# Fractal Control and Synchronization of the Discrete Fractional SIRS Model 

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#### Abstract

SIRS model is one of the most basic models in the dynamic warehouse model of infectious diseases, which describes the temporary immunity after cure. The discrete SIRS models with the Caputo deltas sense and the theories of fractional calculus and fractal theory provide a reasonable and sensible perspective of studying infectious disease phenomenon. After discussing the fixed point of the fractional order system, controllers of Julia sets are designed by utilizing fixed point, which are introduced as a whole and a part in the models. Then, two totally different coupled controllers are introduced to achieve the synchronization of Julia sets of the discrete fractional order systems with different parameters but with the same structure. And new proofs about the synchronization of Julia sets are given. The complexity and irregularity of Julia sets can be seen from the figures, and the correctness of the theoretical analysis is exhibited by the simulation results.


## 1. Introduction

The control and prevention of infectious disease in practice are always the integral part in ensuring human health and security. Infectious diseases not only endanger human fitness, but also lead to huge disaster to the national economy and people's livelihood. Due to the inability of large scale experiments on infectious diseases, human beings are suffering hardships on the way to resist infectious diseases. Therefore, it is particularly vital to analyze them through theoretical quantitative research. Epidemic dynamic is a branch of biological mathematics with momentous practical significance, which establishes a model to reflect the quantitative change of infectious diseases in the process of epidemic according to the relationship between disease infection and immunity among populations. Since the classical SIS and SIR compartment models of infectious diseases were put forward in 1927, they have been widely used in the study of infectious disease dynamics. Based on qualitative and quantitative analysis and simulation of the dynamic model of infectious diseases, the key factors of disease transmission are analyzed. And the
optimal strategies for controlling diseases can be targeted according to these theoretical results. This provides some useful information and reference value for people to prevent and treat diseases. Hence, it is of great actual meaning to study the dynamic model of infectious diseases theoretically [1, 2].

A large number of studies show that fractional calculus has more reasonable results than integral calculus in describing the systems with memory and genetic characteristics. Fractional models can describe complex physics problems more clearly and concisely, especially the nonlinear model and their physical meaning. Research has shown that the fractional order equation offers a possibility for the situation that the traditional integer order equation cannot model [3]. Most importantly, the most prominent feature of the body's immune system is about its memory. Coincidentally, the fractional order equations have the characteristics of memory [4]. Integer order equations cannot express the memory characteristics of the human immune system and its dependence on past history, while fractional order systems contain all information from the start moment to current.

For a long time, the infectious disease model has been principally about the ordinary differential equation model [5-7]. With the rapid development of fractional order differential equation, many researchers try to utilize them to study the dynamics of infectious diseases. In recent years, with the continuous development of mathematical theory, infectious disease models have been constantly enriched. A series of theoretical tools such as stochastic difference equation, functional differential equation, partial differential equation, stochastic differential equation, and fractional differential equation [8-12] have emerged, broadening the types of infectious disease models. Moreover, there are many research achievements on the applications of fractional order calculus in discrete systems [13-15], and the discrete models are more practical because epidemiological statistics are collected in discrete time. The fractional order differential equation can be converted into the fractional order difference equation through discretization. As the discrete system is obviously different from the continuous system, the conclusion in the continuous fractional order system cannot be simply extended to the discrete system. It is particularly important to study infectious diseases through rational analysis of discrete fractional difference systems.

The controlling problem of the fractional order system is considered as a new branch of control field because the fractional order system has the characteristics of noninteger, which cannot be simply analyzed by using the traditional classical control theory. Moreover, the combination of the control field and fractional order systems further promotes the evolution of fractional order theory. Therefore, the fractional order control system has achieved considerable research results in both theoretical system and practical application [16-18]. The analysis and control of nonlinear systems are prominent to understand dynamic systems. Among them, there is a class of problems related to the Julia set, see [19, 20]. Julia set can be regarded as a tool to judge the stability of the system, and the analysis of Julia sets can put us a favorable position to better understand the system. In addition, people can control infectious diseases by controlling the Julia sets of the systems. Therefore, how to effectively control Julia set becomes very critical. Scholars have performed a wide range of studies on controlling the model of integer order infectious diseases [21, 22], but there are relatively few studies on controlling the model of the fractional order. We attempt to extend the control methods of the integer order to the fractional order system and utilize the fractional order difference equation to study the control of diseases. For example, by referring to the fixed point analysis method of the integer order nonlinear system, the fixed point method of the fractional order is found, so as to control the Julia set of the fractional order infectious disease system. Recently, synchronization problems in chaotic systems have been widely studied [23-25]. Besides, synchronization control fosters a reasonable and sensible perspective of comprehending fractional order systems [26-29]. Master-slave synchronization of chaotic fractionalorder Ikeda delay systems with linear coupling is studied in [30]. Wang and Song [31] analyze the synchronization
conditions of the fractional order chaotic systems with the activation feedback control method. In [32], the authors utilize active control technique to synchronize different fractional order chaotic dynamical systems.

