Research Article

Codimension-2 Bifurcation Analysis and Control of a Discrete Mosquito Model with a Proportional Release Rate of Sterile Mosquitoes

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This paper concerns a discrete wild and sterile mosquito model with a proportional release rate of sterile mosquitoes. It is shown that the discrete model undergoes codimension-2 bifurcations with 1:2, 1:3, and 1:4 strong resonances by applying the bifurcation theory. Some numerical simulations, including codimension-2 bifurcation diagrams, maximum Lyapunov exponents diagrams, and phase portraits, are also presented to illustrate the validity of theoretical results and display the complex dynamical behaviors. Moreover, two control strategies are applied to the model.

1. Introduction

In recent years, the mosquito-borne diseases, including West Nile virus, dengue, malaria, Japanese encephalitis, and chikungunya, have attracted much attention. Until now, there are no available vaccines for many of these viruses, which constitute a significant threat to global health. To control the transmissions of mosquito-borne diseases, it is necessary to adopt some strategies to reduce the number of wild mosquitoes. One of effective and environment-friendly approaches is sterile insect technique (SIT). Generally, the mass male mosquitoes are exposed to radiation and thereby rendered sexually sterile. After being released into the desired area, the treated male mosquitoes will mate with wide female mosquitoes and their offspring is nonviable. In this process, choosing the number of sterile mosquitoes and suitable releasing strategy are particularly important.

To understand the interactive dynamics of wide mosquitoes and sterile mosquitoes, some mathematical models have been introduced, for example, [1–9]. Recently, Cai et al. [7] proposed three mathematical models which represent

different releasing strategies of sterile mosquitoes. The existence and stability of equilibria were investigated, and the threshold dynamics were obtained. One of the models considered in [7] is

$$\begin{cases} \frac{dw}{dt} = \left(\frac{aw}{1+w+g} - (\mu_1 + \xi_1(w+g)) \right) w, \\ \frac{dg}{dt} = bw - (\mu_2 + \xi_2(w+g))g, \end{cases} \quad (1)$$

where w and g are the numbers of wild mosquitoes and sterile mosquitoes, respectively. All the coefficients are positive constants, a is the number of offspring produced by per mate, b is the linear release rate of sterile mosquitoes with coefficient b , and μ_i and ξ_i ($i = 1, 2$) represent the density independent and dependent death rates of wild and sterile mosquitoes, respectively. It was shown in [10, 11] that continuous model (1) undergoes a sequence of codimensional-1 bifurcations (including saddle-node bifurcation, supercritical and subcritical Hopf bifurcation, and

homoclinic bifurcation) and codimensional-2 bifurcation (Bogdanov–Takens bifurcation). When bw in (1) is replaced by $bw/(1+w)$, the system becomes a mosquito population model with nonlinear saturated releasing rate of sterile mosquitoes. Some complex bifurcation phenomena, such as Hopf bifurcation of codimension-3, nilpotent cusp of codimension-4, and Bogdanov–Takens bifurcation of codimension-2, were provided in [12] if $\mu_1 \neq \mu_2$ and $\xi_1 \neq \xi_2$.

In mathematical modelling, difference equations (that is, discrete-time models) are also widely used to describe the

population dynamics. Compared with continuous models, discrete-time cases have many advantages. The most obvious one is that discrete models are more convenient to formulate and simulate, especially when the populations have non-overlapping generations [13–15]. Moreover, for several biological models, their discrete-time cases may produce much richer dynamical behaviors than the continuous cases.

In this paper, we study the complex dynamics of the following discrete-time model:

$$\begin{cases} w(n+1) = w(n) \exp \left[\frac{aw(n)}{1+w(n)+g(n)} - (\mu_1 + \xi_1(w(n)+g(n))) \right], \\ g(n+1) = g(n) + bw(n) - (\mu_2 + \xi_2(w(n)+g(n)))g(n), \end{cases} \quad (2)$$

which is established from (1) by the Euler forward difference method. w, g, a, b, μ_i, ξ_i ($i = 1, 2$) are defined as in model (1). By the biological meaning of the model, we only consider system (2) in the region $\Omega = \{(w, g) : w \geq 0, g \geq 0\}$ in the (w, g) -plane. We will study the existence of codimension-2 bifurcations associated with 1:2, 1:3, and 1:4 strong resonances of model (2). The codimension-2 bifurcation diagrams, maximum Lyapunov exponent diagrams, and phase portraits are also presented to illustrate the validity of theoretical results and display the complex dynamical behaviors.

The bifurcation and chaotic behavior of discrete models have been widely investigated by many researchers, but most of them focus on codimensional-1 bifurcations, for example, [16–21]. The codimension-2 bifurcations for discrete models are considered very few. The codimension-2 bifurcations of a delayed discrete-time Hopfield neural network, discrete epidemic model with nonlinear incidence rate, and discrete SIS epidemic model with standard incidence are analyzed in [22–24]. A discrete Hindmarsh–Rose model undergoes codimension-2 bifurcations with 1:2 and 1:4 strong resonances [25], and 1:3 strong resonance [26]. 1:1 strong resonance bifurcation of a discrete predator-prey model with non-monotonic functional response has also been studied in [27]. Moreover, Ren and Yu [28] investigated codimension-2 bifurcations of a discrete information diffusion model.

The main goal of this paper is to investigate the impact of the mating rate a and the release rate coefficient b on the interactive dynamics of wide mosquitoes and sterile mosquitoes. The rest of this paper is organized as follows. Preliminaries are given in Section 2. In Sections 3–5, some sufficient conditions of codimension-2 bifurcations associated with 1:2, 1:3, and 1:4 strong resonances are established, respectively. The numerical simulations and controller designs are presented in Sections 6 and 7. A brief discussion is given in Section 8.

2. Preliminaries

In this section, we first briefly introduce some known results on the existence of positive equilibria, which have been obtained in [7], and then provide some sufficient conditions for the existence of codimension-2 bifurcations with 1:2, 1:3, and 1:4 strong resonances.

Define the function

$$\begin{aligned} F(N) &= (1+N)(\mu_1 + \xi_1 N)(b + \mu_2 + \xi_2 N) - aN(\mu_2 + \xi_2 N) \\ &= \xi_1 \xi_2 N^3 + (\xi_1(b + \mu_2) + \xi_2(\mu_1 + \xi_1 - a))N^2 \\ &\quad + (\mu_1 \xi_2 + b(\mu_1 + \xi_1) + (\mu_1 + \xi_1 - a)\mu_2)N \\ &\quad + \mu_1(b + \mu_2). \end{aligned} \quad (3)$$

We can check that model (2) admits a positive equilibrium:

$$(w, g) = \left(\frac{(1+\bar{N})(\mu_1 + \xi_1 \bar{N})}{a}, \frac{b(1+\bar{N})(\mu_1 + \xi_1 \bar{N})}{a(\mu_2 + \xi_2 \bar{N})} \right), \quad (4)$$

if and only if $F(N) = 0$ has a positive root $\bar{N}$, which is also equivalent to the existence of positive solution $\bar{N}$ to the equation $G(N) = b$ with

$$G(N) := \frac{(aN - (1+N)(\mu_1 + \xi_1 N))(\mu_2 + \xi_2 N)}{(1+N)(\mu_1 + \xi_1 N)}. \quad (5)$$

Assuming that $a > \mu_1 + \xi_1 + 2\sqrt{\mu_1 \xi_1}$, we know the equation $G(N) = 0$ has two positive roots:

$$N_{1,2} = \frac{1}{2\xi_1} \left((a - \mu_1 - \xi_1) \mp \sqrt{(a - \mu_1 - \xi_1)^2 - 4\mu_1 \xi_1} \right). \quad (6)$$

It follows that there exists a unique $N^* \in (N_1, N_2)$ such that $G'(N^*) = 0$. Let

$$b_0 := G(N^*), \quad (7)$$

then the equation $G(N) = b$ has two positive solutions $N_{1,2}^* \in (N_1, N_2)$ with $N_1^* < N_2^*$ if and only if $b < b_0$, and only one positive solution $N^* \in (N_1, N_2)$ when $b = b_0$. In summary, we have the following results on the existence of equilibria of model (2).

Lemma 1. Assume that $a > \mu_1 + \xi_1 + 2\sqrt{\mu_1 \xi_1}$. Then,

- (i) model (2) always has a boundary equilibrium $E(0, 0)$
- (ii) if $b > b_0$, then model (2) has no any positive equilibrium
- (iii) if $b = b_0$, then model (2) has a unique positive equilibrium $E^*(w^*, g^*)$, where

$$(w^*, g^*) = \left(\frac{(1 + N^*)(\mu_1 + \xi_1 N^*)}{a}, \frac{b_0(1 + N^*)(\mu_1 + \xi_1 N^*)}{a(\mu_2 + \xi_2 N^*)} \right). \quad (8)$$

Moreover, $F(N^*) = 0$ and $F'(N^*) = 0$.

