Hindawi
Complexity
Volume 2020, Article ID 2474278, 8 pages
https://doi.org/10.1155/2020/2474278

Research Article

M-Eigenvalues-Based Sufficient Conditions for the Positive Definiteness of Fourth-Order Partially Symmetric Tensors

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Received 24 July 2019; Revised 16 November 2019; Accepted 7 December 2019; Published 8 January 2020

Academic Editor: Xianming Zhang

M-eigenvalues of fourth-order partially symmetric tensors play important roles in the nonlinear elastic material analysis and the entanglement problem of quantum physics. In this paper, we introduce M-identity tensor and establish two M-eigenvalue inclusion intervals with n parameters for fourth-order partially symmetric tensors, which are sharper than some existing results. Numerical examples are proposed to verify the efficiency of the obtained results. As applications, we provide some checkable sufficient conditions for the positive definiteness and establish bound estimations for the M-spectral radius of fourth-order partially symmetric nonnegative tensors.

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^n$ be the set of all dimension n real vectors, and $[n] = \{1, 2, \ldots, n\}$ a fourth-order real tensor, denoted by $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[n_1]\times[n_2]\times[n_3]\times[n_4]}$, consists of $[n_1] \times [n_2] \times [n_3] \times [n_4]$ components:

$$a_{ijkl} \in \mathbb{R}, \quad i \in [n_1], j \in [n_2], k \in [n_3], l \in [n_4]. \tag{1}$$

Specifically, $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ is called partially symmetric, if its components are invariant under the following permutation of subscripts:

$$a_{ijkl} = a_{kjil} = a_{ilkj} = a_{klij}, \quad i, k \in [m],\ j, l \in [n]. \tag{2}$$

In fact, the tensor of elastic moduli for elastic materials exactly is partially symmetric [1], and the components of such tensor are regarded as the coefficients of the following biquadratic homogeneous polynomial optimization problem:

$$\begin{cases} \min f_{\mathscr{A}}(x, y) = \mathscr{A}xyxy = \sum_{i,k\in[m]} \sum_{j,l\in[n]} a_{ijkl} x_i y_j x_k y_l \\ \text{s.t. } x^T x = 1, y^T y = 1, \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n. \end{cases} \tag{3}$$

This optimization problem induced by tensor $\mathscr{A}$, finds applications in nonlinear elastic materials analysis [2], the ordinary ellipticity and strong ellipticity [1, 3], and stability study of nonlinear autonomous systems [4, 5]. As we know, the strong ellipticity condition is essential in theory of elasticity, which guarantees the existence of solutions of basic boundary-value problems of elastostatics and ensures an elastic material to satisfy some mechanical properties. Qi et al. [6] pointed out that the strong ellipticity condition holds if and only if the optimal value of problem (3) is positive. To establish the criteria in identifying the strong ellipticity in elastic mechanics, Qi et al. [6, 7] introduced the following definition.

Definition 1. Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric real tensor. For $\lambda \in \mathbb{R}, x \in \mathbb{R}^m, y \in \mathbb{R}^n$, if

$$\begin{cases} \mathscr{A}\cdot yxy=\lambda x, \\ \mathscr{A}xyx\cdot=\lambda y, \\ x^Tx=1, \\ y^Ty=1, \end{cases} \tag{4}$$

where $(\mathscr{A}\cdot yxy)_i=\sum_{k\in[m],j,l\in[n]}a_{ijkl}y_jx_ky_l$ and $(\mathscr{A}xyx\cdot)_l=\sum_{i,k\in[m],j\in[n]}a_{ijkl}x_iy_jx_k$, then the scalar λ is called an M-eigenvalue of the tensor $\mathscr{A}$ and x and y are called left and right M-eigenvectors of $\mathscr{A}$, respectively, which are associated with the M-eigenvalue λ. Denote $\sigma_M(\mathscr{A})$ as the set of all M-eigenvalues of $\mathscr{A}$. Then, the M-spectral radius of $\mathscr{A}$ is denoted by

$$\rho_M(\mathscr{A})=\max\{|\lambda|:\ \lambda\in\sigma_M(\mathscr{A})\}. \tag{5}$$

Note that $f_{\mathscr{A}}(x,y)$ is positive definite if and only if M-eigenvalues of $\mathscr{A}$ are positive [7]. Hence, effective algorithms for finding M-eigenvalue and the corresponding eigenvector have been implemented [8–16]. Due to the complexity of the tensor eigenvalue problem [17, 18], it is difficult to compute all M-eigenvalues. Thus, some researchers turned to investigating the inclusion sets of M-eigenvalue [19–21]. For example, Che et al. [19] proposed a Gershgorin-type M-inclusion set as follows.

Lemma 1 (Theorem 2.1 of [19]). *Suppose $\mathscr{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric real tensor.* Then,

$$\sigma_M(\mathscr{A})\subseteq\Gamma(\mathscr{A}):=\bigcup_{i\in[m]}\Gamma_i(\mathscr{A}), \tag{6}$$

where

$$\Gamma_i(\mathscr{A})=\{z\in\mathbb{C}:\ |z|\le R_i(\mathscr{A})\},\qquad R_i(\mathscr{A})=\sum_{k\in[m];j,l\in[n]}|a_{ijkl}|. \tag{7}$$

Unfortunately, the mentioned inclusion sets always include zero and cannot identify the positive definiteness of $f_{\mathscr{A}}(x,y)$.

Example 1. Consider the following partially symmetric tensor $\mathscr{A}=(a_{ijkl})\in\mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl}=\begin{cases} a_{1111}=20, a_{1122}=a_{1221}=1, a_{1212}=8; \\ a_{2222}=10, a_{2112}=a_{2211}=1, a_{2121}=7; \\ a_{ijkl}=0, \text{otherwise}. \end{cases} \tag{8}$$

From Lemma 1, it holds that

$$\Gamma(\mathscr{A})=\bigcup_{i\in[2]}\Gamma_i(\mathscr{A})=\{\lambda\in\mathbb{C}:\ |\lambda|\le 30\}. \tag{9}$$

By computation, we can obtain that the corresponding M-eigenvalues are $7, 20$. Hence, $\mathscr{A}$ is positive definite. However, we could not use $\Gamma(\mathscr{A})$ to identify the positive definiteness of $\mathscr{A}$. To overcome the drawback above, we present new M-eigenvalue inclusion intervals with n parameters, which can be used to identify the positive definiteness of fourth-order partially symmetric tensors.

This paper is organized as follows. In Section 2, we establish two M-eigenvalue inclusion intervals for fourth-order partially symmetric tensors. In Section 3, we propose some checkable sufficient conditions of the positive definiteness and establish bound estimations for the M-spectral radius of fourth-order partially symmetric nonnegative tensors. Numerical examples are proposed to verify the efficiency of the obtained results.

