Research Article

Lie Symmetry Analysis, Traveling Wave Solutions, and Conservation Laws to the $(3 + 1)$ -Dimensional Generalized B-Type Kadomtsev-Petviashvili Equation

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In this paper, the $(3 + 1)$ -dimensional generalized B-type Kadomtsev-Petviashvili(BKP) equation is studied applying Lie symmetry analysis. We apply the Lie symmetry method to the $(3 + 1)$ -dimensional generalized BKP equation and derive its symmetry reductions. Based on these symmetry reductions, some exact traveling wave solutions are obtained by using the tanh method and Kudryashov method. Finally, the conservation laws to the $(3 + 1)$ -dimensional generalized BKP equation are presented by invoking the multiplier method.

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used as models to describe a great number of complex nonlinear phenomena which appear in many fields, such as hydrodynamics, biology, plasma physics, fluid dynamics, solid state physics, optics, and applied mathematics. To really understand such phenomena describing in nature, searching for exact solutions of NPDEs plays an important role in the study of nonlinear science. In recent years, more and more methods have been proposed, such as the inverse scattering method [1], the Bäcklund transform method [2], the Darboux transform method [3], the Hirota bilinear transformation method [4], the Exp-function method [5], the tanh-function method [6], the sine-Gordon equation expansion technique [7], and the Kudryashov method [8].

The Lie symmetry method presented by Lie [9] is one of the well-known methods for obtaining exact solutions of nonlinear PDEs. Up to now, the Lie symmetry method has been applied to a number of mathematical and physical models, see [10–14] and references therein. This method is effective to get similarity solutions and solitary wave solutions of NPDEs.

The study of different BKP equations has attracted a considerable size of research. Different forms of BKP equations were studied by using the Hirota method, the multiple exp-function algorithm, the Pfaffian technique, the Wronskian technique, and the Bäcklund transformation [15–21]. In this paper, by means of the Lie symmetry group method, we consider the following $(3 + 1)$ -dimensional generalized BKP equation:

$$u_{yt} - u_{xxx}y - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz} = 0, \quad (1)$$

where u is a real differentiable function of the scaled spatial coordinates x , y , and z and temporal coordinate t , while the subscripts denote the partial derivatives. This equation was first presented by Ma et al. [15]. When $z = 0$, the $(3 + 1)$ -dimensional generalized BKP equation can be reduced to the following $(2 + 1)$ -dimensional BKP equation:

$$u_{yt} - u_{xxx}y - 3(u_x u_y)_x + 3u_{xx} = 0. \quad (2)$$

Equation (1) has been investigated by different methods. Following the linear superposition principle of exponential waves, Ma et al. obtained an N-wave solution for equation (1) in [15]. A bilinear Bäcklund transformation and a class of

exact Pfaffian solutions for equation (1) were established using the Pfaffian technique and Hirota's bilinear operator identities in [16]. Moreover, Wazwaz [17] established two sets of distinct kinds of multiple-soliton solutions under specific conditions for equation (1) using the simplified form of the Hirota method. Soliton solutions in Wronskian form of (1) were also presented in [18, 19]. Multiple wave solutions and auto-Bäcklund transformation for the (1) were obtained in [20]. The lump solutions, periodic waves, and rogue waves as well as interaction solutions of (1) were obtained in [21].

As far as we know, the Lie symmetry analysis and conservation laws to the (3+1)-dimensional generalized BKP equation (1) have not been discussed. The main purpose of this paper is to study the Lie symmetry analysis method [22–24], exact traveling wave solutions, and conservation laws of equation (1). The rest paper is arranged as follows: Section 2 is devoted to describe the Lie symmetry vectors and symmetry reductions using Lie symmetry analysis. In Section 3, various exact traveling wave solutions are attained after reductions process by using the tanh method and Kudryashov method. Section 4 is devoted to find the conservation laws of equation (1) by utilizing the multiplier method. Some conclusions are made in Section 5.

