A^\xi_{(\alpha,\delta)}$ and $m_o \notin B^\xi_{(\alpha,\delta)}$, which contradicts our assumption.
(2) In a similar way, we can obtain the required equality. □

Theorem 6. *Any two ξ-CFS A^ξ and B^ξ satisfy the following characteristics:*

(1) $A^\xi \subseteq B^\xi$ *if and only if* $A^\xi_{+(\alpha,\delta)} \subseteq B^\xi_{+(\alpha,\delta)}$
(2) $A^\xi = B^\xi$ *if and only if* $A^\xi_{+(\alpha,\delta)} = B^\xi_{+(\alpha,\delta)}$

Theorem 7. *Every ξ-CFS A^ξ satisfies the following relations:*

(1) $A^\xi_{(\alpha,\delta)} = \bigcap_{\substack{\alpha'<\alpha\\ \delta'<\delta}} A^\xi_{(\alpha',\delta')}$
(2) $A^\xi_{+(\alpha,\delta)} = \bigcup_{\substack{\alpha<\alpha'\\ \delta<\delta'}} A^\xi_{(\alpha',\delta')}$
(3) $A^\xi_{(\alpha,\delta)} = \bigcap_{\substack{\alpha'<\alpha\\ \delta'<\delta}} A^\xi_{+(\alpha',\delta')}$
(4) $A^\xi_{+(\alpha,\delta)} = \bigcup_{\substack{\alpha<\alpha'\\ \delta<\delta'}} A^\xi_{+(\alpha',\delta')}$

Proof

(1) In view of Theorem 1 (2),

$$A^\xi_{(\alpha,\delta)} \subseteq \bigcap_{\substack{\alpha'<\alpha\\ \delta'<\delta}} A^\xi_{(\alpha',\delta')}, \tag{14}$$

for all $\alpha' < \alpha$ and $\delta' < \delta$.

Next, suppose $m \in \bigcap_{\substack{\alpha'<\alpha\\ \delta'<\delta}} A^\xi_{(\alpha',\delta')}$. Again, by applying Theorem 1 (2), we have $r_{A^\xi}(m) \geq \alpha'$ and $\omega_{A^\xi}(m) \geq \delta'$.

The application of the given condition in the above relation yields that $r_{A^\xi}(m) \geq \alpha$ and $\omega_{A^\xi}(m) \geq \delta$. This implies that $m \in A^\xi_{(\alpha,\delta)}$.

Hence,

$$\bigcap_{\substack{\alpha'<\alpha\\ \delta'<\delta}} A^\xi_{(\alpha',\delta')} \subseteq A^\xi_{(\alpha,\delta)}. \tag{15}$$

From (14) and (15), the required equality holds. The remaining parts can be proved in a similar manner. In the following definitions, we present a new approach to define ξ-CFS which is quite necessary to establish the proofs of decomposition theorems. □

Definition 13. Let $A^\xi_{(\alpha,\delta)}$ be a (α,δ)-cut set of $A^\xi \in F^\xi(U)$. Then, the ξ-CFS $A^{\xi^*}_{(\alpha,\delta)}$ with respect to $A^\xi_{(\alpha,\delta)}$ is defined as follows:

$$A^{\xi^*}_{(\alpha,\delta)}(m) = \begin{cases} \alpha e^{i\delta}, \\ 0, \\ \text{if } m \in A^\xi_{(\alpha,\delta)}, \\ \text{otherwise}. \end{cases} \tag{16}$$

Definition 14. Let $A^\xi_{+(\alpha,\delta)}$ be a strong (α,δ) cut set of $A^\xi \in F^\xi(U)$. Then, the ξ-CFS $A^{\xi^*}_{+(\alpha,\delta)}$ with respect to $A^\xi_{+(\alpha,\delta)}$ can be described as follows:

$$A^{\xi^*}_{+(\alpha,\delta)}(m) = \begin{cases} \alpha e^{i\delta}, \\ 0, \\ \text{if } m \in A^\xi_{+(\alpha,\delta)}, \\ \text{otherwise}. \end{cases} \tag{17}$$

The subsequent result illustrates the decomposition of an ξ-CFS as a union of ξ-CFS $A^{\xi^*}_{(\alpha,\delta)}$.

Theorem 10 (first decomposition theorem). For every $A^\xi \in F^\xi(U)$, $A^\xi = \bigcup_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{(\alpha,\delta)}$.

Proof. Suppose $r_{A^\xi}(m) = u$ and $\omega_{A^\xi}(m) = v$ for a particular $m \in U$; then,

$$\left(\bigcup_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{(\alpha,\delta)}\right)(m) = \operatorname*{Sup}_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{(\alpha,\delta)}(m) = \max\left[\operatorname*{Sup}_{\substack{\alpha\in[0,u]\\ \delta\in[0,v]}} A^{\xi^*}_{(\alpha,\delta)}(m), \operatorname*{Sup}_{\substack{\alpha\in(u,1]\\ \delta\in(v,2\pi]}} A^{\xi^*}_{(\alpha,\delta)}(m)\right]. \tag{18}$$

Choose any $\alpha \in (u,1]$ and $\delta \in (v,2\pi]$, then $r_{A^\xi}(m) = u < \alpha$ and $\omega_{A^\xi}(m) = v < \delta$. Therefore, $A^{\xi^*}_{(\alpha,\delta)}(m) = 0e^{i0}$. On the contrary, for any choice of $\alpha \in [0,u]$ and $\beta \in [0,v]$, we have $r_{A^\xi}(m) = u \geq \alpha$ and $\omega_{A^\xi}(m) = v \geq \delta$.

Therefore, $A^{\xi^*}_{(\alpha,\delta)}(m) = \alpha e^{i\delta}$, which implies that $\left(\bigcup_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{(\alpha,\delta)}\right)(m) = \operatorname{Sup}_{\substack{\alpha\in[0,u]\\ \delta\in[0,v]}} \alpha e^{i\delta} = ue^{iv} = \mu_{A^\xi}(m).$
The following result describes the decomposition of A^ξ as a union of $A^{\xi^*}_{+(\alpha,\delta)}$. □

Table 1: Performance of all artists after initial screening.

