Research Article

Flocking Behavior of Cucker–Smale Model with Processing Delay

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Received 5 May 2020; Accepted 23 June 2020; Published 23 July 2020

Guest Editor: Xiaodi Li

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The dynamics of a delay multiparticle swarm, which contains symmetric and asymmetric pairwise influence functions, are analyzed. Two different sufficient conditions to achieve conditional flocking are obtained. One does not have a clear relationship with this delay, and the other proposes a range of processing delays that affect the emergence of a flock. It is also pointed out that if the interparticle communication function has tail dissipation, unconditional flocking can be guaranteed. Compared with the previous results, the range of the communication rate β that allows a flock to emerge has been expanded from $1/4$ to $1/2$.

1. Introduction

There are many survival-oriented clusters in nature, such as ant colonies that coordinate food transportation, birds that increase the success rate of foraging, and fish that unite against danger, and so on. The research on the colony of biological groups should be traced back to Reynolds' simulation experiments on birds in [1]; further some scholars have proposed many motion models to mathematically characterize them. Among them, a second-order model proposed by Cucker and Smale in [2, 3] to explain self-organizing behavior in complex adaptive systems has been favored by researchers and continuously improved. For example, Motsch and Tadmor in [4] modified the symmetry of the influence intensity between particles to be asymmetric to explore the aggregation behavior of nonuniformly distributed particle swarms. Some scholars have carried out the impact of the time delay on flocking or consensus of the system in [5–7] and the references therein.

Liu and Wu in [5] proposed a model with processing delay, which is described as

$$\begin{cases} \dot{x}_i(t) = v_i(t), & i = 1, 2, \dots, N, \\ \dot{v}_i(t) = \alpha \sum_{j=1}^N I\left(\|x_i(t-\tau) - x_j(t-\tau)\|\right) (v_j(t-\tau) - v_i(t)), \end{cases} \quad (1)$$

where $x_i, v_i \in R^d$ and d is a positive integer, $\alpha > 0$ indicates the intensity of the influence between particles, and $\tau > 0$ represents the time lag, which includes the response time of the particle i and the communication time between particles i and j . The communication function can be defined as

$$\begin{aligned} I(r_{ij}) &= I^{\text{CS}}(r_{ij}) := \frac{\psi(r_{ij})}{N} \text{ or } I(r_{ij}) \\ &= I^{\text{MT}}(r_{ij}) := \frac{\psi(r_{ij})}{\sum_{k=1}^N \psi(r_{ik})}, \end{aligned} \quad (2)$$

which further satisfies $\sum_{l=1}^N I(r_{kl}) = 1$ for all $k \in \Gamma$, where $\psi(r) = (1 + r^2)^{-\beta}$, $\beta \geq 0$, $r_{ij}(t) = \|x_i(t) - x_j(t)\|$, $i, j \in \Gamma$. Further, system (1) can be simplified to

$$\begin{cases} \dot{x}_i(t) = v_i(t), & i \in \Gamma, \\ \dot{v}_i(t) = \alpha(\bar{v}_i(t) - v_i(t)), \end{cases} \quad (3)$$

where $\bar{v}_i(t) = \sum_{j=1}^N I(r_{ij}(t-\tau))v_j(t-\tau)$. The initial conditions are

$$\begin{aligned} x_i(\theta) &= \varphi_i(\theta), \\ v_i(\theta) &= \phi_i(\theta), \\ \theta &\in [-\tau, 0], \end{aligned} \quad (4)$$

where $(\varphi_i, \phi_i) \in C^2 = C \times C$ and $C := C([- \tau, 0], R^2)$ is the Banach space of all continuous functions.

In this study, we further consider the flocking conditions of the delayed model proposed in [5]. The significant contributions of our results are reflected in the following three aspects. (1) Compared with Theorem 3.1 in [5], the unconditional flocking condition $\int_{-\infty}^{\infty} \psi^2(r) dr = \infty$ is improved to $\int_{-\infty}^{\infty} \psi(r) dr = \infty$, that is, the communication rate β is expanded from 1/4 to 1/2. (2) Note that with $\tau = 0$ in (1), the communication rate β of unconditional flocking in [4] has also been expanded from 1/4 to 1/2. (3) It is clearly pointed out that processing delay can affect the occurrence of aggregation behavior, which is specifically manifested in the controllable range of the delay in flocking conditions.

