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Research Article

Network Similarity Measure and Ediz Eccentric Connectivity Index

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Network similarity measures have proven essential in the field of network analysis. Also, topological indices have been used to quantify the topology of networks and have been well studied. In this paper, we employ a new topological index which we call the Ediz eccentric connectivity index. We use this quantity to define network similarity measures as well. First, we determine the extremal value of the Ediz eccentric connectivity index on some network classes. Second, we compare the network similarity measure based on the Ediz eccentric connectivity index with other well-known topological indices such as Wiener index, graph energy, Randić index, the largest eigenvalue, the largest Laplacian eigenvalue, and connectivity eccentric index. Numerical results underpin the usefulness of the chosen measures. They show that our new measure outperforms all others, except the one based on Wiener index. This means that the measure based on Wiener index is still the best, but the new one has certain advantage to some extent.

1. Introduction

Similarity measures for networks have been studied extensively. Up to now, exploring methods to measure the similarity/distance between networks has been a current research item [1]. However, not every measure is generally applicable as these methods rely on different concepts, e.g., graph ismomorphism. Also, different measure methods may be used for different kind of networks.

In this paper, we employ a distance measure [1, 2]:

$$d(x, y) = 1 - e^{-(x- y/\sigma)^2}, \tag{1}$$

for $x, y \in \mathbb{R}$, which leads to a graph distance measure based on topological indices [1, 3]:

$$d_I(G, H) \coloneqq d(I(G), I(H)) = 1 - e^{-(I(G)- I(H)/\sigma)^2}, \tag{2}$$

where G and H are two networks and $I(G)$ and $I(H)$ are topological indices applied to both G and H. In this paper, we set $\sigma = 1$. Dehmer et al. [1, 3–6] explored the interrelations between the graph similarity measures based on some well-known topological indices such as Wiener index, Randić index, graph entropy, and eigenvalue-based quantities. Triggered by this, we continue to investigate the network similarity measure d_I by employing a new topological graph measures, namely, the Ediz eccentric connectivity index [7]. Furthermore, we compare the similarity measures based on different topological indices.

2. Topological Indices

A topological index [8] is a network/graph invariant, which maps networks to the reals. Topological indices for graphs have been well studied. They have been used for examining quantitative structure-activity relationships (QSARs) [9]. Also, they have been applied in ecology [10], biology [11], and network physics [12, 13]. In the following, we list several topological indices which will be used.

Wiener index $W(G)$ of a graph $G = \{V, E\}$ is defined as $W(G) = \sum_{x,y \in V} D(x, y)$, where $D(x, y)$ is the distance between two vertices x and y of G, which is introduced by Wiener [14]. Some properties of Wiener index were discussed, see [15, 16] and the references cited therein.

Randić index $R(G)$ of a graph G is defined as $R(G) = \sum_{x,y\in E} 1/\sqrt{d(x)d(y)}$, where $d(x)$ is the degree of the vertex $v \in V$, which was proposed by Randić [17]. Some properties of Randić index were discussed, see [18, 19] and the references cited therein.

Graph energy $\mathscr{E}(G)$ of a graph G on n vertices is defined as $\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i $(1 \le i \le n)$ is the eigenvalues of the adjacency matrix of G, which is introduced by Gutman [20]. Some properties of graph energy were discussed, see [21] and the references cited therein.

Connective eccentricity index $\xi^{ce}(G)$ of a graph G [22] is defined as $\xi^{ce}(G) = \sum_{v\in V(G)} d(v)/\varepsilon(v)$, where $d(x)$ and $\varepsilon(v)$ are the degree and the eccentricity of the vertex $v \in V$, respectively. Some properties of the connective eccentricity index were discussed, see [23] and the references cited therein.

The Ediz eccentric connectivity index $\xi_E^{ec}(G)$ of a graph [7] G is defined by $\xi_E^{ec}(G) = \sum_{v\in V(G)} S(v)/\varepsilon(v)$, where $S(v)$ is the sum of the degrees of the vertices adjacent to the vertex v and $\varepsilon(v)$ is the eccentricity of the vertex v. Some mathematical properties of Ediz eccentric connectivity index are discussed in [7]. The physico-chemical properties of certain molecular structures were investigated by computing the Ediz eccentric connectivity index [24]. In [25], the Ediz eccentric connectivity index for Circumcoronene Series of Benzenoid by Ring-Cut method.

3. Main Results

3.1. Extremal Values of Ediz Eccentric Connectivity Index. Let G be an unweighted, undirected graph with vertex set $V(G)$. For vertices $u, v \in V(G)$, the distance $d(u,v)$ is defined as the length of the shortest path between u and v in G. The eccentricity $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. Obviously, $\varepsilon(v) \ge 1$, for any vertex $v \in V(G)$.

Theorem 1. *Let G be a graph on n vertices with the decreasing degree sequence $\{d_1, d_2, \ldots, d_n\}$. Then,*

$$\xi_E^{ce}(G) \le \sum_{i=1}^{n} d_i^2, \tag{3}$$

with equality if and only if G is a complete graph.