2. Lie Symmetry Analysis and Symmetry Reductions for Equation (1)

We consider the one-parameter Lie group of infinitesimal transformations in x, t, y, z , and u given by

$$\begin{aligned} x &\longrightarrow x + \varepsilon \xi(x, t, y, z, u), \\ t &\longrightarrow t + \varepsilon \tau(x, t, y, z, u), \\ y &\longrightarrow y + \varepsilon \eta(x, t, y, z, u), \\ z &\longrightarrow z + \varepsilon \mu(x, t, y, z, u), \\ u &\longrightarrow u + \varepsilon \phi(x, t, y, z, u), \end{aligned} \quad (3)$$

where ε is a group parameter. The associated vector field of equation (1) can be written as

$$\begin{aligned} V = & \xi(x, t, y, z, u) \frac{\partial}{\partial x} + \tau(x, t, y, z, u) \frac{\partial}{\partial t} + \eta(x, t, y, z, u) \frac{\partial}{\partial y} \\ & + \mu(x, t, y, z, u) \frac{\partial}{\partial z} + \phi(x, t, y, z, u) \frac{\partial}{\partial u}, \end{aligned} \quad (4)$$

where the coefficient functions $\xi(x, t, y, z, u)$, $\tau(x, t, y, z, u)$, $\eta(x, t, y, z, u)$, $\mu(x, t, y, z, u)$, and $\phi(x, t, y, z, u)$ are to be determined later.

The vector field (4) is a symmetry of equation (1) if and only if

$$\text{pr}^{(4)}V(\Delta)|_{\Delta=0} = 0, \quad (5)$$

where $\Delta = u_{yt} - u_{xxx}y - 3u_{xx}u_y - 3u_xu_{xy} + 3u_{xx} + 3u_{zz}$ and $\text{pr}^{(4)}V$ denote the fourth prolongation of V . Based on the Lie theory, we obtain an equivalent condition for (5) as

$$\begin{aligned} \text{pr}^{(4)}V(\Delta) = & \phi^{yt} - \phi^{xxx}y - 3u_y\phi^{xx} - 3u_{xx}\phi^y - 3u_{xy}\phi^x \\ & - 3u_x\phi^{xy} + 3\phi^{xx} + 3\phi^{zz} = 0, \end{aligned} \quad (6)$$

with

$$\begin{aligned} \phi^i = & D_i(\phi - \xi u_x - \eta u_y - \mu u_z - \tau u_t) + \xi u_{ix} + \eta u_{iy} + \mu u_{iz} \\ & + \tau u_{it}, \quad i = \{x, y, t, y, xx, xy, zz, xxx, xxx, y\}, \end{aligned} \quad (7)$$

where D_t, D_x, D_y , and D_z are total derivative operators.

Substituting (7) into (6), we can obtain the determining equations for the symmetry group of equation (1). Solving these determining equations, we conclude that the general infinitesimal symmetry of (1) has the following forms of the coefficient functions:

$$\begin{aligned} \xi &= -c_1x - 3h_1(t) + c_3, \\ \tau &= -3c_1t + c_4, \\ \eta &= (c_1 + c_2)y - c_5z + c_6, \\ \mu &= \left(-c_1 + \frac{1}{2}c_2\right)z + 6c_5t + c_7, \\ \phi &= c_1u + h_1'(t)x + c_2y + h_2(t)z + h_3(t), \end{aligned} \quad (8)$$

where $c_i (i = 1, \dots, 7)$ are arbitrary constants, $h_1(t)$, $h_2(t)$, and $h_3(t)$ are arbitrary functions of t , and “ $'$ ” is the derivative with respect to t . Therefore, the infinitesimal symmetries of equation (1) are

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial y}, \\ V_4 &= \frac{\partial}{\partial z}, \\ V_5 &= h_2(t)z \frac{\partial}{\partial u}, \\ V_6 &= h_3(t) \frac{\partial}{\partial u}, \\ V_7 &= -z \frac{\partial}{\partial y} + 6t \frac{\partial}{\partial z}, \\ V_8 &= -3h_1(t) \frac{\partial}{\partial x} + h_1'(t)x \frac{\partial}{\partial u}, \\ V_9 &= 2y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + 2y \frac{\partial}{\partial u}, \\ V_{10} &= -x \frac{\partial}{\partial x} - 3t \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}. \end{aligned} \quad (9)$$

It can be verified that $\{V_1, \dots, V_9\}$ is closed under the Lie bracket. The commutator table is shown in Tables 1 and 2.