The following two variables (D_x and D_v) are used to analyze the evolution of the aggregation behavior of systems (3) and (4), for $t \geq -\tau$:

$$\begin{aligned} D_x(t) &= \max_{i,j \in \Gamma} \left\{ \|x_i(t) - x_j(t)\| \right\}, \\ D_v(t) &= \max_{i,j \in \Gamma} \left\{ \|v_i(t) - v_j(t)\| \right\}. \end{aligned} \quad (5)$$

Thus, for both $I^{CS}(r)$ and $I^{MT}(r)$, it follows from a few simple calculations that a uniform result can be directly verified about the estimation of influence function $I(r)$ as

$$I(r_{ij}(t-\tau)) \geq \frac{\psi(D_x(t-\tau))}{N}, \quad \text{for all } i, j \in \Gamma. \quad (6)$$

We still adopt the definition of time-asymptotic flocking proposed in [4].

Definition 1. Let $\{x_i(t), v_i(t)\}_{i=1}^N$ be a solution to systems (3) and (4); a time-asymptotic flocking can be achieved if and only if the solution satisfies $\sup_{t>0} D_x(t) < +\infty$ and $\lim_{t \rightarrow +\infty} D_v(t) = 0$, where $D_x(t)$ and $D_v(t)$ are given in (5).

2. Main Results

This section proposes two different sufficient conditions for systems (3) and (4) with I^{CS} or I^{MT} to achieve the conditional flocking in Theorem 1 and Theorem 2. We have also established certain conditions for the completion of unconditional flocking in Theorem 3.

2.1. Conditional Flocking. To establish the flocking solution of systems (3) and (4), the following important auxiliary lemmas are introduced first.

Lemma 1 (see [8]). Let $x(t)$ be the solution of the linear functional differential equation, $\dot{x}(t) = \lambda - \delta_1 x(t) + \delta_2 x(t-\tau)$. If $|\delta_2| < \delta_1$, then

$$\lim_{t \rightarrow +\infty} x(t) = x^* = \frac{\lambda}{\delta_1 - \delta_2}. \quad (7)$$

Lemma 2. Let $\{x_i(t), v_i(t)\}_{i=1}^N$ be a solution to systems (3) and (4); then, we have

$$\begin{aligned} &\langle v_i(t) - v_j(t), \bar{v}_i(t) - \bar{v}_j(t) \rangle \\ &\leq \left(1 - \frac{\psi(D_x(t-\tau))}{N} \right) D_v(t) D_v(t-\tau), \end{aligned} \quad (8)$$

where D_x and D_v are defined in (5).

Proof. Making use of system (3) yields

$$\begin{aligned} &\langle v_i(t) - v_j(t), \bar{v}_i(t) - \bar{v}_j(t) \rangle \\ &= \sum_{p=1}^N I(r_{ip}(t-\tau)) \sum_{q \neq p} I(r_{jq}(t-\tau)) \\ &\quad \cdot \langle v_i(t) - v_j(t), v_p(t-\tau) - v_q(t-\tau) \rangle \\ &\leq \sum_{p=1}^N I(r_{ip}(t-\tau)) \sum_{q \neq p} I(r_{jq}(t-\tau)) D_v(t) D_v(t-\tau). \end{aligned} \quad (9)$$

Using inequality (6) and the normalization assumptions for communication functions, that is, $\sum_{j=1}^N I(r_{ij}(t-\tau)) = 1$, we get

$$\begin{aligned} &\langle v_i(t) - v_j(t), \bar{v}_i(t) - \bar{v}_j(t) \rangle \\ &\leq \sum_{p=1}^N I(r_{ip}(t-\tau)) \sum_{q \neq p} I(r_{jq}(t-\tau)) D_v(t) D_v(t-\tau) \\ &= \sum_{p=1}^N I(r_{ip}(t-\tau)) (1 - I(r_{jp}(t-\tau))) D_v(t) D_v(t-\tau) \\ &\leq \left(1 - \frac{\psi(D_x(t-\tau))}{N} \right) D_v(t) D_v(t-\tau), \end{aligned} \quad (10)$$

and this proof is completed. $\square$

Lemma 3. Let $\{x_i(t), v_i(t)\}_{i=1}^N$ be a solution of systems (3) and (4); then, the upper Dini derivative of $D_x(t)$ and $D_v(t)$ satisfies

$$\begin{aligned} D^+ D_x(t) &\leq D_v(t), \text{ a.e. } t \geq -\tau, \\ D^+ D_v(t) &\leq \alpha \left(1 - \frac{\psi(D_x(t-\tau))}{N} \right) D_v(t-\tau) - \alpha D_v(t). \end{aligned} \quad (11)$$