2. *M*-Eigenvalue Inclusion Intervals for Fourth-Order Partially Symmetric Tensors

In this section, inspired by H-eigenvalue inclusion theorems [22–26] and Z-eigenvalue inclusion intervals [27–31], we establish two M-eigenvalue inclusion intervals for fourth-order partially symmetric tensors. We begin our work by introducing M-identity tensor.

Definition 2. We call $\mathscr{I}_M\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ an M-identity tensor if its entries are

$$(\mathscr{I}_M)_{ijkl}=\begin{cases} 1, & \text{if } i=k,\ j=l, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $i,k\in[m]$, $j,l\in[n]$.

Obviously, $\mathscr{I}_M\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ is a partially symmetric tensor and has the following property:

$$\begin{cases} \mathscr{I}_M\cdot yxy=x, \\ \mathscr{I}_Mxyx\cdot=y, \end{cases} \tag{11}$$

with $x^Tx=1$, $y^Ty=1$ for all $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$.

Theorem 1. *Let $\mathscr{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathscr{I}_M$ be an M-identity tensor. For any $\alpha=(\alpha_1,\cdots,\alpha_m)^T\in\mathbb{R}^m$, then*

$$\sigma_M(\mathscr{A})\subseteq\mathscr{G}(\mathscr{A},\alpha)=\bigcup_{i\in[m]}\mathscr{G}_i(\mathscr{A},\alpha), \tag{12}$$

where

$$\mathscr{G}_i(\mathscr{A},\alpha)=\{z\in\mathbb{R}:\ |z-\alpha_i|\le R_i(\mathscr{A},\alpha_i)\},\qquad R_i(\mathscr{A},\alpha_i)=\sum_{\substack{k\in[m]\\ j,l\in[n]}}|a_{ijkl}-\alpha_i(\mathscr{I}_M)_{ijkl}|. \tag{13}$$

Further, $\sigma_M(\mathscr{A})\subseteq\cap_{\alpha\in\mathbb{R}^m}\mathscr{G}(\mathscr{A},\alpha)$.

Proof. Let (λ,x,y) be an M-eigenpair of $\mathscr{A}$ and $\mathscr{I}_M$ be an M-identity tensor. From the definition of M-identity tensor and (11), it holds

$$\mathscr{A}yxy=\lambda x=\lambda\mathscr{I}_Myxy. \tag{14}$$

Setting $|x_t|=\max_{i\in[m]}|x_i|$, by $x^Tx=1$, one has $0<|x_t|\le 1$. From the tth equality of (14), we obtain

$$\sum_{\substack{k\in[m]\\ j,l\in[n]}}\lambda(\mathscr{I}_M)_{tjkl}y_jx_ky_l=\sum_{\substack{k\in[m]\\ j,l\in[n]}}a_{tjkl}y_jx_ky_l. \tag{15}$$

Hence, for any real number α_t, it follows that

$$(\lambda-\alpha_t)x_t = \sum_{\substack{k\in[m]\\ j,l\in[n]}} (\lambda-\alpha_t)(\mathcal{I}_M)_{tjkl}y_jx_ky_l = \sum_{\substack{k\in[m]\\ j,l\in[n]}} \left(a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right)y_jx_ky_l. \quad (16)$$

Taking modulus in the above equation, one has

$$\begin{aligned}|\lambda-\alpha_t||x_t| &= \left|\sum_{\substack{k\in[m]\\ j,l\in[n]}} \left(a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right)y_jx_ky_l\right| \\ &\le \sum_{\substack{k\in[m]\\ j,l\in[n]}} \left|a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right||y_j||x_k||y_l| \\ &\le \sum_{\substack{k\in[m]\\ j,l\in[n]}} \left|a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right||x_t|.\end{aligned} \quad (17)$$

Therefore,

$$|\lambda-\alpha_t| \le \sum_{\substack{k\in[m]\\ j,l\in[n]}} \left|a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right|, \quad (18)$$

which implies that $\lambda\in\mathcal{G}_t(\mathcal{A},\alpha)\subseteq\mathcal{G}(\mathcal{A},\alpha)$. From the arbitrariness of α, the conclusion follows. □

Theorem 2. *Let* $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor and* $\mathcal{I}_M$ *be an M-identity tensor. For any* $\alpha=(\alpha_1,\dots,\alpha_m)^T\in\mathbb{R}^m$, *then*

$$\sigma_M(\mathcal{A})\subseteq\mathcal{K}(\mathcal{A},\alpha)=\bigcup_{i\in[m]}\left(\bigcap_{v\ne i,v\in[m]}\mathcal{K}_{i,v}(\mathcal{A},\alpha)\right), \quad (19)$$

where

$$\begin{aligned}R_i^v(\mathcal{A},\alpha_i) &= \sum_{j,l\in[n]}\left|a_{ijvl}-\alpha_i(\mathcal{I}_M)_{ijvl}\right| \\ \mathcal{K}_{i,v}(\mathcal{A},\alpha) &= \Big\{\lambda\in\mathbb{R}: \left[|\lambda-\alpha_i|-\left(R_i(\mathcal{A},\alpha_i)-R_i^v(\mathcal{A},\alpha_i)\right)\right] \\ &\quad\cdot|\lambda-\alpha_v|\le R_i^v(\mathcal{A},\alpha_i)R_v(\mathcal{A},\alpha_v)\Big\}.\end{aligned} \quad (20)$$

Further, $\sigma_M(\mathcal{A})\subseteq\cap_{\alpha\in\mathbb{R}^m}\mathcal{K}(\mathcal{A},\alpha)$.

Proof. Let (λ,x,y) be an M-eigenpair of $\mathcal{A}$ and $\mathcal{I}_M$ be an M-identity tensor. Set $|x_t|=\max_{i\in[m]}|x_i|$. Since $x^Tx=1$, it holds that $0<|x_t|\le 1$. From the tth equation of $\mathcal{A}\cdot yxy=\lambda x$ in (4), for any $p\in[m]$, $p\ne t$ and any real number α_t, we have

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{\substack{k\in[m]\\ j,l\in[n]}} a_{tjkl}y_jx_ky_l - \sum_{\substack{k\in[m]\\ j,l\in[n]}}\alpha_t(\mathcal{I}_M)_{tjkl}y_jx_ky_l \\ &= \sum_{\substack{k\ne p,k\in[m]\\ j,l\in[n]}} \left(a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right)y_jx_ky_l \\ &\quad + \sum_{j,l\in[n]}\left(a_{tjpl}-\alpha_t(\mathcal{I}_M)_{tjpl}\right)y_jx_py_l.\end{aligned} \quad (21)$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}|\lambda-\alpha_t||x_t| &\le \sum_{\substack{k\ne p,k\in[m]\\ j,l\in[n]}} \left|\left(a_{tjkl}-\alpha_t(\mathcal{I}_M)_{tjkl}\right)\right||y_j||x_t||y_l| \\ &\quad + \sum_{j,l\in[n]}\left|\left(a_{tjpl}-\alpha_t(\mathcal{I}_M)_{tjpl}\right)\right||y_j||x_p||y_l| \\ &\le \left(R_t(\mathcal{A},\alpha_t)-R_t^p(\mathcal{A},\alpha_t)\right)|x_t| + R_t^p(\mathcal{A},\alpha_t)|x_p|.\end{aligned} \quad (22)$$