Proof. Since $\varepsilon(v) \ge 1$ for any $v \in V(G)$, then

$$\xi_E^{ce}(G) \le \sum_{v\in V(G)} \frac{S(v)}{1} = \sum_{v\in V(G)} S(v) = \sum_{v\in V(G)} \sum_{u\sim v} d(u) = \sum_{i=1}^{n} d_i^2, \tag{4}$$

where $u \sim v$ means that u is adjacent to v. The above equalities hold if and only if $\varepsilon(v) = 1$ for any $v \in V(G)$, i.e., G is a complete graph. □

Remark 1. Since the first Zagreb index [26] of a graph G is defined as $M_1(G) = \sum_{u\in V(G)} [d(u)]^2$, the result $\xi_E^{ce}(G) \le M_1(G)$ holds. The equality holds if and only if G is a complete graph.

Average eccentricity of a graph G on n vertices is defined as $\zeta(G) = 1/n \sum_{v\in V(G)} \varepsilon(v)$. A relation between Ediz eccentric connectivity index and average eccentricity is present as follows.

Theorem 2. *Let G be a graph on n vertices. Then,*

$$\xi_E^{ce}(G) \le n(n-1)^2 \zeta(G), \tag{5}$$

with equality if and only if G is a complete graph.

Proof. Since $\varepsilon(v) \ge 1$ for any vertex $v \in V(G)$, then $1/\varepsilon(v) \le \varepsilon(v)$ with equality if and only if $\varepsilon(v) = 1$. Moreover, $d_v \le n-1$. Therefore,

$$\begin{aligned} \xi_E^{ce}(G) &= \sum_{v\in V(G)} \frac{S(v)}{\varepsilon(v)} \le \sum_{v\in V(G)} \varepsilon(v) S(v) \le \sum_{v\in V(G)} (n-1)^2 \varepsilon(v) \\ &= n(n-1)^2 \zeta(G). \end{aligned} \tag{6}$$

The first equality holds if and only if $1/\varepsilon(v) = \varepsilon(v)$, i.e., G is a complete graph. The second equality holds if and only if $d(v) = n-1$ for any $v \in V(G)$, i.e., G is a complete graph. □

Theorem 3. *Let G be a graph on n vertices with independence number α. Then,*

$$\xi_E^{ce}(G) \le (n-\alpha)[(n-1)(n+\alpha-1) + (n-\alpha)\alpha], \tag{7}$$

with equality if and only if $G \cong K_{n-\alpha} \vee \overline{K_\alpha}$.

Proof. Let $\mathbb{G}$ be the set of graphs on n vertices with independence number α. Assume that $G_0 \in \mathbb{G}$ has the maximal Ediz eccentric connectivity index. Let $V(G_0)$ be the vertex set of G_0 and $S = \{v_1, v_2, \ldots, v_\alpha\}$ be the maximal independence set. Let G^* be the subgraph, induced by $V(G_0)/S$, of G_0. Since adding an edge shall increase the Ediz eccentric connectivity index, G^* is complete. Moreover, there must exit an edge joining any two vertices between S and $V(G_0)/S$ in G_0. So, $G_0 \cong K_{n-\alpha} \vee \overline{K_\alpha}$. By directed computation, we have $\xi_E^{ce}(K_{n-\alpha} \vee \overline{K_\alpha}) = (n-\alpha)[(n-1)(n+\alpha-1) + (n-\alpha)\alpha]$.

Let $\gamma(G)$ be the covering number of a graph G. It is well known that $\alpha(G) + \gamma(G) = n$. From Theorem 3, the following holds. □

Corollary 1. *Let G be a connected graph on n vertices with covering number γ. Then,*

$$\xi_E^{ce}(G) \leq \gamma[(n-1)(2n-\gamma-1)+\gamma(n-\gamma)], \tag{8}$$

with equality if and only if $G \cong K_\gamma \vee \overline{K_{n-\gamma}}$.

In [23, 27], authors investigated the eccentric distance sum and connective eccentricity index of graphs with vertex connectivity, respectively. By modifying their methods, we have the following.

Theorem 4. *Let G be a connected graph on n vertices with the vertex connectivity κ. Then, $\xi_E^{ce}(G) \leq \kappa(\kappa-1)(2n-\kappa-1) + (3/2\kappa - n1)[(n-\kappa)^2 - 2(n-\kappa-1)] + 1/2\kappa(n-1)(n-\kappa) + 1/2[(n-\kappa)^3 - 3(n-\kappa)(n-\kappa-1)]$ with equality if and only if $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.*

