Research Article

New Results on Synchronization of Fractional-Order Memristor-Based Neural Networks via State Feedback Control

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This paper addresses the synchronization issue for the drive-response fractional-order memristor-based neural networks (FOMNNs) via state feedback control. To achieve the synchronization for considered drive-response FOMNNs, two feedback controllers are introduced. Then, by adopting nonsmooth analysis, fractional Lyapunov's direct method, Young inequality, and fractional-order differential inclusions, several algebraic sufficient criteria are obtained for guaranteeing the synchronization of the drive-response FOMNNs. Lastly, for illustrating the effectiveness of the obtained theoretical results, an example is given.

1. Introduction

In recent years, fractional calculus has become a useful tool in the analysis of slow relaxation phenomena. As we all know, fractional derivative has two main advantages: infinite memory and more degrees of freedom [1, 2]. Hence, fractional derivative plays a critical part in the depiction of memory and hereditary characteristics of multifarious processes. Compared with dynamic systems described by the classical integer-order derivative, dynamic systems described by fractional derivative can accurately reflect the actual dynamic properties of real systems due to their memory and hereditary characteristics. Recently, as an extension of the classical integer-order calculus, fractional calculus has many practical applications in many interdisciplinary areas, such as fractional-order sinusoidal oscillators [3], transient wave propagation [4], fractional relaxation-oscillation and fractional diffusion-wave phenomena [5], drug release and absorption [6], and so on. Moreover, dynamic behaviors of fractional-order systems have attracted the attention of many researchers because of

their practical applications. In the past decade, the dynamic analysis of the fractional-order systems has achieved many outstanding results [7–10].

In the past few years, dynamic behaviors of neural networks (NNs) have gained many attentions [11–21], since NNs have lots of applications [22, 23]. In addition, fractional derivative has been introduced to NNs, and dynamic analysis of fractional-order neural networks (FONNs) has become a focus of research and many results have been obtained [24, 25]. Among these dynamic behaviors, as a significant dynamic characteristic, synchronization was firstly introduced [26]. Since then, research on synchronization of NNs has become a hot topic because of their wide potential applications in a large number of real systems [27, 28].

On the other hand, memristor was firstly predicted by Chua [29], and a practical memristor device was successfully obtained [30]. The memristor exhibits the characteristics of pinch hysteresis, which is possessed by the human brain. It is more practical to construct artificial NNs by replacing the resistor with the memristor, that is, memristor-based neural

networks (MNNs). In recent years, dynamic behaviors of MNNs have caused widespread concern around the world [31–34]. Accordingly, it is very valuable to study the synchronization problem for fractional-order memristor-based neural networks (FOMNNs), and many excellent works have been conducted [35, 36].

Inspired by the discussions given above, this paper studies the synchronization issue for FOMNNs via feedback control. Firstly, to achieve the synchronization for considered drive-response FOMNNs, two feedback controllers are introduced. Then, by adopting nonsmooth analysis, fractional Lyapunov's direct method and Young inequality, and fractional-order differential inclusions, several algebraic sufficient criteria are obtained for guaranteeing the synchronization for the drive-response FOMNNs.

2. Preliminaries and Model Description

The following preliminaries on fractional calculus are recalled.

Definition 1 (see [37]). Given an arbitrary integrable function $\chi(t)$, its Riemann–Liouville fractional integral with fractional order $\alpha > 0$ is defined as

$$I^\alpha \chi(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \chi(s) ds, \quad (1)$$

where $t \geq t_0$, $\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} e^{-s} ds$.

Definition 2 (see [37]). Given an arbitrary differentiable function $\chi(t)$, its Caputo fractional derivative is defined as

$${}_t^C D_t^\alpha \chi(t) = \frac{1}{\Gamma(k-\alpha)} \int_{t_0}^t (t-s)^{k-\alpha-1} \chi^{(k)}(s) ds, \quad (2)$$

where the fractional order $\alpha > 0$, $t \geq t_0$, $\alpha \in (k-1, k)$, k is a positive integer.

Consider an FOMNN as follows:

$${}_0^C D_t^\alpha r_l(t) = -a_l r_l(t) + \sum_{m=1}^n b_{lm}(r_m(t)) g_m(r_m(t)) + J_l, \quad (3)$$

where $a_l > 0$ is the self-inhibition, ${}_0^C D_t^\alpha r_l(t)$ refers to the Caputo fractional derivative, $0 < \alpha < 1$, $r_l(t)$ refers to the state, $g_m(r_m(t))$ refers to the activation function, J_l is an external input or bias, and $b_{lm}(r_m(t))$ is the memristor connection weight. The initial value of (3) is $r(0) = (r_1(0), r_2(0), \dots, r_n(0))^T \in \mathbb{R}^n$ and $l \in \mathbb{N}$.