Artists	Performance of the artists
$r_A(a)$	0.2
$r_A(b)$	0.91
$r_A(c)$	0.5
$r_A(d)$	0.7
$r_A(e)$	0.4
$r_A(f)$	0.61
$r_A(g)$	0.72
$r_A(h)$	0.3
$r_A(i)$	0.8
$r_A(j)$	0.9

Theorem 11 (second decomposition theorem). For every $A^\xi \in F^\xi(U)$, $A^\xi = \bigcup_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{+(\alpha,\delta)}$.

Proof. Suppose $r_{A^\xi}(m) = u$ and $\omega_{A^\xi}(m) = v$ for a particular $m \in U$, then

$$\left(\bigcup_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{+(\alpha,\delta)}\right)(m) = \operatorname{Sup}_{\substack{\alpha\in[0,1]\\ \delta\in[0,2\pi]}} A^{\xi^*}_{+(\alpha,\delta)}(m) = \max\left[\operatorname{Sup}_{\substack{\alpha\in[0,u]\\ \delta\in[0,v]}} A^{\xi^*}_{+(\alpha,\delta)}(m), \operatorname{Sup}_{\substack{\alpha\in[u,1]\\ \delta\in[v,2\pi]}} A^{\xi^*}_{+(\alpha,\delta)}(m)\right] = \operatorname{Sup}_{\substack{\alpha\in[0,u)\\ \delta\in[0,v)}} \alpha e^{i\delta} = ue^{iv} = \mu_{A^\xi}(m). \tag{19}$$

The decomposition of ξ CFS A^ξ as a union of level sets can be established by the following result. □

Theorem 12 (third decomposition theorem). For every $A^\xi \in F^\xi(U)$, $A^\xi = \bigcup_{\substack{\alpha\in\Omega_{A^\xi}\\ \delta\in\Omega_{A^\xi}}} A^{\xi^*}_{(\alpha,\delta)}$.

Proof. The proof is analogous to that of Theorem 10.

The following example illustrates the algebraic fact stated in first decomposition theorem. □

Example 1. Consider the ξ-CFS:

$$A^\xi = \left\{\frac{0.2e^{i0.5\pi}}{m_1} + \frac{0.4e^{i\pi}}{m_2} + \frac{0.6e^{i1.2\pi}}{m_3} + \frac{0.8e^{i1.8\pi}}{m_4}\right\}. \tag{20}$$

For $\alpha = 0.2$ and $\delta = 0.5\pi$, ξ-CFS $A^{\xi^*}_{(\alpha,\delta)}$ with respect to ξ-CFS A^ξ is given by

$$A^{\xi^*}_{(0.2,0.5\pi)} = \left\{\frac{0.2e^{i0.5\pi}}{m_1} + \frac{0.2e^{i0.5\pi}}{m_2} + \frac{0.2e^{i0.5\pi}}{m_3} + \frac{0.2e^{i0.5\pi}}{m_4}\right\}. \tag{21}$$

Corresponding to $\alpha = 0.4$ and $\delta = \pi$, we have

$$A^{\xi^*}_{(0.4,\pi)} = \left\{\frac{0e^{i0}}{m_1} + \frac{0.4e^{i\pi}}{m_2} + \frac{0.4e^{i\pi}}{m_3} + \frac{0.4e^{i\pi}}{m_4}\right\}. \tag{22}$$

Corresponding to $\alpha = 0.6$ and $\delta = 1.2\pi$,

$$A^{\xi^*}_{(0.6,1.2\pi)} = \left\{\frac{0e^{i0}}{m_1} + \frac{0e^{i0}}{m_2} + \frac{0.6e^{i1.2\pi}}{m_3} + \frac{0.6e^{i1.2\pi}}{m_4}\right\}. \tag{23}$$

Also, for $\alpha = 0.8$ and $\delta = 1.8\pi$,

$$A^{\xi^*}_{(0.8,1.8\pi)} = \left\{\frac{0e^{i0}}{m_1} + \frac{0e^{i0}}{m_2} + \frac{0e^{i0}}{m_3} + \frac{0.8e^{i1.8\pi}}{m_4}\right\}. \tag{24}$$

Consequently, $A^\xi = \bigcup A^{\xi^*}_{(\alpha,\delta)}$.

In the following example, we apply the concept of ξ-CFS to judge the performance of an artist in an art competition.

Example 2. Let $X = \{a, b, c, d, e, f\}$ be the list of 10 artists competing in an art competition. After the initial screening based on sketch designing, the performance of each artist is given in Table 1.

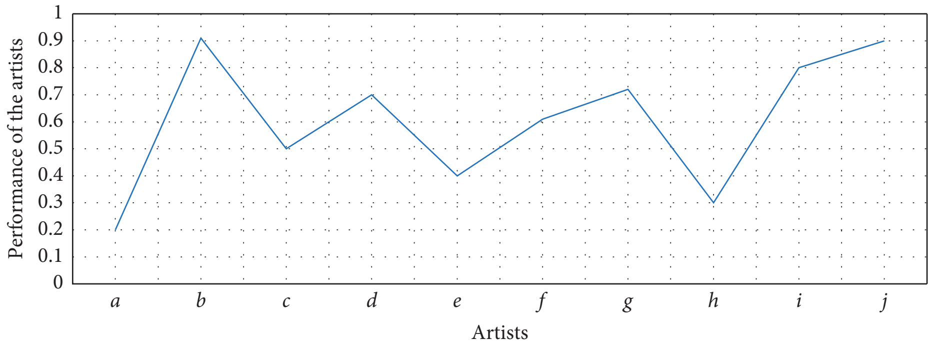

FIGURE 1: A graphical overview of Table 1.