The main content of this paper is the follow-up work of the literature [33], which applies the ideas of Julia set in fractal theory and basic theoretical knowledge of fractional calculus to the SIRS model. The remainder of this article is arranged as follows. In Section 2, the Julia set of the discrete fractional difference SIRS model is introduced, fixed points of the system are calculated, and effective control methods are proposed. In Section 3, the synchronization of Julia sets between systems is realized by designing two totally different controllers and novel proofs about synchronization are presented. Besides, the simulation results are given, and the images reveal the complexity and irregularity of the Julia set. Section 4 summarizes and anticipates the work of this paper.

Moreover, in [33], the authors researched the Julia set of the discrete fractional SIRS model and designed three different controllers, which are added to different parts of the model as a whole, a part, and a product factor, respectively, to change the Julia set. Nevertheless, in this paper, two kinds of control items including fixed points are designed for the fractional order model, which are added into the system as a whole and a part, respectively. In addition, we also introduce two different synchronization methods and exhibit novel proofs about the synchronization between systems. The methods to control Julia sets are completely different in these two articles.

## 2. Preliminaries

Definition 1 (see [34]). Hausdorff distance is a measure of similarity between two sets of points. It is a definition of distance between two sets of points. Suppose there are two sets $A=\{a 1, \ldots, a p\}$ and $B=\{b 1, \ldots, b q\}$, and then the Hausdorff distance between the two sets of points is defined as $H(A, B)=\max (h(A, B), h(B, A))$. Among them, $h(A, B)=$ $\max (a \in A) \min (b \in B)\|a-b\|$, and $h(B, A)=\max (b \in B)$ $\min (a \in A)\|b-a\|,\|\cdot\|$ is the distance paradigm (e.g., $L 2$ or Euclidean distance) between the set of points $A$ and $B$.

Here, $H(A, B)$ is called the two-way Hausdorff distance, which is the most basic form of the Hausdorff distance. And $h(A, B)$ and $h(B, A)$ are, respectively, called the one-way Hausdorff distance from set $A$ to set $B$ and from set $B$ to set $A$. That is, $h(A, B)$ actually sorts the distance between each point $a i$ in point set $A$ and the point $b j$ in set $B$ which is closest to the point ai. The two-way Hausdorff distance $H(A, B)$ is the larger of the one-way distance $h(A, B)$ and $h(B, A)$, which measures the maximum mismatch between the two sets of points.

## 3. Julia Set of Discrete Fractional Order SIRS Model and Fixed Point Control

Consider system (2) of the SIRS model in [33], in which the effective contact rate coefficient is affected by seasonal factors:

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} \tau_{1}}{\mathrm{~d} t}=\beta_{0}(1+\varphi \sin (\omega t)) \tau_{1}^{2}\left(1-\tau_{1}-\tau_{2}\right)-(d+v) \tau_{1}  \tag{1}\\
\frac{\mathrm{~d} \tau_{2}}{\mathrm{~d} t}=p+v \tau_{1}-(d+\varepsilon) \tau_{2}
\end{array}\right.
$$

where $\tau_{1}$ represents the number of persons susceptible to infection and $\tau_{2}$ expresses the number of infected person. Suppose the birth and natural death coefficients of the population are equivalent to the constant $d$ during epidemics, $p$ signifies the inoculation rate, the removal rate coefficient is shown by $v$, the immune loss rate coefficient is denoted by $\varepsilon, \beta_{0}$ means the effective contact rate coefficient, and the seasonal influence factor is $\varphi$.

Take $\tau_{3}=\sin (\omega t)$ and $\tau_{4}=\cos (\omega t)$; thus, $\left(\mathrm{d} \tau_{3} / \mathrm{d} t\right)=$ $\omega \tau_{4}$ and $\left(\mathrm{d} \tau_{4} / \mathrm{d} t\right)=-\omega \tau_{3}$. The following system (2) is obtained:

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} \tau_{1}}{\mathrm{~d} t}=\beta_{0}\left(1+\varphi \tau_{3}\right) \tau_{1}^{2}\left(1-\tau_{1}-\tau_{2}\right)-(d+v) \tau_{1} \\
\frac{\mathrm{~d} \tau_{2}}{\mathrm{~d} t}=p+v \tau_{1}-(d+\varepsilon) \tau_{2} \\
\frac{\mathrm{~d} \tau_{3}}{\mathrm{~d} t}=\omega \tau_{4} \\
\frac{\mathrm{~d} \tau_{4}}{\mathrm{~d} t}=-\omega \tau_{3}
\end{array}\right.
$$

where the initial values of $\tau_{3}$ and $\tau_{4}$ are $\tau_{3}(0)=0$ and $\tau_{4}(0)=1$. According to the method and discretization process of the literature [33], discretize the equation by using

$$
\begin{equation*}
\dot{\tau}_{i} \longrightarrow \frac{\tau_{i}(t+\Delta t)-\tau_{i}(t)}{\Delta t}, \quad i=1,2,3,4 \tag{3}
\end{equation*}
$$