Proof. Let G_0 be a graph having the maximal Ediz eccentric connectivity index among all connected graphs on n vertices with vertex connectivity κ. Then, there exists a vertex cut $C \subset V(G_0)$ with $|C| = \kappa$ such that $G_0 - C = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_1, G_2, \ldots, G_t$ are $t\,(\geq 2)$ connected components of $G_0 - C$. Since adding an edge shall increase Ediz eccentric connectivity index, $t = 2$ and G_1, G_2 and $G[C]$ are complete graphs, any vertex of G_1 and G_2 is adjacent to any vertex in C. So, $G_0 \cong K_\kappa \vee (K_{n_1} \cup K_{n_2})$ and $n_1 + n_2 = n - \kappa$ for $n_i = |G_i|$ $(i = 1, 2)$. By directly computing, we have

$$\begin{aligned}\xi_E^{ce}\left(K_\kappa \vee \left(K_{n_1} \cup K_{n_2}\right)\right) &= \left(\frac{3}{2}\kappa - 1\right)\left[(n-\kappa)^2 - 2n_1n_2\right] \\ &\quad + \frac{1}{2}\left[(n-\kappa)^3 - 3(n-\kappa)n_1n_2\right].\end{aligned} \tag{9}$$

Note that $n_1n_2 = n_1(n-\kappa-n_1) \geq (n-k-1)$ with equality if and only if $n_1 = 1$, $n_2 = n-\kappa-1$, or $n_2 = 1$, $n_1 = n-\kappa-1$. Then, $\xi_E^{ce}(G) \leq \kappa(\kappa-1)(2n-\kappa-1) + (3/2\kappa t - n1)[(n-\kappa)^2 - 2(n-\kappa-1)] + 1/2\kappa(n-1)(n-\kappa) + 1/2[(n-\kappa)^3 - 3(n-\kappa)(n-\kappa-1)]$, with equality if and only if $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.

$T_{k,n}$, which is called Turán graph, is the complete k-partite graph on n vertices in which all parts are as equal in size as possible. □

Lemma 1. *(see [28]). Let G be a graph which contains no K_k $(k \geq 2)$. Then, $|E(G)| \leq |E(T_{k-1,n})|$ with equality if and only if $G \cong T_{k-1,n}$.*

The following is immediate from Lemma 1.

Lemma 2. *A graph G is connected with $|E(G)| > 1/4|V(G)|^2$, and then G contains at least one triangle.*

Theorem 5 *Let G be a triangle-free connected graph on n vertices. Then, $\xi_E^{ce}(G) \leq n^2(n-2)/4$ if $\varepsilon(u) > 1$ for any vertex $v \in V(G)$, with equality if and only if $G \cong K_{2,2}$. Otherwise, $\xi_E^{ce}(G) \leq n^2 - 1/2$, with equality if and only if $G \cong K_{1,n-1}$.*

Proof. Let G be a triangle-free graph with the maximal Ediz eccentric connectivity index. If there exists a vertex $u \in V(G)$ such that $\varepsilon(u) = 1$, then $d(u) = n-1$. Hence, $G \cong K_{1,n-1}$ since G is triangle-free. Assume that $\varepsilon(v) > 1$ for $v \in V(G)$. Then, $d(v) \leq n-2$ and

$$\begin{aligned}\xi_E^{ce}(G) &= \sum_{v\in V(G)} \frac{S(v)}{\varepsilon(v)} \leq \frac{1}{2}\sum_{v\in V(G)} S(v) \leq \frac{1}{2}\sum_{v\in V(G)} (n-2)d(v) \\ &= (n-2)|E(G)| \leq \frac{n^2(n-2)}{4}, \quad \text{by Lemma 2.}\end{aligned} \tag{10}$$

The above equalities holds if and only if $\varepsilon(v) = 2$, $d(v) = n-2$ for all $v \in V(G)$ and $|E(G)| = n^2/4$, i.e., $G \cong K_{2,2}$. By computing, $\xi_E^{ce}(K_{1,n-1}) = n^2 - 1/2$. □

Theorem 6. *Let G be a connected graph on n vertices with k pendent vertices. Then,*

$$\xi_E^{ce}(G) \leq \frac{(n-1)^2}{2} + \frac{(n-k)(n-k-1)^2}{2} + k, \tag{11}$$

with equality if and only if $G \cong K_n^k$, where K_n^k is a graph obtained from K_{n-k} by attaching k pendent edges to one vertex of K_{n-k}.

Proof. Let G_0 be the graph on n vertices with k pendent vertices and the maximal Ediz eccentric connectivity index. Assume that $\{v_1, v_2, \ldots, v_k\}$ be the set of pendent vertices in G_0. The subgraph G_0' induced by $V(G_0)/\{v_1, v_2, \ldots, v_k\}$ is a complete graph. In what follows, we would prove that $G_0 \cong K_n^k$.

Suppose that there exist at least two vertices of G_0' such that their degrees are more than $n-k-1$. Set $v_i, v_j \in V(G_0')$ and $d(v_i) \geq d(v_j) > n-k-1$. Let G_1 be the new graph obtained from G_0 by removing all pendent vertices attached to v_j and attaching them to v_i. Let V_1 (V_2) be the set of pendent vertices attached to v_i (v_j) in G_0'. Let V_3 be the set of pendent vertices not in V_1 or V_2.