- (iv) if $b < b_0$, then model (2) has two positive equilibria $E_1^*(w_1^*, g_1^*)$ and $E_2^*(w_2^*, g_2^*)$, where

$$(w_1^*, g_1^*) = \left(\frac{(1 + N_1^*)(\mu_1 + \xi_1 N_1^*)}{a}, \frac{b(1 + N_1^*)(\mu_1 + \xi_1 N_1^*)}{a(\mu_2 + \xi_2 N_1^*)} \right),$$

$$(w_2^*, g_2^*) = \left(\frac{(1 + N_2^*)(\mu_1 + \xi_1 N_2^*)}{a}, \frac{b(1 + N_2^*)(\mu_1 + \xi_1 N_2^*)}{a(\mu_2 + \xi_2 N_2^*)} \right). \quad (9)$$

Moreover, $F'(N_1^*) < 0$ and $F'(N_2^*) > 0$.

Denote by $E(w, g)$, the equilibrium of model (2), then the Jacobian matrix at $E(w, g)$ is

$$J(E) = \begin{pmatrix} 1 + w \left(\frac{a(1+g)}{(1+N)^2} - \xi_1 \right) & -w \left(\frac{aw}{(1+N)^2} + \xi_1 \right) \\ b - \xi_2 g & 1 - \mu_2 - \xi_2(N + g) \end{pmatrix}, \quad (10)$$

its trace is $\text{Tr}(J(E)) = 2 + w((a(1+g)/(1+N)^2 - \xi_1) - \mu_2 - \xi_2(N + g)) = 2 + 1/aA(N)$, and its determinant is $\text{Det}(J(E)) = 1 - w/1 + NF'(N) + 1/aA(N)$, where

$$A(N) := \xi_1(\xi_2 - 2\xi_1)N^2 + (a(\xi_1 - 2\xi_2) + \xi_1(\xi_2 - \xi_1) + \mu_1(\xi_2 - 3\xi_1))N + a(\mu_1 - \mu_2) + \mu_1(\xi_2 - \xi_1 - \mu_1). \quad (11)$$

If the matrix $J(E)$ has eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ with $\theta = \pi$, $\theta = (2/3)\pi$, or $\theta = (1/2)\pi$, then there may present a codimension-2 bifurcation with 1:2, 1:3, or 1:4 strong resonances at $E(w, g)$. For the equilibrium $E^*(w^*, g^*)$ in Lemma 1 (iii), if codimension-2 bifurcation with 1:2, 1:3, or 1:4 strong resonance occurs, then $\text{Det}(J(E^*)) = 1$ and $\text{Tr}(J(E^*)) = -2, -1$, or 0 , which imply that $(1/a)A(N^*) < 0$

and $(w^*/(1 + N^*))F'(N^*) = (1/a)A(N^*) < 0$. It is a contradiction with $F'(N^*) = 0$. Similarly, since $F'(N_2^*) > 0$, we can also show that there does not exist the above codimension-2 bifurcations at the equilibrium $E_2^*(w_2^*, g_2^*)$. Thus, in this paper, we discuss the existence of codimension-2 bifurcation at $E_1^*(w_1^*, g_1^*)$.

Choosing a and b as bifurcation parameters, if $(a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in F_1$, where

$$F_1 := \left\{ (a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in \mathbb{R}_+^6 : a > \mu_1 + \xi_1 + 2\sqrt{\mu_1 \xi_1}, \right. \\ \left. b < b_0, \frac{w_1^*}{1 + N_1^*} F'(N_1^*) = \frac{1}{a} A(N_1^*) = -4 \right\}, \quad (12)$$

then two eigenvalues of Jacobian matrix $J(E_1^*)$ are $\lambda_{1,2} = e^{\pm \pi i} = -1$. This shows that there may present a codimension-2 bifurcation with 1:2 strong resonance at $E_1^*(w_1^*, g_1^*)$.

If $(a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in F_2$, where

$$F_2 := \left\{ (a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in \mathbb{R}_+^6 : a > \mu_1 + \xi_1 + 2\sqrt{\mu_1 \xi_1}, \right. \\ \left. b < b_0, \frac{w_1^*}{1 + N_1^*} F'(N_1^*) = \frac{1}{a} A(N_1^*) = -3 \right\}, \quad (13)$$

then two eigenvalues of the Jacobian matrix $J(E_1^*)$ are $\lambda_{1,2} = e^{\pm (2\pi/3)i} = -(1/2) \pm (\sqrt{3}/2)i$. This shows that there may present a codimension-2 bifurcation with 1:3 strong resonance at $E_1^*(w_1^*, g_1^*)$.

Furthermore, if $(a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in F_3$, where

$$F_3 := \left\{ (a, b, \mu_1, \mu_2, \xi_1, \xi_2) \in \mathbb{R}_+^6 : a > \mu_1 + \xi_1 + 2\sqrt{\mu_1 \xi_1}, \right. \\ \left. b < b_0, \frac{w_1^*}{1 + N_1^*} F'(N_1^*) = \frac{1}{a} A(N_1^*) = -2 \right\}, \quad (14)$$

then two eigenvalues of the Jacobian matrix $J(E_1^*)$ are $\lambda_{1,2} = e^{\pm (\pi/2)i} = \pm i$. This shows that there may present a codimension-2 bifurcation with 1:4 strong resonance at $E_1^*(w_1^*, g_1^*)$. In the following, we will give the theoretical analysis and numerical simulations.

3. 1:2 Strong Resonance

In this section, by applying the bifurcation theory in [29], we analyze the codimension-2 bifurcation associated with 1:2 strong resonance of model (2) at the equilibrium $E_1^*(w_1^*, g_1^*)$.

We choose a and b as bifurcation parameters and assume that $(a_1, b_1, \mu_1, \mu_2, \xi_1, \xi_2) \in F_1$. Consider model (2) with $a = a_1 + a_1^*$ and $b = b_1 + b_1^*$, where a_1^* and b_1^* are small perturbations of the parameters a_1 and b_1 . Let $\hat{w}(n) = w(n) - w_1^*$ and $\hat{g}(n) = g(n) - g_1^*$, and model (2) becomes into the following form which has the equilibrium $(0, 0)$:

$$\begin{cases} \hat{w}(n+1) = (\hat{w}(n) + w_1^*) \exp \left[\frac{a(\hat{w}(n) + w_1^*)}{1 + \hat{w}(n) + \hat{g}(n) + w_1^* + g_1^*} - (\mu_1 + \xi_1(\hat{w}(n) + \hat{g}(n) + w_1^* + g_1^*)) \right] - w_1^*, \\ \hat{g}(n+1) = \hat{g}(n) + b(\hat{w}(n) + w_1^*) - (\mu_2 + \xi_2(\hat{w}(n) + \hat{g}(n) + w_1^* + g_1^*))(\hat{g}(n) + g_1^*). \end{cases} \quad (15)$$

By expanding the right-hand side of (15) into the Taylor series at $(0, 0)$, we have

$$\begin{cases} \hat{w}(n+1) = (1 + w_1^* \alpha) \hat{w}(n) - w_1^* \beta \hat{g}(n) + \sum_{2 \leq i+j \leq 3} k_{ij}(a, b) \hat{w}^i(n) \hat{g}^j(n) + O(|\hat{w}(n)| + |\hat{g}(n)|^4), \\ \hat{g}(n+1) = (b - \xi_2 g_1^*) \hat{w}(n) + [1 - (\mu_2 + \xi_2 g_1^* + \xi_2 N_1^*)] \hat{g}(n) - \xi_2 \hat{w}(n) \hat{g}(n) - \xi_2 \hat{g}^2(n) + O(|\hat{w}(n)| + |\hat{g}(n)|^4), \end{cases} \quad (16)$$

where

$$\begin{aligned} \alpha &= \frac{a(1 + g_1^*)}{(1 + N_1^*)^2} - \xi_1, \\ \beta &= \frac{aw_1^*}{(1 + N_1^*)^2} + \xi_1, \\ k_{20}(a, b) &= \alpha + \frac{1}{2} w_1^* \alpha^2 - \frac{w_1^*}{1 + N_1^*} (\alpha + \xi_1), \\ k_{11}(a, b) &= -\beta - w_1^* \alpha \beta - \frac{aw_1^*}{(1 + N_1^*)^2} + \frac{2a(w_1^*)^2}{(1 + N_1^*)^3}, \\ k_{02}(a, b) &= \frac{1}{2} w_1^* \beta^2 + \frac{a(w_1^*)^2}{(1 + N_1^*)^3}, \\ k_{30}(a, b) &= \frac{1}{2} \alpha^2 + \frac{1}{6} w_1^* \alpha^3 - \frac{w_1^*}{1 + N_1^*} \alpha (\alpha + \xi_1) - \frac{1}{a} (\alpha + \xi_1)^2, \\ k_{03}(a, b) &= -\frac{1}{6} w_1^* \beta^3 - \frac{a(w_1^*)^2}{(1 + N_1^*)^3} \beta - \frac{a(w_1^*)^2}{(1 + N_1^*)^4}, \\ k_{21}(a, b) &= -\alpha \beta - \frac{1}{2} w_1^* \alpha^2 \beta + \frac{aw_1^* (1 + g_1^*)}{(1 + N_1^*)^3} \beta \\ &\quad - \frac{aw_1^* (2 + g_1^* - 3w_1^*)}{2(1 + N_1^*)^3} \alpha \\ &\quad - \frac{a}{(1 + N_1^*)^2} + \frac{4aw_1^*}{(1 + N_1^*)^3} - \frac{3a(w_1^*)^2}{(1 + N_1^*)^4}, \\ k_{12}(a, b) &= \frac{aw_1^* (1 + g_1^* - w_1^*)}{(1 + N_1^*)^3} \beta + \frac{a(w_1^*)^2}{(1 + N_1^*)^3} \alpha + \frac{1}{2} \beta^2 \\ &\quad + \frac{1}{2} w_1^* \alpha \beta^2 \\ &\quad + \frac{aw_1^* (2 + 2N_1^* - 3w_1^*)}{(1 + N_1^*)^4}. \end{aligned} \quad (17)$$