TABLE 2: Performance of the artists after the first phase.

Artists	Roughness of the sketch
$r_{A^\xi}(a)$	0.2
$r_{A^\xi}(c)$	0.5
$r_{A^\xi}(d)$	0.7
$r_{A^\xi}(e)$	0.4
$r_{A^\xi}(f)$	0.61
$r_{A^\xi}(h)$	0.3

The graphical interpretation of the above performance of the artists is displayed in Figure 1.

Let $\xi = \alpha e^{i\beta}$ be a parameter to select a candidate based on the performance to draw a sketch of a healthy environment. The judgment procedure has two phases α and β, where α denotes the level of roughness of the drawing and β denotes the use of inappropriate color in the drawing. Table 2 indicates the performance of the artists after the first phase for $\alpha = 0.7$.

The graphical interpretation of the qualified artists of stage one is presented in Figure 2.

Table 3 indicates the performances of the qualified artists for phase two.

The graphical interpretation of the artists of stage two is presented in Figure 3.

The following outcomes indicate the performance of the qualified artists after phase two for $\beta = 0.5\pi$ are $\omega_{A^\xi}(a) = 0.5\pi$a dm $\omega_{A^\xi}(h) = 0.4\pi$ and final score is $\mu_{A^\xi}(a) = 0.2e^{i0.5\pi}$ and $\mu_{A^\xi}(h) = 0.3e^{i0.4\pi}$. At this stage, we will use the score function to compare the performance of the artists. For this, we may take $|a| = 0.2276$ and $|h| = 0.2998$. The above information shows that artist "h" may be considered the best of all the artists.

4. Algebraic Attributes of ξ-Complex Fuzzy Subgroups

In this section, we innovate the notion of ξ-CFSG defined on ξ-CFS and establish fundamental algebraic characteristics of this phenomenon.

Definition 15. A homogeneous ξ-CFS A^ξ of a group G is called ξ-complex fuzzy subgroup (ξ-CFSG) if A^ξ admits the following conditions:

(1) $\mu_{A^\xi}(\mathrm{mn}) \geq \min\{\mu_{A^\xi}(m), \mu_{A^\xi}(n)\}$
(2) $\mu_{A^\xi}(m^{-1}) \geq \mu_{A^\xi}(m)$, $\forall m, n \in G$

The family of all ξ-CFSG defined on the group G is denoted by $F^\xi(G)$.

Proposition 2. *Each ξ-CFSG (G) A^ξ satisfies the following properties:*

(1) $\mu_{A^\xi}(m) \leq \mu_{A^\xi}(e)$
(2) $\mu_{A^\xi}(mn^{-1}) = \mu_{A^\xi}(e) \Longrightarrow \mu_{A^\xi}(m) = \mu_{A^\xi}(n)$, $\forall m, n \in G$

Proof

(1) Let $m \in G$; then,

$$\begin{aligned}\mu_{A^\xi}(e) &= \mu_{A^\xi}\left(mm^{-1}\right)\\ &\geq \min\left(\mu_{A^\xi}(m), \mu_{A^\xi}\left(m^{-1}\right)\right)\\ &= \mu_{A^\xi}(m).\end{aligned} \tag{25}$$

(2) Let m, n $\in G$; then,

$$\begin{aligned}\mu_{A^\xi}(m) &= \mu_{A^\xi}\left(mn^{-1}n\right)\\ &\geq \min\left\{\mu_{A^\xi}\left(mn^{-1}\right), \mu_{A^\xi}(n)\right\}\\ &= \min\{\mu_{A^\xi}(e), \mu_{A^\xi}(n)\}\\ &= \mu_{A^\xi}(n).\end{aligned} \tag{26}$$

In the following result, we investigate the condition under which a given CFS is ξ-CFSG. □

Proposition 3. *Let A be a CFS (G), such that $\mu_A(m^{-1}) = \mu_A(m)$ $\forall m \in G$.*

Moreover, $\xi \leq q$, where $q = inf\{\mu_A(m)\colon m \in G\}$. Then, A^ξ is ξ-CFSG (G).

Proof. By using the given conditions for any $m \in G$, we obtain $\mu_A(m) \geq \xi$. The application of Definition 7 in the above inequality yields that $\mu_{A^\xi}(m) = \xi$. Therefore, $\mu_{A^\xi}(\mathrm{mn}) = \min\{\mu_A(\mathrm{mn}), \xi\}$ and $\mu_{A^\xi}(\mathrm{mn}) \geq \min\{\mu_{A^\xi}(m), \mu_{A^\xi}(n)\}$, for all $m, n \in G$. Moreover, by using the given condition $\mu_A(m^{-1}) = \mu_A(m)$, we get $\mu_{A^\xi}(m^{-1}) = \mu_{A^\xi}(m)$.

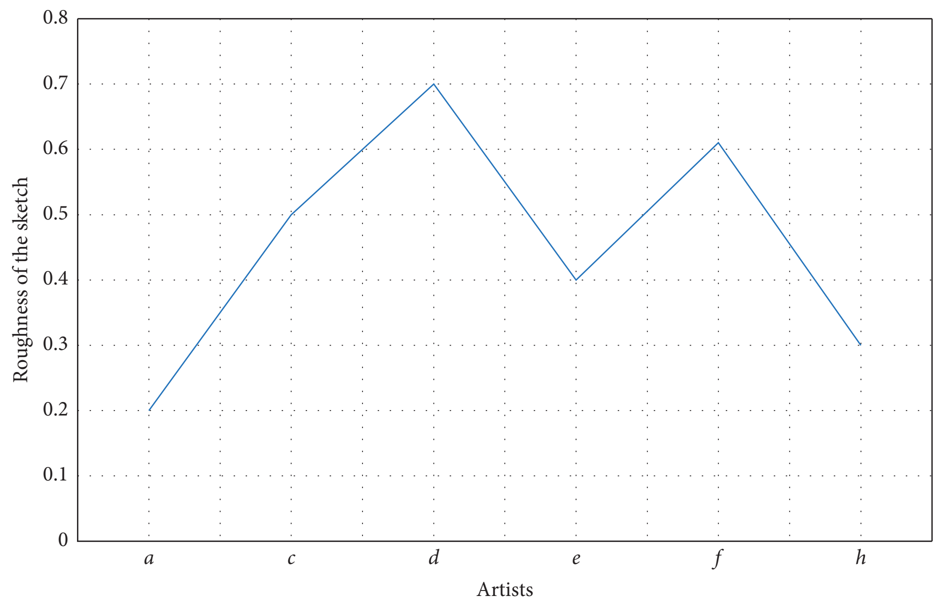

FIGURE 2: A graphical interpretation of Table 2.