Note that $\varepsilon_{G_0}(v_i) = \varepsilon_{G_1}(v_i) = 2$, $\varepsilon_{G_0}(v_j) = \varepsilon_{G_1}(v_j) = 2$, and $\varepsilon_{G_0}(v) = \varepsilon_{G_1}(v) = 2$ for $v \in V(G_0')/\{v_i\}$ and $\varepsilon_{G_0}(v) = \varepsilon_{G_1}(v) = 3$ for $v \in V_1 \cup V_2 \cup V_3$. The degree of vertex in $V(G_0)/\{v_i, v_j\}$ does not change, but the degree of v_i increases by $|V_2|$ and the one of v_j decreases by $|V_2|$. Let C be the sum of the degrees of vertices in $V(G_0')/\{v_i, v_j\}$. Note that

C is a constant under G_0 and G_1. By the definition of Ediz eccentric connectivity index, we have

$$\xi_E^{ce}(G_0) = \sum_{v\in V(G_0')/\{v_i,v_j\}} \frac{C + d_{G_0}(v_i) + d_{G_0}(v_j)}{2} + \sum_{v\in V_1} \frac{d_{G_0}(v_i)}{3} + \sum_{v\in V_2} \frac{d_{G_0}(v_j)}{3} + \sum_{v\in V_3} \frac{S_{G_0}(v)}{3} + \frac{S_{G_0}(v_i)}{2} + \frac{S_{G_0}(v_j)}{2},$$

$$\begin{aligned}\xi_E^{ce}(G_1) &= \sum_{v\in V(G_1')/\{v_i,v_j\}} \frac{C + d_{G_1}(v_i) + d_{G_1}(v_j)}{2} + \sum_{v\in V_1} \frac{d_{G_1}(v_i)}{3} + \sum_{v\in V_2} \frac{d_{G_1}(v_j)}{3} + \sum_{v\in V_3} \frac{S_{G_1}(v)}{3} + \frac{S_{G_1}(v_i)}{2} + \frac{S_{G_1}(v_j)}{2} \\ &= \sum_{v\in V(G_0')/\{v_i,v_j\}} \frac{C + d_{G_0}(v_i) + d_{G_0}(v_j)}{2} + \sum_{v\in V_1} \frac{d_{G_0} + |V_2|}{3} + \sum_{v\in V_2} \frac{d_{G_0} + |V_2|}{3} + \sum_{v\in V_3} \frac{S_{G_0}(v)}{3} + \frac{S_{G_0}(v_i)}{2} + \frac{S_{G_0}(v_j)}{2}.\end{aligned} \tag{12}$$

Then,

$$\xi_E^{ce}(G_0) - \xi_E^{ce}(G_1) = -\sum_{v\in V_1} \frac{|V_2|}{3} + \sum_{v\in V_2} \frac{d_{G_0}(v_j) - d_{G_0}(v_i) - |V_2|}{3} < 0, \tag{13}$$

and this is a contradiction. So, $G_0 \cong K_n^k$.

By computing, it follows that $\xi_E^{ce}(K_n^k) = (n-1)^2/2 + (n-k)(n-k-1)^2/2 + k$. □

Lemma 3. *Let H_1 and H_2 be two disjoint connected graphs on at least 2 vertices with $u \in V(H_1)$ and $v \in V(H_2)$. G_1 is the graph obtained from $H_1 \cup H_2$ by adding an edge uv. G_2 is the graph obtained from $H_1 \cup H_2$ by identifying u and v (to be a new vertex, say, u) and adding a pendent edge, say uv without confusion. Then, $\xi_E^{ce}(G_1) < \xi_E^{ce}(G_2)$.*

Proof. Let V_1 be the set of neighbors of u in H_1 and C_1 be the sum of degrees of vertices in V_1. Let V_2 be the set of neighbors of v in H_2 and C_2 be the sum of degrees of vertices in V_2. Note that $\varepsilon_{G_1}(x) \geq \varepsilon_{G_2}(x)$ for all $x \in (V(H_1) \cup V(H_2))/\{u,v\}$ and $S_{G_1}(x) = S_{G_2}(s)$ for $x \in (V(H_1) \cup V(H_2))/(V_1 \cup V_2 \cup \{u,v\})$. So, we have

$$\begin{aligned}\xi_E^{ce}(G_1) &= \sum_{x\in V(H_1)/(V_1\cup\{u\})} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V(H_2)/(V_2\cup\{v\})} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V_1} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V_2} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \frac{S_{G_1}(u)}{\varepsilon_{G_1}(u)} + \frac{S_{G_1}(v)}{\varepsilon_{G_1}(v)} \\ &= \sum_{x\in V(H_1)/(V_1\cup\{u\})} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V(H_2)/(V_2\cup\{v\})} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V_1} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \sum_{x\in V_2} \frac{S_{G_1}(x)}{\varepsilon_{G_1}(x)} + \frac{C_1 + d_{G_1}(v)}{\varepsilon_{G_1}(u)} + \frac{C_2 + d_{G_1}(u)}{\varepsilon_{G_1}(v)}.\end{aligned} \tag{14}$$