So, the discrete version of system (2) is acquired:

$$
\left\{\begin{array}{l}
x_{n+1}=(1-c-q) x_{n}+\eta\left(1+b s_{n}\right)\left(1-x_{n}-y_{n}\right) x_{n}^{2}  \tag{4}\\
y_{n+1}=q x_{n}+(1-c-e) y_{n}+z \\
s_{n+1}=s_{n}+\delta r_{n} \\
r_{n+1}=r_{n}-\delta s_{n}
\end{array}\right.
$$

where $s_{0}=0, r_{0}=1$, and $\eta, b, c, q, e, z$, and $\delta$ are system parameters. Now, refer to the method of converting the integer order equation into fractional order equation in [33], subtract $x_{n}, y_{n}, s_{n}$, and $r_{n}$ from both sides of the 4 equations of system (4), rewrite the equation as a fractional one from the perspective of discrete fractional calculus, and take the momentums $x(n), y(n), s(n)$, and $r(n)$ into account. Therefore, the numerical equations can be accurately proposed to the following one, the fractional difference about 4 equations is introduced:

$$
\left\{\begin{array}{l}
x(n)=x(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x(j-1)+\eta(1+b s(j-1))(1-x(j-1)-y(j-1)) x(j-1)^{2}\right),  \tag{5}\\
y(n)=y(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q x(j-1)-(c+e) y(j-1)+z), \\
s(n)=s(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta r(j-1)), \\
r(n)=r(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta s(j-1)) .
\end{array}\right.
$$

Next, the thought of Julia set in fractal theory is used to study fractional order dynamic systems (5).

$$
\begin{align*}
& F(x, y, s, r)=x+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x+\eta(1+b s)(1-x-y) x^{2}\right), \\
& G(x, y, s, r)=y+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q x-(c+e) y+z), \\
& S(x, y, s, r)=s+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta r),  \tag{6}\\
& R(x, y, s, r)=r+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta s), \\
& H(x, y, s, r)=(F(x, y, s, r), G(x, y, s, r), S(x, y, s, r), R(x, y, s, r)) .
\end{align*}
$$

The set
$D=\left\{(x, y, 0,1) \mid\left\{H^{n}(x, y, 0,1)\right\}_{n=1}^{\infty}\right.$ remains bounded $\}$,
is referred to the filled Julia set which is correspondent with the map $H(x, y, 0,1)$. Besides, the boundary of $D$ is the Julia set of the map $H(x, y, 0,1)$, and it is indicated by $J_{H}$, which means, $J_{H}=\partial D$.

Obviously, simulating system (5) is difficult because it has four dimensions. Fortunately, the equation can be
viewed as a mapping on the $x-y$ plane because the initial values of the third and fourth variables are already settled.

The fractional order of the model are taken to be $\mu=0.8$ and the system parameters are taken to be $\eta=0.08, b=$ $0.02, c=0.005, q=0.005, e=0.05, z=0.005$, and $\delta=0.001$, Figure 1 shows the original correlative Julia set of system (5).

Now, the fixed point is utilized to consider the stability of system (5). Let the fixed point of the equation be $\left(x^{*}, y^{*}, s^{*}, r^{*}\right)$, so the following equation is presented:

$$
\left\{\begin{array}{l}
x^{*}=x^{*}+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x^{*}+\eta\left(1+b s^{*}\right)\left(1-x^{*}-y^{*}\right) x^{* 2}\right) \\
y^{*}=y^{*}+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(q x^{*}-(c+e) y^{*}+z\right)  \tag{8}\\
s^{*}=s^{*}+\frac{\delta}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} r^{*} \\
r^{*}=r^{*}-\frac{\delta}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} s^{*} .
\end{array}\right.
$$



Figure 1: Original Julia sets of system (5) when $\mu=0.8, \eta=0.08, b=0.02, c=0.005, q=0.005, e=0.05, z=0.005, \delta=0.001$.
From [35], for $\alpha \in R$ and $n \in N_{0}$ one has $\sum_{j=0}^{n}\binom{j-\alpha-1}{j}=\binom{n-\alpha}{n}$. Besides, this equation is equivalent to $(1 / \Gamma(\mu)) \sum_{j=0}^{n}(\Gamma(n-j+\mu) / \Gamma(n-j+1))=$ of $-\alpha=\mu$ because $\binom{a}{b}:=\Gamma(a+1) /(\Gamma(b+1) \Gamma(a-b+1))$. Also, referring to [36], this most striking conclusion can be reached. So, the equation below is derived: $\binom{n+\mu}{n}=(\Gamma(n+\mu+1) / \Gamma(n+1) \Gamma(\mu+1))$ on the condition

$$
\left\{\begin{array}{l}
x^{*}=x^{*}+\frac{\Gamma(n+\mu)}{\Gamma(n) \Gamma(\mu+1)}\left(-(c+q) x^{*}+\eta\left(1+b s^{*}\right)\left(1-x^{*}-y^{*}\right) x^{* 2}\right)  \tag{9}\\
y^{*}=y^{*}+\frac{\Gamma(n+\mu)}{\Gamma(n) \Gamma(\mu+1)}\left(q x^{*}-(c+e) y^{*}+z\right) \\
s^{*}=s^{*}+\delta \frac{\Gamma(n+\mu)}{\Gamma(n) \Gamma(\mu+1)} r^{*} \\
r^{*}=r^{*}-\delta \frac{\Gamma(n+\mu)}{\Gamma(n) \Gamma(\mu+1)} s^{*}
\end{array}\right.
$$

