Research Article

All Solutions of the Yang–Baxter-Like Matrix Equation for Nilpotent Matrices of Index Two

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Let A be a nilpotent matrix of index two, and consider the Yang–Baxter-like matrix equation $AXA = XAX$. We first obtain a system of matrix equations of smaller sizes to find all the solutions of the original matrix equation. When A is a nilpotent matrix with rank 1 and rank 2, we get all solutions of the Yang–Baxter-like matrix equation.

1. Introduction

We are interested in finding all solutions of the quadratic matrix equation:

$$AXA = XAX, \quad (1)$$

where the given A and the unknown X are $n \times n$ square complex matrices with $n \geq 2$. The above equation (1) is called the Yang–Baxter-like matrix equation because it is similar to the classical parameter-free Yang–Baxter equation [1, 2]. The Yang–Baxter equation was first introduced by Yang in 1967 and then by Baxter in 1972 in the study of statistical mechanics. The Yang–Baxter equation has been extensively researched by mathematicians and physicists in knot theory, braid group theory, and so on [3–6].

Obviously, the Yang–Baxter matrix equation has two trivial solutions $X = 0$ and $X = A$. However, we are interested in finding nontrivial solutions. Finding all solutions of equation (1) is a hard work for a general matrix A . Indeed, we can reformulate (1) into a system of polynomial equations, so it is equivalent to solving a system of n^2 quadratic polynomial equations in n^2 variables. To find all solutions is not an easy task even if for a 3×3 matrix [7]. Most solutions obtained so far are commuting ones for particular choices of matrices A . See, for example, [8] for diagonalizable matrix and [9, 10] for

nilpotent matrix. In [11], infinitely many solutions of (1) were obtained for any semisimple eigenvalues of the given matrix. A family of commuting solutions of (1) were constructed for those eigenvalues of A that are non-semisimple in [12, 13]. Some researchers have also proposed some numerical methods for finding commuting solutions. For instance, in [14], when A is a nonsingular matrix such that its inverse is a stochastic matrix, Ding and Rhee found nontrivial solutions of (1) via Brouwer's fixed point theorem. In [15], when A is a diagonalisable matrix, the authors proposed numerical methods to calculate solutions of (1) by applying the mean ergodic theorem. When A is a low rank matrix, all solutions of (1) have been found in [16–18] for the noncommuting case. In [19], the authors have obtained explicit solutions when A is an idempotent matrix. However, for a general matrix A , it is difficult to characterize and determine all the solutions of (1), even if for nilpotent matrix.

The purpose of this paper is to find all the solutions of equation (1) under the assumption that A is a nilpotent matrix of index 2. We first give a system of matrix equations of smaller sizes to find all solutions of the original matrix equation in Section 2. In the next two sections, we study all solutions for $AXA = XAX$ when A is rank one and rank two, respectively. Finally, we present two examples of our solution result in Section 5 and conclude with Section 6.

2. All Solutions for the Nilpotent Matrix of Index 2

Let $A \neq 0$ be a nilpotent matrix with index 2. Clearly, matrix A has 0 as its only eigenvalue and the minimal polynomial of A is $m(t) = t^2$. So, we can write the Jordan canonical form J of A as

$$J = \begin{bmatrix} 0 & & & \\ & J_0 & & \\ & & \ddots & \\ & & & J_0 \end{bmatrix}, \quad (2)$$

in which the 2×2 Jordan block

$$J_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (3)$$

appears k times with k the rank of A , and 0 denotes the $(n-2k) \times (n-2k)$ zero matrix. So, there exists a nonsingular matrix W such that

$$W^{-1}AW = J \equiv \text{diag}(0, J_0, \dots, J_0). \quad (4)$$

Lemma 1. *Let two matrices A and X satisfy $AXA = XAX$. Then, for any nonsingular matrix U , the matrices $B = U^{-1}AU$ and $Y = U^{-1}XU$ satisfy*

$$BYB = YBY. \quad (5)$$

Conversely, if Y satisfies the above equations for a given $B = U^{-1}AU$, then $X = UYU^{-1}$ satisfies $AXA = XAX$.