Case 1. $\varepsilon_{H_1}(u) \geq 1 + \varepsilon_{H_2}(v)$

In this case, $\varepsilon_{G_1}(u) = \varepsilon_{H_1}(u) = \varepsilon_{G_2}(u)$ and $\varepsilon_{G_1}(v) = 1 + \varepsilon_{H_1}(u) = \varepsilon_{G_2}(v)$. It follows that

$$\begin{aligned}\xi_E^{ce}(G_2) &= \sum_{x\in V(H_1)/(V_1\cup\{u\})} \frac{S_{G_2}(x)}{\varepsilon_{G_2}(x)} + \sum_{x\in V(H_2)/(V_2\cup\{v\})} \frac{S_{G_2}(x)}{\varepsilon_{G_2}(x)} + \sum_{x\in V_1} \frac{S_{G_1}(x) + d_{G_1}(v) - 1}{\varepsilon_{G_2}(x)} + \sum_{x\in V_2} \frac{S_{G_1}(x) + d_{G_1}(u) - 1}{\varepsilon_{G_2}(x)} + \frac{C_1 + C_2 + 1}{\varepsilon_{G_2}(u)} \\ &\quad + \frac{d_{G_1}(u) + d_{G_1}(v) - 1}{\varepsilon_{G_2}(v)}.\end{aligned} \tag{15}$$

Then,

$$\begin{aligned}\xi_E^{ce}(G_2)-\xi_E^{ce}(G_1) &\geq \sum_{x\in V(H_1)/(V_1\cup\{u\})}\frac{S_{G_2}(x)-S_{G_1}(x)}{\varepsilon_{G_1}(x)}+\sum_{x\in V(H_2)/(V_2\cup\{v\})}\frac{S_{G_2}(x)-S_{G_1}(x)}{\varepsilon_{G_1}(x)}+\sum_{x\in V_1}\frac{d_{G_1}(v)-1}{\varepsilon_{G_1}(x)}\\ &+\sum_{x\in V_2}\frac{d_{G_1}(u)-1}{\varepsilon_{G_1}(x)}+\frac{C_1+C_2+1}{\varepsilon_{G_1}(u)}-\frac{C_1+d_{G_1}(v)}{\varepsilon_{G_1}(u)}+\frac{d_{G_1}(u)+d_{G_1}(v)-1}{\varepsilon_{G_1}(v)}-\frac{C_2+d_{G_1}(u)}{\varepsilon_{G_1}(v)}\\ &>\frac{1+C_2-d_{G_1}(v)}{\varepsilon_{G_1}(u)}+\frac{d_{G_1}(v)-1-C_2}{\varepsilon_{G_1}(v)}>\frac{1+C_2-d_{G_1}(v)}{\varepsilon_{G_1}(u)}+\frac{d_{G_1}(v)-1-C_2}{\varepsilon_{G_1}(u)}=0,\end{aligned}\tag{16}$$

Case 2. $\varepsilon_{H_1}(u)=\varepsilon_{H_2}(v)$

In this case, $\varepsilon_{G_1}(u)=1+\varepsilon_{H_1}(u)$, $\varepsilon_{G_2}(u)=\varepsilon_{H_1}(u)$, $\varepsilon_{G_1}(v)=1+\varepsilon_{H_2}(v)$, and $\varepsilon_{G_2}(v)=1+\varepsilon_{H_1}(v)$. It follows that

$$\begin{aligned}\xi_E^{ce}(G_2)-\xi_E^{ce}(G_1) &>\frac{C_1+C_2+1}{\varepsilon_{G_2}(u)}-\frac{C_1+d_{G_1}(v)}{\varepsilon_{G_1}(u)}+\frac{d_{G_1}(u)+d_{G_1}(v)-1}{\varepsilon_{G_2}(v)}-\frac{C_2+d_{G_1}(u)}{\varepsilon_{G_1}(v)}\\ &=\frac{C_1+C_2+1}{\varepsilon_{H_1}(u)}-\frac{C_1+d_{G_1}(v)}{1+\varepsilon_{H_1}(u)}+\frac{d_{G_1}(u)+d_{G_1}(v)-1}{1+\varepsilon_{H_1}(u)}-\frac{C_2+d_{G_1}(u)}{1+\varepsilon_{H_1}(u)}=(1+C_1+C_2)\left(\frac{1}{\varepsilon_{H_1}(u)}-\frac{1}{1+\varepsilon_{H_1}(u)}\right)>0.\end{aligned}\tag{17}$$

Case 3. $\varepsilon_{H_1}(u)<\varepsilon_{H_2}(v)$.