Proof. Without loss of generality, let $D_x(t) = \|x_p(t) - x_q(t)\|$ at time t , where $p, q \in \Gamma$. One can obtain

$$D^+ D_x(t) \leq \|\dot{x}_p(t) - \dot{x}_q(t)\| = \|v_p(t) - v_q(t)\| \leq D_v(t). \quad (12)$$

Similarly, without loss of generality, let $D_v(t)$ satisfy $D_v(t) = \|v_p(t) - v_q(t)\|$ at time t , where $p, q \in 1, 2, \dots, N$; it follows from Lemma 2 that

$$\begin{aligned}
D^+ D_v^2(t) &= 2\alpha \langle v_p(t) - v_q(t), \dot{v}_p(t) - \dot{v}_q(t) \rangle \\
&= 2\alpha \langle v_p(t) - v_q(t), \bar{v}_p(t) - \bar{v}_q(t) \rangle - 2\alpha D_v^2(t) \\
&\leq 2\alpha \left(1 - \frac{\psi(D_x(t-\tau))}{N} \right) D_v(t) D_v(t-\tau) - 2\alpha D_v^2(t).
\end{aligned} \tag{13}$$

Therefore, (11) is proven.

To establish the flocking conditions, we define a set containing all the initial configurations of asymptotic flocking allowed, that is,

$$\mathcal{S} := \begin{cases} (x_i(\theta), v_i(\theta)), \\ \theta \in [-\tau, 0], \\ i = 1, 2, \dots, N. \end{cases} \tag{14}$$

$$D_v(0) + \alpha \int_{-\tau}^0 D_v(s) ds < \frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr \Big\},$$

where D_x and D_v are shown in (5) and $\psi(r) = (1+r^2)^{-\beta}$, $\beta \geq 0$ is defined in [2, 3]. $\square$

Theorem 1. For $\beta > (1/2)$, suppose that the initial conditions (4) are selected from the set $\mathcal{S}$; then systems (3) and (4) with $I^{MT}(r)$ or $I^{CS}(r)$ can complete the conditional flocking.

Proof. Inspired by the work in [5, 9], we take the following Lyapunov function:

$$\begin{aligned}
E(D_x, D_v)(t) &= D_v(t) + \frac{\alpha}{N} \int_{D_x(-\tau)}^{D_x(t-\tau)} \psi(r) dr \\
&\quad + \alpha \int_{-\tau}^0 D_v(t+s) ds.
\end{aligned} \tag{15}$$

Thus, the upper Dini derivative of $E(D_x, D_v)(t)$ along (D_x, D_v) with respect to t is shown below.

$$\begin{aligned}
D^+ E(D_x(t), D_v(t)) \\
= D^+ D_v(t) + \frac{\alpha}{N} D^+ \int_{D_x(-\tau)}^{D_x(t-\tau)} \psi(r) dr + \alpha D^+ \int_{-\tau}^0 D_v(t+s) ds.
\end{aligned} \tag{16}$$

Furthermore, combining with Lemma 3, we have $D^+ E(D_x(t), D_v(t)) \leq 0$, which means that $E(D_x, D_v)(t)$ is nonincreasing and then $E(D_x, D_v)(t) \leq E(D_x, D_v)(0)$ for all $t > 0$. Thus,

$$\frac{\alpha}{N} \int_{D_x(-\tau)}^{D_x(t-\tau)} \psi(r) dr \leq D_v(0) + \alpha \int_{-\tau}^0 D_v(s) ds. \tag{17}$$

Since the initial conditions (4) are selected from the set $\mathcal{S}$, it follows from (16) that

$$\frac{\alpha}{N} \int_{D_x(-\tau)}^{D_x(t-\tau)} \psi(r) dr < \frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr. \tag{18}$$

Due to the fact that ψ has a divergent tail, there must be a constant $D^* < \infty$ such that $D_x(t-\tau) \leq D^*$ for $t \geq 0$. Considering inequality (6) yields

$$I(r_{ij}(t-\tau)) \geq \frac{\psi(D_x(t-\tau))}{N} \geq \frac{\psi(D^*)}{N}, \quad \text{for all } j \in \Gamma. \tag{19}$$

Using the second inequality in (11) in Lemma 3, we can further derive that

$$D^+ D_v(t) \leq \left(1 - \frac{\psi(D^*)}{N} \right) D_v(t-\tau) - \alpha D_v(t). \tag{20}$$

Making use of Lemma 1, we can show that $D_v(t) \rightarrow 0$ as $t \rightarrow \infty$ and systems (3) and (4) converge to a flock as shown in Definition 1. The proof is completed.