For $(a_1, b_1, \mu_1, \mu_2, \xi_1, \xi_2) \in F_1$, the Jacobian matrix at E_1^* is

$$J(E_1^*) = \begin{pmatrix} 1 + w_1^* \alpha_1 & -w_1^* \beta_1 \\ b_1 - \xi_2 g_1^* & 1 - \mu_2 - \xi_2 (N_1^* + g_1^*) \end{pmatrix}, \quad (18)$$

where

$$\begin{aligned} \alpha_1 &:= \frac{a_1(1 + g_1^*)}{(1 + N_1^*)^2} - \xi_1, \\ \beta_1 &:= \frac{a_1 w_1^*}{(1 + N_1^*)^2} + \xi_1. \end{aligned} \quad (19)$$

Take two linearly independent (generalized) eigenvectors of $J(E_1^*)$,

$$\begin{aligned} p_1 &= \begin{pmatrix} \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} \\ 1 \end{pmatrix}, \\ p_2 &= \begin{pmatrix} 1 \\ b_1 - \xi_2 g_1^* \\ 0 \end{pmatrix}, \end{aligned} \quad (20)$$

and that of the transposed matrix $J^T(E_1^*)$,

$$\begin{aligned} q_1 &= \begin{pmatrix} b_1 - \xi_2 g_1^* \\ -(2 + \alpha_1 w_1^*) \end{pmatrix}, \\ q_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (21)$$

It is easy to check that $\langle p_1, q_2 \rangle = \langle p_2, q_1 \rangle = 1$, $\langle p_1, q_1 \rangle = \langle p_2, q_2 \rangle = 0$, where $\langle x, y \rangle := x^T y = x_1 y_1 + x_2 y_2$ for any $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T \in \mathbb{R}^2$. Therefore, any vector $(\hat{w}(n), \hat{g}(n))^T \in \mathbb{R}^2$ can be represented in the form

$$(\hat{w}(n), \hat{g}(n))^T = x(n)p_1 + y(n)p_2, \quad (22)$$

which implies that

$$\begin{cases} x(n) = \langle (\hat{w}(n), \hat{g}(n))^T, q_2 \rangle, \\ y(n) = \langle (\hat{w}(n), \hat{g}(n))^T, q_1 \rangle. \end{cases} \quad (23)$$

Under new coordinates $(x(n), y(n))$, map (16) can be written as

$$\begin{cases} x(n+1) = (-1 + \Theta_1(a, b))x(n) + \left(1 - \frac{1}{2}\Theta_1(a, b)\right)y(n) - \xi_2 \left(1 + \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*}\right)x^2(n) \\ \quad - \frac{\xi_2}{b_1 - \xi_2 g_1^*} x(n)y(n) + O(|x(n)| + |y(n)|^4), \\ y(n+1) = \Theta_2(a, b)x(n) + \left(-1 - \frac{1}{2}\Theta_2(a, b)\right)y(n) + \sum_{2 \leq i+j \leq 3} r_{ij}(a, b)x^i(n)y^j(n) \\ \quad + O(|x(n)| + |y(n)|^4), \end{cases} \quad (24)$$

where

$$\Theta_1(a, b) = 2 \left(1 - \frac{b - \xi_2 g_1^*}{b_1 - \xi_2 g_1^*}\right),$$

$$\Theta_2(a, b) = -2 \left[(2 + w_1^* \alpha) - (2 + w_1^* \alpha_1) \frac{b - \xi_2 g_1^*}{b_1 - \xi_2 g_1^*} \right],$$

$$r_{20} = (b_1 - \xi_2 g_1^*) \left[k_{11} \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} + k_{20} \left(\frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} \right)^2 + k_{02} \right] \\ + (2 + \alpha_1 w_1^*) \xi_2 + \beta_1 w_1^* \xi_2,$$

$$r_{02} = \frac{k_{20}}{b_1 - \xi_2 g_1^*},$$

$$r_{11} = k_{11} + 2k_{20} \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} + \frac{(2 + \alpha_1 w_1^*) \xi_2}{b_1 - \xi_2 g_1^*},$$

$$r_{30} = (b_1 - \xi_2 g_1^*) \left[k_{30} \left(\frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} \right)^3 + k_{21} \left(\frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} \right)^2 \right. \\ \left. + k_{12} \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} + k_{03} \right],$$

$$r_{21} = 3k_{30} \left(\frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} \right)^2 + 2k_{21} \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*} + k_{12},$$

$$r_{12} = 3k_{30} \frac{\beta_1 w_1^*}{(2 + \alpha_1 w_1^*)(b_1 - \xi_2 g_1^*)} + k_{21} \frac{1}{b_1 - \xi_2 g_1^*},$$

$$r_{03} = k_{30} \left(\frac{1}{b_1 - \xi_2 g_1^*} \right)^2.$$

(25)

By introducing a coordinate transformation

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\Theta_1(a, b) & 0 \\ -\Theta_1(a, b) & 1 \end{pmatrix} \begin{pmatrix} \hat{x}(n) \\ \hat{y}(n) \end{pmatrix}, \quad (26)$$

map (24) can be reduced into the following form:

$$\begin{cases} \hat{x}(n+1) = -\hat{x}(n) + \hat{y}(n) + \hat{k}_{20} \hat{x}^2(n) + \hat{k}_{11} \hat{x}(n) \hat{y}(n) \\ \quad + O(|\hat{x}(n)| + |\hat{y}(n)|^4), \\ \hat{y}(n+1) = \Theta_2 \hat{x}(n) + (-1 + \Theta_3) \hat{y}(n) + \sum_{2 \leq i+j \leq 3} \hat{r}_{ij}(a, b) \hat{x}^i(n) \hat{y}^j(n) \\ \quad + O(|\hat{x}(n)| + |\hat{y}(n)|^4), \end{cases} \quad (27)$$

where

$$\Theta_3(a, b) = \Theta_1(a, b) - \frac{1}{2}\Theta_2(a, b),$$

$$\hat{k}_{11}(k, r) = \frac{\xi_2}{b_1 - \xi_2 g_1^*},$$

$$\hat{r}_{20}(a, b) = \Theta_1 \hat{k}_{20} + \left(1 - \frac{\Theta_1}{2}\right)^2 r_{20} + \Theta_1^2 r_{02} - \left(1 - \frac{\Theta_1}{2}\right) \Theta_1 r_{11},$$

$$\hat{k}_{20}(k, r) = -\left(1 - \frac{\Theta_1}{2}\right) \xi_2 \left(1 + \frac{\beta_1 w_1^*}{2 + \alpha_1 w_1^*}\right) + \Theta_1 \frac{\xi_2}{b_1 - \xi_2 g_1^*},$$

$$\hat{r}_{02}(a, b) = r_{02},$$

$$\hat{r}_{11}(a, b) = \Theta_1 \hat{k}_{11} + \left(1 - \frac{\Theta_1}{2}\right) r_{11} - 2\Theta_1 r_{02},$$

$$\begin{aligned}
\hat{r}_{30}(a, b) &= \left(1 - \frac{\Theta_1}{2}\right)^3 r_{30} - \Theta_1^3 r_{03} - \Theta_1 \left(1 - \frac{\Theta_1}{2}\right)^2 r_{21} \\
&\quad + \Theta_1^2 \left(1 - \frac{\Theta_1}{2}\right) r_{12}, \\
\hat{r}_{21}(a, b) &= 3\Theta_1^2 r_{03} + \left(1 - \frac{\Theta_1}{2}\right)^2 r_{21} - 2\Theta_1 \left(1 - \frac{\Theta_1}{2}\right) r_{12}, \\
\hat{r}_{12}(a, b) &= \left(1 - \frac{\Theta_1}{2}\right) r_{12} - 3\Theta_1 r_{03}, \\
\hat{r}_{03}(a, b) &= r_{03}.
\end{aligned} \tag{28}$$

Next, we aim to eliminate all quadratic terms and some cubic terms in (27). To achieve it, we introduce the following transformation:

$$\begin{cases} \hat{x}(n) = u(n) + \sum_{2 \leq i+j \leq 3} \varphi_{ij}(a, b) u^i(n) v^j(n), \\ \hat{y}(n) = v(n) + \sum_{2 \leq i+j \leq 3} \psi_{ij}(a, b) u^i(n) v^j(n), \end{cases} \tag{29}$$

where φ_{ij} and ψ_{ij} ($2 \leq i+j \leq 3$) will be determined later.