In this case, $\varepsilon_{G_1}(u)=1+\varepsilon_{H_2}(v)$, $\varepsilon_{G_2}(u)=\varepsilon_{H_2}(v)$, $\varepsilon_{G_1}(v)=\varepsilon_{H_2}(v)$, and $\varepsilon_{G_2}(v)=1+\varepsilon_{H_2}(v)$. It follows that

$$\begin{aligned}\xi_E^{ce}(G_2)-\xi_E^{ce}(G_1) &>\frac{C_1+C_2+1}{\varepsilon_{G_2}(u)}-\frac{C_1+d_{G_1}(v)}{\varepsilon_{G_1}(u)}+\frac{d_{G_1}(u)+d_{G_1}(v)-1}{\varepsilon_{G_2}(v)}-\frac{C_2+d_{G_1}(u)}{\varepsilon_{G_1}(v)}=\frac{C_1+C_2+1}{\varepsilon_{H_2}(v)}-\frac{C_1+d_{G_1}(v)}{1+\varepsilon_{H_1}(v)}\\ &+\frac{d_{G_1}(u)+d_{G_1}(v)-1}{1+\varepsilon_{H_2}(v)}-\frac{C_2+d_{G_1}(u)}{\varepsilon_{H_2}(v)}=(1+C_1+C_2)\left(\frac{1}{\varepsilon_{H_2}(v)}-\frac{1}{1+\varepsilon_{H_2}(v)}\right)>0.\end{aligned}\tag{18}$$

This completes the proof. □

Theorem 7. *Let G be a connected graph on n vertices with $k\geq 1$ cut edges. Then,*

$$\xi_E^{ce}(G_2)\leq\frac{(n-1)^2}{2}+\frac{(n-k)(n-k-1)^2}{2}+k,\tag{19}$$

with equality if and only if $G\cong K_n^k$.

Proof. Suppose that G_0 is a connected graph on n vertices with $k\geq 1$ cut edges and the maximal Ediz eccentric connectivity index. By Lemma 3, all cut edges in G_0 must be pendent edges. From the proof of Theorem 6, the proof is completed. □

3.2. Numerical Result and Analysis. As we know, infinitely many topological indices exist. The number of similarity measures like the ones employed in this paper is infinite too. So, finding a suitable topological index to obtain a better measure is an interesting question. This section contributes to the results obtained by comparing the similarity measures based on topological indices by means of two different perspectives. We shall discuss numerical results to underpin our analytical findings.The first method is to comparing the similarity measures based on topological indices and graph edit distance. Graph edit distance serves as a benchmark similarity measure, but it is generally NP-hard. Here, we shall compare how close the similarity measures based on topological indices (Wiener index, graph energy, Randić, the largest adjacent eigenvalue, the largest Laplacian eigenvalue, connective eccentricity index, and Ediz eccentric connectivity index) and graph edit distance are. We choose all nonisomorphic trees whose number of vertices is from 6 to 9, respectively. To ensure the comparability of the two measures, we revise the graph edit distance for trees as

$$\operatorname{ged}(T_1, T_2) = 1 - \frac{1}{2n-6}\operatorname{GED}(T_1, T_2), \quad (20)$$

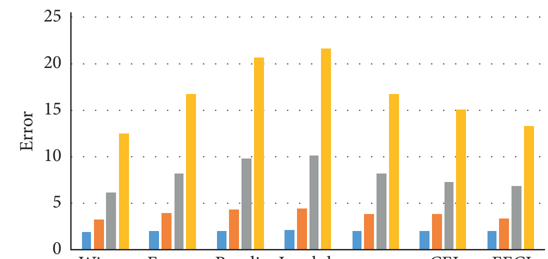
where $\operatorname{GED}(T_1, T_2)$ is the graph edit distance of trees T_1, T_2. The number $1/(2n-6)$ is the standardization coefficient which is used to standardize the value of graph edit distance into the interval $[0, 1]$. This revised graph edit distance is still a benchmark similarity measure. To compare the similarity measure with the graph edit distance, we employ the least square method as follows to compute the error between each similarity measure and revised graph edit distance:

$$\sum_{T_i, T_j} \sqrt{\left(d_I\left(T_i, T_j\right) - \operatorname{ged}\left(T_i, T_j\right)\right)^2}, \quad (21)$$

where the summation is over all pairs of the different nonisomorphic trees with the same number of vertices.

Numerical results are shown in Figure 1. It shows that the similarity measure based on Wiener index has more advantage than the one based on Ediz eccentric connectivity index, but the measure based on Ediz eccentric connectivity index has more advantage than the ones based on other topological indices (graph energy, Randić index, the largest adjacent eigenvalue, the largest Laplacian eigenvalue, and connective eccentricity index).

Distribution of the ranked values of the similarity measure based on Wiener index (W), graph energy ($\mathscr{E}$), Randić index (R), the largest adjacent eigenvalue (λ), the largest Laplacian eigenvalue (μ), connective eccentricity index (CEI), and Ediz eccentric connectivity index (EECI). The X-axis represents the values of the distance measure. The Y-axis represents the percentage rate of all graph pairs studied.



	Wiener	Energy	Randic	Lambda	mu	CEI	EECI
Six vertices	1.9123	2.0104	2.0535	2.1262	2.0117	2.0213	2.0056
Seven vertices	3.3382	3.9409	4.3055	4.4498	3.8349	3.8645	3.4187
Eight vertices	6.1808	8.2293	9.7964	10.1535	8.2016	7.3276	6.8201
Nine vertices	12.5563	16.7632	20.6999	21.6302	16.7256	15.1056	13.269

Figure 1: Errors between graph edit distance and similarity measures based on Wiener index (Wiener), graph energy (Energy), Randić index (Randic), the largest adjacent eigenvalue (lambda), the largest Laplacian eigenvalue (mu), connective eccentricity index (CEI), and Ediz eccentric connectivity index (EECI). The X-axis represents different topological indices. The Y-axis represents the distributions of error values between graph edit distance and similarity measures.

