Research Article

Characteristic Model-Based Adaptive Control for a Class of MIMO Uncertain Nonaffine Nonlinear Systems Governed by Differential Equations

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This paper addresses the difficulty of designing a controller for a class of multi-input multi-output uncertain nonaffine nonlinear systems governed by differential equations. We first derive the first-order characteristic model composed of a linear time-varying uncertain system for such nonaffine systems and then design an adaptive controller based on this first-order characteristic model for position tracking control. The designed controller exhibits a simple structure that can effectively avoid the controller singularity problem. The stability of the closed-loop system is analyzed using the Lyapunov method. The effectiveness of our proposed method is validated with a numerical example.

1. Introduction

The study of complex nonlinear systems governed by differential equations has attracted considerable attention [1–4]. It is worth noting that in the real world, there are many nonaffine nonlinear systems, such as certain flight control systems [5, 6], tank reactor system [7, 8], biochemical processes [9, 10], and carrier landing control systems for unmanned aerial vehicles [11]. The study of nonaffine systems is often much more difficult and complicated than that of affine ones because control inputs always appear implicitly in the nonlinear functions [7], and finding the explicit inversion of nonaffine functions is quite difficult, even when the existence of such nonlinear function inversion can be proved using the implicit function theorem [5, 9]. On the other hand, in practical applications, uncertainties often arise owing to the presence of unknown parameter variations, modeling simplifications, unmodeled dynamics, and external disturbances, among others [12–14]. In certain practical situations, establishing accurate and

detailed models of nonlinear systems may be difficult, costly, and time consuming. Therefore, the control design for uncertain nonaffine systems is an interesting yet challenging problem. Over the past decades, nonaffine nonlinear system control has received considerable attention, and many important results have been presented on the design of such controllers [5–12, 14, 15]. Owing to the universal approximation theorem, some adaptive control schemes based on fuzzy logic systems [16–22] and neural networks [8, 10, 23–25] have been developed for uncertain nonaffine nonlinear systems. Nevertheless, these approaches typically incur heavy computational costs. Apart from these approaches, other adaptive control methods have also been considered for uncertain nonaffine-in-control nonlinear systems, such as adaptive dynamic surface control [7] and generalized PI control [26].

Recently, a majority of representative adaptive control schemes focuses on the continuous-time domain for nonaffine nonlinear systems described by differential equations. However, in engineering implementations, computers are

usually utilized to produce digital control signals, which require controllers to be designed in discrete time [27].

The above discussion motivates us to use a novel method, called characteristic modeling proposed by Wu [28–35], to solve the control design problem for uncertain nonaffine systems. The key idea of the characteristic modeling is to use a lower-order discrete time-varying linear system to express an original continuous system equivalently based on both the dynamic characteristics of controlled plants and performance specifications. This discrete system is called the “characteristic model of the original system.” It is worth mentioning that the characteristic modeling differs from the conventional model reduction methods in that all information of the original dynamic model, including uncertainties and nonlinearities, is compressed/integrated into the time-varying coefficients of the characteristic model instead of truncating parts of the plant model [28, 32, 33]. The time-varying coefficients of the characteristic model are referred to as the characteristic parameters [33, 35, 36], and the characteristic parameters are identified online adaptively [31, 33, 35, 37]. The characteristic model-based adaptive control might be a fresh perspective in the field of adaptive control: creating simplicity out of complexity [31], and it has already been applied successfully to more than 400 engineering systems including spacecraft, e.g., the reentry control of a manned spaceship [28, 29, 32], the control of servo systems with backlash and friction [30], and the control for the swing arm in a Fourier transform spectrometer [38].

However, in the existing literature, the main focus is on the second-order characteristic model; the first-order model has received scarce attention for related results on controller design based on the characteristic model, and few studies have reported on the control method for nonaffine nonlinear systems based on the characteristic model. In the controller design method based on the second-order characteristic model, almost all the existing control methods belong to the indirect method in adaptive control. In other words, the model parameters are identified by the input and output of the system, and then the control law is designed. For the second-order characteristic model, the most successful application in practical engineering is the characteristic model-based golden section adaptive control method, and it must also be combined with the integral controller and the differential controller to improve the control performance [30, 33, 35]. This makes it very difficult to analyze the stability of the control system. Stability analysis, especially for MIMO systems, remains a challenging problem for adaptive control based on the second-order characteristic model despite its practical successes. Compared with the second-order characteristic model, the first-order model has the advantages of simpler form, fewer identified parameters, and more convenient applicability for practical systems.