The memristor connection weight $b_{lm}(r_m(t))$ switches among different numbers, which can be simply modeled as follows:

$$b_{lm}(r_m(t)) = \begin{cases} b_{lm}^*, & |r_m(t)| \leq Y_l, \\ b_{lm}^{**}, & |r_m(t)| > Y_l, \end{cases} \quad (4)$$

where the switching jump $Y_l > 0$ and b_{lm}^* and b_{lm}^{**} are constant numbers.

Let $\underline{b}_{lm} = \min\{b_{lm}^*, b_{lm}^{**}\}$, $\bar{b}_{lm} = \max\{b_{lm}^*, b_{lm}^{**}\}$, and

$$\text{co}[b_{lm}(r_m(t))] = \begin{cases} b_{lm}^*, & |r_m(t)| < Y_l, \\ b_{lm}^{**}, & |r_m(t)| > Y_l, \\ [\underline{b}_{lm}, \bar{b}_{lm}], & |r_m(t)| = Y_l. \end{cases} \quad (5)$$

Then, according to the theories of differential inclusion and set-valued map, for (3), we have

$${}_0^C D_t^\alpha r_l(t) \in -a_l r_l(t) + \sum_{m=1}^n \text{co}[b_{lm}(r_m(t))] g_m(r_m(t)) + J_l. \quad (6)$$

Or equivalently, there exists $\tilde{b}_{lm}(r_m(t)) \in \text{co}[b_{lm}(r_m(t))]$, such that

$${}_0^C D_t^\alpha r_l(t) = -a_l r_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(r_m(t)) g_m(r_m(t)) + J_l. \quad (7)$$

The drive-response synchronization is considered. The corresponding response system of the drive system (3) is

$${}_0^C D_t^\alpha w_l(t) = -a_l w_l(t) + \sum_{m=1}^n b_{lm}(w_m(t)) g_m(w_m(t)) + J_l + u_l, \quad (8)$$

where u_l denotes the state feedback controller, and the initial condition $w(s) = (w_1(0), w_2(0), \dots, w_n(0))^T$ and $l \in \mathbb{N}$. Similarly,

$${}_0^C D_t^\alpha w_l(t) \in -a_l w_l(t) + \sum_{m=1}^n \text{co}[b_{lm}(w_m(t))] g_m(w_m(t)) + J_l + u_l. \quad (9)$$

Or equivalently, there exists $\tilde{b}_{lm}(w_m(t)) \in \text{co}[b_{lm}(w_m(t))]$, such that

$${}_0^C D_t^\alpha w_l(t) = -a_l w_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(w_m(t)) g_m(w_m(t)) + J_l + u_l. \quad (10)$$

Assumption 1. For $\forall x, y \in \mathbb{R}$, and $l \in \mathbb{N}$, the function $g_l(\cdot): \mathbb{R} \rightarrow \mathbb{R}$ is monotone nondecreasing and satisfies $g_l(0) = 0$, $|g_l(\cdot)| \leq \bar{M}_l$, and $|g_l(x) - g_l(y)| \leq \eta_l |x - y|$, where $\bar{M}_l$ and η_l are positive constants.

Next, let the synchronization error $e_l(t) = r_l(t) - w_l(t)$. For (3) and (8), the following synchronization error dynamics system can be obtained as

$$\begin{aligned}
{}_0^C D_t^\alpha e_l(t) &= -a_l e_l(t) + \sum_{m=1}^n \left[\tilde{b}_{lm}(r_m(t)) g_m(r_m(t)) - \tilde{b}_{lm}(w_m(t)) g_m(w_m(t)) \right] - u_l, \\
&= -a_l e_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(r_m(t)) \tilde{\zeta}_m(e_m(t)) + \sum_{m=1}^n \left[\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t)) \right] g_m(w_m(t)) - u_l,
\end{aligned} \tag{11}$$

where $\tilde{\zeta}_m(e_m(t)) = g_m(r_m(t)) - g_m(w_m(t))$. According to Assumption 1, we can know that $\tilde{\zeta}_m(e_m)$ is also monotone nondecreasing, bounded, and $\text{sgn}(e_m(t)) \tilde{\zeta}_m(e_m(t)) = |\tilde{\zeta}_m(e_m(t))| \leq \eta_m |e_m(t)|$.

Definition 3 (see [37]). The drive FOMNN (3) is globally synchronized with the response FOMNN (8), if there are two positive constants ν_1 and ν_2 such that

$$\|e(t)\|_l \leq \{\mathcal{M}(e(t_0)) E_\alpha(-\nu_1 t^\alpha)\}^{\nu_2}, \tag{12}$$

where t_0 denotes the initial time, $\mathcal{M}(e(t))$ is locally Lipschitz on $\mathbb{R}^n$, $\mathcal{M}(0) = 0$, and $\mathcal{M}(e(t)) > 0$.