By applying transformation (29) and its inverse, map (27) becomes

$$\begin{cases} u(n+1) = -u(n) + v(n) + \sum_{2 \leq i+j \leq 3} \eta_{ij}(a, b) u^i(n) v^j(n) \\ \quad + O(|u(n)| + |v(n)|^4), \\ v(n+1) = \Theta_2(a, b) u(n) + (-1 + \Theta_3(a, b)) v(n) \\ \quad + \sum_{2 \leq i+j \leq 3} \zeta_{ij}(a, b) u^i(n) v^j(n) + O(|u(n)| + |v(n)|^4), \end{cases} \tag{30}$$

where

$$\begin{aligned}
\eta_{20}(a, b) &= \hat{k}_{20} + \psi_{20} - 2\varphi_{20} - \Theta_2^2 \varphi_{02} + \Theta_2 \varphi_{11}, \\
\eta_{11}(a, b) &= \hat{k}_{11} + \psi_{11} - 2\Theta_2(1 + \Theta_3) \varphi_{02} \\
&\quad + (\Theta_3 - \Theta_2) \varphi_{11} + 2\varphi_{20}, \\
\eta_{02}(a, b) &= \psi_{02} - (1 + (1 + \Theta_3)^2) \varphi_{02} + (1 + \Theta_3) \varphi_{11} - \varphi_{20}, \\
\zeta_{20}(a, b) &= \hat{r}_{20} - \Theta_2^2 \psi_{02} + \Theta_2 \psi_{11} + (\Theta_2 + \Theta_3) \psi_{20}, \\
\zeta_{11}(a, b) &= \hat{r}_{11} - 2\Theta_2(1 + \Theta_3) \psi_{02} + (2 - \Theta_2 + 2\Theta_3) \psi_{11} \\
&\quad + 2\psi_{20} + \Theta_2 \varphi_{11}, \\
\zeta_{02}(a, b) &= \hat{r}_{20} - \Theta_3(1 + \Theta_3) \psi_{02} - (1 + \Theta_3) \psi_{11} - \psi_{20} + \Theta_2 \varphi_{02}.
\end{aligned} \tag{31}$$

Assume that the following vanishing conditions hold:

$$\eta_{20} = \eta_{11} = \eta_{02} = \zeta_{20} = \zeta_{11} = \zeta_{02} = 0, \tag{32}$$

and we can obtain all the coefficients φ_{ij} and ψ_{ij} satisfying $i+j=2$. Moreover, by taking $\eta_{30} = \eta_{12} = \eta_{21} = \eta_{03} = \zeta_{12} = \zeta_{03} = 0$, we can determine the coefficients φ_{ij} and ψ_{ij} for $i+j=3$, see more details in [24, 25]. Thus, model (15) can be transformed into the normal form for 1:2 strong resonance bifurcation as follows:

$$\begin{cases} u(n+1) = -u(n) + v(n) + O(|u(n)| + |v(n)|^4), \\ v(n+1) = \Theta_2(a, b) u(n) + (-1 + \Theta_3(a, b)) v(n) + C(a, b) u^3(n) \\ \quad + D(a, b) u^2(n) v(n) + O(|u(n)| + |v(n)|^4), \end{cases} \tag{33}$$

where $C(a, b)$ and $D(a, b)$ satisfy

$$\begin{aligned}
C(a_1, b_1) &= \hat{r}_{30}(a_1, b_1) + \hat{k}_{20}(a_1, b_1) \hat{r}_{20}(a_1, b_1) + \frac{1}{2} \hat{k}_{20}^2(a_1, b_1) + \frac{1}{2} \hat{r}_{20}(a_1, b_1) \hat{r}_{11}(a_1, b_1), \\
D(a_1, b_1) &= \hat{r}_{21}(a_1, b_1) + \frac{1}{2} \hat{k}_{20}(a_1, b_1) \hat{r}_{11}(a_1, b_1) + \frac{5}{4} \hat{r}_{20}(a_1, b_1) \hat{r}_{11}(a_1, b_1) \\
&\quad + \hat{r}_{20}(a_1, b_1) \hat{r}_{02}(a_1, b_1) + 3\hat{k}_{20}^2(a_1, b_1) + \frac{5}{2} \hat{k}_{20}(a_1, b_1) \hat{r}_{20}(a_1, b_1) \\
&\quad + \frac{5}{2} \hat{k}_{11}(a_1, b_1) \hat{r}_{20}(a_1, b_1) + \hat{r}_{20}^2(a_1, b_1) + \frac{1}{2} \hat{r}_{11}^2(a_1, b_1).
\end{aligned} \tag{34}$$

Applying the results established in [29], we can get the existence of 1:2 strong resonance bifurcation.

Theorem 1. Assume that $C(a_1, b_1) \neq 0$ and $D(a_1, b_1) + 3C(a_1, b_1) \neq 0$. Then, model (2) has a 1:2 strong resonance bifurcation at $E_1^*(w_1^*, g_1^*)$ when the parameters a and b vary in a sufficiently small neighborhood of F_1 with fixed μ_1, μ_2, ξ_1 , and ξ_2 . If we further assume $C(a_1, b_1) < 0$, then $E_1^*(w_1^*, g_1^*)$ is a saddle; if $C(a_1, b_1) > 0$, then $E_1^*(w_1^*, g_1^*)$ is elliptic. $D(a_1, b_1) + 3C(a_1, b_1)$ determines the bifurcation scenario under small perturbations. Moreover, model (2) has the following bifurcation behaviors:

- (i) There is a pitchfork bifurcation curve $P = \{(\theta_1, \theta_2): \theta_1 = 0\}$, and there exist nontrivial equilibria for $\theta_1 < 0$
- (ii) There is a nondegenerate Neimark-Sacker bifurcation curve $H = \{(\theta_1, \theta_2): \theta_1 = -\theta_2 + O((|\theta_1| + |\theta_2|)^2), \theta_1 < 0\}$
- (iii) There is a heteroclinic bifurcation curve $HL = \{(\theta_1, \theta_2): \theta_1 = -(5/3)\theta_2 + O((|\theta_1| + |\theta_2|)^2), \theta_1 < 0\}$

4. 1:3 Strong Resonance

In this section, we prove that there exists a codimension-2 bifurcation with 1:3 strong resonance of model (2) at $E_1^*(w_1^*, g_1^*)$. We choose a and b as bifurcation parameters of model (2) and assume that $(a_2, b_2, \mu_1, \mu_2, \xi_1, \xi_2) \in F_2$.

Similar to Section 3, we can also transform $E_1^*(w_1^*, g_1^*)$ into $(0, 0)$ and then expand the right-hand side of the map into Taylor series. For $(a_2, b_2, \mu_1, \mu_2, \xi_1, \xi_2) \in F_2$, the Jacobian matrix at $E_1^*(w_1^*, g_1^*)$ is

$$J(E_1^*) = \begin{pmatrix} 1 + w_1^* \alpha_2 & -w_1^* \beta_2 \\ b_2 - \xi_2 g_1^* & 1 - \mu_2 - \xi_2 (N_1^* + g_1^*) \end{pmatrix}, \quad (35)$$

with

$$\begin{aligned} \alpha_2 &:= \frac{a_2(1 + g_1^*)}{(1 + N_1^*)^2} - \xi_1, \\ \beta_2 &:= \frac{a_2 w_1^*}{(1 + N_1^*)^2} + \xi_1, \end{aligned} \quad (36)$$

and it has two complex eigenvalues $\lambda_{1,2} = -(1/2) \pm (\sqrt{3}/2)i$. The corresponding eigenvector $\zeta_1 \in \mathbb{C}^2$ and adjoint eigenvector $\zeta_2 \in \mathbb{C}^2$ of $J(E_1^*)$ are defined as

$$\begin{aligned} \zeta_1 &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} := \begin{pmatrix} w_1^* \beta_2 \\ \frac{3}{2} + w_1^* \alpha_2 - \frac{\sqrt{3}}{2}i \end{pmatrix}, \\ \zeta_2 &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} \frac{3 + \sqrt{3}(3 + 2w_1^* \alpha_2)i}{6w_1^* \beta_2} \\ -\frac{\sqrt{3}}{3}i \end{pmatrix}. \end{aligned} \quad (37)$$

