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Research Article

Linguistic Interval-Valued Intuitionistic Fuzzy Archimedean Power Muirhead Mean Operators for Multiattribute Group Decision-Making

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Two important tasks in multiattribute group decision-making (MAGDM) are to describe the attribute values and to generate a ranking of all alternatives. A superior tool for the first task is linguistic interval-valued intuitionistic fuzzy number (LIVIFN), and an effective tool for the second task is aggregation operator (AO). To date, nearly ten AOs of LIVIFNs have been presented. Each AO has its own features and can work well in its specific context. But there is not yet an AO of LIVIFNs that can offer desirable generality and flexibility in aggregating attribute values and capturing attribute interrelationships and concurrently reduce the influence of unreasonable attribute values. To this end, a linguistic interval-valued intuitionistic fuzzy Archimedean power Muirhead mean operator and its weighted form, which have such capabilities, are presented in this paper. Firstly, the generalised expressions of the AOs are established by a combination of the Muirhead mean operator and the power average operator under the Archimedean T-norm and T-conorm operations of LIVIFNs. Then the properties of the AOs are explored and proved, their specific expressions are constructed, and the special cases of the specific expressions are discussed. After that, a new method for solving the MAGDM problems based on LIVIFNs is designed on the basis of the weighted AO. Finally, the designed method is illustrated via a practical example, and the presented AOs are evaluated via experiments and comparisons.

1. Introduction

Multiattribute group decision-making (MAGDM) or multiattribute group decision analysis refers to a process of finding the most desirable alternatives from a set of finite alternatives on the basis of a ranking or the collective attribute values of all alternatives, in which the value of each attribute is provided by a group of experts [1]. This process has two critical tasks. This first task is to describe the values of attributes, while the second task is to generate a ranking of all alternatives.

For the description of attribute values, real number is a default tool. But it is usually difficult for an expert to provide the assessment results in the form of crisp values because of ambiguity and incomplete information. To this end, fuzzy set is naturally introduced in MAGDM and becomes a popular tool in this field [2, 3]. To date, more than twenty different types of fuzzy sets have been presented with academia [4], where Atanassov's intuitionistic fuzzy set (IFS) [5] and interval-valued IFS (IVIFS) [6] are two representative examples. IFS and IVIFS are powerful extensions of Zadeh's fuzzy set [7] for dealing with vagueness. Both of them have a membership degree (MD) and a nonmembership degree (NMD), which can quantify the degrees of satisfaction and dissatisfaction, respectively. Due to such strong expressiveness, IFSs and IVIFSs have been widely used to express the values of attributes in MAGDM [8]. Many research topics about IFSs and IVIFSs for MAGDM, such as operational rules for the sets [9–13], fuzzy calculus for the sets [14–17], score and accuracy functions of the sets [18–22], preference relations of the sets [23–26], similarity and distance measures of the sets [27–30], aggregation operators (AOs) for the sets [31–35], and decision-making approaches based on the sets [36–40], have received widespread attention during the past few decades.

Although IFSs and IVIFSs have gained importance and popularity in MAGDM, they can only be leveraged to express the rating values in quantitative aspect (i.e., the numerical rating values). In practical decision-making problems, the rating values may be denoted by other kinds of variables, where linguistic variables are one of the most important types [41–45]. For example, selection of a proper 3D printer from a certain number of alternatives to print a specific component is a classic decision-making problem in manufacturing domain. In this problem, the performance parameters of a 3D printer, such as surface roughness, strength, elongation, and hardness, may be described by some of linguistic terms like "small", "medium", and "large". Under such case, IFSs and IVIFSs are not applicable. To address this issue, Chen et al. [46] extended the IFS to the linguistic IFS (LIFS), in which the MD and NMD are represented by linguistic variables. As a result, the rating values in the form of single-valued linguistic terms can be expressed by LIFSs. Because of such expressiveness, LIFSs have been applied to solve decision-making problems by a number of researchers. For example, Li et al. [47] developed a set of new operational rules and entropy for LIFSs; Zhang et al. [48] designed an extended outranking approach for decision-making problems with LIFSs; Liu et al. [49], Garg and Kumar [50], Peng et al. [51], and Teng and Liu [52] presented some AOs for LIFSs; Jin et al. [53] established a decision support model for MAGDM with the preferences relations of LIFSs.

To further improve the expressiveness of LIFS, Garg and Kumar [54, 55] extended it to the linguistic interval-valued IFS (LIVIFS), in which the MD and NMD are quantified by the intervals of linguistic variables. LIVIFS can provide more freedom to experts, because it allows them to describe their preferences using intervals of linguistic terms. Due to such characteristic, Garg and Kumar [55] presented a weighted average (WA) operator, an ordered WA (OWA) operator, a hybrid average (HA) operator, a weighted geometric (WG) operator, an ordered WG (OWG) operator, and a hybrid geometric (HG) operator of linguistic interval-valued intuitionistic fuzzy numbers (LIVIFNs) and applied these operators to solve MAGDM problems; Kumar and Garg [56] presented a prioritised weighted averaging (PWA) operator, a prioritised ordered weighted averaging (POWA) operator, a prioritised weighted geometric (PWG) operator, and a prioritised ordered weighted geometric (POWG) operator and study their applications in MAGDM problems; Liu and Qin [57] presented a weighted Maclaurin symmetric mean (WMSM) operator of LIVIFNs and proposed a new decision-making method based on it; Garg and Kumar presented [58] an extended TOPSIS group decision-making method under LIVIFS environment; Tang et al. [59] developed a procedure for MAGDM with LIVIFSs that can cope with inconsistent and incomplete preference relations of LIVIFSs.

For the generation of a ranking of all alternatives, there are usually two ways. One way is to use conventional decision-making methods (e.g., TOPSIS, VIKOR, and ELECTRE), and the other way is to adopt AOs. Generally, AOs can resolve the MAGDM problems more effectively than conventional methods, because they can generate both the collective attribute values and a ranking of all alternatives, while conventional methods can only provide a ranking [60]. So far, over ten AOs of LIVIFNs have been presented. Representative examples are the WA, OWA, HA, WG, OWG, and HG operators presented by Garg and Kumar [54, 55], the PWA, POWA, PWG, and POWG operators presented by Kumar and Garg [56], and the WMSM operator presented by Liu and Qin [57]. Each of these AOs can work well under its specific circumstance, but none of them can provide desirable generality and flexibility in aggregating attribute values and capturing attribute interrelationships and concurrently reduce the negative influence of extreme attribute values on aggregation result.

In practical decision-making problems, the aggregation of attribute values is a complicated process, in which a set of general and versatile AOs is needed. Further, the attributes considered in the problems are always not independent of each other, but are interrelated. Thus, it is important and useful to use a general, flexible, and effective AO to capture the interrelationships of different attributes for making a reasonable decision [60]. The seven existing AOs, however, are sometimes not versatile and flexible, since all of them are based on a specific type of T-norm and T-conorm (i.e., Algebraic T-norm and T-conorm). Except the WMSM operator, all of the AOs can only deal with the situation in which the attributes are independent of each other or have priority relationships. In addition, the values of attributes are generally evaluated by domain experts. It is usually difficult to ensure the absolute objectivity of this way, which means that some biased experts will provide extreme attribute values [61]. To get a reasonable aggregation result under such circumstance, it is of necessity to reduce the effect of unduly high or unduly low attribute values. But none of the seven existing operators can achieve this. Based on these considerations, the motivations and objectives of the present paper are as follows:

(1) To develop a versatile and flexible AO of LIVIFNs, the Archimedean T-norm and T-conorm (ATNTC) operations [62], which can generate versatile and flexible operational rules for fuzzy numbers, are introduced to establish a set of operational rules of LIVIFNs

(2) To make the AO have the capability to capture the interrelationships among attributes, the Muirhead mean (MM) operator [63], which is suitable for the situations where all aggregated arguments are independent of each other, where there are interrelationships between any two arguments, and

where there are interrelationships among any multiple arguments [64, 65], is selected as the core component of the AO

(3) To make the AO capable to reduce the negative influence of biased attribute values on the aggregation result, the power average (PA) operator [66], which has the capability to reduce the negative effect of unreasonable argument values, is combined with the MM operator

Based on the analysis above, this paper aims to present a linguistic interval-valued intuitionistic fuzzy weighted Archimedean power MM operator for MAGDM. This aim is achieved via combining the ATNTC operations, the MM operator, and the PA operator with weights in the context of LIVIFSs. The major contributions of the paper are as follows:

(1) A set of general and flexible operational rules of LIVIFNs based on ATNTC operations is developed. The operational rules of LIVIFNs in the existing AOs of LIVIFNs are based on Algebraic T-norm and T-conorm, which are sometimes not versatile and flexible enough. The developed operational rules are based on any types of T-norm and T-conorm. They have satisfying generality and flexibility.

(2) A weighted Archimedean power MM operator of LIVIFNs is presented to solve the MAGDM problems based on LIVIFNs. Compared to the existing AOs of LIVIFNs, the presented AO has generality and flexibility in aggregating attribute values and capturing attribute interrelationships and concurrently can reduce the influence of extreme attribute values.

The rest of the paper is organised as follows. A brief introduction of some prerequisites is provided Section 2. Section 3 explains the details of the presented AO of LIVIFNs. A MAGDM method based on the AO is designed in Section 4. Section 5 demonstrates the method and evaluates the AOs. Section 6 ends the paper with a conclusion.

2. Preliminaries

In this section, some prerequisites in LIVIFS theory, operational rules for LIVIFNs, PA operator, and MM operator are briefly introduced to facilitate the understanding of the paper.

2.1. LIVIFS Theory. LIVIFS was extended from IVIFS and LIFS by Garg and Kumar [54, 55]. Its formal definition is as follows.

Definition 1 (see [55]). Let $S_{[o,h]} = \{s_t \mid s_o \leq s_h\}$ be a continuous linguistic term set (where s_h is a possible value for a variable and h is a positive integer, and for any $s_x, s_y \in S_{[0,h]}, s_x > s_y$ iff $x > y$). A LIVIFS A in a finite universe of discourse X is $A = \{< x, s\lfloor \mu(x) \rfloor, s\lfloor \nu(x) \rfloor > | x \in X\}$ (in this paper, the symbol in $\lfloor s \rfloor \lfloor r \rfloor$ denotes that the subscript of s is r), where $s\lfloor \mu(x) \rfloor = [s\lfloor \mu_L(x) \rfloor, s\lfloor \mu_U(x) \rfloor]$ and $s\lfloor \nu(x) \rfloor = [s\lfloor \nu_L(x) \rfloor, s\lfloor \nu_U(x) \rfloor]$ are subsets of $[s_0, s_h]$ and, respectively, stand for the linguistic MD and NMD of x to A, and $s\lfloor \mu_U(x) \rfloor + s\lfloor \nu_U(x) \rfloor \leq s_h$ (i.e., $\mu_U(x) + \nu_U(x) \leq h$) for any $x \in X$. The linguistic intuitionistic index of x to A is $s\lfloor \pi(x) \rfloor = [s\lfloor \pi_L(x) \rfloor, \quad s\lfloor \pi_U(x) \rfloor] = [s\lfloor h - \mu_U(x) - \nu_U(x) \rfloor, s\lfloor h - \mu_L(x) - \nu_L(x) \rfloor]$.

A pair, $([s\lfloor \mu_L(x) \rfloor + s\lfloor \mu_U(x) \rfloor], [s\lfloor \nu_L(x) \rfloor + s\lfloor \nu_U(x) \rfloor])$, is called a LIVIFN. For convenience, a LIVIFN is denoted as $\alpha = ([s_a, s_b], [s_c, s_d])$, where $s_a, s_b, s_c, s_d \in S_{[0,h]}, [s_a, s_b] \subseteq [s_0, s_h], [s_c, s_d] \subseteq [s_0, s_h]$, and $b + d \leq h$. To compare two LIVIFNs, their scores and accuracies are required, which can be calculated according to the following definitions.

Definition 2 (see [55]). Let $\alpha = ([s_a, s_b], [s_c, s_d])$ be a LIVIFN. Then its score is

$$S(\alpha) = s\lfloor \frac{(2h + a + b - d)}{4} \rfloor. \tag{1}$$

Definition 3 (see [55]). Let $\alpha = ([s_a, s_b], [s_c, s_d])$ be a LIVIFN. Then its accuracy is

$$A(\alpha) = S\lfloor \frac{(a + b + c + d)}{2} \rfloor. \tag{2}$$

Using $S(\alpha)$ and $A(\alpha)$, two LIVIFNs can be compared via the following definition.

Definition 4 (see [55]). Let $\alpha_1 = ([s\lfloor a_1 \rfloor, s\lfloor b_1 \rfloor], [s\lfloor c_1 \rfloor, s\lfloor d_1 \rfloor])$, and $\alpha_2 = ([s\lfloor a_2 \rfloor, s\lfloor b_2 \rfloor], [s\lfloor c_2 \rfloor, s\lfloor d_2 \rfloor])$ and be any two LIVIFNs, $S(\alpha_1)$ and $S(\alpha_2)$ be, respectively, the scores of α_1 and α_2, and $A(\alpha_1)$ and $A(\alpha_2)$ be, respectively, the accuracies of α_1 and α_2. Then, (1) if $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$; (2) if $S(\alpha_1) = S(\alpha_2)$ and $A(\alpha_1) > A(\alpha_2)$, then $\alpha_1 > \alpha_2$; (3) if $S(\alpha_1) = S(\alpha_2)$ and $A(\alpha_1) = A(\alpha_2)$, then $\alpha_1 = \alpha_2$.

To calculate the distance between two LIVIFNs, a distance measure of LIVIFNs is required. On the basis of the distance measure of IVIFNs introduced by Xu [67] and the distance measure of LIFNs introduced by Liu and Liu [68], the following distance measure for LIVIFNs is defined.

Definition 5. Let $\alpha_1 = ([s\lfloor a_1 \rfloor, s\lfloor b_1 \rfloor], [s\lfloor c_1 \rfloor, s\lfloor d_1 \rfloor])$, and $\alpha_2 = ([s\lfloor a_2 \rfloor, s\lfloor b_2 \rfloor], [s\lfloor c_2 \rfloor, s\lfloor d_2 \rfloor])$ be any two LIVIFNs. Then, the distance of α_1 and α_2 is

$$D(\alpha_1, \alpha_2) = \frac{|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|}{4h}. \tag{3}$$

Obviously, $D(\alpha_1, \alpha_2)$ satisfies (1) $D(\alpha_1, \alpha_2) = D(\alpha_2, \alpha_1) \geq 0$; (2) if $\alpha_3 = ([s\lfloor a_3 \rfloor, s\lfloor b_3 \rfloor], [s\lfloor c_3 \rfloor, s\lfloor d_3 \rfloor])$ is an arbitrary LIVIFN, then $D(\alpha_1, \alpha_3) \leq D(\alpha_1, \alpha_2) + D(\alpha_2, \alpha_3)$.