Lemma 1 (see [38]). Let $V(t, e(t))$ be a continuous function: $\mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ and satisfy ${}_0^C D_t^\alpha V(t, e(t)) \leq \varrho V(t, e(t))$, where $0 < \alpha < 1$ and ϱ is a constant. Then, the following inequality holds:

$$V(t, e(t)) \leq V(0, e(0)) E_\alpha(\varrho t^\alpha). \tag{13}$$

Lemma 2 (see [39]). If $z_3 > 1, z_4 > 1$ and $(z_3 - 1)(z_4 - 1) = 1$, the following inequality holds:

$$z_1 z_2 \leq \left(\frac{\kappa^{z_3}}{z_3} \right) |z_1|^{z_3} + \left(\frac{1}{z_4 \kappa^{z_4}} \right) |z_2|^{z_4}, \tag{14}$$

where κ is an arbitrary positive number and $(z_1, z_2) \in \mathbb{R}^2$.

3. Main Results

In this section, two state feedback controllers are provided for achieving synchronization of FOMNNs.

The following two state feedback controllers are provided:

$$u_l(t) = \lambda_l e_l(t), \tag{15}$$

$$u_l(t) = \sigma_l e_l(t) + \gamma_l \text{sign}(e_l(t)), \tag{16}$$

where λ_l, σ_l and γ_l denote the control parameters, $l \in \mathbb{N}$.

Theorem 1. By holding Assumption 1, the drive FOMNN (3) is globally synchronized with the response FOMNN (8) via the controller (15), if there are two positive constants ς_1 and ς_2 so that

$$\begin{aligned}
&2(a_l + \lambda_l) - \sum_{m=1}^n \dot{b}_{lm} \eta_m \varsigma_1 + \frac{\dot{b}_{ml} \eta_l}{\varsigma_1} + |b_{lm}^* - b_{lm}^*| \eta_m \varsigma_2 \\
&+ \frac{|b_{ml}^* - b_{ml}^*| \eta_l}{\varsigma_2} > 0, \quad l \in \mathbb{N}.
\end{aligned} \tag{17}$$

Proof. Consider the Lyapunov function,

$$V(t, e(t)) = \sum_{l=1}^n \frac{1}{2} e_l^2(t). \tag{18}$$

Also, by calculating the Caputo fractional derivative along the trajectory of (11) with $0 < \alpha < 1$, we get

$$\begin{aligned}
&{}_0^C D_t^\alpha V(t, e(t)) \\
&\leq \sum_{l=1}^n e_l(t) {}_0^C D_t^\alpha e_l(t), \\
&= \sum_{l=1}^n e_l(t) \left[-a_l e_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(r_m(t)) \tilde{\zeta}_m(e_m(t)) \right. \\
&\quad \left. + \sum_{m=1}^n [\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))] g_m(w_m(t)) - \lambda_l e_l(t) \right], \\
&\leq \sum_{l=1}^n -(a_l + \lambda_l) e_l^2(t) + \sum_{l=1}^n \sum_{m=1}^n \dot{b}_{lm} |e_l(t)| |\tilde{\zeta}_m(e_m(t))| \\
&\quad + \sum_{l=1}^n \sum_{m=1}^n |\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))| |g_m(w_m(t))| |e_l(t)|.
\end{aligned} \tag{19}$$

By utilizing Lemma 2, we get

$$\begin{aligned}
&\sum_{l=1}^n \sum_{m=1}^n \dot{b}_{lm} |e_l(t)| |\tilde{\zeta}_m(e_m(t))| \\
&\leq \sum_{l=1}^n \sum_{m=1}^n \dot{b}_{lm} |e_l(t)| \eta_m |e_m(t)| \\
&\leq \sum_{l=1}^n \sum_{m=1}^n \dot{b}_{lm} \eta_m \left[\frac{1}{2} (e_l(t) \varsigma_1^{(1/2)})^2 + \frac{1}{2} (e_m(t) \varsigma_1^{(-1/2)})^2 \right], \\
&= \sum_{l=1}^n \sum_{m=1}^n \frac{\dot{b}_{lm} \eta_m \varsigma_1}{2} e_l^2(t) + \sum_{l=1}^n \sum_{m=1}^n \frac{\dot{b}_{lm} \eta_m}{2 \varsigma_1} e_m^2(t), \\
&= \sum_{l=1}^n \sum_{m=1}^n \left(\frac{\dot{b}_{lm} \eta_m \varsigma_1}{2} + \frac{\dot{b}_{ml} \eta_l}{2 \varsigma_1} \right) e_l^2(t).
\end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
& \sum_{l=1}^n \sum_{m=1}^n \left| \tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t)) \right| \left| g_m(w_m(t)) \right| |e_l(t)| \\
& \leq \sum_{l=1}^n \sum_{m=1}^n |b_{lm}^{**} - b_{lm}^*| |e_l(t)| \eta_m |e_m(t)| \\
& \leq \sum_{l=1}^n \sum_{m=1}^n \left(\left(\frac{|b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2}{2} \right) + \left(\frac{|b_{ml}^{**} - b_{ml}^*| \eta_l}{2\varsigma_2} \right) \right) e_l^2(t).
\end{aligned} \tag{21}$$