TABLE 3: Performance of the artists after the second phase.

Artists	Performance of the artists
$\omega_A(a)$	0.5π
$\omega_A(c)$	0.6π
$\omega_A(d)$	1.5π
$\omega_A(e)$	0.6π
$\omega_A(f)$	0.7π
$\omega_A(h)$	0.4π

The following result shows that every CFSG is always ξ-CFSG. □

Proposition 4. *Every CFSG A is ξ-CFSG of a group G.*

Proof. By using Definition 7, for all $m, n \in G$, we have $\mu_{A^\xi}(\mathrm{mn}) = \min\{\mu_A(\mathrm{mn}), \xi\}$. The application of Definition 13 in the above relation gives us $\mu_{A^\xi}(\mathrm{mn}) \geq \min\{\mu_{A^\xi}(m), \mu_{A^\xi}(n)\}$.

Moreover,

$$\begin{aligned}\mu_{A^\xi}\left(m^{-1}\right) &= \min\left\{\mu_A\left(m^{-1}\right), \xi\right\} \\ &\geq \min\{\mu_A(m), \xi\} \\ &= \mu_{A^\xi}(m).\end{aligned} \tag{27}$$

Hence, A is a ξ-CFSG (G). □

Remark 2. The converse of Proposition 4 does not hold in general. This algebraic fact may be viewed in the following example.

Example 3. The CFS A defined on a $G = \{1, -1, i - i\}$ is given as

$$A(m) = \left\{\frac{0.2e^{i\pi}}{1} + \frac{0.4e^{i\pi}}{-1} + \frac{0.4e^{i1.2\pi}}{-i} + \frac{0.3e^{i0.9\pi}}{i}\right\}. \tag{28}$$

The ξ-CFSG (G) corresponding to the value $\xi = 0.1e^{i0.5\pi}$ is given by

$$A^\xi(m) = \left\{\frac{0.1e^{i0.5\pi}}{1} + \frac{0.1e^{i0.5\pi}}{-1} + \frac{0.1e^{i0.5\pi}}{-i} + \frac{0.1e^{i0.5\pi}}{i}\right\}. \tag{29}$$

Moreover, A is not CFSG (G) as A does not satisfy Definition 4.

The following result indicates that intersection of any two ξ-CFSG is also ξ-CFSG.

Proposition 5. *For any two A^ξ, $B^\xi \in F^\xi(G)$, $(A \cap B)^\xi = A^\xi \cap B^\xi$.*

Proof. By using Proposition 1, for any two elements $\mathrm{m}, \mathrm{n} \in G$,

$$\begin{aligned}\mu_{(A\cap B)^\xi}(mn) &= \mu_{A^\xi \cap B^\xi}(mn) \\ &= \min\{\mu_{A^\xi}(mn), \mu_{B^\xi}(mn)\}.\end{aligned} \tag{30}$$

The application of Definition 13 in the above relation gives that

$$\mu_{(A\cap B)^\xi}(mn) = \min\left\{\mu_{(A\cap B)^\xi}(m), \mu_{(A\cap B)^\xi}(n)\right\}. \tag{31}$$

Moreover,

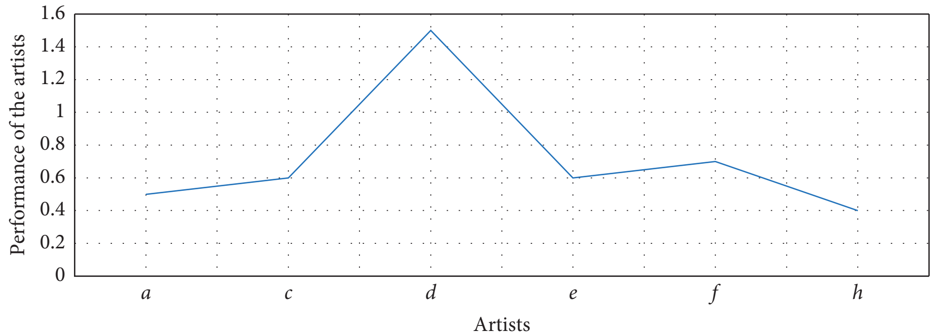

FIGURE 3: A graphical interpretation of Table 3.

$$\begin{aligned}\mu_{(A\cap B)^{\xi}}\left(m^{-1}\right) &= \mu_{A^{\xi}\cap B^{\xi}}\left(m^{-1}\right)\\ &\geq \min\{\mu_{A^{\xi}}(m), \mu_{B^{\xi}}(m)\}\\ &= \mu_{(A\cap B)^{\xi}}(m).\end{aligned} \tag{32}$$

This concludes the proof. □

Theorem 13. *A^{ξ} is a ξ-CFSG (G) if and only if $A^{\xi'}$ is a ξ-CFSG (G).*

Proof. Let A^{ξ} be a ξ-CFSG. Then,

$$\mu_{A^{\xi'}}(mn) = 1 - r_{A^{\xi}}(mn)e^{i\left(2\pi - \omega_{A^{\xi}}(\mathrm{mn})\right)}. \tag{33}$$