In this paper, inspired by [28, 39], we derive the first-order characteristic model for multi-input multi-output (MIMO) uncertain nonaffine nonlinear systems. Then, we develop a characteristic model-based adaptive control law for the position tracking problem. By constructing a suitable Lyapunov functional, we investigate the stability of the

closed-loop system involving the first-order characteristic model-based control law.

Compared with the existing results, the primary contributions and novelties of our study can be highlighted as follows:

- (1) The first-order characteristic model composed of a linear time-varying uncertain system is derived for a class of MIMO uncertain nonaffine nonlinear systems governed by differential equations. To the best of the authors’ knowledge, first-order characteristic models have only been presented by Xu and Hu [39] and Sun et al. [40]. We note that SISO systems are only considered in [39, 40] and linear systems are only considered in [40].
- (2) An adaptive controller based on this first-order characteristic model for position tracking control (see formula (22)) is designed by adaptively estimating the time-varying coefficients of the characteristic model (see formula (23)). In the existing literature, no results have been reported on the adaptive control design based on the first-order characteristic model for our considered class of nonaffine multivariable systems.
- (3) The ranges of controller parameters, obtained (see Lemma 1) by utilizing some mathematical technique, guarantee the stability of the closed-loop system (see Theorem 2). In [39], no intervals of controller parameters are presented.
- (4) The designed controller exhibits a simple structure and can effectively avoid possible controller singularity problems.

The paper is organized as follows. The problem formulation and the characteristic model are given in Section 2. Then, we present the design of the adaptive controller based on the characteristic model and the stability analysis by using the Lyapunov theory in Section 3. The effectiveness of the proposed method is illustrated by a numerical example in Section 4. Section 5 concludes this paper.

2. Problem Formulation and Characteristic Modeling

The following notation is used in the paper. $\|x\|$ denotes the Euclidean norm for a vector x . $\|A\|$ denotes the Frobenius norm for a matrix A . R^n denotes the n -dimensional Euclidean space. I is the n -dimensional identity matrix. Let $T_s > 0$ be the sampling period.

Consider a class of MIMO uncertain nonaffine nonlinear systems:

$$\dot{x} = f(x, u), \quad (1)$$

where $x = [x_1, \dots, x_n]^T \in R^n$ and $u = [u_1, \dots, u_n]^T \in R^n$ represent the state (output) and the input vector of the controlled plant, respectively. $f(x, u) = [f_1(x, u), \dots, f_n(x, u)]^T$ is an unknown smooth nonlinear function vector.

Let $r_i \in (0, 1)$ for $i = 1, 2, \dots, n$. For Jacobian matrix $\partial f(x, u)/\partial u$, u in the i th row is replaced by $r_i u$, and the resulting matrix is denoted as $f_u(x, \cdot)$. We suppose that system (1) satisfies the following assumptions.

Assumption 1. $f_u(x, \cdot)$ is nonsingular.

Assumption 2. Each $f_i(x, 0)$ and $\|f_u(x, \cdot)^{-1}\|$ are bounded.

The control objective is to design the sampled data controller $u_i(t) = u_i(kT_s)$, $kT_s \leq t \leq (k+1)T_s$, such that the output $x_i(t)$ of the controlled system (1) tracks the preset value x_{ir} , $i = 1, 2, \dots, n$.

We first define the following set:

$$\Omega_i = \{x_i \in R \mid |x_i| \leq \eta_i\}, \quad i = 1, 2, \dots, n, \quad (2)$$

where η_i is a small positive constant.

Denote $x_{i,k} = x_i(kT_s)$ and $u_{i,k} = u_i(kT_s)$ for $i = 1, 2, \dots, n$. Let

$$y_{i,k} = x_{i,k} + D_i d_i, \quad i = 1, 2, \dots, n, \quad (3)$$

where $d_i > \eta_i$ is a constant; $D_i = 1$ if $x_{i,k} \in \Omega_i$ and $D_i = 0$ if $x_{i,k} \notin \Omega_i$; we refer to $y_{i,k}$ as the unified variable. From (3), we can see that unified variables $y_{i,k}$ are not close to zero.

Theorem 1. Given an MIMO nonaffine nonlinear system (1) satisfying Assumption 1, if position keeping or position tracking control is desired, then its characteristic model can be described by the following system of first-order difference equations:

$$y_{i,k+1} = \theta_{i,k} y_{i,k} + T_s u_{i,k}, \quad i = 1, 2, \dots, n, \quad (4)$$

where $\theta_{i,k}$ is the so-called characteristic parameter as follows:

$$\theta_{i,k} = 1 + \varepsilon_{i,k}, \quad \varepsilon_{i,k} = \frac{g_{i,k}}{y_{i,k}}, \quad (5)$$

where $g_{i,k}$ is the i th component of the vector $\int_{kT_s}^{(k+1)T_s} [(I - f_u(x, \cdot)^{-1})\dot{x} + f_u(x, \cdot)^{-1}f(x, 0)]ds$.