Substituting (20), (21) into (19),

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t, e(t)) \\
& \leq \sum_{l=1}^n -(a_l + \lambda_l) e_l^2(t) + \sum_{l=1}^n \sum_{m=1}^n \left(\frac{\dot{b}_{lm} \eta_m \varsigma_1}{2} \right) + \left(\frac{\dot{b}_{ml} \eta_l}{2\varsigma_1} \right) e_l^2(t) \\
& \quad + \sum_{l=1}^n \sum_{m=1}^n \left(\frac{|b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2}{2} \right) + \left(\frac{|b_{ml}^{**} - b_{ml}^*| \eta_l}{2\varsigma_2} \right) e_l^2(t), \\
& = \sum_{l=1}^n \left[-a_l - \lambda_l + \sum_{m=1}^n \left(\frac{\dot{b}_{lm} \eta_m \varsigma_1}{2} + \frac{\dot{b}_{ml} \eta_l}{2\varsigma_1} \right) \right. \\
& \quad \left. + \frac{|b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2}{2} + \frac{|b_{ml}^{**} - b_{ml}^*| \eta_l}{2\varsigma_2} \right] e_l^2(t) \\
& \leq -\min_{1 \leq l \leq n} \left\{ 2(a_l + \lambda_l) - \sum_{m=1}^n \left(\dot{b}_{lm} \eta_m \varsigma_1 + \frac{\dot{b}_{ml} \eta_l}{\varsigma_1} \right) \right. \\
& \quad \left. + |b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2 + \frac{|b_{ml}^{**} - b_{ml}^*| \eta_l}{\varsigma_2} \right\} V(t, e(t)).
\end{aligned} \tag{22}$$

According to (17), we choose a constant $q > 0$ so that

$$\begin{aligned}
& -\min_{1 \leq l \leq n} \left\{ 2(a_l + \lambda_l) - \sum_{m=1}^n \left(\dot{b}_{lm} \eta_m \varsigma_1 + \frac{\dot{b}_{ml} \eta_l}{\varsigma_1} \right) \right. \\
& \quad \left. + |b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2 + \frac{|b_{ml}^{**} - b_{ml}^*| \eta_l}{\varsigma_2} \right\} \leq -q < 0.
\end{aligned} \tag{23}$$

From (22) and (23), we get

$${}_0^C D_t^\alpha V(t, e(t)) \leq -qV(t, e(t)). \tag{24}$$

By utilizing Lemma 1, we can know

$$V(t, e(t)) \leq V(0, e(0)) E_\alpha(-qt^\alpha), \tag{25}$$

namely,

$$\|e(t)\|_2 \leq [\|r(s) - w(s)\|_2 E_\alpha(-qt^\alpha)]^{(1/2)}. \tag{26}$$

According to (26) and Definition 3, we can obtain that the drive FOMNN (3) is globally synchronized with the response FOMNN (8) via controller (15). The proof of Theorem 1 is completed. $\square$

Theorem 2. By holding Assumption 1, the drive FOMNN (3) is globally synchronized with the response FOMNN (8) via controller (16), if there are positive constants ρ_l so that

$$\sum_{m=1}^n \rho_l |b_{lm}^{**} - b_{lm}^*| \tilde{M}_m - \rho_l \gamma_l \leq 0, \tag{27}$$

$$a_l + \sigma_l - \sum_{m=1}^n \left(\frac{\rho_m \dot{b}_{ml} \eta_l}{\rho_l} \right) > 0. \tag{28}$$

Proof. Firstly, we can know that, on $t \in [0, +\infty)$, $e_l(t)$ is a differentiable and continuous function. Therefore, $(d|e_l(t)|)/dt$ is piecewise continuous, and $\lim_{s \rightarrow t^+} ((d|e_l(s)|)/ds)$ exists for $\forall t \in \mathbb{R}^+$. Then, we choose the Lyapunov function:

$$V(t, e(t)) = \sum_{l=1}^n \rho_l |e_l(t)|. \tag{29}$$

Now, by calculating the Caputo fractional derivative along the trajectory of (11) with $0 < \alpha < 1$, we get