For the above equation, if $\delta=0, s^{*}$ and $r^{*}$ can be arbitrary numbers, and if $\delta \neq 0$, thus $s^{*}=0$ and $r^{*}=0$. There is no significance of studying in the case of $\delta=0$, so just consider the fixed point that both $s^{*}$ and $t^{*}$ are zero. The fixed point $\left(x^{*}, y^{*}\right)$ is as follows:
(i) If $\quad x^{*}=0 \quad$ and $\quad y^{*} \neq 0, \quad(c+e) y^{*}=v \quad$ and $\left(x^{*}, y^{*}\right)=(0,(v / c+e))$.
(ii) If $x^{*} \neq 0$ and $y^{*}=0, d x^{*}+v=0,(c+d) x^{*}+a$ $\left(x^{*}-1\right) x^{* 2}=0, v d^{3}+c v d^{2}+a v^{2} d+a v^{3}=0 \quad$ and $\left(x^{*}, y^{*}\right)=(((-v) / d), 0)$.
(iii) If $x^{*} \neq 0$ and $y^{*} \neq 0$,

$$
\begin{align*}
& x^{*}=\frac{(c+e-v)}{2(c+e+d)}-\frac{\sqrt{a^{2}(v-c-e)^{2}-4 a(c+e+d)(c+d)(c+e)}}{2 a(c+e+d)} \\
& y^{*}=\frac{d x^{*}+v}{c+e} \tag{10}
\end{align*}
$$

In general, cases (i) and (ii) have no actual significance. The common phenomenon is that the infected coexist the recovered when an outbreak occurs. Furthermore, they can coexist for a long time and neither of them will suddenly go to zero. Therefore, we only consider case (iii). Take the same system parameters; thus, the fixed point of the system is $\left(x^{*}, y^{*}, s^{*}, r^{*}\right)=(0.6596$, $0.1509,0,0)$. There are three ways to import control items, see [33]. Here, in order to control Julia set of model (5), the control items by use of the fixed points are introduced into different parts of the model as a whole and a part. So, the following controlled system (11) and system (12) are obtained.

Firstly, control items with the fixed points $l\left(x^{2}(j-1)-x^{* 2}\right)\left(y(j-1)-y^{*}\right), l\left(y(j-1)-y^{*}\right), l(s(j-$ $\left.1)-s^{*}\right)$, and $l\left(r(j-1)-r^{*}\right)$ are introduced into the model as a whole, which means that the controllers are added to the inside of the summation sign to control the Julia set:

$$
\left\{\begin{array}{l}
x(n)=x(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}-\left((c+q) x(j-1)+\eta(1+b s(j-1))(1-x(j-1)-y(j-1)) x(j-1)^{2}\right.  \tag{11}\\
\left.\quad+l\left(x^{2}(j-1)-x^{* 2}\right)\left(y(j-1)-y^{*}\right)\right), \\
y(n)=y(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(q x(j-1)-(c+e) y(j-1)+z+l\left(y(j-1)-y^{*}\right)\right), \\
s(n)=s(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(\delta r(j-1)+l\left(s(j-1)-s^{*}\right)\right), \\
r(n)=r(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-\delta s(j-1)+l\left(r(j-1)-r^{*}\right)\right) .
\end{array}\right.
$$

Take the fractional order $\mu$ as $\mu=0.8$ and the control parameter $l$ as (a) $l=0.02$, (b) $l=0.04$, (c) $l=0.05$, (d) $l=$ 0.06 , $(e) l=0.07$, and $(f) l=0.08$. Figure 2 shows the relevant Julia set with different control parameters $l$.

With the constant tiny increase of parameter $l$, the structure of Julia set becomes more and more delicate.

Compared with Figure 2(e), figure 2(f) shrinks apparently, and, the control of Julia set has been achieved.

Secondly, the fixed point control items $l\left(x^{2}(n-1)-x^{* 2}\right)\left(y(n-1)-y^{*}\right), l\left(y(n-1)-y^{*}\right), l(s(n-$ $\left.1)-s^{*}\right)$, and $l\left(r(n-1)-r^{*}\right)$ are added into the model as a part; more precisely, the controller is imported to the outside of the summation sign to implement control over Julia set:

$$
\left\{\begin{array}{l}
x(n)=x(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x(j-1)+\eta(1+b s(j-1))(1-x(j-1)-y(j-1)) x(j-1)^{2}\right)  \tag{12}\\
\quad+l\left(x^{2}(n-1)-x^{* 2}\right)\left(y(n-1)-y^{*}\right), \\
y(n)=y(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q x(j-1)-(c+e) y(j-1)+z)+l\left(y(n-1)-y^{*}\right), \\
s(n)=s(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta r(j-1))+l\left(s(n-1)-s^{*}\right), \\
r(n)=r(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta s(j-1))+l\left(r(n-1)-r^{*}\right) .
\end{array}\right.
$$

Also, the fractional order $\mu$ is taken as $\mu=0.8$ and the control parameters $l$ are fetched as (a) $l=0.001,(b) l=$ 0.002 , (c) $l=0.005,(d) l=0.008$, (e) $l=0.015$, and $(f) l=$ 0.017 , and the images about the correlative Julia set with disparate control parameters $l$ are exhibited in Figure 3.

In Figure 3, as the parameter $l$ increases, the area of the filled-in Julia set becomes smaller and smaller. In Figures 3(e) and 3(f), some points are not in the interior of the Julia set, which means that the connectivity of the Julia set is closely interrelated with the control parameter $l$.