Proof. For a fixed i ($i = 1, \dots, n$), the following equation can be obtained based on the differential mean value theorem:

$$\dot{x}_i = f_i(x, 0) + \bar{f}_i(x, \bar{u}_i)u, \quad i = 1, 2, \dots, n, \quad (6)$$

where $\bar{u}_i = r_i u$, $0 < r_i < 1$,

$$\bar{f}_i(x, \bar{u}_i) = [\bar{f}_{i1}(x, \bar{u}_i), \bar{f}_{i2}(x, \bar{u}_i), \dots, \bar{f}_{in}(x, \bar{u}_i)] = \frac{\partial f_i(x, u)}{\partial u} \Big|_{u=\bar{u}_i}, \quad \bar{f}_{il}(x, \bar{u}_i) = \frac{\partial f_{il}(x, u)}{\partial u_l} \Big|_{u=\bar{u}_i}. \quad (7)$$

Therefore, we have

$$\dot{x} = f(x, 0) + \bar{f}u, \quad (8)$$

where $\bar{f} = f_u(x, \cdot) = [\bar{f}_1(x, \bar{u}_1)^T, \bar{f}_2(x, \bar{u}_2)^T, \dots, \bar{f}_n(x, \bar{u}_n)^T]^T$.

According to Assumption 1, the inverse function $\bar{f}^{-1}$ must exist. Thus, multiplying both sides of (8) with $\bar{f}^{-1}$ yields

$$\bar{f}^{-1}\dot{x} = \bar{f}^{-1}f(x, 0) + u. \quad (9)$$

Therefore, we have

$$\dot{x} = (I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0) + u. \quad (10)$$

Integrating both sides of the above equation yields

$$x(t) = x(0) + \int_0^t [(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)]ds + \int_0^t u(s)ds. \quad (11)$$

Now set $t = (k+1)T_s$; then, (11) can be rewritten as

$$x((k+1)T_s) = x(0) + \int_0^{(k+1)T_s} [(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)]ds + \int_0^{(k+1)T_s} u(s)ds. \quad (12)$$

Setting $t = kT_s$, (11) can also be rewritten as

$$x(kT_s) = x(0) + \int_0^{kT_s} [(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)]ds + \int_0^{kT_s} u(s)ds. \quad (13)$$

Subtracting (13) from (12) yields

$$\begin{aligned} x((k+1)T_s) &= x(kT_s) + \int_{kT_s}^{(k+1)T_s} [(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)]ds + \int_{kT_s}^{(k+1)T_s} u(s)ds \\ &= x(kT_s) + \int_{kT_s}^{(k+1)T_s} [(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)]ds + T_s u(kT_s). \end{aligned} \quad (14)$$

Denote

$$\begin{aligned} g_k &= (g_{1,k}, g_{2,k}, \dots, g_{n,k})^T \\ &= \int_{kT_s}^{(k+1)T_s} \left[(I - \bar{f}^{-1}) \dot{x} + \bar{f}^{-1} f(x, 0) \right] ds, \end{aligned} \quad (15)$$

$$u_k = (u_{1,k}, u_{2,k}, \dots, u_{n,k})^T = u(kT_s). \quad (16)$$

Then, we have $x((k+1)T_s) = x(kT_s) + g_k + u_k$, or equivalently,

$$x_i((k+1)T_s) = x_i(kT_s) + g_{i,k} + T_s u_{i,k}, \quad i = 1, 2, \dots, n. \quad (17)$$

Next, we analyze two cases individually.

Case 1. If $x_i(k) \notin \Omega_i$, then the i th output from the controlled object is far away from zero. Therefore, (17) can be rewritten as

$$x_i((k+1)T_s) = \left[1 + \frac{g_{i,k}}{x_i(kT_s)} \right] x_i(kT_s) + T_s u_{i,k}. \quad (18)$$

Case 2. If $x_i(k) \in \Omega_i$, then the i th output from the controlled object is close to zero. In this case, (17) can be rewritten as

$$\begin{aligned} x_i((k+1)T_s) + d_i &= x_i(kT_s) + d_i + g_{i,k} + T_s u_{i,k} \\ &= \left[1 + \frac{g_{i,k}}{x_i(kT_s) + d_i} \right] (x_i(kT_s) + d_i) + T_s u_{i,k}. \end{aligned} \quad (19)$$

Combining Cases 1 and 2 from (3), we can obtain the characteristic model described by (4).