By using Definition 13, we obtain

$$\begin{aligned}\mu_{A^{\xi'}}(mn) &\geq 1 - \min\{r_{A^{\xi}}(m), r_{A^{\xi}}(n)\}e^{i\left(2\pi - \min\{\omega_{A^{\xi}}(m), \omega_{A^{\xi}}(n)\}\right)}\\ &= \max\{(1 - r_{A^{\xi}}(m)), (1 - r_{A^{\xi}}(n))\}e^{i\max\{(2\pi - \omega_{A^{\xi}}(m)), (2\pi - \omega_{A^{\xi}}(n))\}}\\ &\geq \min\{(1 - r_{A^{\xi}}(m)), (1 - r_{A^{\xi}}(n))\}e^{i\min\{(2\pi - \omega_{A^{\xi}}(m)), (2\pi - \omega_{A^{\xi}}(n))\}}.\end{aligned} \tag{34}$$

By using Definition 8 (3) in the above relation, we get $\mu_{A^{\xi'}}(mn) = \min\left\{\mu_{A^{\xi'}}(m), \mu_{A^{\xi'}}(n)\right\}$. Moreover,

$$\begin{aligned}\mu_{A^{\xi'}}\left(m^{-1}\right) &= 1 - r_{A^{\xi}}\left(m^{-1}\right)e^{i\left(2\pi - \omega_{A^{\xi}}\left(m^{-1}\right)\right)}\\ &= 1 - r_{A^{\xi}}(m)e^{i\left(2\pi - \omega_{A^{\xi}}(m)\right)}\\ &= \mu_{A^{\xi'}}(m).\end{aligned} \tag{35}$$

Conversely, let $A^{\xi'}$ be a ξ-CFSG. Assume that

$$\begin{aligned}\mu_{A^{\xi}}(mn) &= r_{A^{\xi}}(mn)e^{i\omega_{A^{\xi}}(mn)}\\ &= 1 - \left(1 - r_{A^{\xi}}(mn)e^{i\left(2\pi - \left(2\pi - \omega_{A^{\xi}}(mn)\right)\right)}\right)\\ &\geq 1 - \min\{(1 - r_{A^{\xi}}(m)), (1 - r_{A^{\xi}}(n))\}e^{i2\pi - \min\{(2\pi - \omega_{A^{\xi}}(m)), (2\pi - \omega_{A^{\xi}}(n))\}}\\ &= \max\{(r_{A^{\xi}}(m), r_{A^{\xi}}(n))\}e^{i\max\{\omega_{A^{\xi}}(m), \omega_{A^{\xi}}(n)\}}\\ &\geq \min\{(r_{A^{\xi}}(m), r_{A^{\xi}}(n))\}e^{i\min\{\omega_{A^{\xi}}(m), \omega_{A^{\xi}}(n)\}}\\ \mu_{A^{\xi}}(mn) &= \min\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}(n)\}.\end{aligned} \tag{36}$$

Moreover,

$$\mu_{A^{\xi}}\left(m^{-1}\right) = 1 - \left(1 - r_{A^{\xi}}\left(m^{-1}\right)e^{i2\pi - \left(2\pi - \omega_{A^{\xi}}\left(m^{-1}\right)\right)}\right) = r_{A^{\xi}}(m)e^{i\omega_{A^{\xi}}(m)} = \mu_{A^{\xi}}(m). \quad (37)$$

Definition 16. Let $A^{\xi} \in F^{\xi}(G)$, $\alpha \in [0,1]$, and $\delta \in [0,2\pi]$. Then, the subgroup $A^{\xi}_{(\alpha,\delta)}$ with $r_{A^{\xi}}(e) \geq \alpha$ and $\omega_{A^{\xi}}(e) \geq \delta$ is called the level subgroup of ξ-CFSG A^{ξ}.

In the subsequent result, we establish necessary and sufficient condition for an ξ-CFS to be ξ-CFSG.

Theorem 14. *A ξ-CFS A^{ξ} of G is a ξ-CFSG (G) if and only if each of its level set $\Omega_{A^{\xi}}$ with $r_{A^{\xi}}(e) \geq \alpha$ and $\omega_{A^{\xi}}(e) \geq \delta$ is a subgroup of G.*

Proof. Suppose $\Omega_{A^{\xi}}$ with $r_{A^{\xi}}(e) \geq \alpha$ and $\omega_{A^{\xi}}(e) \geq \delta$ is a subgroup of G. Assume that $r_{A^{\xi}}(m) = \alpha$, $r_{A^{\xi}}(n) = \alpha_1$, $\omega_{A^{\xi}}(m) = \delta$, and $\omega_{A^{\xi}}(n) = \delta_1$, for any elements m, n $\in G$. By using Definition 10 in the above relations, we have $m \in \Omega_{A^{\xi}}$ and $n \in \Omega_{A_1^{\xi}}$. By applying Theorem 1 (2) for $\alpha < \alpha_1$ and $\delta < \delta_1$, we have $n \in \Omega_{A^{\xi}}$, since $\Omega_{A^{\xi}}$ is a subgroup of G. Therefore, $mn \in \Omega_{A^{\xi}}$. It shows that $r_{A^{\xi}}(mn) \geq \min\{r_{A^{\xi}}(m), r_{A^{\xi}}(n)\} = \alpha$ and $\omega_{A^{\xi}}(mn) \geq \min\{\omega_{A^{\xi}}(m), \omega_{A^{\xi}}(n)\} = \delta$.