If the values are inside the Julia set, the trajectory is bounded and the number of infected people is limited. Relatively, if the values are outside the Julia set, the trajectory is unbounded; at this moment, the number of infected people is unlimited and the epidemic is out of control. In this section, the fixed point control methods are designed for the discrete fractional order SIRS model; from these figures, some initial values are iterated from points outside the set into the set, which means the infected person is controlled. The control of infectious diseases is realized by controlling Julia sets of systems.


FIGURE 2: The changing process of Julia sets of the controlled system (11) when (a) $l=0.02$; (b) $l=0.04$; (c) $l=0.05$; (d) $l=0.06$; (e) $l=0.07$; (f) $l=0.08$.

## 4. Synchronization of Julia Sets of the Discrete Fractional Order SIRS Model

Different biological systems sometimes require similar or even identical characteristics. Consider synchronization in different
systems is necessary. In this section, the synchronization methods are applied to the investigation of Julia set.

Theorem 1. If $|C(n)-x(n)|,|D(n)-y(n)|,|A(n)-s(n)|$, and $|B(n)-r(n)|$ are bounded for any finite $l$ and $n \in N$, the


FIGURE 3: The changing process of Julia sets of the controlled system (12) when (a) $l=0.001$; (b) $l=0.002$; (c) $l=0.005$; (d) $l=0.008$; (e) $l=0.015$; (f) $l=0.017$.
synchronization between system (5) and system (14) or (27) is realized.

Proof. Take $C(n)$ and $x(n)$ as an example. If $|C(n)-x(n)|$ is bounded, then $x(n)$ is bounded if $C(n)$ is bounded, and $C(n)$ is bounded if $x(n)$ is bounded. Therefore, $C(n)$ and
$x(n)$ have the same boundedness. And according to the definition of Julia set, $C(n)$ is synchronized with $x(n)$. Similarly, $D(n)$ is synchronized with $y(n), A(n)$ is synchronized with $s(n)$, and $B(n)$ is synchronized with $r(n)$. Therefore, the synchronization of Julia sets between system (5) and system (14) or (27) is acquired.

In this section, different coupling terms are designed to achieve synchronization between the initial system and the target system. By using the idea of the Julia set, the synchronization process is shown in our simulation figures.
4.1. The First Synchronization of Julia Sets. Consider system (13) with the same structure of equation form (5) but with different parameter $\hat{e}$ :

$$
\left\{\begin{array}{l}
C(n)=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-(c+q) C(j-1) \\
\left.\quad+\eta(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}\right), \\
D(n)=D(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q C(j-1) \\
\quad-(c+\widehat{e}) D(j-1)+z) \\
A(n)=A(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta B(j-1)) \\
B(n)=B(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta A(j-1))
\end{array}\right.
$$

where $\hat{e} \neq e>0$. In an effort to tie systems (5) and (13) together, nonlinear coupling terms are devised to enable one Julia set of the SIRS model change to be another. We added coupling terms $h_{1}$ and $h_{2}$ into equation (13), so we obtain

$$
\left\{\begin{array}{l}
C(n)=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-(c+q) C(j-1) \\
\left.\quad+\eta(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}\right)+h_{1}, \\
D(n)=D(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q C(j-1)) \\
\quad-(c+\widehat{e}) D(j-1)+z)+h_{2}, \\
A(n)=A(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta B(j-1)),
\end{array} \quad \begin{array}{l}
B(n)=B(0)-\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta A(j-1)),
\end{array}\right.
$$

where

$$
\begin{align*}
h_{1}= & l\left(\frac { 1 } { \Gamma ( \mu ) } \sum _ { j = 1 } ^ { n } \frac { \Gamma ( n - j + \mu ) } { \Gamma ( n - j + 1 ) } \left(-\eta(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}+\eta(1+b s(j-1))(1-x(j-1)\right.\right. \\
& \left.\left.-y(j-1)) x(j-1)^{2}\right)\right), \\
h_{2}= & l\left(\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}((c+\widehat{e}) D(j-1)-(c+e) y(j-1))\right) . \tag{15}
\end{align*}
$$

It is evident that different Julia sets are obtained by selecting different coupling parameters $l$. Taking the control parameter as (a) $l=0.09,(b) l=0.095,(c) l=0.1,(d) l=$ $0.2,(e) l=0.5$, and $(f) l=0.7$, other system parameters remain unchanged, and Figure 4 is obtained.

In Figure 4, Figure 4(a) is shown in blue, which is the background, and the synchronization process diagrams of Julia set of this method are shown in red. As can be seen from Figure 4, Julia set in Figure 4(a) looks like a star with four horns. In addition, with the increase of the control parameter $l$, the graph lines become mellow and full and the area of the Julia sets becomes larger and larger, eventually it overlaps with original Figure 1.

We use Hausdorff distance to measure the matching degree between the images, as shown in Table 1; the Hausdorff distances gradually decrease to 0 , and two images are getting closer and closer. And Hausdorff distance has turned into 0 since Figure 4(d), seen from Figure 4(d), some of the points have coincided with the original figure. By the definition of Hausdorff distance, it becomes 0 as long as two points overlap. Therefore, it also reflects the synchronization between system (5) and system (14).