Thus,

$$|\lambda-\alpha_t|-\left(R_t(\mathcal{A},\alpha_t)-R_t^p(\mathcal{A},\alpha_t)\right)\le R_t^p(\mathcal{A},\alpha_t)\left|\frac{|x_p|}{|x_t|}\right| \quad (23)$$

If $|x_p|=0$, then

$$|\lambda-\alpha_t|\le R_t(\mathcal{A},\alpha_t)-R_t^p(\mathcal{A},\alpha_t), \quad (24)$$

which shows $\lambda\in\mathcal{K}_{t,p}(\mathcal{A},\alpha)$. Otherwise, for $|x_p|>0$, we obtain

$$\begin{aligned}|\lambda-\alpha_p||x_p| &\le \sum_{\substack{k\in[m]\\ j,l\in[n]}}\left|\left(a_{pjkl}-\alpha_p(\mathcal{I}_M)_{pjkl}\right)\right||y_j||x_k||y_l| \\ &\le R_p(\mathcal{A},\alpha_p)|x_t|.\end{aligned} \quad (25)$$

That is,

$$|\lambda-\alpha_p|\le R_p(\mathcal{A},\alpha_p)\frac{|x_t|}{|x_p|}. \quad (26)$$

Multiplying (23) with (26) yields

$$\begin{aligned}&\left[|\lambda-\alpha_t|-\left(R_t(\mathcal{A},\alpha_t)-R_t^p(\mathcal{A},\alpha_t)\right)\right] \\ &\quad\cdot|\lambda-\alpha_p|\le R_t^p(\mathcal{A},\alpha_t)R_p(\mathcal{A},\alpha_p),\end{aligned} \quad (27)$$

which implies that $\lambda\in\mathcal{K}_{t,p}(\mathcal{A},\alpha)$. From the arbitrariness of p, it follows that $\lambda\in\cap_{v\ne t,v\in[m]}\mathcal{K}_{t,v}(\mathcal{A},\alpha)$. Further, $\lambda\in\cup_{i\in[m]}(\cap_{v\ne i,v\in[m]}\mathcal{K}_{i,v}(\mathcal{A},\alpha))$. It follows from the arbitrariness of α that $\sigma_M(\mathcal{A})\subseteq\cap_{\alpha\in\mathbb{R}^m}\mathcal{K}(\mathcal{A},\alpha)$. □

Remark 1.

(i) It is clear that Theorems 1 and 2 reduce to Theorems 2.1 and 2.2 of [19] if one takes $\alpha = 0$, respectively. Consequently, the upper bounds of $\rho_M(\mathscr{A})$ in Theorems 1 and 2 are smaller than those in Theorems 2.1 and 2.2 of [19].

(ii) By using the equation $\mathscr{A}xyx = \lambda y$, we can establish some conclusions similar to Theorems 1 and 2.

Corollary 1. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathscr{I}_M$ be an M-identity tensor. For any $\alpha = (\alpha_1, \cdots, \alpha_m)^T \in \mathbb{R}^m$, then*

$$\sigma_M(\mathscr{A}) \subseteq \mathscr{K}(\mathscr{A},\alpha) \subseteq \mathscr{G}(\mathscr{A},\alpha). \tag{28}$$

Proof. For any $\lambda \in \mathscr{K}(\mathscr{A},\alpha)$, without loss of generality, there exists $t \in [m]$ such that $\lambda \in \mathscr{K}_{t,k}(\mathscr{A},\alpha)$, for all $t \neq k$. Thus,

$$\begin{aligned}&\left(|\lambda-\alpha_t| - \left(R_t(\mathscr{A},\alpha_t) - R_t^k(\mathscr{A},\alpha_t)\right)\right)|\lambda-\alpha_k| \\ &\quad \leq R_t^k(\mathscr{A},\alpha_t)R_k(\mathscr{A},\alpha_k).\end{aligned} \tag{29}$$

We now break up the argument into two cases. □

Case 1. If $R_t^k(\mathscr{A},\alpha_t)R_k(\mathscr{A},\alpha_k) = 0$, then

$$|\lambda-\alpha_t| - \left(R_t(\mathscr{A},\alpha_t) - R_t^k(\mathscr{A},\alpha_t)\right) \leq 0 \text{ or } \lambda = \alpha_k. \tag{30}$$

Hence,

$$|\lambda-\alpha_t| \leq R_t(\mathscr{A},\alpha_t) - R_t^k(\mathscr{A},\alpha_t) \leq R_t(\mathscr{A},\alpha_t) \text{ or } \lambda = \alpha_k. \tag{31}$$

Therefore, $\lambda \in \mathscr{G}_t(\mathscr{A},\alpha) \cup \mathscr{G}_k(\mathscr{A},\alpha) \subseteq \mathscr{G}(\mathscr{A},\alpha)$.

Case 2. If $R_t^k(\mathscr{A},\alpha_t)R_k(\mathscr{A},\alpha_k) > 0$, then

$$\frac{|\lambda-\alpha_t| - \left(R_t(\mathscr{A},\alpha_t) - R_t^k(\mathscr{A},\alpha_t)\right)}{R_t^k(\mathscr{A},\alpha_t)} \frac{|\lambda-\alpha_k|}{R_k(\mathscr{A},\alpha_k)} \leq 1, \tag{32}$$

which implies that

$$\frac{|\lambda-\alpha_t| - \left(R_t(\mathscr{A},\alpha_t) - R_t^k(\mathscr{A},\alpha_t)\right)}{R_t^k(\mathscr{A},\alpha_t)} \leq 1, \tag{33}$$

or

$$\frac{|\lambda-\alpha_k|}{R_k(\mathscr{A},\alpha_k)} \leq 1. \tag{34}$$

Thus, $\lambda \in \mathscr{G}_t(\mathscr{A},\alpha) \cup \mathscr{G}_k(\mathscr{A},\alpha) \subseteq \mathscr{G}(\mathscr{A},\alpha)$.

In summary, $\sigma_M(\mathscr{A}) \subseteq \mathscr{K}(\mathscr{A},\alpha) \subseteq \mathscr{G}(\mathscr{A},\alpha)$ and the desired result follows.

The following example exhibits the superiority of the results given in Theorems 1 and 2.

Example 2. Consider the tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ in Example 1.

Set $\alpha = (14, 8.5)^T$. For this tensor, the bounds via different estimations given in the literature are shown in Table 1.

It is easy to see that the results given in Theorems 1 and 2 are sharper than some existing results.