Any vector $Z(n) = (\hat{w}(n), \hat{g}(n))^T \in \mathbb{R}^2$ can be represented as $Z(n) = z(n)\zeta_1 + \bar{z}(n)\bar{\zeta}_1$. Thus, for $(a_2, b_2, \mu_1, \mu_2, \xi_1, \xi_2) \in F_2$, model (16) can be written in the complex form as follows:

$$z(n+1) = \lambda_1 z(n) + \sum_{2 \leq i+j \leq 3} \frac{\sigma_{ij}}{i!j!} z^i(n) \bar{z}^j(n) + O(|z(n)|^4), \quad (38)$$

where $\lambda_1 = -(1/2) + (\sqrt{3}/2)i$ and

$$\begin{aligned} \sigma_{20} &= 2\bar{x}_2(k_{20}(a_2, b_2)x_1^2 + k_{11}(a_2, b_2)x_1 y_1 + k_{02}(a_2, b_2)y_1^2) \\ &\quad - 2\xi_2 \bar{y}_2(x_1 y_1 + y_1^2), \\ \sigma_{02} &= 2\bar{x}_2(k_{20}(a_2, b_2)\bar{x}_1^2 + k_{11}(a_2, b_2)\bar{x}_1 \bar{y}_1 + k_{02}(a_2, b_2)\bar{y}_1^2) \\ &\quad - 2\xi_2 \bar{y}_2(\bar{x}_1 \bar{y}_1 + \bar{y}_1^2), \\ \sigma_{11} &= \bar{x}_2[2k_{20}(a_2, b_2)x_1 \bar{x}_1 + k_{11}(a_2, b_2)(x_1 \bar{y}_1 + \bar{x}_1 y_1) \\ &\quad + 2k_{02}(a_2, b_2)y_1 \bar{y}_1] \\ &\quad - \xi_2 \bar{y}_2(x_1 \bar{y}_1 + \bar{x}_1 y_1 + 2y_1 \bar{y}_1), \\ \sigma_{30} &= 6\bar{x}_2(k_{30}(a_2, b_2)x_1^3 + k_{21}(a_2, b_2)x_1^2 y_1 + k_{12}(a_2, b_2)x_1 y_1^2 \\ &\quad + k_{03}(a_2, b_2)y_1^3), \\ \sigma_{03} &= 6\bar{x}_2(k_{30}(a_2, b_2)\bar{x}_1^3 + k_{21}(a_2, b_2)\bar{x}_1^2 \bar{y}_1 + k_{12}(a_2, b_2)\bar{x}_1 \bar{y}_1^2 \\ &\quad + k_{03}(a_2, b_2)\bar{y}_1^3), \\ \sigma_{21} &= 2\bar{x}_2[3k_{30}(a_2, b_2)x_1^2 \bar{x}_1 + k_{21}(a_2, b_2)x_1^2 \bar{y}_1 \\ &\quad + 2k_{21}(a_2, b_2)x_1 \bar{x}_1 y_1 \\ &\quad + k_{12}(a_2, b_2)(y_1^2 \bar{x}_1 + 2x_1 y_1 \bar{y}_1) + 3k_{03}(a_2, b_2)y_1^2 \bar{y}_1], \\ \sigma_{12} &= 2\bar{x}_2[3k_{30}(a_2, b_2)x_1 \bar{x}_1^2 + k_{21}(a_2, b_2)\bar{x}_1^2 y_1 \\ &\quad + 2k_{21}(a_2, b_2)x_1 \bar{x}_1 \bar{y}_1 \end{aligned} \quad (39)$$

Next, we shall establish the normal form for 1:3 strong resonance bifurcation. We first eliminate some quadratic terms in (38) through the transformation

$$z(n) = \phi(n) + \frac{h_{20}}{2}\phi^2(n) + h_{11}\phi(n)\bar{\phi}(n) + \frac{h_{02}}{2}\bar{\phi}^2(n), \quad (40)$$

where the coefficients h_{ij} satisfying $i + j = 2$ will be determined later.

Using transformation (40) and its inverse transformation,

$$\begin{aligned}
\phi(n) = & z(n) - \frac{h_{20}}{2} z^2(n) - h_{11} z(n) \bar{z}(n) - \frac{h_{02}}{2} \bar{z}^2(n) \\
& + \frac{1}{2} (h_{20}^2 + h_{11} \bar{h}_{02}) z^3(n) \\
& + \left(|h_{11}|^2 + \frac{3}{2} h_{11} h_{20} + \frac{1}{2} |h_{02}|^2 \right) z^2(n) \bar{z}(n) \\
& + \left(h_{02} \bar{h}_{11} + h_{11}^2 + \frac{1}{2} h_{11} \bar{h}_{20} + \frac{1}{2} h_{20} h_{02} \right) z(n) \bar{z}^2(n) \\
& + \frac{1}{2} (h_{11} h_{02} + h_{02} \bar{h}_{20}) \bar{z}^3(n) + O(|z(n)|^4),
\end{aligned} \tag{41}$$

map (41) is changed into the following form:

$$\phi(n+1) = \lambda_1 \phi(n) + \sum_{2 \leq i+j \leq 3} \frac{\hat{\sigma}_{ij}}{i!j!} \phi^i(n) \bar{\phi}^j(n) + O(|\phi(n)|^4), \tag{42}$$

where

$$\begin{aligned}
\hat{\sigma}_{20} &= \sqrt{3} i h_{20} + \sigma_{20}, \\
\hat{\sigma}_{11} &= \left(-\frac{3}{2} + \frac{\sqrt{3}}{2} i \right) h_{11} + \sigma_{11}, \\
\hat{\sigma}_{02} &= \sigma_{02}, \\
\hat{\sigma}_{30} &= \frac{9-3\sqrt{3}i}{2} \sigma_{20} h_{20} + 3\sigma_{11} \bar{h}_{02} + \sigma_{30} + \frac{9+3\sqrt{3}i}{2} h_{20}^2 \\
&\quad - 3\lambda_1 \bar{\sigma}_{02} h_{11}, \\
\hat{\sigma}_{21} &= 2\sigma_{11} \bar{h}_{11} + \sigma_{11} h_{20} + \sigma_{02} \bar{h}_{02} + \sigma_{21} + (1-\sqrt{3}i) \sigma_{11} h_{20} \\
&\quad + \frac{1+\sqrt{3}i}{2} \bar{\sigma}_{02} h_{02} \\
&\quad + (-3+\sqrt{3}i) |h_{11}|^2 + \frac{-3+5\sqrt{3}i}{2} h_{11} h_{20} \\
&\quad + (1-\sqrt{3}i) h_{11} \bar{\sigma}_{11} + \frac{5+\sqrt{3}i}{2} \sigma_{20} h_{11}, \\
\hat{\sigma}_{12} &= (3+\sqrt{3}i) \sigma_{11} h_{11} + \sigma_{11} \bar{h}_{20} + 2\sigma_{02} \bar{h}_{11} + \sigma_{20} h_{02} + \sigma_{12} \\
&\quad + (1-\sqrt{3}i) \sigma_{02} h_{20} \\
&\quad - (3+\sqrt{3}i) h_{11}^2 - \frac{3+\sqrt{3}i}{2} h_{11} \bar{h}_{20} + \frac{1-\sqrt{3}i}{2} h_{11} \bar{\sigma}_{20} \\
&\quad - 2\sqrt{3} i h_{02} \bar{h}_{11} + (1+\sqrt{3}i) \bar{\sigma}_{11} h_{02}, \\
\hat{\sigma}_{03} &= 3\sigma_{11} h_{02} + 3\sigma_{02} \bar{h}_{20} + \sigma_{03} + \frac{3(1+\sqrt{3}i)}{2} \sigma_{02} h_{11} \\
&\quad + \frac{9-3\sqrt{3}i}{2} h_{02} \bar{h}_{20} + \frac{3(1+\sqrt{3}i)}{2} \bar{\sigma}_{20} h_{02}.
\end{aligned} \tag{43}$$

Take

$$\begin{aligned}
h_{20} &= -\frac{\sigma_{20}}{\sqrt{3}i}, \\
h_{11} &= \frac{2\sigma_{11}}{3-\sqrt{3}i}, \\
h_{02} &= 0,
\end{aligned} \tag{44}$$

then transformation (40) is determined and map (42) becomes

$$\begin{aligned}
\phi(n+1) = & \lambda_1 \phi(n) + \frac{\sigma_{02}}{2} \bar{\phi}^2(n) + \sum_{i+j=3} \frac{\hat{\sigma}_{ij}}{i!j!} \phi^i(n) \bar{\phi}^j(n) \\
& + O(|\phi(n)|^4).
\end{aligned} \tag{45}$$

In order to eliminate some cubic terms, we take the transformation

$$\begin{aligned}
\phi(n) = & \chi(n) + \frac{1}{6} h_{30} \chi^3(n) + \frac{1}{2} h_{21} \chi^2(n) \bar{\chi}(n) \\
& + \frac{1}{2} h_{12} \chi(n) \bar{\chi}^2(n) + \frac{1}{6} h_{03} \bar{\chi}^3(n).
\end{aligned} \tag{46}$$