Next, we give a proof about this method to achieve the synchronization of trajectories between systems (5) and (14). In order to promote the expression of this demonstration, take


FIGure 4: The synchronization process of Julia sets between system (5) and system (14) when (a) $l=0.09$; (b) $l=0.095$; (c) $l=0.1$; (d) $l=0.2$; (e) $l=0.5$; (f) $l=0.7$.

Table 1: The Hausdorff distance of Figure 4.

| $l$ | 0.09 | 0.095 | 0.1 | 0.2 | 0.5 | 0.7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Hausdorff distance | 5.0687 | 0.2642 | 0.2118 | 0 | 0 | 0 |

$$
\left\{\begin{array}{l}
E_{1, j-1}=\eta(1+b s(j-1))(1-x(j-1)-y(j-1)) x(j-1)^{2},  \tag{16}\\
E_{2, j-1}=\eta(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2} .
\end{array}\right.
$$

Theorem 2. The synchronization of system (5) and system (14) can be achieved if $|C(n)-x(n)|,|D(n)-y(n)|$, $|A(n)-s(n)|$, and $|B(n)-r(n)|$ are bounded when $0<c+q<1, n \in N$, and $l$ is limited.

Proof. Here, take the synchronization between $C(n)$ and $x(n)$ for an example. Substitute equation (16) into system (5) and system (14), so we have
$C(n)=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) C(j-1)+E_{2, j-1}\right)$

$$
+\frac{l}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(E_{1, j-1}-E_{2, j-1}\right),
$$

$$
\begin{equation*}
x(n)=x(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x(j-1)+E_{1, j-1}\right) . \tag{17}
\end{equation*}
$$

Observe that if we utilize the general binomial coefficient $\binom{a}{b}:=(\Gamma(a+1) / \Gamma(b+1) \Gamma(a-b+1))$, see [35], then $(\Gamma(n-j+\mu) / \Gamma(n-j+1) \Gamma(\mu)) \quad$ can be rewritten as $\binom{n-1-j+\mu}{n-j}$. Thus, the above equations can be rewritten as

$$
\left\{\begin{array}{l}
C(n)=C(0)+\sum_{j=1}^{n}\binom{n-j-1+\mu}{n-j}\left(-(c+q) C(j-1)+E_{2, j-1}\right)  \tag{18}\\
\quad+\quad l \sum_{j=1}^{n}\binom{n-j-1+\mu}{n-j}\left(E_{1, j-1}-E_{2, j-1}\right), \\
x(n)=x(0)+\sum_{j=1}^{n}\binom{n-j-1+\mu}{n-j}\left(-(c+q) x(j-1)+E_{1, j-1}\right) .
\end{array}\right.
$$

Therefore, take the item as $n-1$, and the $C(n-1)$ and $x(n-1)$ can be expressed as

$$
\left\{\begin{array}{l}
C(n-1)=C(0)+\sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-1-j}\left(-(c+q) C(j-1)+E_{2, j-1}\right) \\
\quad+l \sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-1-j}\left(E_{1, j-1}-E_{2, j-1}\right), \\
x(n-1)=x(0)+\sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-1-j}\left(-(c+q) x(j-1)+E_{1, j-1}\right) . \tag{19}
\end{array}\right.
$$

Next, $C(n)$ in system (18) minus $C(n-1)$ in system (19) and $x(n-1)$ is subtracted by $x(n)$. And see [35], for $\alpha \in R$ and $k \in N_{1}$ one has $\binom{\alpha}{k}-\binom{\alpha-1}{k-1}=\binom{\alpha-1}{k}$, and set $\alpha=n-1-j+\mu \quad$ and $\quad k=n-j, \quad$ distinctly, $\binom{n-1-j+\mu}{n-j}-\binom{n-2-j+\mu}{n-1-j}=\binom{n-2-j+\mu}{n-j}$.
Then, we obtain

$$
\left\{\begin{array}{l}
C(n)-C(n-1)=-(c+q) \sum_{j=1}^{n}\binom{n-j-2+\mu}{n-j} C(j-1) \\
\quad+\sum_{j=1}^{n}\binom{n-j-2+\mu}{n-j}\left(l E_{1, j-1}+(1-l) E_{2, j-1}\right), \\
x(n)-x(n-1)=-(c+q) \sum_{j=1}^{n}\binom{n-j-2+\mu}{n-j} x(j-1) \\
\quad+\sum_{j=1}^{n}\binom{n-j-2+\mu}{n-j} E_{1, j-1}, \tag{20}
\end{array}\right.
$$

where $j=1,2, \ldots, n-1,\binom{n-2-j+\mu}{n-j}=((n-2-j+$ $\mu)(n-3-j+\mu) \cdots \mu(\mu-1) /(n-j)!)<0, \quad$ and $\quad j=n$, $\binom{\mu-2}{0}=1$.