According to Lemma 1, we know that solving (1) can be reduced to solving the simplified matrix equation:

$$YJY = JYJ. \quad (6)$$

So, solution X to (1) can be expressed as $X = WYW^{-1}$, where W satisfies $A = WJW^{-1}$. Thus, in our analysis below, we find all solutions of (6). Let Y be partitioned the same way as J into the $(k+1) \times (k+1)$ block matrix:

$$Y = \begin{bmatrix} M & Q_1 & \cdots & Q_k \\ P_1 & Y_{11} & \cdots & Y_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ P_k & Y_{k1} & \cdots & Y_{kk} \end{bmatrix}, \quad (7)$$

where M is $(n-2k) \times (n-2k)$, $Q_i = [q_{2i-1}, q_{2i}] \in \mathbb{C}^{(n-2k) \times 2}$ and $P_i = [p_{2i-1}, p_{2i}]^T \in \mathbb{C}^{2 \times (n-2k)}$ for $i = 1, \dots, k$, and

$$Y_{ij} = \begin{bmatrix} b_{ij} & c_{ij} \\ s_{ij} & t_{ij} \end{bmatrix}, \quad i, j = 1, 2, \dots, k. \quad (8)$$

Then, the equation $JYJ = YJY$ becomes

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & J_0 Y_{11} J_0 & \cdots & J_0 Y_{1k} J_0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & J_0 Y_{k1} J_0 & \cdots & J_0 Y_{kk} J_0 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^k Q_i J_0 P_i & \sum_{i=1}^k Q_i J_0 Y_{i1} & \cdots & \sum_{i=1}^k Q_i J_0 Y_{ik} \\ \sum_{i=1}^k Y_{1i} J_0 P_i & \sum_{i=1}^k Y_{1i} J_0 Y_{i1} & \cdots & \sum_{i=1}^k Y_{1i} J_0 Y_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^k Y_{ki} J_0 P_i & \sum_{i=1}^k Y_{ki} J_0 Y_{i1} & \cdots & \sum_{i=1}^k Y_{ki} J_0 Y_{ik} \end{bmatrix}, \quad (9)$$

which is equivalent to the system

$$\begin{cases} \sum_{i=1}^k Q_i J_0 P_i = 0, \\ \sum_{i=1}^k Y_{ji} J_0 P_i = 0, & j = 1, 2, \dots, k, \\ \sum_{i=1}^k Q_i J_0 Y_{ij} = 0, & j = 1, 2, \dots, k, \\ \sum_{i=1}^k Y_{ii} J_0 Y_{ij} = J_0 Y_{ij} J_0, & i, j = 1, 2, \dots, k. \end{cases} \quad (10)$$

Note that the above system does not contain M . This means that M is arbitrary. Because finding all solutions of (10) is very difficult, we focus on finding all solutions of (10) when $k = 1$ and $k = 2$.

3. All Solutions for the Nilpotent Matrix of Index 2 with Rank 1

When $k = 1$, this means that the rank of A is 1. Then,

$$J = \begin{bmatrix} 0 & 0 \\ 0 & J_0 \end{bmatrix}, \quad (11)$$

where 0 is the $(n-2) \times (n-2)$ zero matrix. Let Y be partitioned the same way as J into the 2×2 block matrix:

$$Y = \begin{bmatrix} M & Q \\ P & Z \end{bmatrix}, \quad (12)$$

where $M \in \mathbb{C}^{(n-2) \times (n-2)}$, $Q = [q_1 \ q_2] \in \mathbb{C}^{(n-2) \times 2}$, $P = [p_1 \ p_2]^T \in \mathbb{C}^{2 \times (n-2)}$, and $Z \in \mathbb{C}^{2 \times 2}$. According to (4), we obtain

$$\begin{cases} QJ_0P = 0, \\ QJ_0Z = 0, \\ ZJ_0P = 0, \\ ZJ_0Z = J_0ZJ_0. \end{cases} \quad (13)$$

Obviously, the last equation of (13) is also a Yang-Baxter matrix equation. Next, we solve the last equation in (13).

Lemma 2. *The solutions Z of the equation $ZJ_0Z = J_0ZJ_0$ are*

$$\begin{aligned}
Z^{(1)} &= \begin{bmatrix} 0 & z_2 \\ 0 & z_4 \end{bmatrix}, \quad z_4 \neq 0, \\
Z^{(2)} &= \begin{bmatrix} 0 & z_2 \\ 0 & 0 \end{bmatrix}, \\
Z^{(3)} &= \begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix}, \quad z_1 \neq 0.
\end{aligned} \tag{14}$$

Proof. Let

$$Z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}. \tag{15}$$

Then, the equation $ZJ_0Z = J_0ZJ_0$ becomes

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

That is,

$$\begin{bmatrix} z_1 z_3 & z_1 z_4 \\ z_3^2 & z_3 z_4 \end{bmatrix} = \begin{bmatrix} 0 & z_3 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

This leads to $z_3 = 0$ and $z_4 z_1 = 0$, and z_2 is arbitrary.