Another flocking condition closely related to processing delay is proposed in the following theorem. $\square$

Theorem 2. For $\beta > (1/2)$, suppose that the initial configurations (4) are met as follows:

$$0 < D_v(0) < \frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr, \tag{21}$$

and the processing delay τ satisfies

$$0 < \tau < \tau_0 := \frac{1}{\alpha R_\tau} \left(\frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr - D_v(0) \right), \tag{22}$$

where $R_\tau := \max_{\theta \in [-\tau, 0]} D_v(\theta) > 0$ and D_x, D_v are defined in (5); then, systems (3) and (4) with $I^{MT}(r)$ or $I^{CS}(r)$ converge to a flock.

Proof. We only need to prove that the initial conditions which satisfy (22) all exist in the set $\mathcal{S}$ defined in (8). Note that

$$\frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr > D_v(0) + \alpha \tau R_\tau > D_v(0) + \alpha \int_{-\tau}^0 D_v(s) ds, \tag{23}$$

which means that the initial conditions which satisfy (22) all exist in set $\mathcal{S}$. Consequently, systems (3) and (4) converge to a flock. $\square$

Remark 1. The following notes are listed for the above two different results of conditional flocking.

- (1) Note that with $\tau = 0$, Theorem 1 and Theorem 2 will degenerate into $D_v(0) < (\alpha/N) \int_{D_x(-\tau)}^{\infty} \psi(r) dr$. If $\alpha = N$, then the flocking condition is further written as $D_v(0) < \int_{D_x(-\tau)}^{\infty} \psi(r) dr$, thereby improving Theorem 3.1 in [4].
- (2) Comparing Theorem 1 and Theorem 2, we can get the following two points worthy of attention. First, it is clear from the set of allowed initial conditions that the former is larger than the latter. Second, the latter helps us realize that the occurrence of aggregation behavior is indeed affected by the size of τ .

FIGURE 1: $\alpha = 15, \beta = 0.51 > 0.5, \tau = 0$. Systems (3) and (4) converge to a flock.

FIGURE 2: $\alpha = 15, \beta = 0.51, \tau = 1 < \tau_0$. Systems (3) and (4) converge asymptotically to form a flock. The convergence rate is slower than that in Example 1 due to processing delays. The diameter of the population is larger than that in Example 1, which means that the cohesion and aggregation density between particles are not as good as the case in Example 1.

2.2. Unconditional Flocking. The following theorem describes the implementation of unconditional flocking for systems (3) and (4).

Theorem 3. For $\beta \in [0, (1/2)]$, systems (3) and (4) with $I^{MT}(r)$ or $I^{CS}(r)$ converge unconditionally to a flock.

Proof. If $\beta \in [0, (1/2)]$, then the interparticle communication function has a tail dissipation $\int_{-\infty}^{\infty} \psi(r) dr = \infty$. Similar to the proof of Theorem 1, it can directly verify the unconditional flocking result. It will be omitted here. $\square$

Remark 2. The two annotations for Theorem 3 are described below.

- (1) The fundamental reason for the unconditional flocking of systems (3) and (4) is that the following condition always holds for any initial configuration:

$$D_v(0) + \alpha \int_{-\tau}^0 D_v(s) ds < \frac{\alpha}{N} \int_{D_x(-\tau)}^{\infty} \psi(r) dr = \infty. \quad (24)$$

- (2) Note that with $\tau = 0$, the unconditional flocking result in [4] is improved to $\int_{-\infty}^{\infty} \psi(r) dr = \infty$, which means that the communication rate β is expanded from $1/4$ to $1/2$.
- (3) Compared with Theorem 3.1 in [5], the range of communication rate β has been expanded. Specifically, we promote the results from $\int_{-\infty}^{\infty} \psi^2(r) dr = \infty$ to $\int_{-\infty}^{\infty} \psi(r) dr = \infty$, that is, we extend the communication rate β from $1/4$ to $1/2$. Thus, the results in [5] have been improved.

3. Numerical Simulations

Some numerical simulations will be enumerated to illustrate the effect of processing delay on the aggregation behavior of systems (3) and (4). For the convenience of calculation,

FIGURE 3: $\alpha = 15, \beta = 0.51, \tau = 12 > \tau_0$. Systems (3) and (4) fail to the emergence of flocking due to the destruction of (22) in Theorem 2.