2.2. Operational Rules. Motivated by the concept of T-norm and T-conorm, a set of operational rules for LIVIFNs based on the Algebraic T-norm and T-conorm were presented by Garg and Kumar [54, 55], Kumar and Garg [56], and Liu and Qin [57]. These operational rules are sometimes not very versatile and flexible since they are just based on a specific type of ATNTC. Inspired by the general operational rules for IFNs [69] and q-rung orthopair fuzzy numbers [60] (which are based on any types of ATNTCs), a set of operational rules for LIVIFNs based on ATNTCs is developed. The definition of the operational rules is as follows.

Definition 6. Let $\alpha_1 = ([s\lfloor a_1\rfloor, s\lfloor b_1\rfloor], [s\lfloor c_1\rfloor, s\lfloor d_1\rfloor])$, $\alpha_2 = ([s\lfloor a_2\rfloor, s\lfloor b_2\rfloor], [s\lfloor c_2\rfloor, s\lfloor d_2\rfloor])$, and $\alpha = ([s\lfloor a_\alpha\rfloor, s\lfloor b_\alpha\rfloor], [s\lfloor c_\alpha\rfloor, s\lfloor d_\alpha\rfloor])$ be any three LIVIFNs, and r be a real number that satisfies $r > 0$, the sum, product, multiplication, and power operations of LIVIFNs based on the Archimedean T-norm $T(x, y) = f^{-1}(f(x) + f(y))$ and its T-conorm $C(x, y) = g^{-1}(g(x) + g(y))$ can be, respectively, defined as follows:

$$\begin{aligned}\alpha_1 \oplus \alpha_2 &= \left(\left[s_{(C(a_1,a_2))}, s_{(C(b_1,b_2))}\right], \left[s_{(T(c_1,c_2))}, s_{(T(d_1,d_2))}\right]\right)\\ &= \left(\left[s_{(g^{-1}(g(a_1)+g(a_2)))}, s_{(g^{-1}(g(b_1)+g(b_2)))}\right], \left[s_{(f^{-1}(f(c_1)+f(c_2)))}, s_{(f^{-1}(f(d_1)+f(d_2)))}\right]\right),\end{aligned} \tag{4}$$

$$\begin{aligned}\alpha_1 \otimes \alpha_2 &= \left(\left[s_{(T(a_1,a_2))}, s_{(T(b_1,b_2))}\right], \left[s_{(C(c_1,c_2))}, s_{(C(d_1,d_2))}\right]\right)\\ &= \left(\left[s_{(f^{-1}(f(a_1)+f(a_2)))}, s_{(f^{-1}(f(b_1)+f(b_2)))}\right], \left[s_{(g^{-1}(g(c_1)+g(c_2)))}, s_{(g^{-1}(g(d_1)+g(d_2)))}\right]\right),\end{aligned} \tag{5}$$

$$r\alpha = \left(\left[s_{(g^{-1}(rg(a_\alpha)))}, s_{(g^{-1}(rg(b_\alpha)))}\right], \left[s_{(f^{-1}(rf(c_\alpha)))}, s_{(f^{-1}(rf(d_\alpha)))}\right]\right), \tag{6}$$

$$\alpha^r = \left(\left[s_{(f^{-1}(rf(a_\alpha)))}, s_{(f^{-1}(rf(b_\alpha)))}\right], \left[s_{(g^{-1}(rg(c_\alpha)))}, s_{(g^{-1}(rg(d_\alpha)))}\right]\right). \tag{7}$$

Equations (4)–(7) are generalised form of the operational rules for LIVIFNs based on ATNTCs. If f and g are assigned specific functions, then specific operational rules can be achieved. The following are four examples:

(1) If $f(t) = -\mathrm{In}(t/h)$ and $g(t) = -\mathrm{In}(1 - t/h)$, then $f^{-1}(t) = he^{-t}$ and $g^{-1}(t) = h - he^{-t}$. Four operational rules for LIVIFNs based on Algebraic T-norm and T-conorm are obtained as follows:

$$\alpha_1 \oplus \alpha_2 = \left(\left[s_{(a_1+a_2-(a_1a_2)/h)}, s_{(b_1+b_2-(b_1b_2)/h)}\right], \left[s_{((c_1c_2)/h)}, s_{((d_1d_2)/h)}\right]\right), \tag{8}$$

$$\alpha_1 \otimes \alpha_2 = \left(\left[s_{((a_1a_2)/h)}, s_{((b_1b_2)/h)}\right], \left[s_{(c_1+c_2-(c_1c_2)/h)}, s_{(d_1+d_2-(d_1d_2)/h)}\right]\right), \tag{9}$$

$$r\alpha = \left(\left[s_{(h(1-(1-a_\alpha/h)^r))}, s_{(h(1-(1-b_\alpha/h)^r))}\right], \left[s_{(h(c_\alpha/h)^r)}, s_{(h(d_\alpha/h)^r)}\right]\right), \tag{10}$$

$$\alpha^r = \left(\left[s_{(h(a_\alpha/h)^r)}, s_{(h(b_\alpha/h)^r)}\right], \left[s_{(h(1-(1-c_\alpha/h)^r))}, s_{(h(1-(1-d_\alpha/h)^r))}\right]\right). \tag{11}$$

(2) If $f(t) = -\mathrm{In}\{[2h - t]/t\}$ and $g(t) = \mathrm{In}[(h + t)/(h - t)]$, then $f^{-1}(t) = (2h)/(e^t + 1)$ and $g^{-1}(t) = (he^t - h)/(e^t + 1)$. Four operational rules for LIVIFNs based on Einstein T-norm and T-conorm are obtained as follows:

$$\alpha_1 \oplus \alpha_2 = \left(\left[s_{(h^2(a_1+a_2)/(h^2+a_1a_2))}, s_{(h^2(b_1+b_2)/(h^2+b_1b_2))}\right], \left[s_{(hc_1c_2/(2h^2-h(c_1+c_2)+c_1c_2))}, s_{(hd_1d_2/(2h^2-h(d_1+d_2)+d_1d_2))}\right]\right), \tag{12}$$

$$\alpha_1 \otimes \alpha_2 = \left(\left[s_{(ha_1a_2/(2h^2-h(a_1+a_2)+a_1a_2))}, s_{(hb_1b_2/(2h^2-h(b_1+b_2)+b_1b_2))}\right], \left[s_{(h^2(c_1+c_2)/(h^2+c_1c_2))}, s_{(h^2(d_1+d_2)/(h^2+d_1d_2))}\right]\right), \tag{13}$$

$$r\alpha = \left(\left[s_{\left(\left(h\left(h+a_\alpha\right)^r - h\left(h-a_\alpha\right)^r\right)/\left(\left(h+a_\alpha\right)^r+\left(h-a_\alpha\right)^r\right)\right)}, s_{\left(\left(h\left(h+b_\alpha\right)^r - h\left(h-b_\alpha\right)^r\right)/\left(\left(h+b_\alpha\right)^r+\left(h-b_\alpha\right)^r\right)\right)}\right], \left[s_{\left(2hc_\alpha^r/\left(\left(2h-c_\alpha\right)^r+c_\alpha^r\right)\right)}, s_{\left(2hd_\alpha^r/\left(\left(2h-d_\alpha\right)^r+d_\alpha^r\right)\right)}\right]\right), \tag{14}$$

$$\alpha^r = \left(\left[s_{\left(2ha_\alpha^r/\left(\left(2h-a_\alpha\right)^r+a_\alpha^r\right)\right)}, s_{\left(2hb_\alpha^r/\left(\left(2h-b_\alpha\right)^r+b_\alpha^r\right)\right)}\right], \left[s_{\left(\left(h\left(h+c_\alpha\right)^r - h\left(h-c_\alpha\right)^r\right)/\left(\left(h+c_\alpha\right)^r+\left(h-c_\alpha\right)^r\right)\right)}, s_{\left(\left(h\left(h+d_\alpha\right)^r - h\left(h-d_\alpha\right)^r\right)/\left(\left(h+d_\alpha\right)^r+\left(h-d_\alpha\right)^r\right)\right)}\right]\right). \tag{15}$$

(3) If $f(t) = \mathrm{In}\{[h\lambda + (1-\lambda)t]/t\}\,(\lambda > 0)$ and $g(t) = \mathrm{In}\{[h + (\lambda - 1)t]/h - t\}$ then $f^{-1}(t) = (\lambda h)/(e^t + \lambda - 1)$ and $g^{-1}(t) = (he^t - h)/(e^t + \lambda - 1)$. Four operational rules for LIVIFNs based on Hamacher T-norm and T-conorm are obtained as follows:

$$\begin{aligned}\alpha_1 \oplus \alpha_2 = \Big(&\left[s_{\left(\left(h^2\left(a_1+a_2\right)+(\lambda-2)ha_1a_2\right)/\left(h^2+(\lambda-1)a_1a_2\right)\right)}, s_{\left(\left(h^2\left(b_1+b_2\right)+(\lambda-2)hb_1b_2\right)/\left(h^2+(\lambda-1)b_1b_2\right)\right)}\right],\\ &\left[s_{\left(hc_1c_2/\left(\lambda h^2+(h-\lambda h)\left(c_1+c_2\right)+(\lambda-1)c_1c_2\right)\right)}, s_{\left(hd_1d_2/\left(\lambda h^2+(h-\lambda h)\left(d_1+d_2\right)+(\lambda-1)d_1d_2\right)\right)}\right]\Big),\end{aligned} \tag{16}$$

$$\begin{aligned}\alpha_1 \otimes \alpha_2 = \Big(&\left[s_{\left(ha_1a_2/\left(\lambda h^2+(h-\lambda h)\left(a_1+a_2\right)+(\lambda-1)a_1a_2\right)\right)}, s_{\left(hb_1b_2/\left(\lambda h^2+(h-\lambda h)\left(b_1+b_2\right)+(\lambda-1)b_1b_2\right)\right)}\right],\\ &\left[s_{\left(\left(h^2\left(c_1+c_2\right)+(\lambda-2)hc_1c_2\right)/\left(h^2+(\lambda-1)c_1c_2\right)\right)}, s_{\left(\left(h^2\left(d_1+d_2\right)+(\lambda-2)hd_1d_2\right)/\left(h^2+(\lambda-1)d_1d_2\right)\right)}\right]\Big),\end{aligned} \tag{17}$$

$$\begin{aligned}r\alpha = \Big(&\left[s_{\left(\left(h\left(h+(\lambda-1)a_\alpha\right)^r - h\left(h-a_\alpha\right)^r\right)/\left(\left(h+(\lambda-1)a_\alpha\right)^r+(\lambda-1)\left(h-a_\alpha\right)^r\right)\right)}, s_{\left(\left(h\left(h+(\lambda-1)b_\alpha\right)^r - h\left(h-b_\alpha\right)^r\right)/\left(\left(h+(\lambda-1)b_\alpha\right)^r+(\lambda-1)\left(h-b_\alpha\right)^r\right)\right)}\right],\\ &\left[s_{\left(\lambda hc_\alpha^r/\left(\left(h\lambda+(1-\lambda)c_\alpha\right)^r+(\lambda-1)c_\alpha^r\right)\right)}, s_{\left(\lambda hd_\alpha^r/\left(\left(h\lambda+(1-\lambda)d_\alpha\right)^r+(\lambda-1)d_\alpha^r\right)\right)}\right]\Big),\end{aligned} \tag{18}$$

$$\begin{aligned}\alpha^r = \Big(&\left[s_{\left(\lambda ha_\alpha^r/\left(\left(h\lambda+(1-\lambda)a_\alpha\right)^r+(\lambda-1)a_\alpha^r\right)\right)}, s_{\left(\lambda hb_\alpha^r/\left(\left(h\lambda+(1-\lambda)b_\alpha\right)^r+(\lambda-1)b_\alpha^r\right)\right)}\right],\\ &\left[s_{\left(\left(h\left(h+(\lambda-1)c_\alpha\right)^r - h\left(h-c_\alpha\right)^r\right)/\left(\left(h+(\lambda-1)c_\alpha\right)^r+(\lambda-1)\left(h-c_\alpha\right)^r\right)\right)}, s_{\left(\left(h\left(h+(\lambda-1)d_\alpha\right)^r - h\left(h-d_\alpha\right)^r\right)/\left(\left(h+(\lambda-1)d_\alpha\right)^r+(\lambda-1)\left(h-d_\alpha\right)^r\right)\right)}\right]\Big).\end{aligned} \tag{19}$$

(4) If $f(t) = -\mathrm{In}[(\varepsilon - 1/\varepsilon^{t/h} - 1)]\,(\varepsilon > 1)$ and $g(t) = -\mathrm{In}[(\varepsilon - 1/\varepsilon^{1-t/h} - 1)]$, then $f^{-1}(t) = \log_\varepsilon[1 + (\varepsilon - 1/e^{-t})]^h$ and $g^{-1}(t) = h - \log_\varepsilon[1 + (\varepsilon - 1)/e^{-t}]^h$. Four operational rules for LIVIFNs based on Frank T-norm and T-conorm are obtained as follows:

$$\begin{aligned}\alpha_1 \oplus \alpha_2 = \Big(&\left[s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-a_1/h}-1\right)\left(\varepsilon^{1-a_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}, s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-b_1/h}-1\right)\left(\varepsilon^{1-b_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}\right],\\ &\left[s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{c_1/h}-1\right)\left(\varepsilon^{c_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}, s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{d_1/h}-1\right)\left(\varepsilon^{d_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}\right]\Big),\end{aligned} \tag{20}$$

$$\begin{aligned}\alpha_1 \otimes \alpha_2 = \Big(&\left[s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{a_1/h}-1\right)\left(\varepsilon^{a_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}, s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{b_1/h}-1\right)\left(\varepsilon^{b_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}\right],\\ &\left[s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-c_1/h}-1\right)\left(\varepsilon^{1-c_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}, s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-d_1/h}-1\right)\left(\varepsilon^{1-d_2/h}-1\right)/(\varepsilon-1)\right)\right)^h\right)}\right]\Big),\end{aligned} \tag{21}$$

$$\begin{aligned}r\alpha = \Big(&\left[s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-a_\alpha/h}-1\right)^r/(\varepsilon-1)^{r-1}\right)\right)^h\right)}, s_{\left(h-\log_\varepsilon\left(1+\left(\left(\varepsilon^{1-b_\alpha/h}-1\right)^r/(\varepsilon-1)^{r-1}\right)\right)^h\right)}\right],\\ &\left[s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{c_\alpha/h}-1\right)^r/(\varepsilon-1)^{r-1}\right)\right)^h\right)}, s_{\left(\log_\varepsilon\left(1+\left(\left(\varepsilon^{d_\alpha/h}-1\right)^r/(\varepsilon-1)^{r-1}\right)\right)^h\right)}\right]\Big),\end{aligned} \tag{22}$$

$$\alpha^{r}=\left(\left[s_{\left(\log_{\varepsilon}\left(1+\left(\left(\varepsilon^{a_{\alpha}/h}-1\right)^{r}/(\varepsilon-1)^{r-1}\right)\right)^{h}\right)}, s_{\left(\log_{\varepsilon}\left(1+\left(\left(\varepsilon^{b_{\alpha}/h}-1\right)^{r}/(\varepsilon-1)^{r-1}\right)\right)^{h}\right)}\right],\right.$$
$$\left.\left[s_{\left(h-\log_{\varepsilon}\left(1+\left(\left(\varepsilon^{1-c_{\alpha}/h}-1\right)^{r}/(\varepsilon-1)^{r-1}\right)\right)^{h}\right)}, s_{\left(h-\log_{\varepsilon}\left(1+\left(\left(\varepsilon^{1-d_{\alpha}/h}-1\right)^{r}/(\varepsilon-1)^{r-1}\right)\right)^{h}\right)}\right]\right). \tag{23}$$

The operational rules in equations (8)–(11) are the four operational rules in [54–57]. From these examples, it can be seen that the developed operational rules can be used to derive any operational rules based on ATNTCs and the operational rules in [54–57] are just one special case of the developed operational rules. Therefore, the developed operational rules are more general and flexible than the operational rules in [54–57].