TABLE 1: Commutator table.

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	0	0
V_2	0	0	0	0	$V_5(h'_2(t))$
V_3	0	0	0	0	0
V_4	0	0	0	0	$V_6(h_2(t))$
V_5	0	$-V_5(h'_2(t))$	0	$-V_6(h_2(t))$	0
V_6	0	$-V_6(h'_3(t))$	0	0	0
V_7	0	$-6V_4$	0	V_3	$6tV_6(h_2(t))$
V_8	$-V_6(h'_1(t))$	$-V_8(h'_1(t))$	0	0	0
V_9	0	0	$-2[V_3 + V_6/h_3(t)]$	$-V_4$	V_5
V_{10}	V_1	$3V_2$	$-V_3$	V_4	$-2V_5 - 3tV_5(h'_2(t))$

TABLE 2: Commutator table.

$[V_i, V_j]$	V_6	V_7	V_8	V_9	V_{10}
V_1	0	0	$V_6(h'_1(t))$	0	$-V_1$
V_2	$V_6(h'_3(t))$	$6V_4$	$V_8(h'_1(t))$	0	$-3V_2$
V_3	0	0	0	$2[V_3 + V_6/h_3(t)]$	V_3
V_4	0	$-V_3$	0	V_4	$-V_4$
V_5	0	$-6tV_6(h_2(t))$	0	$-V_5$	$2V_5 + 3tV_5(h'_2(t))$
V_6	0	0	0	0	$V_6 + 3tV_6(h'_3(t))$
V_7	0	0	0	$V_7 - 2V_5/h_2(t)$	$2V_7$
V_8	0	0	0	0	$V_8(-h_1(t) + 3th'_1(t))$
V_9	0	$-V_7 + 2V_5/h_2(t)$	0	0	0
V_{10}	$-V_6 - 3tV_6(h'_3(t))$	$-2V_7$	$-V_8(-h_1(t) + 3th'_1(t))$	0	0

In order to derive the traveling wave solutions of (1), we consider a linear combination of the translation symmetries V_1, V_2, V_3 , and V_4 , namely, $V = \alpha V_1 + V_2 + V_3 + V_4$, where α is a constant. Solving the corresponding characteristic equation,

$$\frac{dx}{\alpha} = \frac{dy}{1} = \frac{dz}{1} = \frac{dt}{1} = \frac{du}{0}, \quad (10)$$

we obtain the following invariants:

$$\begin{aligned} f &= x - \alpha t, \\ g &= x - \alpha y, \\ h &= y - z, \\ u &= v(f, g, h). \end{aligned} \quad (11)$$

Substituting (11) into (1), we can reduce the equation (1) to

$$\begin{aligned} &(\alpha^2 + 6)v_{fg} - \alpha v_{fh} + \alpha v_{fffg} + 3\alpha v_{ffgg} + 3\alpha v_{fggg} + \alpha v_{gggg} - v_{fffh} - 3v_{ffgh} - 3v_{fghg} - v_{gggh} + 3\alpha v_{fvfg} + 3\alpha v_{fvfg} \\ &+ 9\alpha v_g v_{fg} + 6\alpha v_g v_{gg} + 3\alpha v_g v_{ff} - 3v_f v_{fh} - 3v_f v_{gh} - 3v_g v_{fh} - 3v_g v_{gh} - 6v_h v_{fg} - 3v_h v_{gg} - 3v_h v_{ff} + 3v_{ff} + 3v_{gg} + 3v_{hh} = 0. \end{aligned} \quad (12)$$

We write the vector field of (12) as

$$\begin{aligned} X &= \xi^1(f, g, h, v) \frac{\partial}{\partial f} + \xi^2(f, g, h, v) \frac{\partial}{\partial g} + \xi^3(f, g, h, v) \frac{\partial}{\partial h} \\ &+ \rho(f, g, h, v) \frac{\partial}{\partial v}, \end{aligned} \quad (13)$$

where the coefficient functions $\xi^1(f, g, h, v)$, $\xi^2(f, g, h, v)$, $\xi^3(f, g, h, v)$, and $\rho(f, g, h, v)$ of the vector field are

$$\begin{aligned} \xi^1 &= -C_1 f + C_2, \\ \xi^2 &= 2C_1 f - 3C_1 g + C_3, \\ \xi^3 &= -3C_1 h + C_4, \\ \rho &= \left[\left(\frac{2\alpha}{3} - \frac{4}{\alpha} \right) C_1 - \frac{C_5}{\alpha} \right] f + \frac{4C_1 + C_5}{\alpha} g + C_5 h + C_1 v + C_6, \end{aligned} \quad (14)$$

and $C_i (i = 1, \dots, 6)$ are arbitrary constants.