By Lemma 2, all solutions of the last equation of (13) are $Z^{(1)}$, $Z^{(2)}$, and $Z^{(3)}$. Substituting such matrices into the first three equations of (13), we obtain the following result. $\square$

Theorem 1. Suppose that A is a nilpotent matrix with rank 1. Then, all solutions of (1) are $X = WYW^{-1}$, where Y is partitioned as (12) in which M is an arbitrary $(n-2) \times (n-2)$ matrix such that $Y =$

$$\begin{bmatrix} M & 0 & q_2 \\ p_1^T & 0 & z_2 \\ p_2^T & 0 & z_4 \end{bmatrix}, \quad z_4 \neq 0, \tag{18}$$

$$\begin{bmatrix} M & 0 & q_2 \\ p_1^T & 0 & z_2 \\ p_2^T & 0 & 0 \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} M & q_1 & q_2 \\ p_1^T & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad z_1 \neq 0. \tag{20}$$

Proof. When $Z = Z^{(1)}$, then $ZJ_0 = 0$. So, we just need to solve the first two equations in (13). The second equation (13) is $QJ_0Z = 0$. That is,

$$[0 \quad q_1 z_4] = 0. \tag{21}$$

Since $z_4 \neq 0$, we have $q_1 = 0$. Thus, $QJ_0 = 0$. This gives (18).

Let $Z = Z^{(2)}$. Then, $ZJ_0 = 0$ and $J_0Z = 0$. So, the second and the third equations of (13) are satisfied. Next, we only have to solve the first equation of (13), i.e.,

$$QJ_0P = [q_1 \quad q_2] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1^T \\ p_2^T \end{bmatrix} = q_1 p_2^T = 0, \tag{22}$$

from which $q_1 = 0$, p_2 is arbitrary or $p_2 = 0$, q_1 is arbitrary. This gives (19).

If $Z = Z^{(3)}$, then $J_0Z = 0$. So, the second equation of (13) is satisfied. From the first equation $ZJ_0P = 0$, we obtain

$$\begin{bmatrix} z_1 p_2^T \\ 0 \end{bmatrix} = 0, \tag{23}$$

which leads to $z_1 p_2^T = 0$. Since $z_1 \neq 0$, we obtain $p_2 = 0$. Thus, $J_0P = 0$. This gives (20). $\square$

4. All Solutions for the Nilpotent Matrix of Index 2 with Rank 2

When $k = 2$, this means

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_0 & 0 \\ 0 & 0 & J_0 \end{bmatrix}, \tag{24}$$

where 0 is the $(n-4) \times (n-4)$ zero matrix. Let $\Lambda = \begin{bmatrix} J_0 & 0 \\ 0 & J_0 \end{bmatrix}$. Then,

$$J = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix}. \tag{25}$$

Let Y be partitioned the same way as J into the 2×2 block matrix:

$$Y = \begin{bmatrix} M & U \\ V^T & K \end{bmatrix}, \tag{26}$$

where M is $(n-4) \times (n-4)$, $U = [u_1, u_2, u_3, u_4] \in \mathbb{C}^{(n-4) \times 4}$, $V = [v_1, v_2, v_3, v_4] \in \mathbb{C}^{(n-4) \times 4}$, and K is 4×4 . Then, $JYJ = YJY$ is equivalent to the system:

$$\begin{cases} U\Lambda V^T = 0, \\ U\Lambda K = 0, \\ K\Lambda V^T = 0, \\ \Lambda K\Lambda = K\Lambda K. \end{cases} \tag{27}$$

All solutions of this system were obtained in [18]. We have the following results.

Lemma 3 (see Lemma 4.1 in [18]). The solutions K of the equation $\Lambda K\Lambda = K\Lambda K$ are

$$\begin{aligned}
K_1 &= \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_{33} \neq 0, \\
K_3 &= \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_{13} \neq 0, \\
K_4 &= \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_{13}a_{31} \neq 0, \\
K_5 &= \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & \frac{a_{33}a_{42}}{a_{31}} & 0 & \frac{a_{33}a_{44}}{a_{31}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \quad a_{31} \neq 0, \\
K_6 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & \frac{a_{13}a_{42}}{a_{11}} & 0 & \frac{a_{13}a_{44}}{a_{11}} \\ a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \quad a_{11} \neq 0, \\
K_7 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \neq 0.
\end{aligned} \tag{28}$$