2.3. PA Operator. The PA operator, introduced by Yager [66], has the capability to assign weights to arguments via computing their support degrees. This makes it possible to reduce the negative effect of the unduly high or unduly low argument values on the aggregation result. The formal definition of this operator is as follows.

Definition 7 (see [66]). Let $(a_1, a_2, \ldots, a_n)$ be a collection of crisp numbers, $S(a_i, a_j) = 1 - D(a_i, a_j)$ (where, $D(a_i, a_j)$ is the distance of a_i and a_j and $i, j = 1, 2, \ldots, n$ and $j \neq i$) be the support degree for a_i from a_j which has the following properties: (1) $0 \leq S(a_i, a_j) \leq 1$; (2) $S(a_i, a_j) = S(a_j, a_i)$; (3) $S(a_i, a_j) \geq S(a_p, a_p)$ if $|a_i - a_j| \leq |a_p - a_p|$, and

$$T(a_i) = \sum_{j=1, j\neq i}^{n} S(a_i, a_j). \tag{24}$$

Then the aggregation function

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}\left((1 + T(a_i))a_i\right)}{\sum_{i=1}^{n}\left(1 + T(a_i)\right)}, \tag{25}$$

is called the PA operator.

2.4. MM Operator. The MM operator was firstly introduced to aggregate crisp numbers by Muirhead [63]. It can capture the interrelationships of arguments and provide a generalised form of several other AOs. The formal definition of the MM operator is as follows.

Definition 8 (see [63]). Let $(a_1, a_2, \ldots, a_n)$ be a collection of crisp numbers, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$ (where $Q_1, Q_2, \ldots, Q_n \geq 0$ but not at the same time $Q_1 = Q_2 = \cdots = Q_n = 0$) be a collection of n real numbers, $p(i)$ be a permutation of $(1, 2, \ldots, n)$, and $\mathbf{P}_n$ be the set of all permutations of $(1, 2, \ldots, n)$. Then the aggregation function

$$\mathrm{MM}^{\mathbf{Q}}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n!}\sum_{p\in\mathbf{P}_n}\prod_{i=1}^{n} a_{p(i)}^{Q_i}\right)^{1/\sum_{i=1}^{n} Q_i}, \tag{26}$$

is called the MM operator.

In this operator, whether the interrelationships are considered depends on the values of Q_i $(i = 1, 2, \ldots, n)$: (1) if $Q_1 = Q > 0$ and $Q_2 = Q_3 = \cdots = Q_n = 0$, then the interrelationships are not considered; (2) if $Q_1 = Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$, then the interrelationships between any two crisp numbers are considered; (3) if $Q_1, Q_2, \ldots Q_k > 0$ $(k = 3, 4, \ldots, n)$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$, then the interrelationships among any k crisp numbers are considered. Further, different AOs can be obtained via assigning different values to $Q_1, Q_2, \ldots, Q_n$:

(1) If $Q_1 = Q > 0$ and $Q_2 = Q_3 = \cdots = Q_n = 0$ then the MM operator will reduce to the generalised arithmetic average (GAA) operator. When $Q = 1$, it will become the arithmetic average (AA) operator.
(2) If $Q_1, Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$, then the MM operator will reduce to the Bonferroni mean (BM) operator.
(3) If $Q_1 = Q_2 = \cdots = Q_k = 1$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$, then the MM operator will reduce to the Maclaurin symmetric mean (MSM) operator.
(4) If $Q_1 = Q_2 = \cdots = Q_n = Q > 0$, then the MM operator will reduce to the generalised geometric average (GGA) operator. When $Q = 1$, it will become the geometric average (GA) operator.

3. Aggregation Operators

In this section, a linguistic interval-valued intuitionistic fuzzy Archimedean power MM (LIVIFAPMM) operator and its weighted form, i.e., a linguistic interval-valued intuitionistic fuzzy Archimedean weighted power MM (LIVIFAWPMM) operator, are presented. The properties of these AOs are explored, and their specific cases are discussed.

3.1. LIVIFAPMM Operator. A LIVIFAPMM operator is an AO of LIVIFNs constructed via combining the PA operator and the MM operator under linguistic interval-valued intuitionistic fuzzy environment. The operations in this operator are based on ATNTCs. The formal definition of the operator is as follows.

Definition 9. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $\alpha_i([s\lfloor a_i\rfloor, s\lfloor b_i\rfloor], [s\lfloor c_i\rfloor, s\lfloor d_i\rfloor]), i = 1, 2, \ldots, n$) be a collection of n LIVIFNs, $\mathbf{Q} = (Q_1 = Q_2 = \cdots = Q_n)$ (where $Q_1 = Q_2 = \cdots = Q_n \geq 0$ but not at the same time $Q_1 = Q_2 = \cdots = Q_n = 0$) be a collection of n real numbers that, respectively, correspond to $\alpha_1, \alpha_2, \ldots, \alpha_n$, $p(i)$ be a permutation of $(1, 2, \ldots, n)$, $\mathbf{P}_n$ be the set of all permutations of $(1, 2, \ldots, n)$, $\alpha_i \oplus \alpha_j$ and

$\alpha_i \otimes \alpha_j$ $(j = 1, 2, \ldots, n)$ be, respectively, the sum and product operations of α_i and α_j based on ATNTCs, $r\alpha_k$ and α_r^k $(k = 1, 2, \ldots, n; r > 0)$ be, respectively, the multiplication and power operations of α_k based on ATNTCs, $S(\alpha_i, \alpha_j) = 1 - D(\alpha_i, \alpha_j)$ (where $D(\alpha_i, \alpha_j)$ is the distance of α_i and α_j and $i, j = 1, 2, \ldots, n$ and $j \neq i$) be the support degree for α_i from α_j which satisfies $0 \leq S(\alpha_i, \alpha_j) \leq 1$, $S(\alpha_i, \alpha_j) = S(\alpha_j, \alpha_i)$, and $S(\alpha_i, \alpha_j) \geq S(\alpha_p, \alpha_p)$ if $S|\alpha_i, \alpha_j| \leq |\alpha_p, \alpha_p|$, and

$$T(\alpha_i) = \sum_{j=1, j\neq i}^{n} S(\alpha_i, \alpha_j). \tag{27}$$

Then the aggregation function

$$\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!} \bigoplus_{p\in \mathbf{P}_n} \bigotimes_{i=1}^{n} \left(\frac{n\left(1 + T\left(\alpha_{p(i)}\right)\right)}{\sum_{j=1}^{n}\left(1 + T\left(\alpha_j\right)\right)}\alpha_{p(i)}\right)^{Q_i}\right)^{1/\sum_{i=1}^{n} Q_i}, \tag{28}$$

is called the LIVIFAPMM operator.

In this operator, the values of Q_i are used to capture the interrelationships of the aggregated LIVIFNs: (1) if $Q_1 > 0$ and $Q_2 = Q_3 = \cdots = Q_n = 0$, then the LIVIFNs are independent of each other; (2) if $Q_1, Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$, then the interrelationships between any two LIVIFNs are considered; (3) if $Q_1 = Q_2 = \cdots = Q_k > 0$ $(k = 3, 4, \ldots, n)$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$, then the interrelationships among any k LIVIFNs are considered.

According to equations (4)–(7) and (28), the following theorem is obtained.

Theorem 1. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$) be a collection of n LIVIFNs. Then*

$$\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s_a, s_b], [s_c, s_d]), \tag{29}$$

and it is still a LIVIFN, where

$$\begin{aligned}
s_a &= s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in \mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)},\\
s_b &= s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in \mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)},\\
s_c &= s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in \mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right) f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)},\\
s_d &= s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in \mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(n\xi_{p(i)}\right) f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)},
\end{aligned} \tag{30}$$

and $\xi_{p(i)}$ is a PA factor that can be computed by

$$\xi_i = \frac{1 + \sum_{p=1, p\neq i}^{n}\left(1 - D\left(\alpha_i, \alpha_p\right)\right)}{\sum_{j=1}^{n}\left(1 + \sum_{q=1, q\neq j}^{n}\left(1 - D\left(\alpha_j, \alpha_q\right)\right)\right)}. \tag{31}$$

For the details regarding the proof of this theorem, please refer to Appendix A. The following three theorems respectively state the idempotency, commutativity, and boundedness of the LIVIFPMM operator.

Theorem 2 (idempotency). *Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$) be a collection of n LIVIFNs. If $\alpha_i = \alpha = ([s\lfloor a_\alpha \rfloor, s\lfloor b_\alpha \rfloor], [s\lfloor c_\alpha \rfloor, s\lfloor d_\alpha \rfloor])$ for all $i = 1, 2, \ldots, n$, then LIVIFAPMM$^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$.*

Theorem 3 (commutativity). *Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $a_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$) be a collection of n LIVIFNs. If $(\beta_1, \beta_2, \ldots, \beta_n)$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then LIVIFAPMM$^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) =$ LIVIFAPMM$^{\mathbf{Q}}(\beta_1, \beta_2, \ldots, \beta_n)$.*

Theorem 4 (boundedness). *Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (where $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$) be a collection of n LIVIFNs, $\alpha^- = ([s\lfloor \min(a_i) \rfloor, s\lfloor \min(b_i) \rfloor], [s\lfloor \max(c_i) c_i \rfloor, s\lfloor \max(d_i) \rfloor])$, and $\alpha^+ = ([s\lfloor \max(a_i) \rfloor, s\lfloor \max(b_i) \rfloor], [s\lfloor \min(c_i) c_i \rfloor, s\lfloor \min(d_i) \rfloor])$, Then $\alpha^- \leq$ LIVIFAPMM$^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.*

For the details regarding the proofs of these three theorems, please refer to Appendices B–D, respectively.

Equation (29) is a generalised form of the LIVIFAPMM operator. If specific functions are assigned to f and g, then specific operators can be constructed. For example, if the additive generators of Algebraic T-norm and T-conorm [60, 69] are, respectively, assigned to f and g, i.e., $f(t) = -\mathrm{In}(t/h)$ and $g(t) = -\mathrm{In}(1-t/h)$, then a linguistic interval-valued intuitionistic fuzzy power MM (LIVIFPMM) operator is constructed:

$$\mathrm{LIVIFPMM}^{\mathbf{Q}}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\left[s_{\left(h\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(a_{p(i)}/h\right)\right)^{n\xi_{p(i)}}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{\left(1/\sum_{i=1}^{n}Q_i\right)}\right)}, s_{\left(h\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(b_{p(i)}/h\right)\right)^{n\xi_{p(i)}}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{\left(1/\sum_{i=1}^{n}Q_i\right)}\right)}\right], \left[s_{\left(h\left(1-\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(c_{p(i)}/h\right)^{n\xi_{p(i)}}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{\left(1/\sum_{i=1}^{n}Q_i\right)}\right)\right)}, s_{\left(h\left(1-\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(d_{p(i)}/h\right)^{n\xi_{p(i)}}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{\left(1/\sum_{i=1}^{n}Q_i\right)}\right)\right)}\right]\right). \tag{32}$$

This operator has the following special cases:

(1) If $Q_1 = Q > 0$ and $Q_2 = Q_3 = \cdots = Q_n = 0$, then the LIVIFPMM operator will reduce to

$$\left(\left[s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)\right)^{n\xi_i}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)}, s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)\right)^{n\xi_i}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)}\right], \left[s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(c_i/h\right)^{n\xi_i}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)}, s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(d_i/h\right)^{n\xi_i}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)}\right]\right) = \mathrm{LIVIFPGAA}^{(Q)}(\alpha_1,\alpha_2,\ldots,\alpha_n), \tag{33}$$

which is a linguistic interval-valued intuitionistic fuzzy power GAA (LIVIFPGAA) operator. When $Q=1$, it will become

$$\left(\left[s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)\right)^{n\xi_i}\right)\right)^{1/n}\right)\right)}, s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)\right)^{n\xi_i}\right)\right)^{1/n}\right)\right)}\right], \left[s_{\left(h\left(\prod_{i=1}^{n}\left(c_i/h\right)^{n\xi_i}\right)^{1/n}\right)}, s_{\left(h\left(\prod_{i=1}^{n}\left(d_i/h\right)^{n\xi_i}\right)^{1/n}\right)}\right]\right) = \mathrm{LIVIFPAA}(\alpha_1,\alpha_2,\ldots,\alpha_n), \tag{34}$$

which is a linguistic interval-valued intuitionistic fuzzy power AA (LIVIFPAA) operator.

(2) If $Q_1, Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$ then the LIVIFPMM operator will reduce to

$$\left(\left[\begin{array}{l} s_{\left(h\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\left(1-\left(1-\left(a_i/h\right)\right)^{n\xi_i}\right)^{Q_1}\left(1-\left(1-\left(a_j/h\right)\right)^{n\xi_j}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/(Q_1+Q_2)}\right)},\\ s_{\left(h\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\left(1-\left(1-\left(b_i/h\right)\right)^{n\xi_i}\right)^{Q_1}\left(1-\left(1-\left(b_j/h\right)\right)^{n\xi_j}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/(Q_1+Q_2)}\right)} \end{array}\right],\left[\begin{array}{l} s_{\left(h\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(1-\left(1-\left(c_i/h\right)^{n\xi_i}\right)^{Q_1}\left(1-\left(c_j/h\right)^{n\xi_j}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/(Q_1+Q_2)}\right)\right)},\\ s_{\left(h\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(1-\left(1-\left(d_i/h\right)^{n\xi_i}\right)^{Q_1}\left(1-\left(d_j/h\right)^{n\xi_j}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/(Q_1+Q_2)}\right)\right)} \end{array}\right]\right) = \mathrm{LIVIFPBM}^{(Q_1,Q_2)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right), \tag{35}$$

which is a linguistic interval-valued intuitionistic fuzzy power BM (LIVIFPBM) operator.