We perform further symmetry reductions by applying Lie symmetries to (12), then (12) has the following six Lie symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial f}, \\
X_2 &= \frac{\partial}{\partial g}, \\
X_3 &= \frac{\partial}{\partial h}, \\
X_4 &= \frac{\partial}{\partial v}, \\
X_5 &= \left(-\frac{f}{\alpha} + \frac{g}{\alpha} + h \right) \frac{\partial}{\partial v}, \\
X_6 &= -f \frac{\partial}{\partial f} + 2f \frac{\partial}{\partial g} - 3h \frac{\partial}{\partial h} + \left[\left(\frac{2\alpha}{3} - \frac{4}{\alpha} \right) C_1 f + \frac{4}{\alpha} g + v \right] \frac{\partial}{\partial v}.
\end{aligned} \tag{15}$$

We choose the symmetry $X = X_1 + \beta X_2 + X_3$, where β is a constant. Three invariants are derived by solving the corresponding characteristic equation, namely,

$$\begin{aligned}
r &= f - h, \\
s &= g - \beta h, \\
v(f, g, h) &= \psi(r, s).
\end{aligned} \tag{16}$$

Substituting (16) into (12), we reduce equation (12) to

$$\begin{aligned}
&(\alpha + 6)\psi_{rr} + (\alpha^2 + \alpha\beta + 6\beta + 6)\psi_{rs} + 3(\beta^2 + 1)\psi_{ss} + 3(\alpha + \beta + 1)\psi_r\psi_{ss} + 3(\alpha + \beta + 1)\psi_s\psi_{rr} + 6\psi_r\psi_{rr} \\
&+ 3(3\alpha + 3\beta + 1)\psi_s\psi_{rs} + 6(\alpha + \beta)\psi_s\psi_{ss} + 3(\alpha + \beta + 3)\psi_r\psi_{rs} + (\alpha + \beta)\psi_{sss} + (3\alpha + 3\beta + 1)\psi_{rss} + 3(\alpha + \beta + 1)\psi_{rrs} \\
&+ (\alpha + \beta + 3)\psi_{rrs} + \psi_{rrrr} = 0.
\end{aligned} \tag{17}$$

Further symmetry reduction will transform (17) into an ODE. The symmetries of (17) include two translation symmetries $\Gamma_1 = \partial/\partial r$ and $\Gamma_2 = \partial/\partial s$. We consider the symmetry $\Gamma = \Gamma_1 + \omega\Gamma_2$, where ω is a constant. We obtain two invariants:

$$\begin{aligned}
\xi &= s - \omega r, \\
\psi(r, s) &= U(\xi).
\end{aligned} \tag{18}$$

Substituting (18) into (17), equation (17) is reduced to the following ODE.

$$AU^{(4)} + BU'U'' + CU'' = 0, \tag{19}$$

where $A = -(\omega - 1)^3(\alpha + \beta - \omega)$, $B = 6(\omega - 1)^2(\alpha + \beta - \omega)$, $C = (\alpha + 6)\omega^2 - (\alpha^2 + \alpha\beta + 6\beta + 6)\omega + 3(\beta^2 + 1)$, and $\xi = \alpha\omega t + (1 - \omega)x - (\alpha + \beta - \omega)y + (\beta - \omega)z$.