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t^+, e(t^+)) \\
& \leq \sum_{l=1}^n \rho_l \text{sign}(e_l(t)) {}_0^C D_t^\alpha e_l(t), \\
& = \sum_{l=1}^n \rho_l \text{sign}(e_l(t)) \left[-a_l e_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(r_m(t)) \tilde{\zeta}_m(e_m(t)) \right. \\
& \quad \left. + \sum_{m=1}^n [\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))] g_m(w_m(t)) \right. \\
& \quad \left. - \sigma_l e_l(t) - \gamma_l \text{sign}(e_l(t)) \right] \\
& \leq \sum_{l=1}^n \left[-\rho_l (a_l + \sigma_l) |e_l(t)| + \sum_{m=1}^n \rho_l \dot{b}_{lm} |\tilde{\zeta}_m(e_m(t))| \right. \\
& \quad \left. + \sum_{m=1}^n \rho_l |\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))| |g_m(w_m(t))| - \rho_l \gamma_l \right] \\
& \leq \sum_{l=1}^n \left[-\rho_l (a_l + \sigma_l) |e_l(t)| + \sum_{m=1}^n \rho_l \dot{b}_{lm} \tilde{\eta}_m |e_m(t)| \right. \\
& \quad \left. + \sum_{m=1}^n \rho_l |b_{lm}^{**} - b_{lm}^*| \tilde{M}_m - \rho_l \gamma_l \right].
\end{aligned} \tag{30}$$

From (27), it follows that

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t^+, e(t^+)) \\
& \leq \sum_{l=1}^n \left[-\rho_l (a_l + \sigma_l) |e_l(t)| + \sum_{m=1}^n \rho_l \dot{b}_{lm} \eta_m |e_m(t)| \right] \\
& \leq \sum_{l=1}^n -\rho_l (a_l + \sigma_l) |e_l(t)| + \sum_{l=1}^n \sum_{m=1}^n \rho_m \dot{b}_{ml} \eta_l |e_l(t)| \\
& = \sum_{l=1}^n -(a_l + \sigma_l) \rho_l |e_l(t)| + \sum_{l=1}^n \sum_{m=1}^n \left(\frac{\rho_m \dot{b}_{ml} \eta_l}{\rho_l} \right) \rho_l |e_l(t)| \\
& \leq -\min_{1 \leq l \leq n} \left\{ a_l + \sigma_l - \sum_{m=1}^n \left(\frac{\rho_m \dot{b}_{ml} \eta_l}{\rho_l} \right) \right\} V(t, e(t)).
\end{aligned} \tag{31}$$

According to (28), we choose a constant $\varrho > 0$ so that

$$-\min_{1 \leq l \leq n} \left\{ a_l + \sigma_l - \sum_{m=1}^n \left(\frac{\rho_m \dot{b}_{ml} \eta_l}{\rho_l} \right) \right\} \leq -\varrho < 0. \tag{32}$$

From (31) and (32),

$${}_0^C D_t^\alpha V(t, e(t)) \leq -\varrho V(t, e(t)). \tag{33}$$

By utilizing Lemma 1, we can know

$$V(t, e(t)) \leq V(0, e(0)) E_\alpha(-\varrho t^\alpha), \tag{34}$$

that is,

$$\sum_{l=1}^n \rho_l |e_l(t)| \leq \sum_{l=1}^n \rho_l |r_l(0) - w_l(0)| E_\alpha(-\varrho t^\alpha), \tag{35}$$

and then,

$$\begin{aligned}
& \sum_{l=1}^n |e_l(t)| \\
& \leq \frac{\max_{1 \leq l \leq n} \{\rho_l\}}{\min_{1 \leq l \leq n} \{\rho_l\}} \sum_{l=1}^n |r_l(0) - w_l(0)| E_\alpha(-\varrho t^\alpha) \\
& = \sum_{l=1}^n \frac{\max_{1 \leq l \leq n} \{\rho_l\}}{\min_{1 \leq l \leq n} \{\rho_l\}} |r_l(0) - w_l(0)| E_\alpha(-\varrho t^\alpha).
\end{aligned} \tag{36}$$

According to (36) and Definition 3, we can obtain that the drive FOMNN (3) is globally synchronized with the response FOMNN (8) under the state feedback controller (16). The proof of Theorem 2 is completed.