It is easy to check that the inverse transformation of (46) is

$$\begin{aligned}
\chi(n) = & \phi(n) - \frac{1}{6} h_{30} \phi^3(n) - \frac{1}{2} h_{21} \phi^2(n) \bar{\phi}(n) - \frac{1}{2} h_{12} \phi(n) \bar{\phi}^2(n) \\
& - \frac{1}{6} h_{03} \bar{\phi}^3(n) + O(|\phi(n)|^4).
\end{aligned} \tag{47}$$

By applying (46) and (47), map (45) becomes

$$\begin{aligned}
\chi(n+1) = & \lambda_1 \chi(n) + \frac{\sigma_{02}}{2} \bar{\chi}^2(n) \\
& + \sum_{i+j=3} \frac{\tilde{\sigma}_{ij}}{i!j!} \chi^i(n) \bar{\chi}^j(n) + O(|\chi(n)|^4),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\tilde{\sigma}_{30} &= -\frac{3-\sqrt{3}i}{2} h_{30} + \hat{\sigma}_{30}, \\
\tilde{\sigma}_{21} &= \hat{\sigma}_{21}, \\
\tilde{\sigma}_{12} &= \sqrt{3} i h_{12} + \hat{\sigma}_{12}, \\
\tilde{\sigma}_{03} &= -\frac{3-\sqrt{3}i}{2} h_{03} + \hat{\sigma}_{03}.
\end{aligned} \tag{49}$$

Take

$$\begin{aligned}
h_{30} &= \frac{2\hat{\sigma}_{30}}{3 - \sqrt{3}i}, \\
h_{12} &= \frac{\sqrt{3}i}{3}\hat{\sigma}_{12}, \\
h_{03} &= \frac{2\hat{\sigma}_{03}}{3 - \sqrt{3}i}, \\
h_{21} &= 0,
\end{aligned} \tag{50}$$

then transformation (46) is determined and map (48) reduces into the following form:

$$\chi(n+1) = \lambda_1 \chi(n) + \frac{\sigma_{02}}{2} \bar{\chi}^2(n) + \frac{\hat{\sigma}_{21}}{2} \chi(n)^2 \bar{\chi}(n) + O(|\chi(n)|^4). \tag{51}$$

We define $C(a_2, b_2) := -(3/4)(\sqrt{3}i + 1)\sigma_{02}$ and $D(a_2, b_2) := -(3/4)|\sigma_{02}|^2 - (3/4)(\sqrt{3}i + 1)\hat{\sigma}_{21}$. From the 1 : 3 strong resonance bifurcation analysis established in [29], we can get the following theorem.

Theorem 2. Assume that $C(a_2, b_2) \neq 0$ and $\text{Re } D(a_2, b_2) \neq 0$. Then, model (2) undergoes a 1 : 3 strong resonance bifurcation at $E_1^*(w_1^*, g_1^*)$. $\text{Re } D(a_2, b_2) \neq 0$ determines the stability of the bifurcation invariant closed curve. Moreover, model (2) admits a number of the following complex codimension-1 bifurcation curves:

- (i) There is a nondegenerate Neimark-Sacker bifurcation at the trivial fixed point E_0 of (51)
- (ii) There is a saddle cycle of periodic three corresponding to saddle fixed points E_k ($k = 1, 2, 3$) of map (51)
- (iii) There is a homoclinic structure formed by the stable and unstable invariant manifolds of the period three cycle intersecting transversally in an exponentially narrow parameter region

5. 1 : 4 Strong Resonance

In this section, we prove that there exists a codimension-2 bifurcation with 1 : 4 strong resonance of model (2) at $E_1^*(w_1^*, g_1^*)$. We choose a and b as bifurcation parameters of model (2) and assume that $(a_3, b_3, \mu_1, \mu_2, \xi_1, \xi_2) \in F_3$.

Similar to Section 3, we can also transform $E_1^*(w_1^*, g_1^*)$ into the origin and then expand the right-hand side of the map into Taylor series. For $(a_3, b_3, \mu_1, \mu_2, \xi_1, \xi_2) \in F_3$, the Jacobian matrix at $E_1^*(w_1^*, g_1^*)$ is

$$J(E_1^*) = \begin{pmatrix} 1 + w_1^* \alpha_3 & -w_1^* \beta_3 \\ b_3 - \xi_2 g_1^* & 1 - \mu_2 - \xi_2(N_1^* + g_1^*) \end{pmatrix}, \tag{52}$$

with

$$\begin{aligned}
\alpha_3 &:= \frac{a_3(1 + g_1^*)}{(1 + N_1^*)^2} - \xi_1, \\
\beta_3 &:= \frac{a_3 w_1^*}{(1 + N_1^*)^2} + \xi_1,
\end{aligned} \tag{53}$$

and it has two complex eigenvalues $\lambda_{1,2} = \pm i$. The corresponding eigenvector $\tilde{\zeta}_1 \in \mathbb{C}^2$ and adjoint eigenvector $\tilde{\zeta}_2 \in \mathbb{C}^2$ of $J(E_1^*)$ are defined as

$$\begin{aligned}
\tilde{\zeta}_1 &= \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \end{pmatrix} := \begin{pmatrix} w_1^* \beta_3 \\ 1 + w_1^* \alpha_3 - i \end{pmatrix}, \\
\tilde{\zeta}_2 &= \begin{pmatrix} \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} := \begin{pmatrix} \frac{1 + (1 + w_1^* \alpha_3)i}{2w_1^* \beta_3} \\ -\frac{1}{2}i \end{pmatrix}.
\end{aligned} \tag{54}$$

Any vector $Z(n) = (\hat{x}(n), \hat{y}(n))^T \in \mathbb{R}^2$ can be represented as $Z(n) = z(n)\tilde{\zeta}_1 + \bar{z}(n)\tilde{\zeta}_1$. Thus, for $(a_3, b_3, \mu_1, \mu_2, \xi_1, \xi_2) \in F_3$, model (16) can be written in the following complex form:

$$z(n+1) = \lambda_1 z(n) + \sum_{2 \leq i+j \leq 3} \frac{\sigma_{ij}}{i!j!} z^i(n) \bar{z}^j(n) + O(|z(n)|^4), \tag{55}$$

where $\lambda_1 = i$ and σ_{ij} ($2 \leq i + j \leq 3$) are defined as in (39) with $x_1, y_1, x_2, y_2, k_{ij}(a_2, b_2)$ replaced by $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, k_{ij}(a_3, b_3)$, respectively.

Similar to Section 4, by (40) and (41) we can transform (55) into the following form:

$$\phi(n+1) = \lambda_1 \phi(n) + \sum_{2 \leq i+j \leq 3} \frac{\hat{\sigma}_{ij}}{i!j!} \phi^i(n) \bar{\phi}^j(n) + O(|\phi(n)|^4), \tag{56}$$

where

$$\begin{aligned}
\hat{\sigma}_{20} &= (i+1)h_{20} + \sigma_{20}, \\
\hat{\sigma}_{11} &= (i-1)h_{11} + \sigma_{11}, \hat{\sigma}_{02} = (i+1)h_{02} + \sigma_{02}, \\
\hat{\sigma}_{30} &= 3(1-i)\sigma_{20}h_{20} + 3\sigma_{11}\bar{h}_{02} + \sigma_{30} - 3(i-1)h_{20}^2 - 3(i+1)h_{11}\bar{h}_{02} - 3i\bar{\sigma}_{02}h_{11}, \\
\hat{\sigma}_{21} &= 2\sigma_{11}\bar{h}_{11} + (1-2i)\sigma_{11}h_{20} + (2+i)\sigma_{20}h_{11} + \sigma_{02}\bar{h}_{02} + \sigma_{21} + (3i+1)h_{20}h_{11} \\
&\quad + 2(i-1)|h_{11}|^2 - 2ih_{11}\bar{\sigma}_{11} + (i+1)|h_{02}|^2 + i\bar{\sigma}_{02}h_{02},
\end{aligned}$$

FIGURE 1: 1 : 2 strong resonance bifurcation diagram at $(0.603987, 0.535162)$: (a) in $(a-w)$ plane with $b = 12.87486$ and (b) in $(b-a-w)$ space.

FIGURE 2: (a) Phase portraits for $a = 230.060692$ and $b = 12.87486$. (b) Phase portraits for $a = 230.060731$ and $b = 12.87486$. (c) Local amplification corresponding to (b). (d) The maximum Lyapunov exponents corresponding to (a).

FIGURE 3: 1:3 strong resonance bifurcation diagram at (0.61636576, 0.605198): (a) in $(a - w)$ plane with $b = 15.1575017$ and (b) in $(b - a - w)$ space.