3. Traveling Wave Solutions to Equation (1)

3.1. Exact Solutions of Equation (19) Using Tanh Method. In this subsection, we apply the tanh method to obtain solutions of equation (19). Suppose that the solution of equation (19) can be expressed as

$$U(\xi) = \sum_{i=0}^n a_i Z^i, \tag{20}$$

where n is a positive integer and a_i ($i = 0, 1, 2, \dots, n$) are all constants. $Z = \tanh(\xi)$ is a new independent variable, the derivatives of U with respect to ξ can be written as

$$\begin{aligned}
U' &= (1 - Z^2) \frac{dU}{dZ}, \\
U'' &= (1 - Z^2)^2 \frac{d^2U}{dZ^2} - 2Z(1 - Z^2) \frac{dU}{dZ}, \\
U''' &= (1 - Z^2)^3 \frac{d^3U}{dZ^3} - 6Z(1 - Z^2)^2 \frac{d^2U}{dZ^2} \\
&\quad + 2Z(1 - Z^2)(3Z^2 - 1) \frac{dU}{dZ}, \\
U^{(4)} &= (1 - Z^2)^4 \frac{d^4U}{dZ^4} - 12Z(1 - Z^2)^3 \frac{d^3U}{dZ^3} \\
&\quad + 4(1 - Z^2)^2(9Z^2 - 2) \frac{d^2U}{dZ^2} - 8Z(1 - Z^2)(3Z^2 - 2) \frac{dU}{dZ}.
\end{aligned} \tag{21}$$

Balancing the linear term of the highest order with the highest order nonlinear term, we can easily obtain $n = 1$ and thus (20) becomes

$$U = a_0 + a_1 Z. \tag{22}$$

Substituting (21) into (19) and using (22), we have an algebraic equation, which on splitting with respect to the powers of Z gives the following system:

$$\begin{cases} 2a_1(-Ba_1 + 12A) = 0, \\ -2a_1(-2Ba_1 + 20A - C) = 0, \\ 2a_1(-Ba_1 + 8A - C) = 0, \end{cases} \quad (23)$$

whose solution is

a_0 is an arbitrary constant,

$$a_1 = \frac{12A}{B} = 2(1 - \omega), \quad (24)$$

$$C = -4A.$$

Substituting (24) into (22), we obtain the traveling wave solution to equation (19)

$$U = a_0 + \frac{12A}{B} \tanh(\xi). \quad (25)$$

Thus, the traveling wave solution of the (3 + 1)-dimensional generalized BKP equation (1) is

$$u(x, y, z, t) = a_0 + 2(1 - \omega) \tanh(\alpha\omega t + (1 - \omega)x - (\alpha + \beta - \omega)y + (\beta - \omega)z), \quad (26)$$

where a_0, α, β , and ω are constants and satisfy the algebraic equation $4A + C = 4(1 - \omega)^3(\alpha + \beta - \omega) + (\alpha + 6)\omega^2 - (\alpha^2 + \alpha\beta + 6\beta + 6)\omega + 3(\beta^2 + 1) = 0$.

3.2. Exact Solutions of Equation (19) Using Kudryashov Method. Suppose that the solution of equation (19) is presented as a finite series:

$$U = \sum_{i=0}^n a_i Q^i(\xi), \quad (27)$$

where n is a positive integer and a_i ($i = 0, 1, 2, \dots, n$) are constants, which can be determined later. Here,

$$Q(\xi) = \frac{1}{1 + da^\xi}, \quad (28)$$

which satisfies the Bernoulli equation:

$$Q'(\xi) = Q(\xi)(Q(\xi) - 1)\ln a. \quad (29)$$

Applying the homogeneous balance method similar as the tanh method, we have $n = 1$. Thus, the solution (27) becomes

$$U = a_0 + a_1 Q(\xi). \quad (30)$$

Substituting (30) into equation (19) together with (29), we have a system of nonlinear algebraic equations:

$$\begin{cases} 2(\ln a)^3 a_1 (Ba_1 + 12A \ln a) = 0, \\ -5(\ln a)^3 a_1 (Ba_1 + 12A \ln a) = 0, \\ 2(\ln a)^2 a_1 (25(\ln a)^2 A + 2a_1 B \ln a + C) = 0, \\ -(\ln a)^2 a_1 (15(\ln a)^2 A + a_1 B \ln a + 3C) = 0, \\ (\ln a)^2 a_1 ((\ln a)^2 A + C) = 0. \end{cases} \quad (31)$$