In view of Definition 10, we have

$$\begin{aligned} r_{A^{\xi}}\left(m^{-1}\right) &\geq \alpha = r_{A^{\xi}}(m), \\ \omega_{A^{\xi}}\left(m^{-1}\right) &\geq \delta = \omega_{A^{\xi}}(m). \end{aligned} \quad (38)$$

Consequently, A^{ξ} is a ξ CFSG (G). Conversely, suppose $A^{\xi} \in F^{\xi}(G)$. Let $\Omega_{A^{\xi}}$ be an arbitrary level subgroup of A^{ξ}. Obviously, $\Omega_{A^{\xi}}$ is nonempty as $e \in \Omega_{A^{\xi}}$, where e is the identity element of G. For any elements $m, n \in \Omega_{A^{\xi}}$ and using the fact that $A^{\xi} \in F^{\xi}(G)$, we have $r_{A^{\xi}}(mn) \geq \min\{r_{A^{\xi}}(m), r_{A^{\xi}}(n)\} = \alpha$ and $\omega_{A^{\xi}}(mn) \geq \min\{\omega_{A^{\xi}}(m), \omega_{A^{\xi}}(n)\} = \delta$. It follows that $mn \in \Omega_{A^{\xi}}$. Moreover, for any element $m \in A^{\xi}$ and using the fact that $A^{\xi} \in F^{\xi}(G)$, we have $r_{A^{\xi}}(m^{-1}) \geq r_{A^{\xi}}(m) = \alpha$ and $\omega_{A^{\xi}}(m^{-1}) \geq \omega_{A^{\xi}}(m) = \delta$. Therefore, $m^{-1} \in \Omega_{A^{\xi}}$ implying that $\Omega_{A^{\xi}}$ is a subgroup of G. □

Definition 17. Let A^{ξ} be a ξ-CFSG (G) and $m \in G$. Then, the ξ-complex fuzzy left coset of A^{ξ} in G is represented by $mA^{\xi}$and is given by $mA^{\xi}(g) = \{\min\{\mu_A(m^{-1}g), \xi\}: g \in G\} = \mu_{A^{\xi}}(m^{-1}g)$.

Similarly, one can define the ξ-complex fuzzy right coset of A^{ξ} in G.

Definition 18. A ξ-CFSG A^{ξ} of a group G is ξ-complex fuzzy normal subgroup (ξ-CFNSG) of G if $mA^{\xi} = A^{\xi}m$, $\forall m \in G$.

The following result illustrates another characteristic of ξ-CFNSG.

Proposition 6. *Every ξ-CFNSG A^{ξ} admits the following property:*

$$\mu_{A^{\xi}}(mn) = \mu_{A^{\xi}}(nm) \text{ for all } m, n \in G. \quad (39)$$

Proof. By using Definition 16, we have $mA^{\xi} = A^{\xi}m, \forall m \in G$

By using Definition 15, the above equation gives that

$$\begin{aligned} \left(mA^{\xi}\right)n^{-1} &= \left(A^{\xi}m\right)n^{-1} \forall n \in G, \\ \mu_{A^{\xi}}(nm)^{-1} &= \mu_{A^{\xi}}(mn)^{-1}. \end{aligned} \quad (40)$$

This shows that $\mu_{A^{\xi}}(nm) = \mu_{A^{\xi}}(mn)$. In the following consequence, we explore the condition under which an ξ-CFSG is ξ-CFNSG (G). □

Proposition 7. *For any $A^{\xi} \in F^{\xi}(G)$ with $\xi \leq q$, where $q = Inf\{\mu_A(m): m \in G\}$, then A^{ξ} is a ξ-CFNSG (G).*

Proof. By using the given condition for any $m \in G$, we have $\mu_A(m) \geq \xi$. The application of Definition 7 in the above inequality yields that $\mu_{A^{\xi}}(m) = \xi$.

Therefore,

$$A^{\xi}m(g) = mA^{\xi}(g) \text{ for all } g \in G. \quad (41)$$

Hence,

$$A^{\xi}\mathrm{m} = \mathrm{m}A^{\xi} \text{ for all } m \in G. \quad (42)$$

The following result shows that every CFNSG (G) is ξ-CFNSG (G). □

Proposition 8. *Every CFNSG (G) A is ξ-CFNSG (G).*

Proof. By using Definition 6 for element $m \in G$, we have $\mathrm{mA}(g) = \mathrm{Am}(g)$. By applying Definition 5, the above relation gives that $\mu_A(m^{-1}g) = \mu_A(gm^{-1})$. So, $\min\{\mu_A(m^{-1}g), \xi\} = \min\{\mu_A(gm^{-1}), \xi\}$ implying that $mA^{\xi}(g) = A^{\xi}m(g)$. Consequently, $mA^{\xi} = A^{\xi}m$. □

Remark 3. The converse of Proposition 8 does not hold in general. This algebraic fact may be viewed in the following example.

Example 4. The CFNSG Adefined on a group $G = \langle a, b: a^2 = b^2 = (ab)^2 = 1, ba = a^2b\rangle$ is given by

$$A(m) = \left\{\frac{1e^{i2\pi}}{1} + \frac{0.9e^{i1.8\pi}}{a} + \frac{0.7e^{i1.7\pi}}{a^2} + \frac{0.5e^{i1.6\pi}}{b} + \frac{0.4e^{i\pi}}{ab} + \frac{0.3e^{i0.7\pi}}{a^2b}\right\}. \quad (43)$$

The ξ-CFNSG (G) corresponding to the value $\xi = 0.3e^{i0.7\pi}$ is given by

$$A^{\xi}(m) = \left\{\frac{0.3e^{i0.7\pi}}{1} + \frac{0.3e^{i0.7\pi}}{a} + \frac{0.3e^{i0.7\pi}}{a^2} + \frac{0.3e^{i0.7\pi}}{b} + \frac{0.3e^{i0.7\pi}}{ab} + \frac{0.3e^{i0.7\pi}}{a^2b}\right\}. \quad (44)$$

Moreover, A is not CFNSG (G) because

$$\mu_A(ab) = 0.4e^{i\pi} \neq 0.3e^{i0.7\pi} = \mu_A(ba). \quad (45)$$

Theorem 15. *For any two ξ-CFNSG* A^{ξ} *and* B^{ξ}, $(A\cap B)^{\xi} = A^{\xi}\cap B^{\xi}$.