(3) If $Q_1 = Q_2 = \cdots = Q_k = 1$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$, then the LIVIFPMM operator will reduce to

$$\left(\left[\begin{array}{l} s_{\left(h\left(1-\prod_{1\leq i_1<\cdots<i_k\leq n}^{n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(a_{i_j}/h\right)\right)^{n\xi_{i_j}}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)},\\ s_{\left(h\left(1-\prod_{1\leq i_1<\cdots<i_k\leq n}^{n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(b_{i_j}/h\right)\right)^{n\xi_{i_j}}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)} \end{array}\right],\left[\begin{array}{l} s_{\left(h\left(1-\left(1-\prod_{1\leq i_1<\cdots<i_k\leq n}^{n}\left(1-\prod_{j=1}^{k}\left(1-\left(c_{i_j}/h\right)^{n\xi_{i_j}}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)\right)},\\ s_{\left(h\left(1-\left(1-\prod_{1\leq i_1<\cdots<i_k\leq n}^{n}\left(1-\prod_{j=1}^{k}\left(1-\left(d_{i_j}/h\right)^{n\xi_{i_j}}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)\right)} \end{array}\right]\right) = \mathrm{LIVIFPMSM}^{(k)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right), \tag{36}$$

which is a linguistic interval-valued intuitionistic fuzzy power MSM (LIVIFPMSM) operator.

(4) If $Q_1 = Q_2 = \cdots = Q_n = Q > 0$, then the LIVIFPMM operator will reduce to

$$\left(\left[\begin{array}{l} s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)^{n\xi_i}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)},\\ s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)^{n\xi_i}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)} \end{array}\right],\left[\begin{array}{l} s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(c_i/h\right)\right)^{n\xi_i}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)},\\ s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(d_i/h\right)\right)^{n\xi_i}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)} \end{array}\right]\right) = \mathrm{LIVIFPGGA}^{(Q)}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right), \tag{37}$$

which is a linguistic interval-valued intuitionistic fuzzy power GGA (LIVIFPGGA) operator. When $Q = 1$, it will become

$$\left(\left[s_{\left(h\left(\prod_{i=1}^{n}(a_i/h)^{n\xi_i}\right)^{1/n}\right)}, s_{\left(h\left(\prod_{i=1}^{n}(b_i/h)^{n\xi_i}\right)^{1/n}\right)}\right], \left[s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-(1-(c_i/h))^{n\xi_i}\right)\right)^{1/n}\right)\right)}, s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-(1-(d_i/h))^{n\xi_i}\right)\right)^{1/n}\right)\right)}\right]\right) = \mathrm{LIVIFPGA}(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{38}$$

which is a linguistic interval-valued intuitionistic fuzzy power GA (LIVIFPGA) operator.

Similarly, if the additive generators of other (e.g., Einstein, Hamacher, Frank) ATNTCs [60, 69] are, respectively, assigned to f and g, then other specific operators can be constructed according to equation (29).

3.2. LIVIFAWPMM Operator. The LIVIFAPMM operator has advantages in having desirable generality and flexibility, capturing the complex interrelationships of LIVIFNs, and reducing the negative effect of unreasonable LIVIFNs on the aggregation result. But it does not consider the relative importance of each aggregated LIVIFN. To this end, weights are introduced and a LIVIFAWPMM operator is presented. The formal definition of this operator is as follows.

Definition 10. On the basis of Definition 9, let $w_1, w_2, \ldots, w_n$ be, respectively, the weights of $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $0 \le w_1, w_2, \ldots, w_n \le 1$ and $w_1 + w_2 + \cdots + w_n = 1$. Then the aggregation function

$$\mathrm{LIVIFAWPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!} \bigoplus_{p \in \mathbf{P}_n} \bigotimes_{i=1}^{n} \left(\frac{n w_{p(i)}\left(1 + T\left(\alpha_{p(i)}\right)\right)}{\sum_{j=1}^{n}\left(1 + T\left(\alpha_j\right)\right)} \alpha_{p(i)}\right)^{Q_i}\right)^{1/\sum_{i=1}^{n} Q_i}, \tag{39}$$

is called the LIVIFAWPMM operator.

In this operator, the function of Q_i is the same as the function of Q_i in the LIVIFAPMM operator (see equation (28)).

According to equations (4)–(7) and (39), the following theorem is obtained.

Theorem 5. *Let* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *(where* $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$*) be a collection of n LIVIFNs. Then*

$$\mathrm{LIVIFAWPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s_a, s_b], [s_c, s_d]), \tag{40}$$

and it is still a LIVIFN, where

$$\begin{aligned}
s_a &= s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\left(w_{p(i)}\xi_{p(i)}\right)/\sum_{t=1}^{n}(w_t\xi_t)\right) g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)}, \\
s_b &= s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\left(w_{p(i)}\xi_{p(i)}\right)/\sum_{t=1}^{n}(w_t\xi_t)\right) g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)}, \\
s_c &= s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\left(w_{p(i)}\xi_{p(i)}\right)/\sum_{t=1}^{n}(w_t\xi_t)\right) f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)}, \\
s_d &= s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\left(w_{p(i)}\xi_{p(i)}\right)/\sum_{t=1}^{n}(w_t\xi_t)\right) f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)},
\end{aligned} \tag{41}$$

and $\xi_{p(i)}$ *is a PA factor that can be computed by equation (31).*

The proof of Theorem 5 is similar to the proof of Theorem 1 (see Appendix A) and is omitted here. The following two theorems, respectively, state the commutativity and boundedness of the LIVIFAWPMM operator (it is worth nothing that the LIVIFAWPMM operator no longer has the property of idempotency due to the introduce of weights).

Theorem 6 (commutativity). *Let* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *(where* $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$*) be a collection of n LIVIFNs. If* $(\beta_1, \beta_2, \ldots, \beta_n)$ *is any permutation of* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$*, then* $\mathit{LIVIFAWPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathit{LIVIFAWPMM}^{\mathbf{Q}}(\beta_1, \beta_2, \ldots, \beta_n)$.

Theorem 7 (boundedness). *Let* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *(where* $\alpha_i = ([s\lfloor a_i \rfloor, s\lfloor b_i \rfloor], [s\lfloor c_i \rfloor, s\lfloor d_i \rfloor]), i = 1, 2, 3, \ldots, n$*) be a collection of n LIVIFNs,* $\alpha^- = ([s\lfloor \min(a_i) \rfloor, s\lfloor \min(b_i) \rfloor], [s\lfloor \max(c_i) c_i \rfloor, s\lfloor \max(d_i) \rfloor])$, *and* $\alpha^+ = ([s\lfloor \max(a_i) \rfloor,$

$s\lfloor\max(b_i)\rfloor]$, $[s\lfloor\min(c_i)c_i\rfloor, s\lfloor\min(d_i)\rfloor]\big)$. *Then* $\alpha^- \leq$ *LIVIFAWPMM*$^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.

The proofs of these theorems are, respectively, similar to the proofs of Theorems 3 (see Appendix C) and 4 (see Appendix D) and are omitted here.

Equation (40) is a generalised form of the LIVIFAWPMM operator. If specific functions are assigned to f and g, then specific operators can be constructed. For example, if the additive generators of Algebraic T-norm and T-conorm [60, 69] are, respectively, assigned to f and g, i.e., $f(t) = -\mathrm{In}(t/h)$ and $g(t) = -\mathrm{In}(1 - t/h)$, then a linguistic interval-valued intuitionistic fuzzy weighted power MM (LIVIFWPMM) operator is constructed:

$$
\begin{aligned}
&\mathrm{LIVIFWPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \left(\left[\begin{array}{l} s_{\left(h\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(a_{p(i)}/h\right)\right)^{\left(n\left(w_{p(i)}\xi_{p(i)}\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{1/\sum_{i=1}^{n}Q_i}\right)} \\ s_{\left(h\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(b_{p(i)}/h\right)\right)^{\left(n\left(w_{p(i)}\xi_{p(i)}\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{1/\sum_{i=1}^{n}Q_i}\right)} \end{array}\right],\right. \\
&\left.\left[\begin{array}{l} s_{\left(h\left(1-\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(c_{p(i)}/h\right)^{\left(n\left(w_{p(i)}\xi_{p(i)}\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{1/\sum_{i=1}^{n}Q_i}\right)\right)} \\ s_{\left(h\left(1-\left(1-\left(\prod_{p\in\mathbf{P}_n}\left(1-\prod_{i=1}^{n}\left(1-\left(d_{p(i)}/h\right)^{\left(n\left(w_{p(i)}\xi_{p(i)}\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q_i}\right)\right)^{(1/n!)}\right)^{1/\sum_{i=1}^{n}Q_i}\right)\right)} \end{array}\right]\right).
\end{aligned}
\tag{42}
$$

This operator has the following special cases:

(1) If $Q_1 = Q > 0$ and $Q_2 = Q_3 = \cdots = Q_n = 0$, then the LIVIFWPMM operator will reduce to

$$
\begin{aligned}
&\left(\left[\begin{array}{l} s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)} \\ s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q}\right)^{1/n}\right)^{1/Q}\right)} \end{array}\right], \left[\begin{array}{l} s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(c_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)} \\ s_{\left(h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(d_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}(w_t\xi_t)}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)\right)} \end{array}\right]\right) \\
&= \mathrm{LIVIFWPGAA}^{(Q)}(\alpha_1, \alpha_2, \ldots, \alpha_n),
\end{aligned}
\tag{43}
$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power GAA (LIVIFWPGAA) operator. When $Q = 1$, it will become

$$
\begin{aligned}
&\left(\left[s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{1/n}\right)\right)}, s_{\left(h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{1/n}\right)\right)}\right],\right.\\
&\left.\left[s_{\left(h\left(\prod_{i=1}^{n}\left(c_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{1/n}\right)}, s_{\left(h\left(\prod_{i=1}^{n}\left(d_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{1/n}\right)}\right]\right)\\
&= \mathrm{LIVIFWPAA}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right),
\end{aligned}
\tag{44}
$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power AA (LIVIFWPAA) operator.

(2) If $Q_1, Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$, then the LIVIFWPMM operator will reduce to

$$
\begin{aligned}
&\left(\left[\begin{array}{l}
s_{\left(h\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\left(1-\left(1-\left(a_i/h\right)\right)^{n\left(w_i\xi_i\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_1}\left(1-\left(1-\left(a_j/h\right)\right)^{n\left(w_j\xi_j\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/\left(Q_1+Q_2\right)}\right)},\\
s_{\left(h\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(\left(1-\left(1-\left(b_i/h\right)\right)^{n\left(w_i\xi_i\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_1}\left(1-\left(1-\left(b_j/h\right)\right)^{n\left(w_j\xi_j\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/\left(Q_1+Q_2\right)}\right)}
\end{array}\right],\right.\\
&\left.\left[\begin{array}{l}
s_{\left(h\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(1-\left(1-\left(c_i/h\right)^{n\left(w_i\xi_i\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_1}\left(1-\left(c_j/h\right)^{n\left(w_j\xi_j\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/\left(Q_1+Q_2\right)}\right)\right)},\\
s_{\left(h\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ j\neq i}}^{n}\left(1-\left(1-\left(d_i/h\right)^{n\left(w_i\xi_i\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_1}\left(1-\left(d_j/h\right)^{n\left(w_j\xi_j\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q_2}\right)\right)^{1/n(n-1)}\right)^{1/\left(Q_1+Q_2\right)}\right)\right)}
\end{array}\right]\right)\\
&= \mathrm{LIVIFWPBM}^{\left(Q_1,Q_2\right)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right),
\end{aligned}
\tag{45}
$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power BM (LIVIFWPBM) operator.

(3) If $Q_1 = Q_2 = \cdots = Q_k = 1$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$, then the LIVIFWPMM operator will reduce to

$$\left(\left[s_{h\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(a_{i_j}/h\right)\right)^{\left(n\left(w_{i_j}\xi_{i_j}\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}}, s_{h\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(b_{i_j}/h\right)\right)^{\left(n\left(w_{i_j}\xi_{i_j}\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}}\right],\right.$$
$$\left[s_{h\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(c_{i_j}/h\right)^{\left(n\left(w_{i_j}\xi_{i_j}\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)}, s_{h\left(1-\left(1-\prod_{1\le i_1<\cdots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(d_{i_j}/h\right)^{\left(n\left(w_{i_j}\xi_{i_j}\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{k!(n-k)!/n!}\right)^{1/k}\right)}\right]\Bigg)$$
$$= \mathrm{LIVIFWPMSM}^{(k)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right), \tag{46}$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power MSM (LIVIFWPMSM) operator.

(4) If $Q_1 = Q_2 = \cdots = Q_n = Q > 0$, then the LIVIFWPMM operator will reduce to

$$\left(\left[s_{h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(a_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)}, s_{h\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(b_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q}\right)\right)^{1/n}\right)^{1/Q}\right)}\right], \left[s_{h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(c_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q}\right)^{1/n}\right)^{1/Q}}, s_{h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(d_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{Q}\right)^{1/n}\right)^{1/Q}}\right]\right)$$
$$= \mathrm{LIVIFWPGGA}^{(Q)}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right), \tag{47}$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power GGA (LIVIFWPGGA) operator. When $Q = 1$, it will become

$$\left(\left[s_{h\left(\prod_{i=1}^{n}\left(a_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{1/n}}, s_{h\left(\prod_{i=1}^{n}\left(b_i/h\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)^{1/n}}\right],\right.$$
$$\left[s_{h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(c_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{1/n}\right)}, s_{h\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\left(d_i/h\right)\right)^{\left(n\left(w_i\xi_i\right)\right)/\sum_{t=1}^{n}\left(w_t\xi_t\right)}\right)\right)^{1/n}\right)}\right]\Bigg)$$
$$= \mathrm{LIVIFWPGA}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right), \tag{48}$$

which is a linguistic interval-valued intuitionistic fuzzy weighted power GA (LIVIFWPGA) operator.

Similarly, if the additive generators of other (e.g., Einstein, Hamacher, and Frank) ATNTCs [60, 69] are, respectively, assigned to f and g, then other specific operators can be constructed according to equation (40).

4. MAGDM Method

In this section, an MAGDM method based on the LIVIFAWPMM operator is designed to resolve the MAGDM problems based on LIVIFNs.