The second method is to compare the values of different similarity measures on the set of 14-vertex trees. The number of trees with 14 vertices is 3159 and the number of pairs is 4988061, see [29]. Here, we compare the measures based on the seven topological indices. From the plots shown by Figure 2, some inequalities are demonstrated as follows:

$$\begin{gathered} d_W(T_1, T_2) \geq d_{\xi_E^{ec}}(T_1, T_2), d_{\xi_E^{ec}}(T_1, T_2) \geq d_R(T_1, T_2), d_{\xi_E^{ec}}(T_1, T_2) \geq d_{\mathscr{E}}(T_1, T_2), \\ d_{\xi_E^{ec}}(T_1, T_2) \geq d_\lambda(T_1, T_2), d_{\xi_E^{ec}}(T_1, T_2) \geq d_\mu(T_1, T_2), d_{\xi_E^{ec}}(T_1, T_2) \geq d_{\xi^{ce}}(T_1, T_2), \end{gathered} \quad (22)$$

where the subscripts W, R, $\mathscr{E}$, λ, μ, ξ^{ce}, and ξ_E^{ec} are the abbreviations of Wiener index, graph energy, Randić index, the largest adjacent eigenvalue , the largest Laplacian eigenvalue, connective eccentricity index, and Ediz eccentric connectivity index, respectively. These inequalities reveal the same result as the one obtained from the first method. So, the similarity measure based on Ediz eccentric connectivity index has certain advantage to some extent.

In fact, we derive the same result by employing two other datasets. One dataset is random regular graphs on 100 vertices with degree 3, another is Watts–Strogatz small-world graphs on 100 vertices with $k = 4$ and $p = 0.2$. The results are shown in Figures 3 and 4.

Next, we shall present an upper bound for the similarity measure based on Ediz eccentric connectivity index among all connected graphs on n vertices.

Theorem 8. *Let G_1 and G_2 be two connected graphs on n vertices. Then, we have*

$$d_{\xi_E^{ec}}(G_1, G_2) \leq d_{\xi_E^{ec}}(K_n, P_n), \quad (23)$$

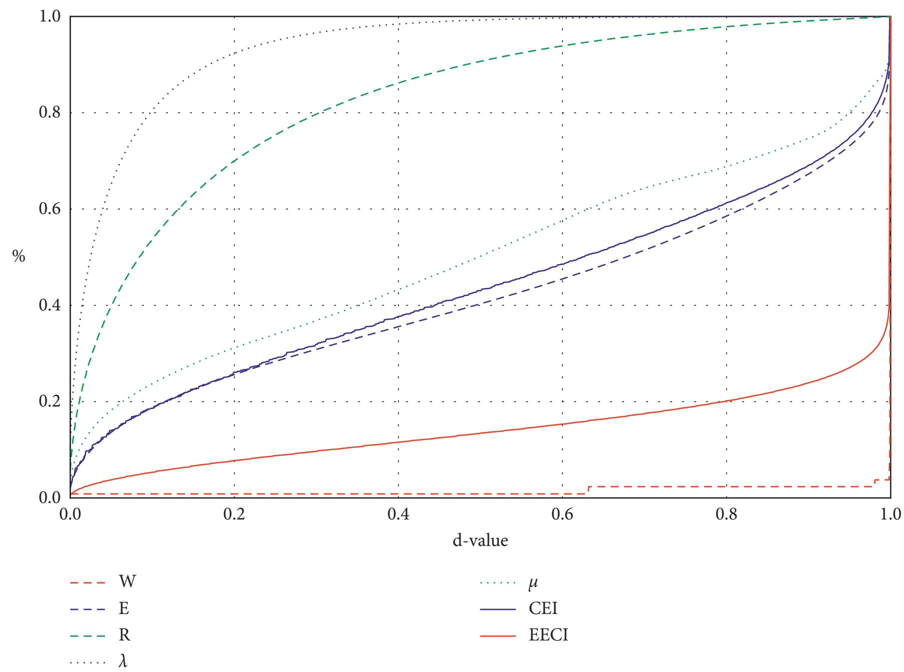


FIGURE 2: Nonisomorphic tree with 14 vertices.