Proof. By using Proposition 1 for any element $m \in G$, we have

$$\begin{aligned}\mu_{(A\cap B)^{\xi}}\left(gm^{-1}\right) &= \mu_{\left(A^{\xi}\cap B^{\xi}\right)}\left(gm^{-1}\right)\\ &= \min\left\{\mu_{A^{\xi}}\left(gm^{-1}\right), \mu_{B^{\xi}}\left(gm^{-1}\right)\right\}.\end{aligned} \quad (46)$$

The application of Definition 16 in the above relation is given as

$$\begin{aligned}\mu_{(A\cap B)^{\xi}}\left(gm^{-1}\right) &= \min\left\{\mu_{A^{\xi}}\left(m^{-1}g\right), \mu_{B^{\xi}}\left(m^{-1}g\right)\right\}\\ &= \mu_{\left(A^{\xi}\cap B^{\xi}\right)}\left(m^{-1}g\right).\end{aligned} \quad (47)$$

Thus, $\mu_{(A\cap B)^{\xi}}(m^{-1}g) = \mu_{(A\cap B)^{\xi}}(gm^{-1})$. □

Theorem 16. A^{ξ} *is a ξ-CFNSG if and only if* $A^{\xi'}$ *is a ξ-CFNSG.*

Proof. Let A^{ξ}be a ξ-CFNSG. Then,

$$\begin{aligned}mA^{\xi'} &= \mu_{A^{\xi'}}\left(m^{-1}g\right)\\ &= 1 - r_{A^{\xi}}\left(m^{-1}g\right)e^{i\left(2\pi-\omega_{A^{\xi}}\left(m^{-1}g\right)\right)}.\end{aligned} \quad (48)$$

By using Definition 16, we obtain

$$\begin{aligned}mA^{\xi'} &= 1 - r_{A^{\xi}}\left(gm^{-1}\right)e^{i\left(2\pi-\omega_{A^{\xi}}\left(gm^{-1}\right)\right)}\\ &= \mu_{A^{\xi'}}\left(gm^{-1}\right).\end{aligned} \quad (49)$$

Hence, $mA^{\xi'} = A^{\xi'}m$.

Conversely, let $A^{\xi'}$ be a ξ-CFNSG. Assume that

$$\begin{aligned}mA^{\xi} &= \mu_{A^{\xi}}\left(m^{-1}g\right)\\ &= r_{A^{\xi}}\left(m^{-1}g\right)e^{i\omega_{A^{\xi}}\left(m^{-1}g\right)}\\ &= 1 - \left(1 - r_{A^{\xi}}\left(m^{-1}g\right)e^{i\left(2\pi-\left(2\pi-\omega_{A^{\xi}}\left(m^{-1}g\right)\right)\right)}\right)\\ &= 1 - \mu_{A^{\xi'}}\left(m^{-1}g\right)\\ &= 1 - \mu_{A^{\xi'}}\left(gm^{-1}\right)\\ &= \mu_{A^{\xi}}\left(gm^{-1}\right).\end{aligned} \quad (50)$$

Thus, $mA^{\xi} = A^{\xi}m$. □

Proposition 9. *Let* A^{ξ}*be a ξ-CFSG (G). Then, the set* $G_{A^{\xi}} = \{m \in G\colon \mu_{A^{\xi}}(m) = \mu_{A^{\xi}}(e)\}$ *is a normal subgroup of G.*

Proof. Obviously, $G_{A^{\xi}} \neq \varnothing$ as $e \in G_{A^{\xi}}$. By applying Definition 13 for any two elements m, n ∈ G, we have

$$\begin{aligned}\mu_{A^{\xi}}\left(mn^{-1}\right) &\geq \min\left\{\mu_{A^{\xi}}(m), \mu_{A^{\xi}}\left(n^{-1}\right)\right\}\\ &= \mu_{A^{\xi}}(e).\end{aligned} \quad (51)$$

This shows that $\mu_{A^{\xi}}(mn^{-1}) \geq \mu_{A^{\xi}}(e)$, but $\mu_{A^{\xi}}(mn^{-1}) \leq \mu_{A^{\xi}}(e)$. Consequently, $mn^{-1} \in G_{A^{\xi}}$. Furthermore, in view of Definition 16 for any element $m \in G_{A^{\xi}}$ and $g \in G$, we obtain

$$\begin{aligned}\mu_{A^{\xi}}\left(g^{-1}mg\right) &= \mu_{A^{\xi}}(m)\\ &= \mu_{A^{\xi}}(e).\end{aligned} \quad (52)$$

This implies that $g^{-1}\text{mg} \in G_{A^{\xi}}$. Hence, $G_{A^{\xi}}$ is a normal subgroup of G. □

Proposition 10. *Every ξ-CFNSG satisfies the following relation;*

If $mA^{\xi} = uA^{\xi}$ *and* $nA^{\xi} = vA^{\xi}$, *then* $mnA^{\xi} = uvA^{\xi}$.

Proof. Since $mA^{\xi} = uA^{\xi}$ and $nA^{\xi} = vA^{\xi}$, therefore $m^{-1}u, n^{-1}v \in G_{A^{\xi}}$.