FIGURE 4: (a) Phase portraits for $a = 248.97819$ and $b = 15.1575$. (b) Phase portraits for $a = 248.97816$ and $b = 15.157509$. (c)-(d) Local amplifications corresponding to (b).

$$\begin{aligned}
 \hat{\sigma}_{12} &= 2(i+1)\sigma_{11}h_{11} + \sigma_{11}\bar{h}_{20} + 2\sigma_{02}\bar{h}_{11} + \sigma_{20}h_{02} + \sigma_{12} - (i-1)h_{20}h_{02} - 2i\sigma_{02}h_{20} \\
 &\quad - 2(i+1)h_{11}^2 - (i+1)h_{11}\bar{h}_{20} - ih_{11}\bar{\sigma}_{20} - 2(i-1)h_{02}\bar{h}_{11} + 2i\bar{\sigma}_{11}h_{02}, \\
 \hat{\sigma}_{03} &= 3\sigma_{11}h_{02} + 3\sigma_{02}\bar{h}_{20} + \sigma_{03} + 3(i-1)h_{11}h_{02} + 3i\sigma_{02}h_{11} \\
 &\quad + 3(i+1)h_{02}\bar{h}_{20} + 3i\bar{\sigma}_{20}h_{02}.
 \end{aligned} \tag{57}$$

FIGURE 5: Maximum Lyapunov exponents of map (2) near (0.61636576, 0.605198) as a and b varying: (a) in $(a - \text{MLE})$ plane with $b = 15.1575017$ and (b) in $(b - a - \text{MLE})$ space.

FIGURE 6: 1 : 4 strong resonance bifurcation diagram at (0.627048, 0.671077): (a) in $(a - w)$ plane with $b = 17.42246$ and (b) in $(b - a - w)$ space.

Take

$$\begin{aligned} h_{20} &= -\frac{\sigma_{20}}{1+i}, \\ h_{11} &= \frac{\sigma_{11}}{1-i}, \\ h_{02} &= -\frac{\sigma_{02}}{1+i}, \end{aligned} \quad (58)$$

then (56) becomes into the following form:

$$\phi(n+1) = \lambda_1 \phi(n) + \sum_{i+j=3} \frac{\hat{\sigma}_{ij}}{i!j!} \phi^i(n) \bar{\phi}^j(n) + O(|\phi(n)|^4). \quad (59)$$

By (46) and (47), we can transform (59) into the following form as (51):

$$\chi(n+1) = \lambda_1 \chi(n) + \frac{\hat{\sigma}_{21}}{2} \chi(n)^2 \bar{\chi}(n) + \frac{\hat{\sigma}_{03}}{6} \bar{\chi}(n)^3 + O(|\chi(n)|^4). \quad (60)$$

Let $C(a_3, b_3) = -2i\hat{\sigma}_{21}$ and $D(a_3, b_3) = -(2/3)i\hat{\sigma}_{03}$. If $D(a_3, b_3) \neq 0$, we denote $B = ((C(a_3, b_3))/(|D(a_3, b_3)|))$. From the 1:4 strong resonance bifurcation analysis established in [29], we can obtain the following results.

Theorem 3. Assume that $D(a_3, b_3) \neq 0$, $\text{Re } B \neq 0$, and $\text{Im } B \neq 0$. Then, model (2) has a 1:4 strong resonance bifurcation at $E_1^*(w_1^*, g_1^*)$. B determines the bifurcation scenario near E_1^* . Near the equilibrium E_1^* , there exist two-parameter families of equilibria of order four bifurcation from E_1^* , and one of them is unstable, or attractive or repelling invariant closed curves, which depends on the choices of parameters a and b . Moreover, there are many complex

FIGURE 7: Phase portraits: (a) $a = 267.2007$ and $b = 17.422455$; (b) $a = 267.2008$ and $b = 17.422455$; (c) $a = 267.20079$ and $b = 17.42246$; (d) $a = 267.20076$ and $b = 17.42246$.

FIGURE 8: Maximum Lyapunov exponents of map (2) near $(0.627048, 0.671077)$ as a and b varying: (a) in $(a - \text{MLE})$ plane with $b = 17.42246$ and (b) in $(b - a - \text{MLE})$ space.

FIGURE 9: (a) Chaotic behavior of w for uncontrolled system (2). (b) Time responses for the state w of controlled system (61). (c) Bifurcation diagram of controlled system (68).

codimension-1 bifurcation curves of map (60) in a sufficiently small neighborhood of (a_3, b_3) .

6. Numerical Simulations

In this section, we give a simulation illustration for codimension-2 bifurcations with 1:2, 1:3, and 1:4 strong resonances. The codimension-2 bifurcation diagrams, maximum Lyapunov exponents, and phase portraits of (2) are presented to illustrate the validity of theoretical results and display the complex dynamical behaviors.

Case 1. Choosing the values of the parameters as $\mu_1 = 8, \mu_2 = 2, \xi_1 = 50$, and $\xi_2 = 11$, we can numerically obtain $a_1 = 230.060722$ and $b_1 = 12.87486$ and equilibrium $E_1^*(w_1^*, g_1^*) = (0.603987, 0.535162)$. The corresponding eigenvalues of $J(E_1^*)$ are $\lambda_{1,2} = -1$. By further calculations, we have $C(a_1, b_1) = 2.1132$ and $D(a_1, b_1) = 4.2543$. From Theorem 1, map (2) is equivalent to the normal form for 1:2 strong resonance bifurcation and E_1^* is a 1:2 strong resonance bifurcation point.

Figure 1(a) shows 2-dimensional bifurcation diagram when $b = b_1 = 12.87486$ and a varies in $[230.060715, 230.06075]$, which is a neighborhood of $a_1 = 230.060722$. Figure 1(b) shows 3-dimensional bifurcation diagram when (a, b) vary in a neighborhood of $(a_1, b_1) = (230.060722, 12.87486)$. Figures 2(a)–2(c) show the phase portraits of (2) near equilibrium E_1^* for different parameters a and b . From the phase portraits, we know that there are periodic orbits and chaos at the corresponding parameter values.

The maximum Lyapunov exponents corresponding to the bifurcation diagram Figure 1(a) are presented in Figure 2(d). From Figure 2(d), we can see that some maximum Lyapunov exponents are positive and some ones are negative, which means that map (2) has chaotic behaviors near the 1:2 strong resonance point E_1^* .

Case 2. Choosing the values of the parameters as $\mu_1 = 8, \mu_2 = 2, \xi_1 = 50$, and $\xi_2 = 11$, we can numerically obtain $a_2 = 248.97816$ and $b_2 = 15.1575017$ and equilibrium $E_1^*(w_1^*, g_1^*) = (0.61636576, 0.605198)$. The corresponding

eigenvalues of $J(E_1^*)$ are $\lambda_{1,2} = -(1/2) \pm (\sqrt{3}/2)i$. By further calculations, we have $C(a_2, b_2) = 1.8958 - 1.1471i$ and $D(a_2, b_2) = -8.362 + 5.4278i$. From Theorem 2, map (2) is equivalent to the normal form for 1:3 strong resonance bifurcation and E_1^* is a 1:3 strong resonance bifurcation point.

Figure 3(a) shows 2-dimensional bifurcation diagram when $b = b_2 = 15.1575017$ and a varies in $[248.97811, 248.97822]$, which is a neighborhood of $a_2 = 248.97816$. Figure 3(b) shows 3-dimensional bifurcation diagram when (a, b) vary in a neighborhood of $(a_2, b_2) = (248.97816, 15.1575017)$. Figures 4(a)–4(d) show phase portraits of map (2) near E_1^* for different a and b . From Figure 4(a), we can see that there is a homoclinic structure formed by the stable and unstable invariant manifolds of the period three cycle intersecting transversally in a narrow parameter region.

The maximum Lyapunov exponents corresponding to the bifurcation diagrams Figures 3(a) and 3(b) are presented in Figures 5(a) and 5(b), respectively. We can see that some maximum Lyapunov exponents are positive and some ones are negative, which means that map (2) has chaotic behaviors near the 1:3 strong resonance point E_1^* .

Case 3. Choosing the values of the parameters as $\mu_1 = 8, \mu_2 = 2, \xi_1 = 50$, and $\xi_2 = 11$, we can numerically obtain $a_3 = 267.20079$ and $b_3 = 17.42246$ and equilibrium $E_1^*(w_1^*, g_1^*) = (0.627048, 0.671077)$. The corresponding eigenvalues of $J(E_1^*)$ are $\lambda_{1,2} = \pm i$. By further calculations, we have $C(a_3, b_3) = -1.3731 - 9.4851i$ and $D(a_3, b_3) = -4.2121 + 2.2817i$. From Theorem 3, map (2) is equivalent to the normal form for 1:4 strong resonance bifurcation and E_1^* is a 1:4 strong resonance bifurcation point.

Figure 6(a) shows 2-dimensional bifurcation diagram when $b = b_3 = 17.42246$ and a varies in $[267.2004, 267.2015]$, which is a neighborhood of $a_3 = 267.20079$. From Figure 6(a), we can see that there are flip bifurcations occur after Neimark–Sacker bifurcation near the 1:4 strong resonance point. Figure 6(b) shows 3-dimensional bifurcation diagram when (a, b) vary in a neighborhood of $(a_3, b_3) = (267.20079, 17.42246)$. Figures 7(a)–7(d) show phase portraits of map (2) near E_1^* for different a and b . When $a = 267.2007$ and $b = 17.422455$, the equilibrium is locally stable. When a and b increase, map (2) has chaotic behaviors.