In general, a MAGDM problem based on LIVIFNs can be formalised by a set of options $\mathbf{O} = \{O_1, O_2, \ldots, O_m\}$, a set of attributes $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$, a vector of weights of attributes $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ such that $0 \le w_1, w_2, \ldots, w_n \le 1$ and $w_1 + w_2 + \ldots + w_n = 1$, a set of experts $\mathbf{E} = \{E_1, E_2, \ldots, E_{nL}\}$, a vector of weights of experts $\varpi = [\varpi_1, \varpi_2, \ldots, \varpi_n]$ such that $0 \le \varpi_1, \varpi_2, \ldots, \varpi_n \le 1$ and $\varpi_1 + \varpi_2 + \ldots + \varpi_n = 1$, and L linguistic interval-valued intuitionistic fuzzy decision matrices $\mathbf{M}_h = [\alpha_{h,i,j}]_{m\times n}$ $(h = 1, 2, \ldots, L; i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ such that $([s\lfloor a_{h,i,j}\rfloor, s\lfloor b_{h,i,j}\rfloor], [s\lfloor c_{h,i,j}\rfloor, s\lfloor d_{h,i,j}\rfloor])$ is a LIVIFN that denotes the evaluation value of A_j with respect to O_i provided by E_h. Based on these components, the MAGDM problem can be formalised as determining the optimal option according to a ranking of all options in $\mathbf{O}$ based on $\mathbf{A}$, $\mathbf{M}_h$, $\mathbf{w}$, and $\boldsymbol{\varpi}$. Using the LIVIFAWPMM operator, the problem is solved according to the following steps:

(1) Normalise the linguistic interval-valued intuitionistic fuzzy decision matrices $\mathbf{M}_h$. To balance the physical dimensions of the rating values in $\mathbf{M}_h$, they are normalised as

$$\mathbf{N}_h = \begin{cases} \left[\left(\left[s_{a_{h,i,j}}, s_{b_{h,i,j}}\right], \left[s_{c_{h,i,j}}, s_{d_{h,i,j}}\right]\right)\right]_{m\times n}, C_1, \\ \left[\left(\left[s_{c_{h,i,j}}, s_{d_{h,i,j}}\right], \left[s_{a_{h,i,j}}, s_{b_{h,i,j}}\right]\right)\right]_{m\times n}, C_2, \end{cases} \tag{49}$$

where C_1 denotes "if C_j is a benefit attribute" and C_2 denotes "if C_j is a cost attribute."

(2) Compute the power weights of $\alpha_{h,i,j}$ The power weights of $\alpha_{h,i,j}$ are calculated using

$$W_{h,i,j} = \frac{(\varpi_h \xi_h)}{\sum_{z=1}^{L} (\varpi_z \xi_z)} = \frac{\varpi_h\left(1 + \sum_{x=1,x\ne h}^{L}\left(1 - D\left(\alpha_{h,i,j}, \alpha_{x,i,j}\right)\right)\right)}{\sum_{z=1}^{L}\left(\varpi_z\left(1 + \sum_{y=1,y\ne z}^{L}\left(1 - D\left(\alpha_{z,i,j}, \alpha_{y,i,j}\right)\right)\right)\right)}, \tag{50}$$

where $D(\alpha_{h,i,j}, \alpha_{x,i,j})\,(D(\alpha_{z,i,j}, \alpha_{y,i,j}))$ is the distance of $\alpha_{h,i,j}$ and $\alpha_{x,i,j}(\alpha_{z,i,j}$ and $\alpha_{y,i,j})$ that can be calculated via equation (3).

(3) Compute the collective values of $\alpha_{h,i,j}$. Taking the matrices $\mathbf{N}_h$ and the set $\boldsymbol{\varpi}$ as input, the collective values of $\alpha_{h,i,j}$ are calculated using

$$\alpha_{i,j} = \left(\left[s_{a_{i,j}}, s_{b_{i,j}}\right], \left[s_{c_{i,j}}, s_{d_{i,j}}\right]\right) = \text{LIVIFAWPMM}^{\mathbf{Q}} \cdot \left(\alpha_{1,i,j}, \alpha_{2,i,j}, \ldots, \alpha_{L,i,j}\right), \tag{51}$$

where LIVIFAWPMM is an arbitrary specific LIVIFAWPMM operator (for example, it can be the LIVIFWPMM operator in equation (42)), and the values of $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_L)$ are assigned as $Q_1 > 0$ and $Q_1 = Q_3 = \cdots = Q_L = 0$ since the rating values of different experts should generally be mutually independent.

(4) Compute the power weights of $\alpha_{i,j}$. The power weights of $\alpha_{i,j}$ are calculated using

$$W_{i,j} = \frac{(w_j \xi_j)}{\sum_{t=1}^{n} (w_t \xi_t)} = \frac{w_j\left(1 + \sum_{r=1,r\ne j}^{n}\left(1 - D\left(\alpha_{i,j}, \alpha_{i,r}\right)\right)\right)}{\sum_{t=1}^{n}\left(w_t\left(1 + \sum_{s=1,s\ne t}^{n}\left(1 - D\left(\alpha_{i,t}, \alpha_{i,s}\right)\right)\right)\right)}, \tag{52}$$

where $D(\alpha_{i,j}, \alpha_{i,r})\,(D(\alpha_{i,t}, \alpha_{i,s}))$ is the distance of $\alpha_{i,j}$ and $\alpha_{i,r}$ $(\alpha_{i,t}$ and $\alpha_{i,s})$ that can be calculated via equation (3).

(5) Calculate the collective values of $\alpha_{i,j}$. The collective values of $\alpha_{i,j}$ are computed using

$$\alpha_i = \left(\left[s_{a_i}, s_{b_i}\right], \left[s_{c_i}, s_{d_i}\right]\right) = \text{LIVIFAWPMM}^{\mathbf{Q}} \cdot \left(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}\right), \tag{53}$$

where LIVIFAWPMM is the same specific operator used in equation (51), and the values of

$\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$ are determined by identifying the interrelationships of the n attributes in $\mathbf{A}$. If n attributes are mutually independent, then $Q_1 > 0$ and $Q_2 = Q_2 = \cdots = Q_n = 0$. If there are interrelationships between any two attributes, then $Q_1, Q_2 > 0$ and $Q_3 = Q_4 = \cdots = Q_n = 0$. If there are interrelationships among any $k\,(k = 3, 4, \ldots, n)$, $Q_1, Q_2, \ldots, Q_k > 0$ and $Q_{k+1} = Q_{k+2} = \cdots = Q_n = 0$.

(6) Compute the scores and accuracies of α_i. The scores and accuracies of α_i are calculated using equations (1) and (2), respectively.

(7) Generate a ranking of O_i. According to the scores and accuracies of α_i and the comparison rules in Definition 4, a ranking of O_i is generated.

(8) Determine the optimal option. The optimal option is determined according to the ranking.

5. Example, Experiments, and Comparisons

In this section, a practical example is firstly used to illustrate the designed MAGDM method. Then a set of test experiments are carried out to validate the method and explore the effects of different values of $\mathbf{Q}$ on the aggregation and ranking results. Finally, qualitative and quantitative comparisons between the presented AO and the existing AOs are reported to show the characteristics and advantages of the presented AO.

5.1. Example. Three-dimensional (3D) printing refers to a series of emerging manufacturing technologies that build 3D physical objects from 3D model data, in which materials are stacked layer by layer through a specific process like sintering, melting, jetting, and lamination. A well-known characteristic of 3D printing technologies compared with traditional manufacturing technologies is that they can be used to fabricate 3D objects with complicated geometric structures and heterogeneous materials without additional cost. Due to such characteristic, the study and application of 3D printing technologies have received extensive attention from the academia and industry. Some even believed that 3D printing technologies would trigger a new round of manufacturing revolution.

The existing 3D printing technologies can be divided into vat photopolymerisation, material jetting, binder jetting, powder bed fusion, material extrusion, directed energy deposition, and sheet lamination. Based on these technologies, over one thousand different industrial 3D printers have been developed and identified in the market so far. A controversy regarding which printer is better than the others is meaningless, as each printer has its own features and application range. However, research on the selection of a suitable printer from a specific number of alternative printers for printing a specific component is nontrivial. This is because such selection needs a comprehensive understanding of the features of all alternative printers and a dynamic interaction with the quality of the built component, while most users lack such knowledge and experience. In addition, different printers could have similarities or overlaps at features, which brings certain difficulties to the selection objectively.

To offer an effective tool for selection of 3D printers, many different kinds of methods have been presented, where methods based on multiattribute decision-making (MADM) are one of the most important kinds. This kind of methods determine appropriate printers by synthetically assessing the values of multiple interrelated attributes of all alternative printers, which are usually achieved from experiments or simulations or provided by domain experts. It is generally difficult to ensure the absolute objectivity of these ways. This means that there could be some attributes values having "noise." To achieve reasonable selection result in this case, it is of necessity to capture the interrelationships of the attributes and concurrently reduce the influence of the attributes values having "noise." But there is yet no evidence that any of the existing MADM-based methods for 3D printer selection have such capabilities. The proposed MAGDM method can meet this requirement when the values of attributes of all alternative printers are quantified by LIVIFNs. The following is an illustrative example about the application of the proposed method in the 3D printer selection.

In this example, a user needs to select a proper printer from four alternative printers, denoted as P_1, P_2, P_3, and P_4, to print a component using a specific material. The user invited three experienced domain experts, denoted as E_1, E_2, and E_3, to evaluate the four alternative printers based on four attributes, which are the surface roughness (A_1), strength (A_2), elongation (A_3), and hardness (A_4) of the built component. The relative importance of the three experts is quantified by $\boldsymbol{\varpi} = [0.4,\ 0.3,\ 0.3]$. The relative importance of the four attributes is measured by $\mathbf{w} = [0.1,\ 0.3,\ 0.3,\ 0.3]$. In the evaluation, the three experts were asked to use LIVIFNs. The available linguistic variables are extremely small (s_0), very small (s_1), small (s_2), slightly small (s_3), medium (s_4), slightly large (s_5), large (s_6), very large (s_7), and extremely large (s_8). The evaluation results of the three experts are, respectively, listed in the following three matrices:

$$
\begin{aligned}
\mathbf{M}_1 &= \begin{bmatrix} ([s_2,s_4],[s_3,s_4]) & ([s_6,s_6],[s_1,s_2]) & ([s_5,s_6],[s_1,s_1]) & ([s_6,s_6],[s_1,s_1]) \\ ([s_3,s_5],[s_2,s_3]) & ([s_5,s_6],[s_1,s_2]) & ([s_4,s_5],[s_1,s_1]) & ([s_4,s_6],[s_1,s_2]) \\ ([s_1,s_1],[s_6,s_7]) & ([s_6,s_6],[s_1,s_2]) & ([s_3,s_4],[s_3,s_3]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_1,s_1],[s_7,s_7]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_5],[s_2,s_3]) & ([s_2,s_3],[s_3,s_4]) \end{bmatrix}, \\
\mathbf{M}_2 &= \begin{bmatrix} ([s_3,s_4],[s_3,s_4]) & ([s_5,s_6],[s_1,s_2]) & ([s_6,s_6],[s_1,s_2]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_3,s_5],[s_1,s_3]) & ([s_6,s_6],[s_1,s_1]) & ([s_5,s_6],[s_1,s_2]) & ([s_3,s_5],[s_1,s_2]) \\ ([s_1,s_2],[s_6,s_6]) & ([s_5,s_6],[s_1,s_1]) & ([s_3,s_5],[s_3,s_3]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_1,s_2],[s_6,s_6]) & ([s_3,s_3],[s_3,s_3]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_4],[s_3,s_4]) \end{bmatrix}, \\
\mathbf{M}_3 &= \begin{bmatrix} ([s_3,s_3],[s_3,s_4]) & ([s_6,s_6],[s_2,s_2]) & ([s_5,s_6],[s_1,s_1]) & ([s_6,s_7],[s_1,s_1]) \\ ([s_3,s_4],[s_2,s_3]) & ([s_6,s_7],[s_1,s_1]) & ([s_5,s_5],[s_1,s_3]) & ([s_5,s_5],[s_1,s_2]) \\ ([s_2,s_3],[s_5,s_5]) & ([s_4,s_5],[s_1,s_2]) & ([s_3,s_4],[s_3,s_4]) & ([s_4,s_5],[s_2,s_2]) \\ ([s_1,s_1],[s_6,s_7]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_5],[s_1,s_2]) & ([s_4,s_5],[s_3,s_3]) \end{bmatrix}.
\end{aligned} \tag{54}
$$

According to the conditions above, the determination can be carried out leveraging the designed MAGDM method. Its process includes the following steps:

(1) Normalise the fuzzy decision matrices $\mathbf{M}_h$ ($h = 1$, 2, 3). Since surface roughness is cost attribute and strength, elongation, and hardness are benefit attributes, the fuzzy decision matrices $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ are, respectively, normalised as follows:

$$
\begin{aligned}
\mathbf{N}_1 &= \begin{bmatrix} ([s_3,s_4],[s_2,s_4]) & ([s_6,s_6],[s_1,s_2]) & ([s_5,s_6],[s_1,s_1]) & ([s_6,s_6],[s_1,s_1]) \\ ([s_2,s_3],[s_3,s_5]) & ([s_5,s_6],[s_1,s_2]) & ([s_4,s_5],[s_1,s_1]) & ([s_4,s_6],[s_1,s_2]) \\ ([s_6,s_7],[s_1,s_1]) & ([s_6,s_6],[s_1,s_2]) & ([s_3,s_4],[s_3,s_3]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_7,s_7],[s_1,s_1]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_5],[s_2,s_3]) & ([s_2,s_3],[s_3,s_4]) \end{bmatrix}, \\
\mathbf{N}_2 &= \begin{bmatrix} ([s_3,s_4],[s_3,s_4]) & ([s_5,s_6],[s_1,s_2]) & ([s_6,s_6],[s_1,s_2]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_1,s_3],[s_3,s_5]) & ([s_6,s_6],[s_1,s_1]) & ([s_5,s_6],[s_1,s_2]) & ([s_3,s_5],[s_1,s_2]) \\ ([s_6,s_6],[s_1,s_2]) & ([s_5,s_6],[s_1,s_1]) & ([s_3,s_5],[s_3,s_3]) & ([s_5,s_6],[s_1,s_2]) \\ ([s_6,s_6],[s_1,s_2]) & ([s_3,s_3],[s_3,s_3]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_4],[s_3,s_4]) \end{bmatrix}, \\
\mathbf{N}_3 &= \begin{bmatrix} ([s_3,s_4],[s_3,s_3]) & ([s_6,s_6],[s_2,s_2]) & ([s_5,s_6],[s_1,s_1]) & ([s_6,s_7],[s_1,s_1]) \\ ([s_2,s_3],[s_3,s_4]) & ([s_6,s_7],[s_1,s_1]) & ([s_5,s_5],[s_1,s_3]) & ([s_5,s_5],[s_1,s_2]) \\ ([s_5,s_5],[s_2,s_3]) & ([s_4,s_5],[s_1,s_2]) & ([s_3,s_4],[s_3,s_4]) & ([s_4,s_5],[s_2,s_2]) \\ ([s_6,s_7],[s_1,s_1]) & ([s_3,s_4],[s_2,s_3]) & ([s_3,s_5],[s_1,s_2]) & ([s_4,s_5],[s_3,s_3]) \end{bmatrix},
\end{aligned} \tag{55}
$$