We observe that the suitable parameter α has a great influence on the numerical effects (Table 2).

Example 3. All testing partially symmetric tensors $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ are generated with $m = n$ as $a_{ijij} = 4i + j$ and other elements are generated randomly in $[-0.5, 0.5]$.

The choice of parameter α is derived as follows: $\alpha_i = \sum_{j\in[n]} a_{ijij}/n$. For the tensors with different dimensions, the values presented in the table are the average values of 10 examples (Table 3).

3. Applications

In this section, based on the inclusion intervals $\mathscr{G}(\mathscr{A},\alpha)$ and $\mathscr{K}(\mathscr{A},\alpha)$ in Theorems 1 and 2, we propose some sufficient conditions for the positive definiteness and make bound estimations on the M-spectral radius of nonnegative fourth-order partially symmetric tensors.

3.1. Positive Definiteness of Fourth-Order Partially Symmetric Tensors

Theorem 3. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathscr{I}_M$ be an M-identity tensor. For $i \in [m]$, if there exists positive real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T$ such that*

$$\alpha_i > R_i(\mathscr{A},\alpha_i), \tag{35}$$

then $\mathscr{A}$ is positive definite and $f_{\mathscr{A}}(x,y)$ defined in (3) is positive definite.

Proof. Suppose on the contrary that $\lambda \leq 0$. From Theorem 1, there exists $i_0 \in [m]$ such that $\lambda \in \mathscr{G}_{i_0}(\mathscr{A},\alpha)$, i.e.,

$$\left|\lambda - \alpha_{i_0}\right| \leq R_{i_0}\left(\mathscr{A},\alpha_{i_0}\right). \tag{36}$$

On the other hand, by $\alpha_{i_0} > 0$ and $\lambda \leq 0$, we have

$$\alpha_{i_0} \leq \left|\lambda - \alpha_{i_0}\right| \leq R_{i_0}\left(\mathscr{A},\alpha_{i_0}\right), \tag{37}$$

which contradicts (35). Hence, $\lambda > 0$. Since $\mathscr{A}$ is partially symmetric and all M-eigenvalues are positive, $\mathscr{A}$ is positive definite and $f_{\mathscr{A}}(x,y)$ defined in (3) is positive definite. □

Theorem 4. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathscr{I}_M$ be an M-identity tensor. For $i \in [m]$, if there exist a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T$ and $k \neq v$ such that*

$$\left(\alpha_i - \left(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i)\right)\right)\alpha_v > R_i^v(\mathscr{A},\alpha_i)R_v(\mathscr{A},\alpha_v), \tag{38}$$

then $\mathscr{A}$ is positive definite and $f_{\mathscr{A}}(x,y)$ defined in (3) is positive definite.

Table 1

References	Inclusion interval
Theorem 2.1 of [19]	$\Gamma(\mathscr{A}) = [-30, 30]$
Theorem 2.2 of [19]	$\mathscr{L}(\mathscr{A}) = [-29.2971, 29.2971]$
Theorem 2.4 of [19]	$\mathscr{M}(\mathscr{A}) = [-28.3523, 28.3523]$
Theorem 2.6 of [19]	$\mathscr{N}(\mathscr{A}) = [-29.2971, 29.2971]$
Theorem 1 of [20]	$\Gamma(\mathscr{A}) = [-29, 29]$
Theorem 2 of [20]	$\Theta(\mathscr{A}) = [-28.4081, 28.4081]$
Theorem 1	$\mathscr{G}(\mathscr{A}, (14, 8.5)) = [0, 28]$
Theorem 2	$\mathscr{K}(\mathscr{A}, (14, 8.5)) = [0.7154, 26.5539]$

Table 2

α	$[20, 10]^T$	$[14, 8.5]^T$	$[7, 6]^T$
Theorem 1	$[5, 34]$	$[0, 28]$	$[-9, 23]$
Theorem 2	$[5.1185, 32.4455]$	$[0.7154, 26.5539]$	$[-6.1521, 20]$

For the medium-sized tensors, we show the validity of the estimations given by our theorems.

Proof. Suppose on the contrary that $\lambda \le 0$. From Theorem 2, there exist $i_0 \in [m]$ such that $\lambda \in \mathscr{K}_{i_0,p}(\mathscr{A}, \alpha)$, i.e.,

$$\left(\left|\lambda - \alpha_{i_0}\right| - \left(R_{i_0}\left(\mathscr{A}, \alpha_{i_0}\right) - R_{i_0}^{p}\left(\mathscr{A}, \alpha_{i_0}\right)\right)\right)\left|\lambda - \alpha_p\right| \le R_{i_0}^{p}\left(\mathscr{A}, \alpha_{i_0}\right) R_p\left(\mathscr{A}, \alpha_p\right), \forall p \ne i_0. \tag{39}$$

Further, it follows from $\alpha_i > 0$ and $\lambda \le 0$ that

$$\begin{aligned}
&\left(\alpha_{i_0} - \left(R_{i_0}\left(\mathscr{A}, \alpha_{i_0}\right) - R_{i_0}^{p}\left(\mathscr{A}, \alpha_{i_0}\right)\right)\right)\alpha_p \\
&\le \left(\left|\lambda - \alpha_{i_0}\right| - \left(R_{i_0}\left(\mathscr{A}, \alpha_{i_0}\right) - R_{i_0}^{p}\left(\mathscr{A}, \alpha_{i_0}\right)\right)\right)\left|\lambda - \alpha_p\right| \\
&\le R_{i_0}^{p}\left(\mathscr{A}, \alpha_{i_0}\right) R_p\left(\mathscr{A}, \alpha_p\right),
\end{aligned} \tag{40}$$

which contradicts (38). Hence, $\lambda > 0$. Since $\mathscr{A}$ is partially symmetric and all M-eigenvalues are positive, $\mathscr{A}$ is positive definite and $f_{\mathscr{A}}(x, y)$ defined in (3) is positive definite.

The following example show Theorems 3 and 4 can judge the positive definiteness of fourth-order partially symmetric tensors. □

Example 4. Consider the partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 10, a_{1122} = a_{1221} = -0.5, a_{1212} = 4; \\ a_{2222} = 3, a_{2112} = a_{2211} = -0.5, a_{2121} = 5; \\ a_{ijkl} = 0, \text{otherwise}. \end{cases} \tag{41}$$

From the calculation method provided in Theorem 7 of [7], we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are $(\overline{\lambda}, \overline{x}, \overline{y}) = (3, (0, 1), (0, 1))$. Hence, $\mathscr{A}$ is positive definite.

Set $\alpha = (8, 4)^T$. According to Theorem 3, we have

$$\alpha_1 = 8 > R_1(\mathscr{A}, \alpha_1) = 7, \alpha_2 = 4 > R_2(\mathscr{A}, \alpha_2) = 3. \tag{42}$$

Hence, $\mathscr{A}$ satisfies all conditions of Theorem 3, which implies that $\mathscr{A}$ is positive definite.