Solving this system with the help of Maple, we obtain

$$a_0 = a_0,$$

$$a_1 = \frac{12A}{B} \ln a = -2(1 - \omega) \ln a, \quad (32)$$

$$C = \frac{B}{12} \ln a.$$

Substituting (32) into (30) together with (29), we can obtain the traveling wave solution to equation (19):

$$U = a_0 - \frac{12A}{B} \frac{\ln a}{1 + da^\xi}. \quad (33)$$

Thus, the traveling wave solution of the (3 + 1)-dimensional generalized BKP equation (1) is

$$u(x, y, z, t) = a_0 - 2(1 - \omega) \frac{\ln a}{1 + da^\xi}, \quad (34)$$

where $\xi = \alpha\omega t + (1 - \omega)x - (\alpha + \beta - \omega)y + (\beta - \omega)z$ and a_0, α, β , and ω are constants and satisfy the algebraic equation $12C - B \ln a = 12(\alpha + 6)\omega^2 - 12(\alpha^2 + \alpha\beta + 6\beta + 6)\omega + 36(\beta^2 + 1) - 6(1 - \omega)^2(\alpha + \beta - \omega) \ln a = 0$.

4. Conservation Laws of Equation (1)

In this section, we study the conservation laws of the generalized (3 + 1)-dimensional BKP equation (1) using the direct multiplier method [25]. We find all the first-order multipliers $\Lambda(t, x, y, z, u, u_t, u_x, u_y, u_z)$ for the equation (1) from the determined equation:

$$\frac{\delta}{\delta u} [\Lambda(u_{yt} - u_{xxx}y - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz})] = 0, \quad (35)$$

where

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} - D_z \frac{\partial}{\partial u_z} + D_t D_y \frac{\partial}{\partial u_{ty}} \\ & + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_z^2 \frac{\partial}{\partial u_{zz}} + D_x^3 D_y \frac{\partial}{\partial u_{xxx}}, \end{aligned} \quad (36)$$

denotes the Euler-Lagrange operator.

In general, for a given PDE (1), if the multiplier Λ yields a divergence expression,

$$\begin{aligned} \Lambda(u_{yt} - u_{xxx}y - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz}) \\ = D_t(T^t) + D_x(T^x) + D_y(T^y) + D_z(T^z), \end{aligned} \quad (37)$$

for every u , then PDE (1) has a local conservation law in the form of $D_t(T^t) + D_x(T^x) + D_y(T^y) + D_z(T^z) = 0$, where $T = (T^t, T^x, T^y, T^z)$ is a conserved vector corresponding to the conservation law.

In order to obtain the multipliers of the equation (1), expanding (35) and splitting with respect to derivative of u , we obtain 10 overdetermined equations as follows:

$$\begin{aligned} \Lambda_u &= 0, \\ \Lambda_{u_t} &= 0, \\ \Lambda_{u_y} &= 0, \\ \Lambda_{u_z} &= 0, \\ \Lambda_{xu_x} &= 0, \\ \Lambda_{yu_x} &= 0, \\ \Lambda_{zu_x} &= 0, \\ \Lambda_{uu} &= 0, \\ -6\Lambda_x + \Lambda_{tu_x} &= 0, \\ \Lambda_{ty} + 3\Lambda_{zz} &= 0, \end{aligned} \quad (38)$$

Using the software Maple, we obtain the solution of the above overdetermined equations:

$$\begin{aligned} \Lambda(t, x, y, z, u, u_t, u_x, u_y, u_z) \\ = (6F_1(t) + C)u_x + F_1'(t)x + F_2\left(y - t - \frac{z}{\sqrt{3}}\right), \end{aligned} \quad (39)$$

where C is an arbitrary constant, $F_1(t)$ is an arbitrary function with respect to t , and $F_2(\xi)$ is an arbitrary function with respect to ξ with $\xi = y - t - z/\sqrt{3}$. We further derive all the first-order multipliers in the following form:

$$\begin{aligned} \Lambda_1 &= u_x, \\ \Lambda_2 &= 6F_1(t) + F_1'(t)x, \\ \Lambda_3 &= F_2(\xi), \\ \xi &= y - t - \frac{z}{\sqrt{3}}. \end{aligned} \quad (40)$$