It is well known that eigenvalue of the system matrix has a tight relation with dynamics. Next, we will obtain synchronization conditions according to it. $\square$

Theorem 3. *By holding Assumption 1, the drive FOMNN (3) is globally synchronized with the response FOMNN (8) via controller (15), if*

$$\begin{aligned}
& a_l + \lambda_l - \sum_{m=1}^n \left(|b_{lm}^{**} - b_{lm}^*| \eta_m \varsigma_2 + \left(\frac{(|b_{ml}^{**} - b_{ml}^*|) \eta_l}{\varsigma_2} \right) \right) \\
& - \lambda_{\max}(\Xi) > 0, \quad l \in \mathbb{N},
\end{aligned} \tag{37}$$

where

$$\Xi = \begin{bmatrix} 0, & (\dot{b}_{lm} \eta_m)_{n \times n}, \\ (\dot{b}_{lm} \eta_m)_{n \times n}^T, & 0, \end{bmatrix}_{2n \times 2n}. \tag{38}$$

Proof. Firstly, according to the matrix theory, we can easily know that $-\theta$ is also the eigenvalue of Ξ if θ is its eigenvalue. Therefore, the maximum eigenvalues of matrix Ξ is greater than zero, that is, $\lambda_{\max}(\Xi) > 0$.

Consider the Lyapunov function,

$$V(t, e(t)) = \sum_{l=1}^n e_l^2(t). \tag{39}$$

Then, by calculating the Caputo fractional derivative along the trajectory of (11) with $0 < \alpha < 1$, we get

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t, e(t)) \\
& \leq \sum_{l=1}^n 2e_l(t) {}_0^C D_t^\alpha e_l(t), \\
& = \sum_{l=1}^n 2e_l(t) \left[-a_l e_l(t) + \sum_{m=1}^n \tilde{b}_{lm}(r_m(t)) \tilde{\zeta}_m(e_m(t)) + \sum_{m=1}^n [\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))] g_m(w_m(t)) - \lambda_l e_l(t) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l=1}^n -2(a_l + \lambda_l)e_l^2(t) + \sum_{l=1}^n \sum_{m=1}^n 2\dot{b}_{lm}|e_l(t)|\|\tilde{\zeta}_m(e_m(t))\| \\
&\quad + \sum_{l=1}^n \sum_{m=1}^n 2|\tilde{b}_{lm}(r_m(t)) - \tilde{b}_{lm}(w_m(t))|g_m(w_m(t))\|e_l(t)\| \leq \sum_{l=1}^n -2(a_l + \lambda_l)e_l^2(t) \\
&\quad + \left(|e(t)|^T, |\tilde{\zeta}'(e(t))|^T\right)^T \Xi \left(|e(t)|^T, |\tilde{\zeta}'(e(t))|^T\right) + \sum_{l=1}^n \sum_{m=1}^n \left(|b_{lm}^{**} - b_{lm}^*|\eta_m\varsigma_2 + \left(\frac{|b_{ml}^{**} - b_{ml}^*|\eta_l}{\varsigma_2}\right)e_l^2(t)\right) \\
&\leq \sum_{l=1}^n \left[-2a_l - 2\lambda_l + \sum_{m=1}^n \left(|b_{lm}^{**} - b_{lm}^*|\eta_m\varsigma_2 + \left(\frac{|b_{ml}^{**} - b_{ml}^*|\eta_l}{\varsigma_2}\right)\right)\right] e_l^2(t) + 2\lambda_{\max}(\Xi) \sum_{l=1}^n e_l^2(t) \\
&\leq -2 \min_{1 \leq l \leq n} \left\{ a_l + \lambda_l - \sum_{m=1}^n \left(|b_{lm}^{**} - b_{lm}^*|\eta_m\varsigma_2 + \left(\frac{|b_{ml}^{**} - b_{ml}^*|\eta_l}{\varsigma_2}\right)\right) - \lambda_{\max}(\Xi) \right\} V(t, e(t)), \tag{40}
\end{aligned}$$

where $|e(t)| = (|e_1(t)|, |e_2(t)|, \dots, |e_n(t)|)$, $|\tilde{\zeta}'(e(t))| = ((1/\eta_1)|\tilde{\zeta}_1(e_1(t))|, (1/\eta_2)|\tilde{\zeta}_2(e_2(t))|, \dots, (1/\eta_n)|\tilde{\zeta}_n(e_n(t))|)$.