According to Theorem 4, it holds

$$\begin{aligned}
&\left(\alpha_1 - \left(R_1(\mathscr{A}, \alpha_1) - R_1^2(\mathscr{A}, \alpha_1)\right)\right)\alpha_2 = 8 > R_1^2(\mathscr{A}, \alpha_1) R_2(\mathscr{A}, \alpha_2) = 4, \\
&\left(\alpha_2 - \left(R_2(\mathscr{A}, \alpha_2) - R_2^1(\mathscr{A}, \alpha_2)\right)\right)\alpha_1 = 16 > R_2^1(\mathscr{A}, \alpha_2) R_1(\mathscr{A}, \alpha_1) = 7.
\end{aligned} \tag{43}$$

Hence, $\mathscr{A}$ satisfies all conditions of Theorem 4, which implies that $\mathscr{A}$ is positive definite.

The following example reveals that Theorem 4 can judge the positive definiteness of partially symmetric tensors more accurately than Theorem 3.

Example 5. Consider the partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 10, a_{1212} = 8, a_{1122} = a_{1221} = 0.5; \\ a_{1222} = -1.5, a_{1112} = a_{1211} = -0.1, a_{1121} = 1.5; \\ a_{2222} = 3, a_{2121} = 5, a_{2112} = a_{2211} = 0.5; \\ a_{2212} = -1.5, a_{2221} = a_{2122} = -0.1, a_{2111} = 1.5. \end{cases} \tag{44}$$

By Theorem 7 of [7], we observe that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are $(\overline{\lambda}, \overline{x}, \overline{y}) = (2.5774, (0.2724, 0.9622), (-0.0452, 0.9990))$. Hence, $\mathscr{A}$ is positive definite. For any positive real number α_2, we have

$$R_2(\mathscr{A}, \alpha_2) = \left|3 - \alpha_2\right| + \left|5 - \alpha_2\right| + 4.2 > \alpha_2, \tag{45}$$

which implies that the condition of Theorem 3 is not satisfied. Thus, Theorem 3 is not suitable to check the positive definiteness of $\mathscr{A}$. However, taking $\alpha = (10, 5)^T$, from Theorem 4, we have

$$\begin{aligned}
\left[\alpha_1 - \left(R_1(\mathscr{A}, \alpha_1) - R_1^2(\mathscr{A}, \alpha_1)\right)\right]\alpha_2 &= 39 > R_1^2(\mathscr{A}, \alpha_1) R_2(\mathscr{A}, \alpha_2) \\
&= 24.8, \\
\left[\alpha_2 - \left(R_2(\mathscr{A}, \alpha_2) - R_2^1(\mathscr{A}, \alpha_2)\right)\right]\alpha_1 &= 28 > R_2^1(\mathscr{A}, \alpha_2) R_1(\mathscr{A}, \alpha_1) \\
&= 24.8,
\end{aligned} \tag{46}$$

which can show the positive definiteness of $\mathscr{A}$.

3.2. Bound Estimations on the M-Spectral Radius. Based on Theorems 1 and 2, we present sharp bound estimations on M-spectral radius of fourth-order partially symmetric nonnegative tensors, which improves the corresponding results in [19, 20]. For M-eigenvalues and associated left and right M-eigenvectors of fourth-order partially symmetric tensors, Qi and Luo [7] provided several related results.

Lemma 2 (Theorem 1 of [7]). *M-eigenvalues always exist. If x and y are left and right M-eigenvectors of $\mathscr{A}$, associated with an M-eigenvalue λ, then $\lambda = \mathscr{A}xyxy$.*

Lemma 3. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric nonnegative tensor. The M-spectral radius of $\mathscr{A}$ is*

Table 3

References	$n = 20$ Inclusion interval	$n = 30$ Inclusion interval	$n = 40$ Inclusion interval
Theorem 2.1 of [1]	[−3392.3, 3392.3]	[−9420.1, 9420.1]	[−19808.2, 19808.2]
Theorem 2.2 of [1]	[−3313.1, 3313.1]	[−9200.3, 9200.3]	[−19345.4, 19345.4]
Theorem 2.4 of [1]	[−3205.7, 3205.7]	[−8901.9, 8901.9]	[−18718.8, 18718.8]
Theorem 2.6 of [1]	[−3391.9, 3391.9]	[−9420.1, 9420.1]	[−19806.2, 19806.2]
Theorem 1 of [14]	[−3279.2, 3279.2]	[−9106.1, 9106.1]	[−19147.5, 19147.5]
Theorem 2 of [14]	[−3211.4, 3211.4]	[−8917.7, 8917.7]	[−18752.5, 18752.5]
Theorem 1	[−1704.2, 1814.4]	[−5649.1, 5793.2]	[−13226.6, 13419.2]
Theorem 2	[−1192.9, 1717.2]	[−3954.4, 5482.8]	[−9258.1, 12700.9]

exactly its largest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.

Proof. Assume that λ^* is the largest M-eigenvalue of $\mathscr{A}$. It is clear that $\lambda^* \leq \rho_M(\mathscr{A})$. In the following, we show $\lambda^* \geq \rho_M(\mathscr{A})$. It follows from Lemma 2 that there exist left and right M-eigenvectors (x^*, y^*) of λ^* such that

$$\lambda^* = \max\left\{f_{\mathscr{A}}(x, y) = \mathscr{A}xyxy\colon x^T x = 1 \text{ and } y^T y = 1\right\}. \tag{47}$$

Obviously, $\lambda^* \geq 0$. Next, we show $(\lambda^*, |x^*|, |y^*|)$ is a M-eigenpair of $\mathscr{A}$. Since $\mathscr{A}$ is nonnegative, $|x^*|^T|x^*| = x^{*T}x^* = 1$ and $|y^*|^T|y^*| = y^{*T}y^* = 1$, we obtain

$$\begin{aligned}\lambda^* &= f_{\mathscr{A}}(x^*, y^*) = \sum_{i,j,k,l\in N} a_{ijkl}x_i^* y_j^* x_k^* y_l^* \\ &\leq \sum_{i,j,k,l\in N} a_{ijkl}|x_i^*||y_j^*||x_k^*||y_l^*| = f_{\mathscr{A}}(|x^*|, |y^*|) \leq \lambda^*,\end{aligned} \tag{48}$$

which implies $\lambda^* = f_{\mathscr{A}}(|x^*|, |y^*|)$. Consequently, $(\lambda^*, |x^*|, |y^*|)$ is a nonnegative M-eigenpair of $\mathscr{A}$. Meanwhile, let $(\overline{x}, \overline{y})$ be a corresponding M-eigenvector of $\overline{\lambda}$ with $|\overline{\lambda}| = \rho_M(\mathscr{A})$. Since $\mathscr{A}$ is nonnegative and λ^* is largest value of $f_{\mathscr{A}}(x, y)$, we have