(2) Compute the power weights of $\alpha_{h,i,j}$ ($i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$). According to equation (50), the power weights of $\alpha_{h,i,j}$ are calculated and listed in the following matrices:

$$\left[W_{1,i,j}\right]_{4\times 4}=\begin{bmatrix}0.3987 & 0.4026 & 0.4026 & 0.4039\\ 0.4026 & 0.3961 & 0.3974 & 0.4000\\ 0.4000 & 0.4013 & 0.4026 & 0.4039\\ 0.4000 & 0.4026 & 0.4039 & 0.3987\end{bmatrix},$$
$$\left[W_{2,i,j}\right]_{4\times 4}=\begin{bmatrix}0.3023 & 0.2987 & 0.2955 & 0.2964\\ 0.2987 & 0.3036 & 0.3013 & 0.3000\\ 0.3068 & 0.3010 & 0.2987 & 0.3029\\ 0.2967 & 0.2955 & 0.2997 & 0.3057\end{bmatrix}, \tag{56}$$
$$\left[W_{3,i,j}\right]_{4\times 4}=\begin{bmatrix}0.2990 & 0.2987 & 0.3019 & 0.2997\\ 0.2987 & 0.3003 & 0.3013 & 0.3000\\ 0.2932 & 0.2977 & 0.2987 & 0.2932\\ 0.3033 & 0.3019 & 0.2964 & 0.2956\end{bmatrix}.$$

(3) Compute the collective values of $\alpha_{h,i,j}$. Taking $\mathbf{N}_h$ and $\boldsymbol{\varpi}$ as input, the collective values of $\alpha_{h,i,j}$ can be calculated using equation (51). Here, the LIVIFWPMM operator in equation (42) (when adapting the operator, $\mathbf{Q}=(1, 0, 0)$) is used in the calculation. The calculated results are listed in the following matrix:

$$\left[\alpha_{i,j}\right]_{4\times 4}=\begin{bmatrix}\left(\left[s_{3.0000}, s_{4.0000}\right], \left[s_{2.5522}, s_{3.6703}\right]\right) & \left(\left[s_{5.7425}, s_{6.0000}\right], \left[s_{1.2300}, s_{2.0000}\right]\right) & \left(\left[s_{5.3387}, s_{6.0000}\right], \left[s_{1.0000}, s_{1.2273}\right]\right) & \left(\left[s_{5.7446}, s_{6.3751}\right], \left[s_{1.0000}, s_{1.2281}\right]\right)\\ \left(\left[s_{1.7173}, s_{3.0000}\right], \left[s_{3.0000}, s_{4.6776}\right]\right) & \left(\left[s_{5.6516}, s_{6.3759}\right], \left[s_{1.0000}, s_{1.3159}\right]\right) & \left(\left[s_{4.6367}, s_{5.3450}\right], \left[s_{1.0000}, s_{1.7158}\right]\right) & \left(\left[s_{4.0767}, s_{5.4492}\right], \left[s_{1.0000}, s_{2.0000}\right]\right)\\ \left(\left[s_{5.7475}, s_{6.2930}\right], \left[s_{1.2253}, s_{1.7070}\right]\right) & \left(\left[s_{5.2226}, s_{5.7434}\right], \left[s_{1.0000}, s_{1.6234}\right]\right) & \left(\left[s_{3.0000}, s_{4.3294}\right], \left[s_{3.0000}, s_{3.2692}\right]\right) & \left(\left[s_{4.7360}, s_{5.7476}\right], \left[s_{1.2253}, s_{2.0000}\right]\right)\\ \left(\left[s_{6.4843}, s_{6.7716}\right], \left[s_{1.0000}, s_{1.2284}\right]\right) & \left(\left[s_{3.0000}, s_{3.7274}\right], \left[s_{2.2546}, s_{3.0000}\right]\right) & \left(\left[s_{3.0000}, s_{4.7299}\right], \left[s_{1.6285}, s_{2.6603}\right]\right) & \left(\left[s_{2.9663}, s_{3.9843}\right], \left[s_{3.0000}, s_{3.6739}\right]\right)\end{bmatrix}. \tag{57}$$

(4) Compute the power weights of $\alpha_{i,j}$. According to equation (52), the power weights of $\alpha_{i,j}$ are calculated and listed in the following matrix:

$$\left[W_{i,j}\right]_{4\times 4}=\begin{bmatrix}0.0888 & 0.3044 & 0.3044 & 0.3024\\ 0.0861 & 0.2962 & 0.3089 & 0.3089\\ 0.1017 & 0.3091 & 0.2784 & 0.3108\\ 0.0888 & 0.3054 & 0.3068 & 0.2990\end{bmatrix}. \tag{58}$$

(5) Calculate the collective values of $\alpha_{i,j}$. The collective values of $\alpha_{i,j}$ can be computed according to equation (53). Since the LIVIFWPMM operator in equation (42) has been leveraged in the third step, this operator (when adapting the operator, $\mathbf{Q}=(1, 2, 3, 4)$, i.e., there are interrelationships among the four attributes) is also used in this step to complete the calculation. The calculated results are listed as follows:

$$\begin{aligned}\alpha_1&=\left(\left[S_{4.6635}, S_{5.2066}\right], \left[S_{1.9801}, S_{2.4910}\right]\right),\\ \alpha_2&=\left(\left[S_{3.8050}, S_{4.7172}\right], \left[S_{2.0661}, S_{2.8436}\right]\right),\\ \alpha_3&=\left(\left[S_{4.3660}, S_{5.2202}\right], \left[S_{1.9944}, S_{2.4892}\right]\right),\\ \alpha_4&=\left(\left[S_{3.4932}, S_{4.4878}\right], \left[S_{2.3124}, S_{2.9583}\right]\right).\end{aligned} \tag{59}$$

(6) Compute the scores and accuracies of α_i. According to equations (1) and (2), the scores and accuracies of α_i are calculated and, respectively, listed as follows:

$$\begin{aligned}S\left(\alpha_1\right)&=S_{5.3498},\\ S\left(\alpha_2\right)&=S_{4.9031},\\ S\left(\alpha_3\right)&=S_{5.2757},\\ S\left(\alpha_4\right)&=S_{4.6776},\\ A\left(\alpha_1\right)&=S_{7.1706},\\ A\left(\alpha_2\right)&=S_{6.7160},\\ A\left(\alpha_3\right)&=S_{7.0349},\\ A\left(\alpha_3\right)&=S_{6.6258}.\end{aligned} \tag{60}$$

(7) Generate a ranking of P_i. According to the scores and accuracies of α_i and the comparison rules in Definition 4, a ranking of P_i is generated as $P_1 \succ P_3 \succ P_2 \succ P_4$.

(8) Determine the best printer. According to the ranking, the best 3D printer is determined as printer P_1.

5.2. Experiments

5.2.1. Validation Experiments. To demonstrate the effectiveness of an MADM method, Wang and Triantaphyllou [70] presented the following three test criteria:

(1) *Criterion 1*: "An effective MADM method should not change the place of the best option in the generated ranking when a nonoptimal option is replaced by a new worse option under the condition that the weight of each attribute remains unchanged"

(2) *Criterion 2*: "The rankings of options generated by an effective MADM method should be transitive"

(3) *Criterion 3*: "If a MADM problem is decomposed into several sub MADM problems and the same MADM method is used in the problem and its sub problems, then a collective ranking of all sub MADM problems should be the same as the ranking of the undecomposed MADM problem."

According to these three criteria, thirteen test experiments using the practical example were designed and carried out. The options included in each of the thirteen experiments are listed in Table 1. In Experiments 1, 2, and 3, the non-optimal options P_2, P_3, and P_4 are, respectively, replaced by the new worse options $P_{2'}, P_{3'}$, and $P_{4'}$, whose evaluation results are listed in Table 2. Using the designed method, the rankings of Experiments 1, 2, and 3 can be generated and are shown in Table 1. From the generated rankings, it can be seen that the best options in them remain unchanged. Thus, the designed method can meet Criterion 1.

The remaining ten test experiments (i.e., Experiments 4–13) aim to solve all sub MAGDM problems of the MAGDM problem in the practical example. Using the designed method, the rankings of these experiments can be generated and are also shown in Table 1. From the rankings of Experiments 2, 4, 7, and 10, Experiments 4, 6, 8, and 11, Experiments 5, 6, 9, and 12, and Experiments 7, 8, 9, and 13, it can be seen that transitivity is satisfied. Therefore, the designed method can meet Criterion 2. In addition, if the rankings of Experiments 4–9 or Experiments 10–13 are combined, then the same collective ranking $P_1 \succ P_3 \succ P_2 \succ P_4$ will be obtained, which is exactly the ranking of the undecomposed MAGDM problem. From this point of view, the designed method can also meet Criterion 3.

In summary, the designed method is effective for resolving the practical MAGDM problems since it can meet all of the three test criteria.

5.2.2. Exploration Experiment. To show the influence of different values of $\mathbf{Q}$ on the aggregation and ranking results, a test experiment using the practical example was designed and carried out. In this experiment, $\mathbf{Q}$ was respectively assigned (1, 0, 0, 0), (2, 0, 0, 0), (3, 0, 0, 0), (4, 0, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0), (1, 3, 0, 0), (1, 4, 0, 0), (1, 1, 1, 0), (1, 2, 3, 0), (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3), (4, 4, 4, 4), and (1, 2, 3, 4) when calculating the collective values of $\alpha_{i,j}$ in Step (5) in the example. Based on these different values of $\mathbf{Q}$, the LIVIFWPMM operator will reduce to different operators, which are, respectively, listed in Table 3. The results of the experiment are the computed scores of α_i and the generated ranking of P_i under each $\mathbf{Q}$, which are also listed in Table 3. As can be seen from the table, the rankings of the four 3D printers generated by the proposed method may have difference, in the situations where (1) all of the four attributes are independent of each other, (2) there are interrelationships between any two attributes, (3) there are interrelationships among any three attributes, and (4) there are interrelationships among the four attributes. This indicates that the presented LIVIFWPMM operator has the capability and generality to capture the interrelationships of attributes. In practical application of the operator, $\mathbf{Q}$ is recommended to take (1, 0, 0, 0), (1, 2, 0, 0), (1, 2, 3, 0), and (1, 2, 3, 4) under the four situations, respectively.

5.3. Comparisons. As mentioned in the introduction, representative AOs of LIVIFNs are the WA, OWA, HA, WG, OWG, and HG operators presented by Garg and Kumar [54, 55], the PWA, POWA, PWG, and POWG operators presented by Kumar and Garg [56], and the WMSM operator presented by Liu and Qin [57]. In this subsection, qualitative and quantitative comparisons between these AOs and the presented AO of LIVIFNs are carried out to evaluate the presented AO:

(1) *Qualitative Comparison*. This comparison was carried out via comparing the characteristics of the AOs. For the twelve AOs above, the generality and flexibility in aggregating the values of attributes, the generality in capturing the interrelationships of attributes, and the capability to reduce the effect of biased attribute values are selected as the comparison characteristics. The results of the comparison, as shown in Table 4, are explained as follows.

(a) *Generality and Flexibility in Aggregating the Values of Attributes*. Among the twelve AOs, the WA, OWA, HA, WG, OWG, HG, PWA, POWA, PWG, POWG, and WMSM operators perform the aggregation using the Algebraic T-norm and T-conorm, their generality and flexibility are relatively limited. Such characteristics of the presented Archimedean weighted power Muirhead mean (AWPMM) operator are satisfying since the aggregation in it can be performed by any types of ATNTCs (e.g., Algebraic, Einstein, Hamacher, and Frank T-norms and T-conorms).

(b) *Generality in Capturing the Relationships of Attributes*. The WA, OWA, HA, WG, OWG, HG, WMSM, and AWPMM operators are suitable for the case where all aggregated attributes are independent of each other. In addition, the WMSM and AWPMM operators can also be applicable in the situations where there are interrelationships between any two attributes and where there are relationships among any multiple attributes, because they, respectively, use the all-in-one MM and MSM operators for capturing interrelationships. It should be noted that MM is more generalised than MSM since MSM is just a special case of MM. From this point of view, the AWPMM operator is more generalised than the WMSM operator. The PWA, POWA, PWG, and POWG operators are specifically presented for dealing with the situation where the attributes are in different priority

Table 1: The details and results of the validation experiments.

Experiment	Included options	Generated ranking
Experiment 1	P_1, $P_{2'}$, P_3, P_4	$P_1 \succ P_3 \succ P_4 \succ P_{2'}$
Experiment 2	P_1, P_2, $P_{3'}$, P_4	$P_1 \succ P_2 \succ P_{3'} \succ P_4$
Experiment 3	P_1, P_2, P_3, $P_{4'}$	$P_1 \succ P_3 \succ P_2 \succ P_{4'}$
Experiment 4	P_1, P_2	$P_1 \succ P_2$
Experiment 5	P_1, P_3	$P_1 \succ P_3$
Experiment 6	P_1, P_4	$P_1 \succ P_4$
Experiment 7	P_2, P_3	$P_3 \succ P_2$
Experiment 8	P_2, P_4	$P_2 \succ P_4$
Experiment 9	P_3, P_4	$P_3 \succ P_4$
Experiment 10	P_1, P_2, P_3	$P_1 \succ P_3 \succ P_2$
Experiment 11	P_1, P_2, P_4	$P_1 \succ P_2 \succ P_4$
Experiment 12	P_1, P_3, P_4	$P_1 \succ P_3 \succ P_4$
Experiment 13	P_2, P_3, P_4	$P_3 \succ P_2 \succ P_4$

Table 2: The evaluation results of the new worse options.