According to (37), we choose a constant $\varrho > 0$ so that

$$\begin{aligned}
&-\min_{1 \leq l \leq n} \left\{ a_l + \lambda_l - \sum_{m=1}^n \left(|b_{lm}^{**} - b_{lm}^*|\eta_m\varsigma_2 + \left(\frac{|b_{ml}^{**} - b_{ml}^*|\eta_l}{\varsigma_2}\right)\right) \right. \\
&\quad \left. - \lambda_{\max}(\Xi) \right\} \leq -\varrho < 0. \tag{41}
\end{aligned}$$

From (40) and (41), we can get

$${}^C_0 D_t^\alpha V(t, e(t)) \leq -2\varrho V(t, e(t)). \tag{42}$$

By utilizing Lemma 1, we can know

$$V(t, e(t)) \leq V(0, e(0))E_\alpha(-2\varrho t^\alpha), \tag{43}$$

namely,

$$\sum_{l=1}^n e_l^2(t) \leq \sum_{l=1}^n (r_l(0) - w_l(0))^2 E_\alpha(-2\varrho t^\alpha), \tag{44}$$

and then,

$$\begin{aligned}
&\left(\sum_{l=1}^n \left(\frac{e_l^2(t)}{2} \right) \right)^{(1/2)} \\
&\leq \left(\sum_{l=1}^n \left(\frac{(r_l(0) - w_l(0))^2}{2} E_\alpha(-2\varrho t^\alpha) \right) \right)^{(1/2)}. \tag{45}
\end{aligned}$$

According to (45) and Definition 3, we can obtain that the drive FOMNN (3) is globally synchronized with the response FOMNN (8) under the state feedback controller (15). The proof of Theorem 3 is completed. $\square$

Remark 1. Free-weighting parameters ς_1 and ς_2 are introduced in synchronization criterion (17) of Theorem 1. Also, free-weighting parameters ς_1 and ς_2 can be used to reduce the conservativeness of the synchronization criterion.

Remark 2. The synchronization criteria obtained in this paper only depend on their system parameters, which are simpler in form than linear matrix inequalities [40]. In addition, these algebraic synchronization criteria are easy to check, quick to calculate, and help greatly reduce the computational burden.

4. Numerical Simulation

Now, we provide a numerical simulation to illustrate the effectiveness of results.

We consider the FOMNN (8) as the response system and the drive FOMNN as follows:

$$\begin{cases} {}^C_0 D_t^\alpha r_1(t) = -r_1(t) + b_{11}(r_1(t))g_1(r_1(t)) + b_{12}(r_2(t))g_2(r_2(t)) + J_1, \\ {}^C_0 D_t^\alpha r_2(t) = -1.2r_2(t) + b_{21}(r_1(t))g_1(r_1(t)) + b_{22}(r_2(t))g_2(r_2(t)) + J_2, \end{cases} \tag{46}$$

FIGURE 1: The chaotic behavior, error signal and state trajectories of the derive FOMNN (46), and the response FOMNN (8) without a controller.

where $t \in [0, +\infty)$, $J_1 = J_2 = 0$, $\alpha = 0.98$, $g_l(r_l(t)) = \tanh(r_l(t))$, $l = 1, 2$, and

$$\begin{aligned}
 b_{11}(r_1(t)) &= \begin{cases} 2, & |r_1(t)| \leq 1, \\ 2.2, & |r_1(t)| > 1, \end{cases} \\
 b_{12}(r_2(t)) &= \begin{cases} -1.2, & |r_2(t)| \leq 1, \\ -0.9, & |r_2(t)| > 1, \end{cases} \\
 b_{21}(r_1(t)) &= \begin{cases} 1.8, & |r_1(t)| \leq 1, \\ 2.1, & |r_1(t)| > 1, \end{cases} \\
 b_{22}(r_2(t)) &= \begin{cases} 1.7, & |r_2(t)| \leq 1, \\ 2.2, & |r_2(t)| > 1. \end{cases}
 \end{aligned} \quad (47)$$

Then, we can obtain $\dot{b}_{11} = 2.2$, $\dot{b}_{12} = 1.2$, $\dot{b}_{21} = 2.1$, $\dot{b}_{22} = 2.2$, $\eta_1 = \eta_2 = 1$, and $\tilde{M}_1 = \tilde{M}_2 = 1$. Let $r_1(0) = 1.1$, $r_2(0) = -0.8$, $w_1(0) = -0.9$, $w_2(0) = 0.6$; Figure 1

depicts the chaotic behavior, error and state trajectories of the derive FOMNN (46), and the response FOMNN (8) without an external controller, which are not synchronous.

Now, according to (17) in Theorem 1, we can get

$$2(1 + \lambda_1) - 3.4\varsigma_1 - \left(\frac{4.3}{\varsigma_1}\right) - 0.5\varsigma_2 - \left(\frac{0.5}{\varsigma_2}\right) > 0, \quad (48)$$

$$2(1.2 + \lambda_2) - 4.3\varsigma_1 - \left(\frac{3.4}{\varsigma_1}\right) - 0.8\varsigma_2 - \left(\frac{0.8}{\varsigma_2}\right) > 0. \quad (49)$$

Next, we choose $\varsigma_1 = \varsigma_2 = 1$ for convenient calculation, then we can obtain $\lambda_1 > 3.35$, $\lambda_2 > 3.45$, and choose $\lambda_1 = 3.4$, $\lambda_2 = 3.5$. The condition of Theorem 1 is satisfied, that is, the drive FOMNN (46) and the response FOMNN (8) are globally synchronized via the controller $u_1(t) = 3.4e_1(t)$, $u_2(t) = 3.5e_2(t)$. Let $r_1(0) = 1.1$, $r_2(0) =$

FIGURE 2: The chaotic behaviors, error and state trajectories of the drive FOMNN (46), and the response FOMNN (8) with the state feedback controller $u_1(t) = 3.4e_1(t)$, $u_2(t) = 3.5e_2(t)$.