$$\begin{aligned}\rho_M(\mathscr{A}) &= \left|f_{\mathscr{A}}(\overline{x}, \overline{y})\right| = \left|\sum_{i,k\in[m],j,l\in[n]} a_{ijkl}\overline{x}_i\overline{y}_j\overline{x}_k\overline{y}_l\right| \\ &\leq \sum_{i,k\in[m],j,l\in[n]} a_{ijkl}|\overline{x}_i||\overline{y}_j||\overline{x}_k||\overline{y}_l| \leq \lambda^*,\end{aligned} \tag{49}$$

which shows

$$\rho_M(\mathscr{A}) \leq \lambda^*. \tag{50}$$

Thus, $\rho_M(\mathscr{A}) = \lambda^*$. □

Lemma 4. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. If $\mathscr{A}$ is nonnegative, then*

$$\rho_M(\mathscr{A}) \geq \max\left\{\max_{i\in[m],j\in[n]} a_{ijij}, \frac{\sum_{i\in[m]} R_i(\mathscr{A})}{mn}\right\}. \tag{51}$$

Proof. Without loss of generality, we assume that $\rho_M(\mathscr{A}) = \lambda^*$ is the largest M-eigenvalue of $\mathscr{A}$ by Lemma 3. It follows from Lemma 2 that

$$\rho_M(\mathscr{A}) = \max_{x,y}\left\{f_{\mathscr{A}}(x, y) = \mathscr{A}xyxy\colon x^T x = 1 \text{ and } y^T y = 1\right\}. \tag{52}$$

Let $a_{i^*j^*i^*j^*} = \max_{i\in[m],j\in[n]}\left\{a_{ijij}\right\}$. Setting a feasible solution of (52)

$$(x^*, y^*) = \begin{cases} x_{i^*} = 1, y_{j^*} = 1, & \text{if } i = i^*, j = j^*; \\ x_i = 0, y_j = 0, & \text{otherwise,}\end{cases} \tag{53}$$

we have

$$\begin{aligned}\rho_M(\mathscr{A}) &= \max_{x,y} f_{\mathscr{A}}(x, y) \geq f_{\mathscr{A}}(x^*, y^*) = a_{i^*j^*i^*j^*} \\ &= \max_{i\in[m],j\in[n]}\left\{a_{ijij}\right\},\end{aligned} \tag{54}$$

which implies $\rho_M(\mathscr{A}) \geq \max_{i\in[m],j\in[n]}\left\{a_{ijij}\right\}$.

Meanwhile, taking a feasible solution $(\overline{x}, \overline{y}) = (1/\sqrt{m}, \ldots, 1/\sqrt{m}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$, from (52), we obtain

$$\rho_M(\mathscr{A}) \geq f_{\mathscr{A}}(\overline{x}, \overline{y}) = \sum_{i,k\in[m]}\sum_{j,l\in[n]} \frac{a_{ijkl}}{mn} = \frac{\sum_{i\in[m]} R_i(\mathscr{A})}{mn}. \tag{55}$$

From (52) and (55), the conclusion follows. □

Theorem 5. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric nonnegative tensor and $\mathscr{I}_M$ be an M-identity tensor. For real vector $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ with $\alpha_i \leq, \max_{i\in[m],j\in[n]}\left\{a_{ijij}\right\}$, then*

$$\rho_M(\mathscr{A}) \leq \max_{i\in[m]}\left\{\alpha_i + R_i(\mathscr{A}, \alpha_i)\right\}. \tag{56}$$

Proof. Without loss of generality, we assume that $\rho(\mathscr{A}) = \lambda^*$ is the largest M-eigenvalue of $\mathscr{A}$ by Lemma 3. It follows from Theorem 1 that there exists $t \in N$ such that

$$\left|\rho_M(\mathscr{A}) - \alpha_t\right| \leq R_t(\mathscr{A}, \alpha_t). \tag{57}$$

Since $\mathscr{A}$ is nonnegative and $\alpha_i \leq \max_{i\in[m],j\in[n]}\left\{a_{ijij}\right\}$, from Lemma 4 and (57), we deduce

$$\alpha_t \leq \max_{i\in[m],j\in[n]}\{a_{ijij}\} \leq \rho_M(\mathscr{A}),$$
$$\rho_M(\mathscr{A}) \leq \alpha_t + R_t(\mathscr{A},\alpha_t). \tag{58}$$

Furthermore,

$$\rho_M(\mathscr{A}) \leq \max_{i\in[m]}\{\alpha_i + R_i(\mathscr{A},\alpha_i)\}. \tag{59}$$

□

Theorem 6. *Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric nonnegative tensor and $\mathscr{I}_M$ be an M-identity tensor. For real vector $\alpha = (\alpha_1,\ldots,\alpha_m)^T \in \mathbb{R}^m$ with $\alpha_i \leq \max_{i\in[m],j\in[n]}\{a_{ijij}\}$, then*

$$\rho_M(\mathscr{A}) \leq \max_{i\in[m]}\min_{v\neq i,v\in[m]}\frac{1}{2}\left(\alpha_i + \alpha_v + \left[\left(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i)\right)\right] + \Delta_{i,v}^{1/2}(\mathscr{A})\right), \tag{60}$$

where $\Delta_{i,v}(\mathscr{A}) = (\alpha_i - \alpha_v + [(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i))])^2 + 4(R_i^v(\mathscr{A},\alpha_i)R_v(\mathscr{A},\alpha_v))$.

Proof. Without loss of generality, we assume that $\rho_M(\mathscr{A}) = \lambda^*$ is the largest M-eigenvalue of $\mathscr{A}$ by Lemma 3. It follows from Theorem 2 that there exists $t \in N$ such that

$$\left(|\lambda - \alpha_i| - \left(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i)\right)\right)|\lambda - \alpha_v| \leq R_i^v(\mathscr{A},\alpha_i)R_v(\mathscr{A},\alpha_v), \quad \forall v \neq t, \tag{61}$$

Noting that $\mathscr{A}$ is nonnegative and $\alpha_i \leq \max_{i\in[m],j\in[n]}\{a_{ijij}\}$, from Lemma 4 and (61), we have

$$\alpha_i \leq \max_{i\in[m],j\in[n]}\{a_{ijij}\} \leq \rho_M(\mathscr{A}), \tag{62}$$

$$\left(\rho_M(\mathscr{A}) - \alpha_t - \left(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i)\right)\right)\left(\rho_M(\mathscr{A}) - \alpha_v\right) \leq R_i^v(\mathscr{A},\alpha_i)R_v(\mathscr{A},\alpha_v), \quad \forall v \neq t. \tag{63}$$