Option	Attribute A_1	Attribute A_2	Attribute A_3	Attribute A_4
$P_{2'}$ in $\mathbf{M}_1$	$([s_3, s_5], [s_1, s_2])$	$([s_4, s_5], [s_1, s_2])$	$([s_3, s_4], [s_1, s_1])$	$([s_3, s_5], [s_1, s_2])$
$P_{2'}$ in $\mathbf{M}_2$	$([s_3, s_5], [s_1, s_3])$	$([s_5, s_5], [s_1, s_1])$	$([s_4, s_5], [s_1, s_2])$	$([s_2, s_4], [s_1, s_2])$
$P_{2'}$ in $\mathbf{M}_3$	$([s_3, s_4], [s_1, s_2])$	$([s_5, s_6], [s_1, s_1])$	$([s_4, s_4], [s_1, s_3])$	$([s_4, s_4], [s_1, s_2])$
$P_{3'}$ in $\mathbf{M}_1$	$([s_1, s_1], [s_5, s_6])$	$([s_5, s_5], [s_1, s_2])$	$([s_2, s_3], [s_3, s_3])$	$([s_4, s_5], [s_1, s_2])$
$P_{3'}$ in $\mathbf{M}_2$	$([s_5, s_5], [s_1, s_2])$	$([s_4, s_5], [s_1, s_1])$	$([s_2, s_4], [s_3, s_3])$	$([s_4, s_5], [s_1, s_2])$
$P_{3'}$ in $\mathbf{M}_3$	$([s_4, s_4], [s_2, s_3])$	$([s_3, s_4], [s_1, s_2])$	$([s_2, s_3], [s_3, s_4])$	$([s_3, s_4], [s_2, s_2])$
$P_{4'}$ in $\boldsymbol{M}_1$	$([s_1, s_1], [s_6, s_6])$	$([s_2, s_3], [s_2, s_3])$	$([s_2, s_4], [s_2, s_3])$	$([s_1, s_2], [s_3, s_4])$
$P_{4'}$ in $\mathbf{M}_2$	$([s_1, s_2], [s_5, s_5])$	$([s_2, s_2], [s_3, s_3])$	$([s_2, s_3], [s_2, s_3])$	$([s_2, s_3], [s_3, s_4])$
$P_{4'}$ in $\mathbf{M}_3$	$([s_1, s_1], [s_5, s_6])$	$([s_2, s_3], [s_2, s_3])$	$([s_2, s_4], [s_1, s_2])$	$([s_3, s_4], [s_3, s_3])$

Table 3: The details and results of the exploration experiment.

Value of Q	Actual operator	Interrelationships	The computed scores of all printers				The generated ranking
			$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
(1, 0, 0, 0)	LIVIFWPAA (equation (44))	Independent of each other	6.1822	5.8369	5.6327	4.7823	$P_1 \succ P_2 \succ P_3 \succ P_4$
(2, 0, 0, 0)	LIVIFWPGAA (equation (43))	Independent of each other	6.3032	5.9960	5.7047	4.8070	$P_1 \succ P_2 \succ P_3 \succ P_4$
(3, 0, 0, 0)	LIVIFWPGAA (equation (43))	Independent of each other	6.3892	6.1016	5.7753	4.8327	$P_1 \succ P_2 \succ P_3 \succ P_4$
(4, 0, 0, 0)	LIVIFWPGAA (equation (43))	Independent of each other	6.4509	6.1753	5.8420	4.8588	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 1, 0, 0)	LIVIFWPBM (equation (45))	Between any 2 attributes	5.7939	5.4072	5.3790	4.7050	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 2, 0, 0)	LIVIFWPBM (equation (45))	Between any 2 attributes	5.9428	5.5941	5.4509	4.7281	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 3, 0, 0)	LIVIFWPBM (equation (45))	Between any 2 attributes	6.0762	5.7504	5.5429	4.7589	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 4, 0, 0)	LIVIFWPBM (equation (45))	Between any 2 attributes	6.1780	5.8670	5.6315	4.7902	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 1, 1, 0)	LIVIFWPMSM (equation (46))	Among any 3 attributes	5.4135	4.9825	5.2503	4.6695	$P_1 \succ P_3 \succ P_2 \succ P_4$
(1, 2, 3, 0)	LIVIFWPMM (equation (42))	Among any 3 attributes	5.7674	5.4217	5.3606	4.7043	$P_1 \succ P_2 \succ P_3 \succ P_4$
(1, 1, 1, 1)	LIVIFWPGA (equation (48))	Among the 4 attributes	4.8961	4.3272	5.1750	4.6449	$P_3 \succ P_1 \succ P_4 \succ P_2$
(2, 2, 2, 2)	LIVIFWPGGA (equation (47))	Among the 4 attributes	4.8961	4.3272	5.1750	4.6449	$P_3 \succ P_1 \succ P_4 \succ P_2$
(3, 3, 3, 3)	LIVIFWPGGA (equation (47))	Among the 4 attributes	4.8961	4.3272	5.1750	4.6449	$P_3 \succ P_1 \succ P_4 \succ P_2$
(4, 4, 4, 4)	LIVIFWPGGA (equation (47))	Among the 4 attributes	4.8961	4.3272	5.1750	4.6449	$P_3 \succ P_1 \succ P_4 \succ P_2$
(1, 2, 3, 4)	LIVIFWPMM (equation (42))	Among the 4 attributes	5.3498	4.9031	5.2757	4.6776	$P_1 \succ P_3 \succ P_2 \succ P_4$

levels. They are totally different from other AOs from the perspective of application range.

(c) *Capability to Reduce the Effect of Biased Attribute Values*. Among the twelve AOs, only the AWPMM operator has this capability due to the combination of the PA operator.

(2) *Quantitative Comparison*. This comparison was carried out using the practical examples in the present paper and in [55, 57] as benchmarks. In the comparison, the WA and WG operators were selected as the AOs for the method of Garg and Kumar [55]. The PWA and PWG operators were chosen as the AOs for the method of Kumar and Garg [56]. The given weights in the three examples were directly used as the priority weights of the PWA and PWG operators to make the input of all comparison operators the same (it should be pointed out that the PWA and PWG operators will, respectively, reduce

Table 4: The results of the qualitative comparison.

AOs presented in the method	Generality and flexibility in aggregation	Generality in capturing interrelationships of attributes				Capability to reduce the effect
		Independent of each other	Between any two	Among any multiple	In different priority levels	
WA [55]	Limited	Yes	No	No	No	No
OWA [55]	Limited	Yes	No	No	No	No
HA [55]	Limited	Yes	No	No	No	No
WG [55]	Limited	Yes	No	No	No	No
OWG [55]	Limited	Yes	No	No	No	No
HG [55]	Limited	Yes	No	No	No	No
PWA [56]	Limited	No	No	No	Yes	No
POWA [56]	Limited	No	No	No	Yes	No
PWG [56]	Limited	No	No	No	Yes	No
POWG [56]	Limited	No	No	No	Yes	No
WMSM [57]	Limited	Yes	Yes	Yes	No	No
AWPMM	Satisfying	Yes	Yes	Yes	No	Yes

Table 5: The details and results of the quantitative comparison.

Benchmark	Decision-making method	Used AOs	Value of arguments	The calculated scores of all options				The generated ranking
				$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	
Example in this paper	Garg and Kumar [55]	WA, WA	—	6.1654	5.8171	5.5985	4.8181	$O_1 \succ O_2 \succ O_3 \succ O_4$
	Garg and Kumar [55]	WG, WG	—	5.9921	5.5132	5.3441	4.5370	$O_1 \succ O_2 \succ O_3 \succ O_4$
	Kumar and Garg [56]	PWA, PWA	—	6.1654	5.8171	5.5985	4.8181	$O_1 \succ O_2 \succ O_3 \succ O_4$
	Kumar and Garg [56]	PWG, PWG	—	5.9921	5.5132	5.3441	4.5370	$O_1 \succ O_2 \succ O_3 \succ O_4$
	Liu and Qin [57]	WMSM, WMSM	$k = 1$, $k = 3$	5.4523	5.0127	5.2577	4.7343	$O_1 \succ O_3 \succ O_2 \succ O_4$
	The designed method	WPMM, WPMM	$\mathbf{Q} = (1, 0, 0)$, $\mathbf{Q} = (1, 2, 3, 0)$	5.7674	5.4217	5.3606	4.7043	$O_1 \succ O_2 \succ O_3 \succ O_4$
Example in [55]	Garg and Kumar [55]	WA, WA	—	5.3457	4.6145	5.1125	4.9687	$O_1 \succ O_3 \succ O_4 \succ O_2$
	Garg and Kumar [55]	WG, WG	—	5.1069	4.3577	4.9218	4.5868	$O_1 \succ O_3 \succ O_4 \succ O_2$
	Kumar and Garg [56]	PWA, PWA	—	5.3457	4.6145	5.1125	4.9687	$O_1 \succ O_3 \succ O_4 \succ O_2$
	Kumar and Garg [56]	PWG, PWG	—	5.1069	4.3577	4.9218	4.5868	$O_1 \succ O_3 \succ O_4 \succ O_2$
	Liu and Qin [57]	WMSM, WMSM	$k = 1$, $k = 3$	5.1768	4.2902	4.7499	4.6151	$O_1 \succ O_3 \succ O_4 \succ O_2$
	The designed method	WPMM, WPMM	$\mathbf{Q} = (1, 0, 0)$, $\mathbf{Q} = (1, 2, 3, 0)$	5.2054	4.3988	4.8582	4.6997	$O_1 \succ O_3 \succ O_4 \succ O_2$
Example in [57]	Garg and Kumar [55]	WA	—	6.2621	6.2709	5.6111	5.5823	$O_2 \succ O_1 \succ O_3 \succ O_4$
	Garg and Kumar [55]	WG	—	6.1407	6.1620	5.3326	5.2541	$O_2 \succ O_1 \succ O_3 \succ O_4$
	Kumar and Garg [56]	PWA, PWA	—	6.2621	6.2709	5.6111	5.5823	$O_2 \succ O_1 \succ O_3 \succ O_4$
	Kumar and Garg [56]	PWG, PWG	—	6.1407	6.1620	5.3326	5.2541	$O_2 \succ O_1 \succ O_3 \succ O_4$
	Liu and Qin [57]	WMSM	$k = 3$	6.1034	6.0611	5.5196	5.1819	$O_1 \succ O_2 \succ O_3 \succ O_4$
	The designed method	WPMM	$\mathbf{Q} = (1, 2, 3, 0, 0)$	6.1584	6.1157	5.5637	5.3056	$O_1 \succ O_2 \succ O_3 \succ O_4$

to the WA and WG operators in this case). The method of Liu and Qin [57] and the proposed method respectively used the WMSM operator and the weighted power MM (WPMM) operator (see equation (42)). Moreover, the same score function (i.e., the function in equation (1)) was used in all of these operators for easy comparison. The details and results of the comparison are listed in Table 5.

As can be seen from Table 5, the best option of the designed method is exactly the same as the best option of the method of Liu and Qin for all examples. This is mainly because the two methods are the most similar in nature (see Table 4). From this result, it can be observed that the designed method is feasible and effective for solving practical decision-making problems based on LIVIFNs. In addition, the rankings of the methods of Garg and Kumar and Kumar and Garg are different from that of the method of Liu and Qin and the designed method at the first and second places for the example in [57]. The reason is that the interrelationships in the former methods are set as "all attributes are independent of each other" and those in the latter two methods are set as "there are interrelationships among any three attributes."

From the qualitative comparison above, the advantage of combination of the PA and MM operators is providing generality in capturing the interrelationships of attributes and capability to reduce the effect of biased attribute values. Such advantage cannot be intuitively seen from the comparison results in Table 5. To explicitly show the advantage,

Table 6: The details and results of the first additional quantitative comparison experiment.

Benchmark	Decision-making method	Generated ranking under assumption (a)	Generated ranking under assumption (b)	Generated ranking under assumption (c)	Generated ranking under assumption (d)	Generated ranking under assumption (e)
Example in this paper	Garg and Kumar [55]	$O_1 \succ O_2 \succ O_3 \succ O_4$	Cannot be applied	Cannot be applied	Cannot be applied	—
	Kumar and Garg [56]	$O_1 \succ O_2 \succ O_3 \succ O_4$	Cannot be applied	Cannot be applied	Cannot be applied	—
	Liu and Qin [57]	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_3 \succ O_2 \succ O_4$	$O_3 \succ O_1 \succ O_4 \succ O_2$	—
	The designed method	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_3 \succ O_2 \succ O_4$	—
Example in [55]	Garg and Kumar [55]	$O_1 \succ O_3 \succ O_4 \succ O_2$	Cannot be applied	Cannot be applied	Cannot be applied	—
	Kumar and Garg [56]	$O_1 \succ O_3 \succ O_4 \succ O_2$	Cannot be applied	Cannot be applied	Cannot be applied	—
	Liu and Qin [57]	$O_1 \succ O_4 \succ O_3 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	—
	The designed method	$O_1 \succ O_4 \succ O_3 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	$O_1 \succ O_3 \succ O_4 \succ O_2$	—
Example in [57]	Garg and Kumar [55]	$O_2 \succ O_1 \succ O_3 \succ O_4$	Cannot be applied	Cannot be applied	Cannot be applied	Cannot be applied
	Kumar and Garg [56]	$O_2 \succ O_1 \succ O_3 \succ O_4$	Cannot be applied	Cannot be applied	Cannot be applied	Cannot be applied
	Liu and Qin [57]	$O_2 \succ O_1 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$
	The designed method	$O_2 \succ O_1 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$	$O_1 \succ O_2 \succ O_3 \succ O_4$

two additional quantitative comparison experiments were carried out. The first experiment aims to show the generality in capturing attribute relationships. In this experiment, the practical examples in the present paper and in [55, 57] were used to compare the proposed method and the methods of Garg and Kumar, Kumar and Garg, and Liu and Qin. Given the following are five assumptions: (a) all attributes are independent of each other; (b) there are interrelationships between any two attributes; (c) there are interrelationships among any three attributes; (d) there are interrelationships among any four attributes; (e) there are interrelationships among any five attributes (only for the example in [57]). The applicability of the four comparison methods and the results of the experiment are shown in Table 6. It can be intuitively seen from the table that the methods of Garg and Kumar and Kumar and Garg are applicable only under assumption a, while both the method of Liu and Qin and the designed method can be applied to generate ranking results under all assumptions. This indicates that both of the latter two methods have the generality in capturing the interrelationships of attributes.

The second experiment aims to show the capability to reduce the influence of extreme attribute values. In this experiment, the practical example in the present paper is used to compare the proposed method and the method of Liu and Qin, which have difference only in whether combining the PA operator when $\mathbf{Q} = (1, 0, 0, 0)$ and $k = 1$ and $\mathbf{Q} = (1, 1, 1, 0)$ and $k = 3$ for the example. It is assumed that the value of attribute A_2 of printer P_1 evaluated by expert E_1 (i.e., $\alpha_{1,1,2}$) is a biased attribute value. This value was constantly adjusted from high to low, as listed in the first column of Table 7. The change of the relative importance (i.e., weight) of $\alpha_{1,1,2}$ with respect to the adjusted value is also shown in Table 7. As can be intuitively seen from the table, the weight of $\alpha_{1,1,2}$ becomes smaller and smaller as the value of $\alpha_{1,1,2}$ changes from high to low in the proposed method, while it remains the same in this process in the method of Liu and Qin. The greater the change of the value of $\alpha_{1,1,2}$, the greater the bias. The relative importance of the value should be dynamically decreased to reduce the effect of this extreme value. From the comparison results, only the proposed method has such capability. That is, the method can reduce the effect of biased attribute values.