Corresponding to the multiplier $\Lambda_1 = u_x$, we obtain the conservation law of equation (1) as

$$D_t(T_1^t) + D_x(T_1^x) + D_y(T_1^y) + D_z(T_1^z) = 0, \quad (41)$$

with the following components:

$$\begin{aligned} T_1^t &= -\frac{1}{4}uu_{xy} + \frac{1}{4}u_x u_y, \\ T_1^x &= \frac{1}{2}uu_{ty} - \frac{1}{8}uu_{xxx}y + \frac{3}{8}u_{xx}u_{xy} - \frac{1}{8}u_{xxx}u_y \\ &\quad - \frac{5}{8}u_x u_{xy} + \frac{3}{2}u_{xx}^2 - \frac{3}{2}u_x^2 u_y - \frac{3}{2}u_z^2, \\ T_1^y &= -\frac{1}{4}uu_{tx} + \frac{1}{8}uu_{xxx}x - \frac{1}{2}u_x^3 - \frac{1}{4}u_x u_{xxx} + \frac{1}{8}u_{xx}^2 + \frac{1}{4}u_t u_x, \\ T_1^z &= 3u_x u_z. \end{aligned} \quad (42)$$

For the multiplier $\Lambda_2 = 6F_1(t) + F_1'(t)x$, we obtain the conservation law of equation (1) as

$$D_t(T_2^t) + D_x(T_2^x) + D_y(T_2^y) + D_z(T_2^z) = 0, \quad (43)$$

with the following components:

$$\begin{aligned} T_2^t &= 3F_1(t)u_x u_y, \\ T_2^x &= -6F_1(t)u_x u_{xy} - 9F_1(t)u_x^2 u_y - 3F_1'(t)u_x u_{xy} \\ &\quad - 3F_1(t)u_t u_y + 3xF_1'(t)u_x - 3F_1'(t)u + 9F_1(t)u_x^2 \\ &\quad - 9F_1(t)u_z^2, \\ T_2^y &= xF_1'(t)u_t + 3F_1(t)u_{xx}^2 - xF_1'(t)u_{xxx} - 3F_1(t)u_x^3 \\ &\quad + 3xF_1(t)u_t u_x, \\ T_2^z &= 18F_1(t)u_x u_z + 3xF_1'(t)u_z. \end{aligned} \quad (44)$$

For the multiplier $\Lambda_3 = F_2(\xi)$, the conservation law of equation (1) is

$$D_t(T_3^t) + D_x(T_3^x) + D_y(T_3^y) + D_z(T_3^z) = 0, \quad (45)$$

and the components are

$$\begin{aligned} T_3^t &= -\frac{1}{2}F_2'(\xi)u + \frac{1}{2}F_2(\xi)u_y, \\ T_3^x &= -F_2(\xi)u_{xy} - 3F_2(\xi)u_x u_y + 3F_2(\xi)u_x, \\ T_3^y &= \frac{1}{2}F_2(\xi)u_t + \frac{1}{2}F_2'(\xi)u, \\ T_3^z &= \frac{3}{2}u_z + \sqrt{3}F_2'(\xi)u, \\ \xi &= y - t - \frac{z}{\sqrt{3}}. \end{aligned} \quad (46)$$

5. Conclusion

The exact solutions of the $(3+1)$ -dimensional generalized BKP equation (1) have been constructed using different methods by many authors. The Pfaffian technique, the Hirota's bilinear method, and the bilinear Bäcklund transformation were applied by many researchers, and various exact solutions such as Pfaffian solutions, lump wave solutions, periodic wave solutions, rogue wave solutions, and interaction solutions were derived. Based on the Hirota bilinear form, the Wronskian method, and the multiple Exp-function method, the Grammian and Pfaffian techniques were used to construct multiple wave solutions to the $(3+1)$ -dimensional generalized BKP equation. Different from these existing results, we choose another approach to study equation (1). In this paper, the combination of Lie symmetry method and the symbolic computation is applied to equation (1). The geometric vector fields of this equation are presented for the first time in the literature. With the aid of Lie symmetry reductions, some exact traveling wave solutions are obtained by using the tanh method and Kudryashov method to the reduced equation of (1). The conservation laws of equation (1) are presented by using the multiplier method at the end of the paper. We can find that these are different approaches to find exact solutions of the $(3+1)$ -dimensional generalized BKP equation, and we hope that some more interesting solutions of equation (1) are shown in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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