Solving (63) for $\rho_M(\mathscr{A})$ gives

$$\rho_M(\mathscr{A}) \leq \frac{1}{2}\left(\alpha_t + \alpha_v + \left[\left(R_t(\mathscr{A},\alpha_t) - R_t^v(\mathscr{A},\alpha_t)\right)\right] + \Delta_{t,v}^{1/2}(\mathscr{A})\right), \quad \forall v \neq t, \tag{64}$$

where $\Delta_{t,v}(\mathscr{A}) = (\alpha_t - \alpha_v + [(R_t(\mathscr{A},\alpha_t) - R_t^v(\mathscr{A},\alpha_t))])^2 + 4(R_t^v(\mathscr{A},\alpha_t)R_v(\mathscr{A},\alpha_v))$. Since $v \in [m]$ is chosen arbitrarily, it holds

$$\rho_M(\mathscr{A}) \leq \min_{v\neq t,v\in[m]}\frac{1}{2}\left(\alpha_t + \alpha_v + \left[\left(R_t(\mathscr{A},\alpha_t) - R_t^v(\mathscr{A},\alpha_t)\right)\right] + \Delta_{t,v}^{1/2}(\mathscr{A})\right). \tag{65}$$

Furthermore,

$$\rho_M(\mathscr{A}) \leq \max_{i\in[m]}\min_{v\neq i,v\in[m]}\frac{1}{2}\left(\alpha_i + \alpha_v + \left[\left(R_i(\mathscr{A},\alpha_i) - R_i^v(\mathscr{A},\alpha_i)\right)\right] + \Delta_{i,v}^{1/2}(\mathscr{A})\right). \tag{66}$$

Table 4

References	Interval
Theorem 3.1 of [19]	$\rho_M(\mathscr{A}) \leq 24$
Theorem 3.3 of [19]	$\rho_M(\mathscr{A}) \leq 24$
Theorem 3.5 of [19]	$\rho_M(\mathscr{A}) \leq 24$
Theorem 1 of [20]	$\rho_M(\mathscr{A}) \leq 26$
Theorem 2 of [20]	$\rho_M(\mathscr{A}) \leq 24$
Lemma 4 and Theorem 5	$11.75 \leq \rho_M(\mathscr{A}) \leq 24$
Lemma 4 and Theorem 6	$11.75 \leq \rho_M(\mathscr{A}) \leq 23.6941$

In the following, we use Example 1 of [20] to show the superiority of our results. □

Example 6. Consider the partially symmetric tensor $\mathscr{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 2, a_{1211} = a_{1112} = 3, a_{1121} = 6, a_{1212} = 2; \\ a_{1222} = 10, a_{2111} = 6, a_{2212} = 10, a_{2222} = 5. \\ a_{ijkl} = 0, \text{otherwise}. \end{cases} \tag{67}$$

In fact, $\sigma_M(\mathscr{A}) = \{-7.6841, 13.8616, -4.2541, 6.6751\}$. From Lemma 4, we compute $11.75 \leq \rho_M(\mathscr{A})$. Set $\alpha = (2,5)^T$. For this tensor, the bounds via different estimations given in the literature are shown in Table 4.

It is easy to see that the result given in Theorem 6 is sharper than some existing results.

4. Conclusions

In this paper, we introduced M-identity tensor to establish sharp M-eigenvalue inclusion intervals. Further, we proposed some sufficient conditions for the positive definiteness of four-order partially symmetric tensors. The given experiments show the validity of the obtained results. It is worth noting that suitable parameter α has a great influence on the numerical effects and positive definiteness. Therefore, how to select the suitable parameter α is our further research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this manuscript.

Acknowledgments

This work was supported by the Natural Science Foundation of China (11671228 and 11801430).

References

[1] D. Han, H. H. Dai, and L. Qi, "Conditions for strong ellipticity of anisotropic elastic materials," *Journal of Elasticity*, vol. 97, no. 1, pp. 1–13, 2009.

[2] J. R. Walton and J. P. Wilber, "Sufficient conditions for strong ellipticity for a class of anisotropic materials," *International Journal of Non-linear Mechanics*, vol. 38, no. 4, pp. 441–455, 2003.

[3] B. Dacorogna, "Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension," *Discrete and Continuous Dynamical Systems-Series B*, vol. 1, no. 2, pp. 257–263, 2001.

[4] L. Gao and D. Wang, "Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching," *Nonlinear Analysis: Hybrid Systems*, vol. 20, pp. 55–71, 2016.

[5] L. Gao, Z. Cao, and G. Wang, "Almost sure stability of discrete-time nonlinear Markovian jump delayed systems with impulsive signals," *Nonlinear Analysis: Hybrid Systems*, vol. 34, pp. 248–263, 2019.

[6] L. Qi, H.-H. Dai, and D. Han, "Conditions for strong ellipticity and *M*-eigenvalues," *Frontiers of Mathematics in China*, vol. 4, no. 2, pp. 349–364, 2009.

[7] L. Qi and Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2017.

[8] H. Chen, Y. Chen, G. Li, and L. Qi, "A semidefinite program approach for computing the maximum eigenvalue of a class of structured tensors and its applications in hypergraphs and copositivity test," *Numerical Linear Algebra with Applications*, vol. 25, no. 1, p. e2125, 2018.

[9] W. Ding, J. Liu, L. Qi, and H. Yan, "Elasticity *M*-tensors and the strong ellipticity condition," http://arxiv.org/abs/1705.09911.

[10] Z.-H. Huang and L. Qi, "Positive definiteness of paired symmetric tensors and elasticity tensors," *Journal of Computational and Applied Mathematics*, vol. 338, pp. 22–43, 2018.

[11] X. Wang, H. Chen, and Y. Wang, "Solution structures of tensor complementarity problem," *Frontiers of Mathematics in China*, vol. 13, no. 4, pp. 935–945, 2018.

[12] Y. Wang and L. Qi, "On the successive supersymmetric rank-1 decomposition of higher-order supersymmetric tensors," *Numerical Linear Algebra with Applications*, vol. 14, no. 6, pp. 503–519, 2007.

[13] Y. Wang, L. Caccetta, and G. Zhou, "Convergence analysis of a block improvement method for polynomial optimization over unit spheres," *Numerical Linear Algebra with Applications*, vol. 22, no. 6, pp. 1059–1076, 2015.

[14] Y. Wang, X. Sun, and F. Meng, "On the conditional and partial trade credit policy with capital constraints: a Stackelberg model," *Applied Mathematical Modelling*, vol. 40, no. 1, pp. 1–18, 2016.

[15] K. Zhang and Y. Wang, "An *H*-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms," *Journal of Computational and Applied Mathematics*, vol. 305, pp. 1–10, 2016.

[16] G. Zhou, G. Wang, L. Qi, and M. Alqahtani, "A fast algorithm for the spectral radii of weakly reducible nonnegative tensors," *Numerical Linear Algebra with Applications*, vol. 25, no. 2, p. e2134, 2018.

[17] C. Hillar and L. H. Lim, "Most tensor problems are NP hard," http://arxiv.org/abs/0911.1393.