(10) in which M is an arbitrary $(n-4) \times (n-4)$ matrix such that $Y =$

$$\begin{aligned}
&\begin{bmatrix} M & 0 & u_2 & 0 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix} \begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_3 \neq 0, \\
&\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ \frac{u_1^H u_3}{\|u_1\|^2} v_4^T & 0 & \frac{u_1^H u_3}{\|u_1\|^2} a_{42} & 0 & \frac{u_1^H u_3}{\|u_1\|^2} a_{44} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \quad u_1 \neq 0, \\
&\begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_1 \neq 0, a_{33} \neq 0, \\
&\begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_1 \neq 0, a_{13} \neq 0, \\
&\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M & u_1 & u_2 & \frac{a_{33}u_1}{a_{31}} & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ \frac{a_{33}v_4^T}{a_{31}} & 0 & \frac{a_{33}a_{42}}{a_{31}} & 0 & \frac{a_{33}a_{44}}{a_{31}} \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \quad a_{13} \neq 0, a_{31} \neq 0, \\
&\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{29}$$

Theorem 2 (see Theorem 4.2 in [18]). Suppose A is a nilpotent matrix with rank 2, and is partitioned as (4). Then, all solutions of (1) are $X = WYW^{-1}$, where Y is partitioned as

with $a_{13} \neq 0$, $u_3 \neq a_{33}u_1/a_{31}$ in the left matrix, and $a_{11} \neq 0$, $u_3 \neq a_{13}u_1/a_{11}$ in the second one,

$$\begin{bmatrix} M & u_1 & u_2 & \frac{a_{13}}{a_{11}}u_1 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ -\frac{a_{13}}{a_{11}}v_4^T & 0 & -\frac{a_{13}a_{42}}{a_{11}} & 0 & -\frac{a_{13}a_{44}}{a_{11}} \\ v_3^T & a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \quad (30)$$

$$\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a_{11} \neq 0,$$

$$a_{33} \neq \frac{a_{13}a_{31}}{a_{11}}.$$

5. Examples

We give two examples to illustrate our results.

Example 1. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}. \quad (31)$$

Then, there exists

$$W = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \quad (32)$$

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

such that $A = WJW^{-1}$. By Theorem 1, we obtain all solutions of (1) as

$$X = \begin{bmatrix} q_2 + z_2 & -m - p_1 & -m - p_1 + q_2 + z_2 \\ z_2 - z_4 & -p_1 + p_2 & -p_1 + p_2 + z_2 - z_4 \\ -q_2 - z_2 + z_4 & m + p_1 - p_2 & m + p_1 - p_2 - q_2 - z_2 + z_4 \end{bmatrix}, \quad z_4 \neq 0,$$

$$X = \begin{bmatrix} q_2 + z_2 & -m - p_1 & -m - p_1 + q_2 + z_2 \\ z_2 & -p_1 + p_2 & -p_1 + p_2 + z_2 \\ -q_2 - z_2 & m + p_1 - p_2 & m + p_1 - p_2 - q_2 - z_2 \end{bmatrix},$$

$$X = \begin{bmatrix} q_1 + q_2 + z_2 & q_1 - p_1 - m & q_1 - p_1 - m + q_2 + z_2 \\ z_2 & -p_1 & z_2 - p_1 \\ -q_1 - q_2 - z_2 & m + p_1 - q_1 & m + p_1 - q_1 - q_2 - z_2 \end{bmatrix},$$

$$X = \begin{bmatrix} q_1 + q_2 + z_1 + z_2 & q_1 - p_1 - m + z_1 & q_1 - p_1 - m + q_2 + z_1 + z_2 \\ z_1 + z_2 & z_1 - p_1 & z_1 - p_1 + z_2 \\ -q_1 - q_2 - z_1 - z_2 & m + p_1 - q_1 - z_1 & m + p_1 - q_1 - q_2 - z_1 - z_2 \end{bmatrix}, \quad z_1 \neq 0. \quad (33)$$

The second example is a matrix with rank 2.