FIGURE 4: $\alpha = 15, \beta = 0.2 < 0.5, \tau = 12$. Systems (3) and (4) form a flock unconditionally.

consider a particle swarm composed of 8 particles and its initial conditions are set as $(\varphi_i(\theta), \phi_i(\theta)) = (8 - i, 4 - i)$, $i = 1, \dots, 8, \theta \in [-\tau, 0]$. The parameters in (3) are fixed as $\alpha = 15$, $\beta = 0.51 > 0.5$, and further we have $\tau_0 = 1.0116$ from (22). To understand the aggregation behavior of the groups (3) and (4) intuitively and conveniently, we used two features in all experiments, namely, the velocity of each particle and the maximum of the relative position between particles.

Example 1. $\alpha = 15, \beta = 0.51, \tau = 0$ (see Figure 1).

It can be seen from Figure 1 that without a time delay, the particle swarm can converge asymptotically to form a flock under the fixed initial configurations and the above parameters.

Example 2. $\alpha = 15, \beta = 0.51, \tau = 1 < \tau_0$ (see Figure 2).

Considering the delay range (22) established in Theorem 2, we can claim that the flocking of the particle population (3) and (4) with $\tau = 1 < \tau_0$ can be maintained. As we can see in Figure 2, systems (3) and (4) can still aggregate and form a

flock after introducing processing time lag τ , which satisfies (22).

Example 3. $\alpha = 15, \beta = 0.51, \tau = 12 > \tau_0$ (see Figure 3).

As shown in Figure 3, under the same initial condition with Example 2, this group cannot converge to a flock, and its fatal factor is that $\tau = 12 > \tau_0$, that is, (22) in Theorem 2 is broken.

To realize the emergence of flocking in this group, the following method can be adopted, that is, to appropriately adjust the communication rate between particles so that $\beta < 0.5$. It may be selected as $\beta = 0.2 < 0.5$ and then combined with the discussion in Theorem 3; the system unconditionally converges to a flock. The simulation results are shown in Figure 4.

4. Conclusions

We study the emergence conditions of flocking of multiple particle swarms with processing delays, establish two

sufficient conditions for conditional flocking in Theorem 1 and Theorem 2, and give an unconditional flocking result in Theorem 3. In particular, Theorem 2 intuitively explains the fact that processing delays affect the emergence of flocking, which is reflected in the time-lag range (22) that affects the emergence of flocking. For $0 \leq \beta \leq 1/2$, we note that $\tau = 0$, which is obtained from the analysis in Remark 1 and Remark 2, and the flocking results in [4] have been improved from $D_v(0) < \int_0^\infty \psi^2(r)dr$ to $D_v(0) < \int_0^\infty \psi(r)dr$. It means that the communication rate β has been expanded from 1/4 to 1/2. Compared with the work on the flocking results in [5], we have also expanded β from 1/4 to 1/2.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (11401577 and 11671011).

References

- [1] C. W. Reynolds, "Flocks, herds and schools: a distributed behavioral model," *ACM SIGGRAPH Computer Graphics*, vol. 21, no. 4, pp. 25–34, 1987.
- [2] F. Cucker and S. Smale, "On the mathematics of emergence," *Japanese Journal of Mathematics*, vol. 2, no. 1, pp. 197–227, 2007.
- [3] F. Cucker and S. Smale, "Emergent behavior in flocks," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 852–862, 2007.
- [4] S. Motsch and E. Tadmor, "A new model for self-organized dynamics and its flocking behavior," *Journal of Statistical Physics*, vol. 144, no. 5, pp. 923–947, 2011.
- [5] Y. Liu and J. Wu, "Flocking and asymptotic velocity of the Cucker-Smale model with processing delay," *Journal of Mathematical Analysis and Applications*, vol. 415, no. 1, pp. 53–61, 2014.
- [6] X. Wang, L. Wang, and J. Wu, "Impacts of time delay on flocking dynamics of a two-agent flock model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 70, pp. 80–88, 2019.
- [7] M. Chen and X. Wang, "Flocking dynamics for multi-agent system with measurement delay," *Mathematics and Computers in Simulation*, vol. 171, pp. 187–200, 2020.
- [8] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, Berlin, Germany, 1993.
- [9] S.-Y. Ha and J.-G. Liu, "A simple proof of the Cucker-Smale flocking dynamics and mean-field limit," *Communications in Mathematical Sciences*, vol. 7, no. 2, pp. 297–325, 2009.