$-0.8, w_1(0) = -0.9, w_2(0) = 0.6$, and then, the chaotic behaviors, error and state trajectories of the drive FOMNN (46), and the response FOMNN (8) via the controller $u_1(t) = 3.4e_1(t)$, $u_2(t) = 3.5e_2(t)$ are shown in Figure 2.

Similarly, according to (27) in Theorem 2 and by selecting $\rho_1 = \rho_2 = 1$, we can get $\gamma_1 \geq 0.5, \gamma_2 \geq 0.7$ and choose $\gamma_1 = 0.5, \gamma_2 = 0.7$. Moreover, according to (28) in Theorem 2, we can get $\sigma_1 \geq 3.3, \sigma_2 \geq 2.2$ and choose $\sigma_1 = 3.4, \sigma_2 = 2.3$. The conditions of Theorem 2 are satisfied, that is, the drive FOMNN (46) and the response FOMNN (8) are globally synchronized via the controller $u_1(t) = 3.4e_1(t) + 0.5\text{sign}(e_1(t))$, $u_2(t) = 2.3e_2(t) + 0.7\text{sign}(e_2(t))$. Let $r_1(0) = -1.9, r_2(0) = -1.8, w_1(0) = 1.2, w_2(0) = -0.6$; then, the chaotic behaviors, error and state trajectories of the drive FOMNN (46), and the response FOMNN (8) via the controller $u_1(t) = 3.4e_1(t) + 0.5\text{sign}(e_1(t))$, $u_2(t) = 2.3e_2(t) + 0.7\text{sign}(e_2(t))$ are shown in Figure 3.

Moreover, in Theorem 3, system parameters matrix Ξ can be easily obtained as

$$\Xi = \begin{bmatrix} 0 & 0 & 2.2 & 1.2 \\ 0 & 0 & 2.1 & 2.2 \\ 2.2 & 1.2 & 0 & 0 \\ 2.1 & 2.2 & 0 & 0 \end{bmatrix}. \quad (50)$$

By easily calculating, the maximal eigenvalue of system parameters matrix Ξ is $\lambda_{\max}(\Xi) = 3.7875$. Next, according to (37) in Theorem 3, choosing $c_1 = c_2 = 1$, we can obtain $\lambda_1 > 3.7875$ and $\lambda_2 > 3.9875$ and choose $\lambda_1 = 3.8, \lambda_2 = 4$. The condition of Theorem 3 is satisfied, that is, the drive FOMNN (46) and the response FOMNN (8) are globally synchronized via the controller $u_1(t) = 3.8e_1(t)$, $u_2(t) = 4e_2(t)$. Let $r_1(0) = 1.5, r_2(0) = -3, w_1(0) = -1.3, w_2(0) = 2.6$; then, the chaotic behaviors, error and state

FIGURE 3: The chaotic behaviors, error and state trajectories of the derive FOMNN (46), and the response FOMNN (8) with the state feedback controller $u_1(t) = 3.4e_1(t) + 0.5\text{sign}(e_1(t))$, $u_2(t) = 2.3e_2(t) + 0.7\text{sign}(e_2(t))$.

FIGURE 4: Continued.

FIGURE 4: The chaotic behaviors, error and state trajectories of the drive FOMNN (46), and the response FOMNN (8) with the state feedback controller $u_1(t) = 3.8e_1(t)$, $u_2(t) = 4e_2(t)$.

trajectories of the drive FOMNN (46), and the response FOMNN (8) via the controller $u_1(t) = 3.8e_1(t)$, $u_2(t) = 4e_2(t)$ are shown in Figure 4.

5. Conclusions

In this paper, the synchronization issue for FOMNNs has been investigated via state feedback control. To achieve the synchronization for the considered drive-response FOMNNs, feedback controllers are first introduced. Then, by adopting nonsmooth analysis, fractional Lyapunov's direct method and Young inequality, and fractional-order differential inclusions, several algebraic sufficient criteria are obtained for guaranteeing the synchronization for the drive-response FOMNNs. Finally, an example is given to illustrate the effectiveness of the theoretical results. In future research, theoretical results here will be used to address the state estimation of FOMNNs.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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