During system (20), $C(n)-C(n-1)$ minus $x(n)-x(n-1)$, so the following equation is presented:

$$
\begin{align*}
C(n)-C(n-1)-(x(n)-x(n-1))= & -(c+q) \sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-j}(C(j-1)-x(j-1))-(c+q)(C(n-1)-x(n-1)) \\
& +\sum_{j=1}^{n}\binom{n-j-2+\mu}{n-j}(1-l)\left(E_{2, j-1}-E_{1, j-1}\right) . \tag{21}
\end{align*}
$$

Move $-(c+q)(C(n-1)-x(n-1))$ to the left-hand side of the above equation, and then take the absolute value of both sides. Make $F_{j}=|C(j)-x(j)|, j=1,2, \ldots, n$, and $0<c+q<1$. Furthermore, from the definition of Julia set [33], apparently, the Julia sets are bounded. Assume the bounded region is $\Omega$, and there is a $m_{0}$ meets $h^{m_{0}}(z) \notin \omega$, in
that way, the Julia set is fixed. In addition, there is always a $T>0$, which is content with $|z|<T$ for any $z \in \omega$ because of the bounded region $\omega$. Therefore, there are positive number $M$ and $N$ which enable $F_{j}<M, j=1,2, \ldots, n-1$, and $\left|E_{2, j-1}-E_{1, j-1}\right|<N, j=1,2, \ldots, n$. So, we get the following inequality:

$$
\begin{equation*}
F_{n}-(1-c-q) F_{n-1} \leq-(c+q) \sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-j} M-|1-l| \sum_{j=1}^{n-1}\binom{n-j-2+\mu}{n-j} N+|1-l| N \tag{22}
\end{equation*}
$$

$\begin{gathered}\text { Due to } \\ n-2-j+\mu\end{gathered} \quad\binom{n-2-j+\mu}{n-j}=\binom{n-1-j+\mu}{n-j}-$
$\binom{n-2-j+\mu}{n-1-j}$, sum from $j=1$ to $j=n-1$, so
$\sum_{j=1}^{n-1}\binom{n-2-j+\mu}{n-j}=\binom{n-2+\mu}{n-1}-1$. Thus,

$$
\begin{equation*}
F_{n}-(1-c-q) F_{n-1} \leq-(c+q)\left[\binom{n-2+\mu}{n-1}-1\right] M-|1-l|\left[\binom{n-2+\mu}{n-1}-1\right] N+|1-l| N . \tag{23}
\end{equation*}
$$

In this equation, $\binom{n-2+\mu}{n-1}=((n-2+\mu)(n-3+$ $\mu) \cdots(1+\mu) \mu) /(n-1)!>0$, thus $-(c+q)\binom{n-2+\mu}{n-1} M<$

0 and $-|1-l|\binom{n-2+\mu}{n-1} N<0$. Iterating the inequality, we gain

$$
\begin{align*}
F_{n} & \leq(1-c-q) F_{n-1}+(c+q) M+2|1-l| N \\
& \leq(1-c-q)\left[(1-c-q) F_{n-2}+(c+q) M+2|1-l| N\right]+(c+q) M+2|1-l| N \\
& =(1-c-q)^{2} F_{n-2}+[(c+q) M+2|1-l| N](1+(1-c-q)) \\
& \leq \cdots \leq(1-c-q)^{n} F_{0}+[(c+q) M+2|1-l| N] \frac{1-(1-c-q)^{n}}{1-(1-c-q)}  \tag{24}\\
& \leq(1-c-q)^{n} F_{0}+M+2|1-l| N \frac{1-(1-c-q)^{n}}{c+q},
\end{align*}
$$

where $0<c+q<1$. Notice that $\left|C_{0}-x_{0}\right|$ is bounded since the initial values $C_{0}$ and $x_{0}$ are fetched inside the bounded region $\omega$, thus $F_{0}$ is limited. Considering $n \in N$, so $(1-c-q)^{n}$ and $\left(1-(1-c-q)^{n}\right) /(c+q)$ are finite. Besides, $l$ is limited, then $2|1-l| N\left(1-(1-c-q)^{n}\right) /(c+q)$ is bounded. Obviously, $F_{n}=|C(n)-x(n)|$ is bounded. Similarly, $|D(n)-y(n)|,|A(n)-s(n)|$, and $|B(n)-r(n)|$ are also bounded. Because the proof is analogous and simple, we do
not expand the proof here. According to Theorem 1, the synchronization of Julia sets between systems (5) and (14) is acquired finally and the proof of theorem 2 is finished.
4.2. The Second Synchronization of Julia Sets. Similarly, consider a system (25) with the same structure of equation form (5) but with different parameters $\hat{\eta}$ and $\widehat{q}$ :

$$
\left\{\begin{array}{l}
C(n)=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+\widehat{q}) C(j-1)+\widehat{\eta}(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}\right),  \tag{25}\\
D(n)=D(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\widehat{q} C(j-1)-(c+e) D(j-1)+z), \\
A(n)=A(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta B(j-1)), \\
B(n)=B(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta A(j-1)),
\end{array}\right.
$$



Figure 5: The synchronization process of Julia sets between system (5) and system (27) when (a) $l=0$; (b) $l=0.2$; (c) $l=0.4$; (d) $l=0.6$; (e) $l=0.8$; (f) $l=0.9$.
where $\hat{\eta} \neq \eta>0$ and $\widehat{q} \neq q>0$. To facilitate the description of this synchronization method, denote

$$
\begin{align*}
& C^{* *}=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+\hat{q}) C(j-1)+\hat{\eta}(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}\right), \\
& C^{*}=C(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+\widehat{q}) C(j-1)+\widehat{\eta}(1+b A(j-1))(1-C(j-1)-D(j-1)) C(j-1)^{2}\right), \\
& x^{* *}=x(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}\left(-(c+q) x(j-1)+\eta(1+b s(j-1))(1-x(j-1)-y(j-1)) x(j-1)^{2}\right),  \tag{26}\\
& D^{* *}=D(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\hat{q} C(j-1)-(c+e) D(j-1)+z), \\
& D^{*}=D(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\hat{q} C(j-1)-(c+e) D(j-1)+z), \\
& y^{* *}=y(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(q x(j-1)-(c+e) y(j-1)+z) .
\end{align*}
$$