[18] C. Ling, J. Nie, and L. Qi, "Bi-quadratic optimization over unit spheres and semidefinite programming relaxations," *SIAM Journal on Matrix Analysis on Optimization*, vol. 20, no. 3, pp. 1286–1310, 2009.

[19] H. Che, H. Chen, and Y. Wang, "On the *M*-eigenvalue estimation of fourth-order partially symmetric tensors," *Journal of Industrial & Management Optimization*, vol. 16, no. 1, pp. 309–324, 2020.

[20] S. Li, C. Li, and Y. Li, "*M*-eigenvalue inclusion intervals for a fourth-order partially symmetric tensor," *Journal of Computational and Applied Mathematics*, vol. 356, pp. 391–401, 2019.

[21] J. He, C. Li, and Y. Wei, "*M*-eigenvalue intervals and checkable sufficient conditions for the strong ellipticity," *Applied Mathematics Letters*, vol. 102, pp. 106–111, 2020.

[22] C. Li, Y. Li, and X. Kong, "New eigenvalue inclusion sets for tensors," *Numerical Linear Algebra with Applications*, vol. 21, no. 1, pp. 39–50, 2014.

[23] C. Li and Y. Li, "An eigenvalue localization set for tensors with applications to determine the positive (semi-) definiteness of tensors," *Linear and Multilinear Algebra*, vol. 64, no. 4, pp. 587–601, 2016.

[24] G. Wang, G. Zhou, and L. Caccetta, "Sharp Brauer-type eigenvalue inclusion theorems for tensors," *Pacific Journal of Optimization*, vol. 14, no. 2, pp. 227–244, 2018.

[25] G. Wang, Y. Wang, and L. Liu, "Bound estimations on the eigenvalues for fan product of *M*-tensors," *Taiwanese Journal of Mathematics*, vol. 23, no. 3, pp. 751–766, 2019.

[26] G. Wang, Y. Wang, and Y. Wang, "Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors," *Linear and Multilinear Algebra*, pp. 1–18, 2019.

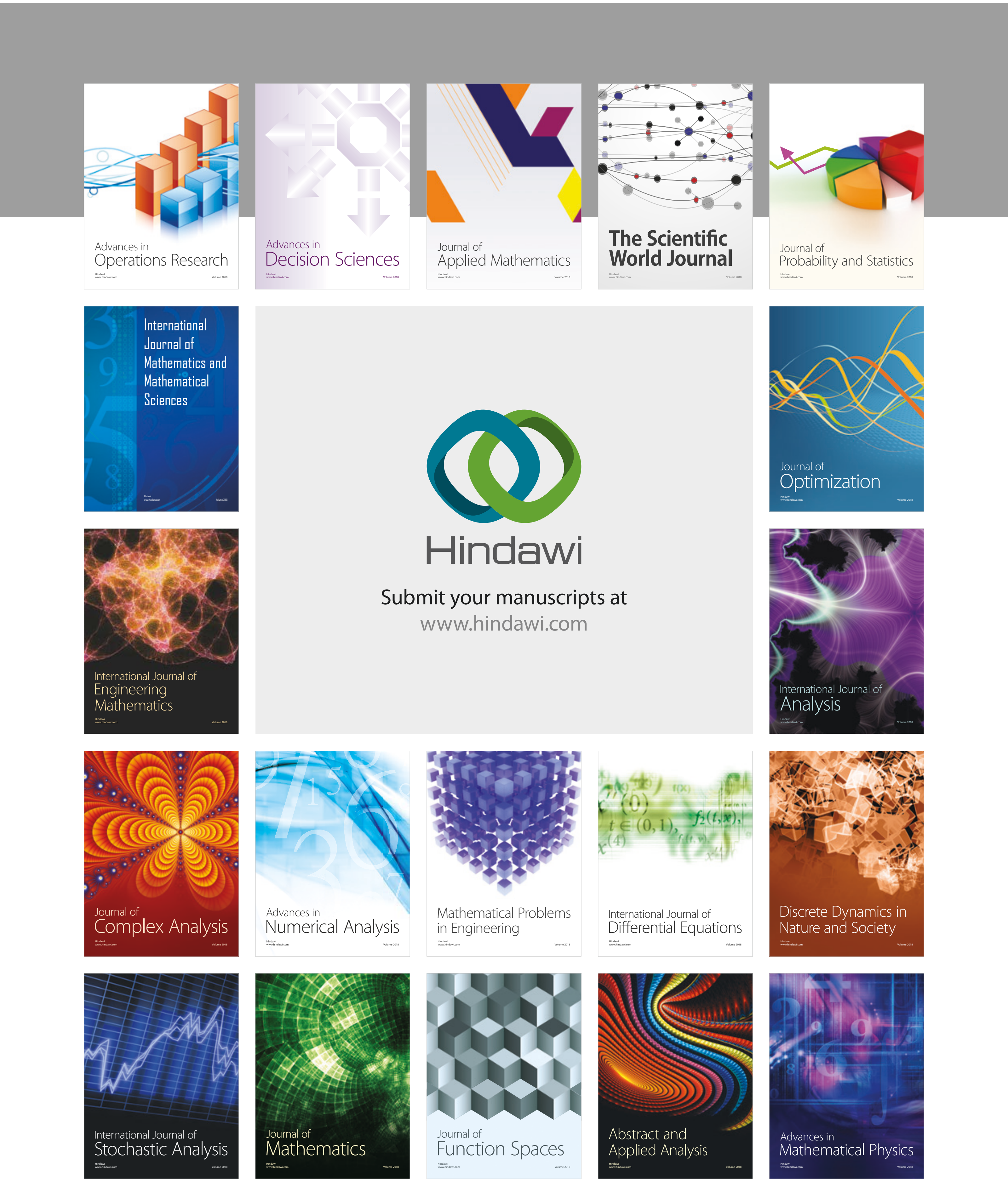
[27] C. Li, Y. Liu, and Y. Li, "Note on *Z*-eigenvalue inclusion theorems for tensors," *Journal of Industrial & Management Optimization*, vol. 13, no. 5, 2017.

[28] C. Sang, "A new Brauer-type *Z*-eigenvalue inclusion set for tensors," *Numerical Algorithms*, vol. 32, pp. 781–794, 2019.

[29] G. Wang, G. Zhou, and L. Caccetta, "*Z*-eigenvalue inclusion theorems for tensors," *Discrete and Continuous Dynamical Systems-Series B*, vol. 22, pp. 187–198, 2017.

[30] J. Zhao, "A new *Z*-eigenvalue localization set for tensors," *Journal of Inequalities and Applications*, vol. 2017, Article ID 85, 2017.

[31] Y. Zhang and Y. Zhang ang, G. Wang, Exclusion sets in the *S*-type eigenvalue localization sets for tensors," *Open Mathematics*, vol. 17, pp. 1136–1146, 2019.



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