On the basis of the comparisons above, the advantages of the designed method over the methods of Garg and Kumar, Kumar and Garg, and Liu and Qin are summarised as follows:

(1) Compared to the methods of Garg and Kumar and Kumar and Garg, the designed method is generalised and flexible for aggregation of attribute values and handling of attribute relationships and concurrently has the capability to reduce the influence of the distortion of attribute values.

(2) Compared to the method of Liu and Qin, the designed method has desirable generality and flexibility in aggregation of attribute values and can reduce the influence of the biased attribute values on aggregation result.

6. Conclusion

In this paper, a LIVIFAPMM operator and a LIVIFAWPMM operator have been presented to solve the MAGDM problems based on LIVIFNs. The generalised expressions of the two operators have been established. Their

Table 7: The details and results of the second additional quantitative comparison experiment.

Value of $\alpha_{1,1,2}$	Change of weights in the designed method		Change of weights in the method of Liu and Qin [57]	
	Weights of expert E_1 ($[W_{1,i,j}]_{4\times4}$)	Weights of attributes ($[W_{i,j}]_{4\times4}$)	Weights of expert E_1 ($[W_{1,i,j}]_{4\times4}$)	Weights of attributes ($[W_{i,j}]_{4\times4}$)
$([s_6, s_6], [s_1, s_2])$	$\begin{bmatrix} 0.3987 & \mathbf{0.4026} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0888 & \mathbf{0.3044} & 0.3044 & 0.3024 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_5, s_5], [s_2, s_3])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3974} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0906 & \mathbf{0.3044} & 0.3045 & 0.3005 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_4, s_4], [s_3, s_4])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3859} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0923 & \mathbf{0.3039} & 0.3039 & 0.2999 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_3, s_3], [s_4, s_5])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3728} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0936 & \mathbf{0.3035} & 0.3035 & 0.2994 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_2, s_2], [s_5, s_6])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3577} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0945 & \mathbf{0.3032} & 0.3032 & 0.2991 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_1, s_1], [s_6, s_7])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3402} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0952 & \mathbf{0.3030} & 0.3030 & 0.2988 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$
$([s_0, s_0], [s_7, s_8])$	$\begin{bmatrix} 0.3987 & \mathbf{0.3195} & 0.4026 & 0.4039 \\ 0.4026 & 0.3961 & 0.3974 & 0.4000 \\ 0.4000 & 0.4013 & 0.4026 & 0.4039 \\ 0.4000 & 0.4026 & 0.4039 & 0.3987 \end{bmatrix}$	$\begin{bmatrix} 0.0955 & \mathbf{0.3029} & 0.3029 & 0.2987 \\ 0.0861 & 0.2962 & 0.3089 & 0.3089 \\ 0.1017 & 0.3091 & 0.2784 & 0.3108 \\ 0.0888 & 0.3054 & 0.3068 & 0.2990 \end{bmatrix}$	$\begin{bmatrix} 0.4000 & \mathbf{0.4000} & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \\ 0.4000 & 0.4000 & 0.4000 & 0.4000 \end{bmatrix}$	$\begin{bmatrix} 0.1000 & \mathbf{0.3000} & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \\ 0.1000 & 0.3000 & 0.3000 & 0.3000 \end{bmatrix}$

properties have been explored and proved and specific expressions have been constructed using the operational rules of LIVIFNs based on the Algebraic T-norm and T-conorm. On the basis of the presented LIVIFAWPMM operator, a method for resolving the LIVIFN-based MAGDM problems has been proposed. The paper has also introduced a practical example to illustrate the proposed method and reported a set of experiments and comparisons to evaluate it. The results of the experiments and comparisons suggest that the method is feasible and effective which has advantages in providing the generality and flexibility in aggregation of attribute values and capturing of attribute interrelationships and the capability to reduce the effect of the deviation of attribute values. Two main limitations of the proposed method are that the method has not captured the risk attitudes of decision makers and it cannot work properly under incomplete attribute information. Future work will focus especially on addressing these limitations. In addition, the application of the method in real MAGDM problems may also be studied.

Appendix

A. Proof of Theorem 1

Proof. To prove $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s_a, s_b], [s_c, s_d])$, we need to prove equation (28) $= ([s_a, s_b], [s_c, s_d])$. The proof process is as follows.

According to the operational rule in equation (6), we have

$$\left(n\xi_{p(i)}\right)\alpha_{p(i)} = \left(\left[s_{\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)}, s_{\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(b_{p(i)}\right)\right)\right)}\right], \left[s_{\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(c_{p(i)}\right)\right)\right)}, s_{\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(d_{p(i)}\right)\right)\right)}\right]\right). \tag{A.1}$$

According to the operational rule in equation (7), we can obtain

$$\begin{aligned}\left(\left(n\xi_{p(i)}\right)\alpha_{p(i)}\right)^{Q_i} = &\left(\left[s_{\left(f^{-1}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)}, s_{\left(f^{-1}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(b_{p(i)}\right)\right)\right)\right)\right)}\right],\right.\\ &\left.\left[s_{\left(g^{-1}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(c_{p(i)}\right)\right)\right)\right)\right)}, s_{\left(g^{-1}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(d_{p(i)}\right)\right)\right)\right)\right)}\right]\right).\end{aligned} \tag{A.2}$$

According to the operational rule in equation (5), we have

$$\begin{aligned}&\bigotimes_{i=1}^{n}\left(\left(n\xi_{p(i)}\right)\alpha_{p(i)}\right)^{\delta_i}\\ &= \left(\left[s_{\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)}, s_{\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)}\right],\right.\\ &\quad\left.\left[s_{\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)}, s_{\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)}\right]\right).\end{aligned} \tag{A.3}$$

According to operational rule in equation (4), we can obtain

$$\bigoplus_{p\in\mathbf{P}_n}\bigotimes_{i=1}^{n}\left(\left(n\xi_{p(i)}\right)\alpha_{p(i)}\right)^{\delta_i} = \left(\begin{bmatrix} s_{\left(g^{-1}\left(\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)}\\ s_{\left(g^{-1}\left(\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)}\end{bmatrix}, \begin{bmatrix} s_{\left(f^{-1}\left(\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)}\\ s_{\left(f^{-1}\left(\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right)f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)}\end{bmatrix}\right). \tag{A.4}$$

According to operational rule in equation (6), we have

$$\frac{1}{n!}\bigoplus_{p\in\mathbf{P}_n}\bigotimes_{i=1}^{n}\left(\left(n\xi_{p(i)}\right)\alpha_{p(i)}\right)^{\delta_i}=\left(\begin{bmatrix} s_{\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} \\ s_{\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} s_{\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right) f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} \\ s_{\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right) f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} \end{bmatrix}\right). \tag{A.5}$$

The following equation is obtained according to the operational rule in equation (7):

$$\left(\frac{1}{n!}\bigoplus_{p\in\mathbf{P}_n}\bigotimes_{i=1}^{n}\left(\left(n\xi_{p(i)}\right)\alpha_{p(i)}\right)^{\delta_i}\right)^{1/\sum_{i=1}^{n}\delta_i}$$
$$=\left(\begin{bmatrix} s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)} \\ s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(b_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)} \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right) f\left(c_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)} \\ s_{\left(g^{-1}\left(\left(1/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(Q_i g\left(f^{-1}\left(\left(n\xi_{p(i)}\right) f\left(d_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)} \end{bmatrix}\right). \tag{A.6}$$

This completes the proof of the theorem. □

B. Proof of Theorem 2

Proof:. Since $\alpha_i = \alpha = ([s_{\lfloor a_\alpha\rfloor}, s_{\lfloor b_\alpha\rfloor}], [s_{\lfloor c_\alpha\rfloor}, s_{\lfloor d_\alpha\rfloor}])$ for all $i = 1, 2, \ldots, n$, we have $D(\alpha_i, \alpha_j) = 0$ for all $j = 1, 2, \ldots, n$ and $j \neq i$. According to equation (27), we further have

$$n\xi_{p(i)}=\frac{n\left(\sum_{p\in\mathbf{P}_n}\sum_{i=1}^{n}\left(1+T\left(\alpha_{p(i)}\right)\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\alpha_j\right)\right)}=\frac{n(1+(n-1))}{(n(1+(n-1)))}=1. \tag{B.1}$$

According to Theorem 1, we can obtain

$$s_a = s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n}Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)}. \tag{B.2}$$

Since $s_{\lfloor a_i\rfloor} = s_{\lfloor a_\alpha\rfloor}$, we have

$$\begin{aligned} s_{\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)} &= s_{\left(\left(\sum_{i=1}^{n}Q_i\right) f\left(a_\alpha\right)\right)}, \\ s_{\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)} &= s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n}Q_i\right) f\left(a_\alpha\right)\right)\right)}. \end{aligned} \tag{B.3}$$

We further have

$$\begin{aligned} s_{\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)\right)} &= s_{\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n}Q_i\right) f\left(a_\alpha\right)\right)\right)\right)}, \\ s_{\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)\right)\right)} &= s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n}Q_i\right) f\left(a_\alpha\right)\right)\right)}. \end{aligned} \tag{B.4}$$

Then, we can obtain

$$
\begin{aligned}
s_a &= s_{\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)} = s_{\left(f\left(a_\alpha\right)\right)},\\
&s_{\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} = s_{a_\alpha}.
\end{aligned}
\tag{B.5}
$$

That is $s\lfloor\alpha\rfloor = s\lfloor a_\alpha\rfloor$. Similarly, we can prove $s\lfloor b\rfloor = s\lfloor b_\alpha\rfloor$, $s\lfloor c\rfloor = s\lfloor c_\alpha\rfloor$, and $s\lfloor d\rfloor = s\lfloor d_\alpha\rfloor$. Thus, we have $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s\lfloor a_\alpha\rfloor, s\lfloor b_\alpha\rfloor], [s\lfloor c_\alpha\rfloor, s\lfloor d_\alpha\rfloor])$, which completes the proof of the theorem. □

C. Proof of Theorem 3

Proof:. Since $(\beta_1, \beta_2, \ldots, \beta_n)$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we have

$$
\begin{aligned}
&\left(\frac{1}{n!}\bigoplus_{p\in\mathbf{P}_n}\otimes_{i=1}^{n}\left(\frac{n\left(1+T\left(\beta_{p(i)}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\beta_j\right)\right)}\beta_{p(i)}\right)^{Q_i}\right)^{1/\Sigma_{i=1}^{n} Q_i}\\
&\quad= \left(\frac{1}{n!}\bigoplus_{p\in\mathbf{P}_n}\bigotimes_{i=1}^{n}\left(\frac{n\left(1+T\left(\alpha_{p(i)}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\alpha_j\right)\right)}\alpha_{p(i)}\right)^{Q_i}\right)^{1/\Sigma_{i=1}^{n} Q_i}.
\end{aligned}
\tag{C.1}
$$

Thus, we can obtain $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\beta_1, \beta_2, \ldots, \beta_n) = \mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, which completes the proof of the theorem. □

D. Proof of Theorem 4

Proof:. According to Theorem 2, we have $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^-, \alpha^-, \ldots, \alpha^-) = \alpha^-$, $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^+, \alpha^+, \ldots, \alpha^+) = \alpha^+$, and $n\xi_p(i) = 1$ for both $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^-, \alpha^-, \ldots, \alpha^-)$ and $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^+, \alpha^+, \ldots, \alpha^+)$. Because $s\lfloor a_{\alpha^-}\rfloor \le s\lfloor a_{p(i)}\rfloor \le s\lfloor a_{\alpha^+}\rfloor$ and $g(x)$ is monotonically increasing, we can obtain

$$
s_{\left(g\left(a_{\alpha-}\right)\right)} \le s_{\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)} \le s_{\left(g\left(a_{\alpha+}\right)\right)}. \tag{D.1}
$$

Because $g^{-1}(t)$ is monotonically increasing, we have

$$
s_{\left(a_{\alpha-}\right)} \le s_{\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)} \le s_{\left(a_{\alpha+}\right)}. \tag{D.2}
$$

Since $f(x)$ is monotonically decreasing, we can obtain

$$
s_{\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha-}\right)\right)} \ge s_{\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)} \ge s_{\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha+}\right)\right)}. \tag{D.3}
$$

Because $f^{-1}(t)$ is monotonically decreasing, we have

$$
s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha-}\right)\right)\right)} \le s_{\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)} \le s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha+}\right)\right)\right)}. \tag{D.4}
$$

Since $g(x)$ is monotonically increasing, we can obtain

$$
s_{\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha-}\right)\right)\right)\right)} \le s_{\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)} \le s_{\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha+}\right)\right)\right)\right)}. \tag{D.5}
$$

Because $g^{-1}(t)$ is monotonically increasing, we have

$$
s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha-}\right)\right)\right)} \le s_{\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)} \le s_{\left(f^{-1}\left(\left(\sum_{i=1}^{n} Q_i\right) f\left(a_{\alpha+}\right)\right)\right)}. \tag{D.6}
$$

Since $f(x)$ is monotonically decreasing, we can obtain

$$
s_{\left(f\left(a_{\alpha-}\right)\right)} \ge s_{\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in\mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right)g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)} \ge s_{\left(f\left(a_{\alpha+}\right)\right)}. \tag{D.7}
$$

Because $f^{-1}(t)$ is monotonically decreasing, we have

$$s\left(a_{\alpha-}\right) \leq s\left(f^{-1}\left(\left(1/\sum_{i=1}^{n} Q_i\right) f\left(g^{-1}\left((1/n!)\sum_{p\in \mathbf{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(Q_i f\left(g^{-1}\left(\left(n\xi_{p(i)}\right) g\left(a_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right) \leq s\left(a_{\alpha+}\right). \tag{D.8}$$

That is $s\lfloor a_{\alpha^-}\rfloor \leq s_a \leq s\lfloor a_{\alpha+1}\rfloor$. Similarly, we can prove $s\lfloor b_{\alpha^-}\rfloor \leq s_b \leq s\lfloor b_{\alpha+1}\rfloor$, $s\lfloor c_{\alpha^-}\rfloor \geq s_c \geq s\lfloor c_{\alpha+1}\rfloor$, and $s\lfloor d_{\alpha^-}\rfloor \geq s_d \geq s\lfloor d_{\alpha+1}\rfloor$. According to Definitions 2 and 4, we can obtain $\mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^-, \alpha^-, \ldots, \alpha^-) \leq \mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha^+, \alpha^+, \ldots, \alpha^+)$, and thus $\alpha^- \leq \mathrm{LIVIFAPMM}^{\mathbf{Q}}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$. This completes the proof of the theorem. □

Data Availability

The Java implementation code of all quantitative comparison methods and related data used to support the findings of this study have been deposited in the GitHub repository (https://github.com/YuchuChingQin/AOsOfLivifns).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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