In order to connect these Julia sets of systems (5) and (25) together, add two coupling terms $-C^{*}+x^{* *}-l\left(\left(C^{* *}-\right.\right.$ $\left.\left.C^{*}\right)-\left(x(n)-x^{* *}\right)\right)$ and $-D^{*}+y^{* *}-l\left(\left(D^{* *}-D^{*}\right)-\right.$ $\left.\left(y(n)-y^{* *}\right)\right)$ into system (25), so we obtain

$$
\left\{\begin{array}{l}
C(n)=C^{* *}-C^{*}+x^{* *}-l\left(\left(C^{* *}-C^{*}\right)-\left(x(n)-x^{* *}\right)\right), \\
D(n)=D^{* *}-D^{*}+y^{* *}-l\left(\left(D^{* *}-D^{*}\right)-\left(y(n)-y^{* *}\right)\right),  \tag{27}\\
A(n)=A(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(\delta B(j-1)), \\
B(n)=B(0)+\frac{1}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)}(-\delta A(j-1)) .
\end{array}\right.
$$

Apparently, by selecting different coupling parameters $l$, disparate Julia sets of system (27) are demonstrated.

Taking Figure 1 as the background frame, shown in blue, the synchronization process diagrams of the Julia set of this method are exhibited, shown in red. From Figure 5, we can find that the Julia set in Figure 5(a) is shaped somewhat like the number 8. Besides, with the increase of control parameters $l$, the graphs expand horizontally and finally synchronize with the initial Figure 1. As can be seen from Figure 5, the points of the initial system have partially overlapped with the points of the target system since Figure 5(a), and then the calculated Hausdorff distance is 0 invariably.

Theorem 3. If $|C(n)-x(n)| \longrightarrow 0,|D(n)-y(n)| \longrightarrow 0$, and $|A(n)-s(n)|,|B(n)-r(n)|$ are bounded when $l \longrightarrow 1$ and $n \in N$, and the synchronization between system (5) and system (27) is gained.

Proof. From equations (5) and (27), $C(n)$ minus $x(n)$, so we have

$$
\begin{equation*}
C(n)-x(n)=(1-l)\left(\left(C^{* *}-C^{*}\right)-\left(x(n)-x^{* *}\right)\right) \tag{28}
\end{equation*}
$$

Take the absolute value of both sides of this equation. As we described in Theorem 1, because of the bounded Julia set, $\left|\left(C^{* *}-C^{*}\right)-\left(x(n)-x^{* *}\right)\right|$ is less than $\left|C^{* *}-C^{*}\right|+\mid x(n)-$ $x^{* *} \mid$ and less than a positive number, thus

$$
\begin{align*}
C(n)-x(n) & =|1-l|\left|\left(C^{* *}-C^{*}\right)-\left(x(n)-x^{* *}\right)\right| \\
& \leq|1-l|\left(\left|C^{* *}-C^{*}\right|+\left|x(n)-x^{* *}\right|\right) . \tag{29}
\end{align*}
$$

The limit of the right side of this equation tends to 0 as $l \longrightarrow 1$. Analogously, $|D(n)-y(n)| \longrightarrow 0,|A(n)-s(n)|$, and $|B(n)-r(n)|$ are bounded if $l \longrightarrow 1$ and $n \in N$. Thus, the synchronization of Julia sets between systems (5) and (27) is implemented and the proof is completed.

## 5. Conclusion

The discrete SIRS model defined by Caputo fractional calculus is mainly analyzed in this paper, the mathematical expression of discrete fractional difference equation is obtained, and then the control and synchronization of Julia set of this model are discussed. We calculate the fixed point of the fractional order system and design the fixed point
control terms for the system. The structure of the control terms is similar, but the positions adding into system are different so that the control results are completely distinct. In addition, two kinds of synchronization controllers are proposed to realize the synchronization between system (5) and system (14) or (27), and novel proofs about synchronization of the two Julia sets are meticulously derived. Simulation results show that the newly designed controllers can realize the synchronization of the discrete fractional order systems.

This paper combines the ideas of fractional calculus and fractal, which will provide the possibility to deeply understand the fractional dynamics and better describe the nonlinear phenomena in nature. More importantly, the control and synchronization methods used in this paper can be applied and extended to other potential applications to solve practical problems of infectious diseases. It was worthwhile noting that we synchronized two systems with different parameters but with the same structure and designed control items and coupling terms to implement the control over Julia sets. However, the structure and size of Julia sets of the fractional order system are not only dependent on system parameters but also on the fractional orders, so an interesting question is whether we can design control items to realize the synchronization between two fractional order systems with different fractional orders. Therefore, there is still a good deal of challenging work to do.

## Data Availability

The figure data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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