The maximum Lyapunov exponents corresponding to the bifurcation diagrams (Figures 6(a) and 6(b)) are presented in Figures 8(a) and 8(b), respectively. We can see that some maximum Lyapunov exponents are positive and some ones are negative, which means that map (2) has chaotic behaviors near the 1:4 strong resonance point E_1^* .

7. Controller Design

In this section, we apply two control strategies (tracking control and hybrid control) to (2). By using tracking control, we can annihilate wide mosquitoes and then prevent the transmission of mosquito-borne diseases. By applying hybrid control, we can advance the codimension-2 bifurcation and extend the range of parameter a from the equilibrium to chaos.

7.1. Tracking Control. In this subsection, we design a tracking controller such that the number of wide mosquitoes vanishes gradually as in [23]. Consider a new controlled system

$$\begin{cases} w(n+1) = w(n) \exp \left[\frac{aw(n)}{1+w(n)+g(n)} - (\mu_1 + \xi_1(w(n) + g(n))) \right], \\ g(n+1) = g(n) + bw(n) - (\mu_2 + \xi_2(w(n) + g(n)))g(n) + R(n), \end{cases} \quad (61)$$

where R is an exterior control signal. When $R = 0$, map (61) has chaos or period orbit under suitable parameter conditions. Let $w^* = 0$ be our control aim, and we shall design a

tracking controller R such that $w(n) \rightarrow w^* = 0$ as the number of iterations is sufficiently large.

Theorem 4. *If the controller of controlled map (61) is*

$$R(n) = \frac{-(\ln \vartheta(n) + \mu_1 + \xi_1) + \sqrt{(\ln \vartheta(n) + \mu_1 + \xi_1)^2 + 4\xi_1(a\Phi_2(n) - \ln \vartheta(n) - \mu_1)}}{2\xi_1} - [\Phi_2(n) + g(n) + bw(n) - (\mu_2 + \xi_2(w(n) + g(n)))g(n)], \quad (62)$$

where

$$\begin{aligned}\vartheta(n) &= k_1 \exp \left[-\frac{aw(n)}{1+w(n)+g(n)} + (\mu_1 + \xi_1(w(n) \right. \\ &\quad \left. + g(n))) \right] + k_2, \\ \Phi_2(n) &= w(n) \exp \left[\frac{aw(n)}{1+w(n)+g(n)} - (\mu_1 + \xi_1(w(n) \right. \\ &\quad \left. + g(n))) \right],\end{aligned}\quad (63)$$

and the constants k_1 and k_2 are positive constants satisfying all roots of $z^2 - k_2z - k_1 = 0$ lie in the unit circle, then $w(n) \rightarrow w^* = 0$ as number of iterations tends to infinity.

Proof. Take the following transformation

$$\begin{cases} \Phi_1(n) = w(n), \\ \Phi_2(n) = w(n) \exp \left[\frac{aw(n)}{1+w(n)+g(n)} - (\mu_1 + \xi_1(w(n) + g(n))) \right], \end{cases}\quad (64)$$

then map (61) becomes

$$\begin{cases} \Phi_1(n+1) = w(n+1) = \Phi_2(n), \\ \Phi_2(n+1) = \Psi(n), \end{cases}\quad (65)$$

where

$$\begin{aligned}\Psi(n) &= w(n+1) \exp \left[\frac{aw(n+1)}{1+w(n+1)+g(n+1)} - (\mu_1 + \xi_1(w(n+1) + g(n+1))) \right] \\ &= \Phi_2(n) \exp \left\{ \frac{a\Phi_2(n)}{1+\Phi_2(n)+g(n)+b\Phi_1(n) - (\mu_2 + \xi_2(\Phi_1(n) + g(n)))g(n) + R(n)} \right. \\ &\quad \left. - [\mu_1 + \xi_1(\Phi_2(n) + g(n) + b\Phi_1(n) - (\mu_2 + \xi_2(\Phi_1(n) + g(n)))g(n) + R(n))] \right\}.\end{aligned}\quad (66)$$

Taking (62) into (66), we can obtain the following map:

$$\begin{cases} \Phi_1(n+1) = \Phi_2(n), \\ \Phi_2(n+1) = k_1\Phi_1(n) + k_2\Phi_2(n). \end{cases}\quad (67)$$

By choosing suitable positive constants k_1 and k_2 such that all roots of $z^2 - k_2z - k_1 = 0$ lie in the unit circle, map (61) after feedback is asymptotically stable. That is, $\Phi_2(n) \rightarrow 0$ as the number of iterations tends to infinity. Therefore, the trajectory of w tracks an ideal state $w^* = 0$. This completes the proof.

We choose the values of the parameters as $\mu_1 = 8$, $\mu_2 = 2$, $\xi_1 = 50$, $\xi_2 = 11$, $a = 267.2014$, and $b = 17.42246$, and

take $k_1 = 0.75$ and $k_2 = 0.125$ in $R(n)$. When the number of iterations is less than 2000, the chaotic behavior of $w(n)$ for uncontrolled system (2) is presented in Figure 9(a). When $n > 2000$, the controller is put on (2). From Figure 9(b), we can see that the trajectory of $w(n)$ tracks ultimately an ideal state $w^* = 0$. $\square$

7.2. Hybrid Control. In this section, we apply hybrid control, which is used in [28, 30], to (2). For any $0 < \tau < 1$, consider a new controlled system

$$\begin{cases} w(n+1) = \tau \left\{ w(n) \exp \left[\frac{aw(n)}{1+w(n)+g(n)} - (\mu_1 + \xi_1(w(n) + g(n))) \right] \right\} + (1-\tau)w(n), \\ g(n+1) = \tau [g(n) + bw(n) - (\mu_2 + \xi_2(w(n) + g(n)))g(n)] + (1-\tau)g(n), \end{cases}\quad (68)$$

which has the same equilibria with (2). The Jacobian matrix at equilibrium $E(w, g)$ is

$$J(E) = \begin{pmatrix} 1 + \tau w \left(\frac{a(1+g)}{(1+N)^2} - \xi_1 \right) & -\tau w \left(\frac{aw}{(1+N)^2} + \xi_1 \right) \\ \tau(b - \xi_2 g) & 1 - \tau\mu_2 - \tau\xi_2(N+g) \end{pmatrix}, \quad (69)$$

its trace is $Tr(J(E)) = 2 + \tau w(((a(1+g)/(1+N)^2)) - \xi_1) - \mu_2 - \xi_2(N+g) = 2 + (\tau/a)A(N)$, and determinant is $Det(J(E)) = 1 - (w/(1+N))\tau^2 F'(N) + (\tau/a)A(N)$. By the analysis in Section 2, we know that only $E_1^*(w_1^*, g_1^*)$ may be a 1:2, 1:3, or 1:4 strong resonance point.

Here, we only consider the codimension-2 bifurcation with 1:4 strong resonance. Taking $\mu_1 = 8, \mu_2 = 2, \xi_1 = 50, \xi_2 = 11$, and $\tau = 0.8$ in (68), we can numerically obtain $a_4 = 258.9192$ and $b_4 = 16.3854$ and equilibrium $E_1^*(w_1^*, g_1^*) = (0.6234, 0.64176)$, which is a 1:4 strong resonance point. The bifurcation diagram of controlled system (68) for fixed $b = b_4 = 16.3854$ is presented in Figure 9(c) and is very similar to Figure 6(a) of the uncontrolled system. Recall that for the uncontrolled system, we have $a_3 = 267.20079$ and $b_3 = 17.42246$. Thus, by using hybrid control, we advance the corresponding codimension-2 bifurcation with 1:4 strong resonance. Compared with these two figures, we can also find the range of parameter a from 1:4 strong resonance point to chaos in Figure 9(c) is about three times larger than that of Figure 6(a).

8. Discussion

In this paper, we analyzed codimension-two bifurcations with 1:2, 1:3, and 1:4 strong resonances of a discrete wild and sterile mosquito model, which is established from continuous case (1) proposed in [7] by the Euler forward difference method. In [10, 11], we have known that continuous model (1) undergoes a sequence of bifurcations such as saddle-node bifurcation, supercritical and subcritical Hopf bifurcation, homoclinic bifurcation, and Bogdanov-Takens bifurcation. Through the theoretical analysis and numerical simulations given in this paper, we find that corresponding discrete mosquito model (2) also has many complex dynamical behaviors, including codimension-1, codimension-2 bifurcations, and chaos. From the impact of the mating rate a and the release rate coefficient b on the dynamics of wide mosquitoes, we know that the release strategy plays an important role in controlling the number of wide mosquitoes. Moreover, two control strategies are applied to model (2). However, whether there is codimension-2 bifurcation with 1:1 strong resonance is an interesting problem worth considering.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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