Example 2. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = WJW^{-1} \quad (34)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

All solutions Y of (11) are $K_1, \dots, K_7$ as given by Lemma 3. Multiplying $X = WYW^{-1} = WK_kW^{-1}$ out with $k = 1, \dots, 7$, we find all solutions of (1) are

$$\begin{aligned}
X &= \begin{bmatrix} m & 0 & 0 & u_2 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & 0 & 0 & a_{32} & a_{34} \\ v_2 & 0 & 0 & a_{22} & a_{24} \\ v_4 & 0 & 0 & a_{42} & a_{44} \end{bmatrix}, \begin{bmatrix} m & 0 & u_3 & u_2 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & 0 & 0 & a_{32} & a_{34} \\ v_2 & 0 & 0 & a_{22} & a_{24} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_3 \neq 0, \\
X &= \begin{bmatrix} m & u_1 & u_2 & u_3 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & 0 & 0 & a_{32} & a_{34} \\ -\frac{u_1^H u_3 v_4^T}{\|u_1\|^2} & 0 & 0 & -\frac{u_1^H u_3}{\|u_1\|^2} a_{42} & -\frac{u_1^H u_3}{\|u_1\|^2} a_{44} \\ v_4^T & 0 & 0 & a_{42} & a_{44} \end{bmatrix}, \quad u_1 \neq 0, \\
X &= \begin{bmatrix} m & 0 & u_3 & u_2 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & 0 & a_{33} & a_{32} & a_{34} \\ v_2 & 0 & 0 & a_{22} & a_{24} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} m & u_1 & u_2 & u_3 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & 0 & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_1 \neq 0, a_{33} \neq 0, \\
X &= \begin{bmatrix} m & 0 & u_3 & u_2 & u_4 \\ v_1 & 0 & a_{13} & a_{12} & a_{14} \\ v_3 & 0 & a_{33} & a_{32} & a_{34} \\ v_2 & 0 & 0 & a_{22} & a_{24} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} m & u_1 & u_3 & u_2 & u_4 \\ v_1 & 0 & a_{13} & a_{12} & a_{14} \\ v_3 & 0 & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad u_1 \neq 0, a_{13} \neq 0, \\
X &= \begin{bmatrix} m & u_1 & u_3 & u_2 & u_4 \\ v_1 & 0 & a_{13} & a_{12} & a_{14} \\ v_3 & a_{31} & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} m & u_1 & \frac{a_{33}}{a_{31}} u_1 & u_2 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & a_{31} & a_{33} & a_{32} & a_{34} \\ \frac{a_{33} v_4^T}{a_{31}} & 0 & 0 & \frac{a_{33} a_{42}}{a_{31}} & \frac{a_{33} a_{44}}{a_{31}} \\ v_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ v_4 & 0 & 0 & a_{42} & a_{44} \end{bmatrix}, \quad a_{13} \neq 0, a_{31} \neq 0, \\
X &= \begin{bmatrix} m & u_1 & u_3 & u_2 & u_4 \\ v_1 & 0 & 0 & a_{12} & a_{14} \\ v_3 & a_{31} & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} m & u_1 & u_3 & u_2 & u_4 \\ v_1 & a_{11} & a_{13} & a_{12} & a_{14} \\ v_3 & a_{31} & a_{33} & a_{32} & a_{34} \\ v_3^T & a_{31} & \frac{a_{13} a_{31}}{a_{11}} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (35)
\end{aligned}$$

with $a_{13} \neq 0$, $u_3 \neq a_{33} u_1 / a_{31}$ in the left matrix, and $a_{11} \neq 0$, $u_3 \neq a_{13} u_1 / a_{11}$ in the second one,

$$X = \begin{bmatrix} m & u_1 & \frac{a_{13} u_1}{a_{11}} & u_2 & u_4 \\ v_1 & a_{11} & a_{13} & a_{12} & a_{14} \\ v_3 & a_{31} & \frac{a_{13} a_{31}}{a_{11}} & a_{32} & a_{34} \\ -\frac{a_{13} v_4^T}{a_{11}} & 0 & 0 & -\frac{a_{13} a_{42}}{a_{11}} & -\frac{a_{13} a_{44}}{a_{11}} \\ v_4 & 0 & 0 & a_{42} & a_{44} \end{bmatrix}, \quad (36)$$

$$\begin{bmatrix} m & u_1 & u_3 & u_2 & u_4 \\ v_1 & a_{11} & a_{13} & a_{12} & a_{14} \\ v_3 & a_{31} & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a_{11} \neq 0,$$

$$a_{33} \neq \frac{a_{13} a_{31}}{a_{11}}.$$

6. Conclusions

In this paper, we obtain all solutions for the Yang–Baxter-like (1) when A is a nilpotent matrix of index 2. We first obtain a system of matrix equations of smaller sizes to find all the solutions of the original matrix equation. For a special case, that is, $k = 1$ and $k = 2$, we derive all solutions of the Yang–Baxter-like matrix equation in detail. However, for an arbitrary nilpotent matrix, finding all solutions of (1) is a hard work. We hope to solve this problem in the future.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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