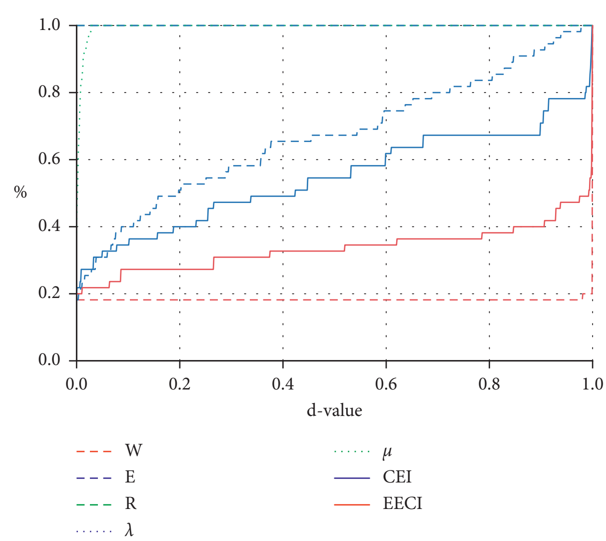


FIGURE 3: Random regular graph with $N = 100$ and $d = 3$.

with equality if and only if G_1 and G_2 are K_n and P_n, respectively. Moreover, $\xi_E^{ec}(K_n) = n(n-1)^2$ and

$$\xi_E^{ec}(P_n) = \begin{cases} 2\left(\frac{2}{n-1} + \frac{3}{n-2} + \frac{4}{n-3} + \frac{4}{n-4} + \cdots + \frac{4}{(n+1)/2}\right) + \frac{4}{(n-1)/2} & \text{for n is odd,} \\ 2\left(\frac{2}{n-1} + \frac{3}{n-2} + \frac{4}{n-3} + \frac{4}{n-4} + \cdots + \frac{4}{n/2}\right) & \text{for n is even.} \end{cases} \quad (24)$$

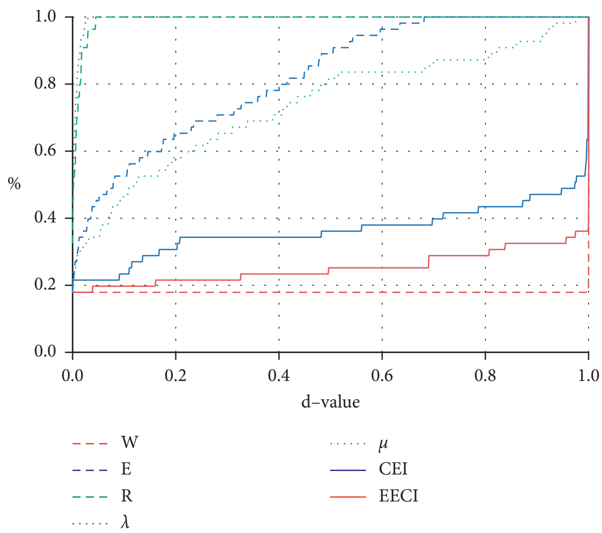


Figure 4: Watts–Strogatz small-world graph with $N = 100$, $k = 4$, and $p = 0.2$.

Proof. From the definition of Ediz eccentric connectivity index, one has $\xi_E^{ec}(G) < \xi_E^{ec}(G+e)$ for a noncomplete graph G and $e \in E(\overline{G})$ since adding an edge should increase the degree of some vertex. This means that adding an edge will increase the Ediz eccentric connectivity index. So, the complete graph K_n has the maximal Ediz eccentric connectivity index among all connected graphs with n vertices. On the contrary, the graph with minimal Ediz eccentric connectivity index among all connected graphs on n vertices must be a tree. We recall that the path P_n has the minimal Ediz eccentric connectivity index among all trees with n vertices [7]. Therefore, the path P_n has the minimal Ediz eccentric connectivity index among all connected graphs with n vertices. So, the following holds.

Note that the function $f(x) = 1 - e^{-x^2}$ is an increasing function when $x \geq 0$. So, among all connected graphs on n vertices $d_{\xi_E^{ec}}(G, H)$ attains the maximal value which are K_n and P_n, respectively. From the definition of Ediz eccentric connectivity index, we can directly compute the values $\xi_E^{ec}(K_n)$ and $\xi_E^{ec}(P_n)$. □

4. Conclusion

The Ediz eccentric connectivity index of a network is a topological invariant. In this paper, we have investigated some mathematical properties thereof. Furthermore, we explored similarity measures for networks by using the Ediz eccentric connectivity index. We compare the network similarity measures based on the Ediz eccentric connectivity index and other well-known topological indices such as Wiener index, Randić index, graph energy, the largest adjacent eigenvalues, the largest Laplacian eigenvalue, and connectivity eccentric index by means of two different methods. Numerical results show that the similarity measure based on Wiener index has more advantage than the one based on Ediz eccentric connectivity index, but the measure based on Ediz eccentric connectivity index has more advantage than the ones based on other topological indices (graph energy, Randić index, the largest adjacent eigenvalue, the largest Laplacian eigenvalue, and connective eccentricity index). This means that the similarity measure based on Ediz eccentric connectivity index has certain advantage to some extent.

Actually, there are many useful topological indices, which are widely investigated, such as Wiener polarity index [30], eccentric connectivity index [31, 32], Estrada index [33], Laplacian Estrada index [34], extended Estrada index [35], and Kirchhoff index [36]. In the future we would like to study the network similarity measure, which is studied in this paper, based on the above indices and explore the relations between them and the one based on Ediz eccentric connectivity index.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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