Consider

$$\begin{aligned}(mn)^{-1}(uv) &= n^{-1}\left(m^{-1}u\right)v = n^{-1}\left(m^{-1}u\right)\left(nn^{-1}\right)v\\ &= \left[n^{-1}\left(m^{-1}u\right)n\right]\left(n^{-1}v\right) \in G_{A^{\xi}}.\end{aligned} \quad (53)$$

It follows that $(mn)^{-1}(uv) \in G_{A^{\xi}}$. Consequently, $mnA^{\xi} = uvA^{\xi}$. □

Proposition 11. *Every ξ CFNSG* A^{ξ} *admits the following characteristics:*

(1) $mA^{\xi} = nA^{\xi}$ *if and only if* $m^{-1}n \in G_{A^{\xi}}$

(2) $A^{\xi}m = A^{\xi}n$ *if and only if* $mn^{-1} \in G_{A^{\xi}}$

Proof

(1) Let $mA^{\xi} = nA^{\xi}$and $m, n \in G_{A^{\xi}}$. Then, by applying Definition 7, we have

$$\begin{aligned}\mu_{A^{\xi}}\left(m^{-1}n\right) &= \min\left\{\mu_A\left(m^{-1}n\right), \xi\right\}\\ &= mA^{\xi}(n).\end{aligned} \quad (54)$$

By using the given condition in the above equations, we obtain

$$\begin{aligned}\mu_{A^{\xi}}\left(m^{-1}n\right) &= nA^{\xi}(n)\\ &= \min\left\{\mu_A\left(n^{-1}n\right), \xi\right\}\\ &= \min\left\{\mu_A(e), \xi\right\}.\end{aligned} \quad (55)$$

So, $\mu_{A^{\xi}}(m^{-1}n) = \mu_{A^{\xi}}(e)$, implying that $m^{-1}n \in G_{A^{\xi}}$. Conversely, suppose that $m^{-1}n \in G_{A^{\xi}}$. This implies that $\mu_{A^{\xi}}(m^{-1}n) = \mu_{A^{\xi}}(e)$. By applying Definition 16 for any element $z \in G$, we have

$$\begin{aligned} mA^{\xi}(z) &= \min\{\mu_A(m^{-1}z), \xi\} \\ &= \mu_{A^{\xi}}(m^{-1}z) \\ &= \mu_{A^{\xi}}((m^{-1}n)(n^{-1}z)). \end{aligned} \tag{56}$$

By using Definition 13 in the above equation, we obtain

$$\begin{aligned} mA^{\xi}(z) &= \min\{\mu_{A^{\xi}}(e), \mu_{A^{\xi}}(n^{-1}z)\} \\ &= \mu_{A^{\xi}}(n^{-1}z). \end{aligned} \tag{57}$$

Consequently, $mA^{\xi}(z) = (nA^{\xi})(z)$. The remaining part can be proved as the first part. □

Definition 19. For any ξ-CFNSG A^{ξ} of G, we define the set of all ξ-complex fuzzy left cosets of G by A^{ξ} as $G/A^{\xi} = \{mA^{\xi}: m \in G\}$. This set forms a group under the following binary operation $(mA^{\xi})(nA^{\xi}) = mnA^{\xi}$. This particular quotient group is called quotient group of G by ξ-CFNSG A^{ξ}.

In the following result, we establish a natural epimorphism between group and its quotient group defined in Definition 17.

Theorem 17. *For any ξ-CFNSG A^{ξ} of G, there exist a natural epimorphism φ: $G \longrightarrow G/A^{\xi}$, defined by $m \longrightarrow mA^{\xi}, m \in G$ with* $\ker\varphi = G_{A^{\xi}}$.

Proof. The surjectivity of the function φ is quite obvious. Moreover, for any elements $m, n \in G$, we have $\varphi(mn) = mA^{\xi}nA^{\xi} = \varphi(m)\varphi(n)$. Therefore, φ is an epimorphism. Moreover, obvious φ is surjective. Now,

$$\begin{aligned} \ker\varphi &= \{m \in G: \varphi(m) = eA^{\xi}\} \\ &= \{m \in G: mA^{\xi} = eA^{\xi}\} \\ &= \{m \in G: me^{-1} \in G_{A^{\xi}}\} \\ &= G_{A^{\xi}}. \end{aligned} \tag{58}$$

In the following result, we establish an isomorphic correspondence between quotient group of G by ξ-CFNSG A^{ξ} and quotient group G by $G_{A^{\xi}}$. □

Theorem 18. *Let A^{ξ} be ξ-CFNSG and $G_{A^{\xi}}$ be normal subgroup of G. Then, there exist an isomorphism between G/A^{ξ} and $G/G_{A^{\xi}}$.*

Proof. Define a mapping φ: $G/A^{\xi} \longrightarrow G/G_{A^{\xi}}$ as $\varphi(mA^{\xi}) = mG_{A^{\xi}}$. For any $xA^{\xi}, yA^{\xi} \in G/A^{\xi}$, we have

$$\begin{aligned} \varphi(mA^{\xi}nA^{\xi}) &= \varphi(mnA^{\xi}) \\ &= mnG_{A^{\xi}} \\ &= mG_{A^{\xi}}nG_{A^{\xi}} \\ &= \varphi(mA^{\xi})\varphi(nA^{\xi}). \end{aligned} \tag{59}$$

This shows that φ is homomorphism.

Moreover, for any $mA^{\xi}, nA^{\xi} \in G/A^{\xi}$, we have

$$\begin{aligned} \varphi(mA^{\xi}) &= \varphi(nA^{\xi}), \\ mG_{A^{\xi}} &= nG_{A^{\xi}}, \\ n^{-1}mG_{A^{\xi}} &= G_{A^{\xi}}, \end{aligned} \tag{60}$$

which shows that $n^{-1}mG_{A^{\xi}} \in G_{A^{\xi}}$.

By applying Theorem 16 in the above relation yields that $mA^{\xi} = nA^{\xi}$. Moreover, the surjective case is quite obvious. Consequently, φ is an isomorphism between G/A^{ξ} and $G/G_{A^{\xi}}$. □

5. Conclusion

In this paper, we first present the ξ-CFS which is completely a new notion. We have utilized this phenomena to define the (α, δ)-cut sets and strong (α, δ)-cut sets and have proved the representation of an ξ-CFS in the framework of these sets. Moreover, the notions of ξ-CFSG and level subgroups of these groups have also been defined in this article. In addition, a necessary and sufficient condition for an ξ-CFS to be a ξ-CFSG has also been investigated. Moreover, an isomorphism has been established between the quotient groups of a group G by its ξ-CFNSG and a normal subgroup $G_{A^{\xi}}$.

Data Availability

Any type of data associated with this work can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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