Remark 1. From Assumption 2, we can derive the range of the characteristic parameter $\theta_{i,k}$. In fact, from Assumption 2, there exist positive constants M_f and M_u such that $\|f(x, 0)\| \leq M_f$ and $\|\bar{f}^{-1}\| = \|f_u(x, \cdot)^{-1}\| \leq M_u$. We note that $\dot{x}$ is continuous. Thus, $\dot{x}$ is bounded on each closed interval $[kT_s, (k+1)T_s]$. That is, there exist positive constants M_{xk} such that $\|\dot{x}\| \leq M_{xk}$ on $[kT_s, (k+1)T_s]$. Further, $\|\dot{x}\| \leq M_x = \text{Supremum}_k \{M_{xk}\}$. This is not restrictive, and the supremum of $\{M_{xk}\}$ exists in practical engineering systems [41, 42]. Therefore, $\|(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)\| \leq \|(I - \bar{f}^{-1})\dot{x}\| + \|\bar{f}^{-1}f(x, 0)\| \leq \|(I - \bar{f}^{-1})\| \cdot \|\dot{x}\| + \|\bar{f}^{-1}\| \cdot \|f(x, 0)\| \leq \|I\| \cdot M_x + M_u \cdot M_x + M_u \cdot M_f$.

We denote $G = \|I\| \cdot M_x + M_u \cdot M_x + M_u \cdot M_f$. Thus, we have $|\bar{g}_{i,k}| \leq G$, where $\bar{g}_{i,k}$ is the i th component of the vector $(I - \bar{f}^{-1})\dot{x} + \bar{f}^{-1}f(x, 0)$; then, $|g_{i,k}| \leq GT_s$, and we obtain $|\varepsilon_{i,k}| < \varepsilon_i$, where $\varepsilon_i = GT_s/\eta_i$ if $x_i(kT_s) \notin \Omega_i$ and $\varepsilon_i = GT_s/(d_i - \eta_i)$ if $x_i(kT_s) \in \Omega_i$. From (5), we get

$$1 - \varepsilon_i = \delta_{i1} < \theta_{i,k} < \delta_{i2} = 1 + \varepsilon_i. \quad (20)$$

3. Controller Design and Stability Analysis

We define the tracking errors as $e_{i,k} = x_{i,k} - x_{ir}$ for $i = 1, 2, \dots, n$. The reference tracking signal of the unified variable $y_{i,k}$ can be expressed by $y_{ir} = x_{ir} + D_i d_i$. Note that the tracking errors of the unified variables equal the ones of the original object, which can be seen from

$$\begin{aligned} y_{i,k} - y_{ir} &= x_{i,k} + D_i d_i - (x_{ir} + D_i d_i) \\ &= x_{i,k} - x_{ir} = e_{i,k}. \end{aligned} \quad (21)$$

We present the basic approach to obtain an adaptive controller that achieves our control objective. If the function f is known, then each $\theta_{i,k}$ is also known. The control laws $u_{i,k}^* = (-\theta_{i,k} y_{i,k} + a_{i0} e_{i,k} + y_{ir})/T_s$ ($i = 1, 2, \dots, n$) applied to (4) result in $y_{i,k+1} = a_{i0} e_{i,k} + y_{ir}$, that is, $e_{i,k+1} - a_{i0} e_{i,k} = 0$. Notice that if one chooses a_{i0} such that $|a_{i0}| < 1$, where $i = 1, 2, \dots, n$, then the roots of the polynomials $H(q_i) = q_i - a_{i0}$ related to the characteristic equation of $e_{i,k+1} - a_{i0} e_{i,k} = 0$ are inside unit circles, which implies that $\lim_{k \rightarrow +\infty} e_{i,k} = 0$. This is main objective of control. Because f is unknown, the optimal control $u_{i,k}^*$ cannot be implemented. Thus, our purpose is to design an adaptive controller to approximate this optimal control.

Based on the above ideas, the adaptive controller is designed as

$$u_{i,k} = \frac{1}{T_s} (-\hat{\theta}_{i,k} y_{i,k} + a_{i0} e_{i,k} + y_{ir}), \quad i = 1, 2, \dots, n, \quad (22)$$

where $a_{i0} \in (0, 1)$ and $\hat{\theta}_{i,k}$ denotes the estimation of the parameter $\theta_{i,k}$.

Note that $a_{i0} \in (-1, 0]$ is not considered for the convenience of theoretical analysis.

The adaptive law of parameter $\theta_{i,k}$ is designed as

$$\hat{\theta}_{i,k} = \frac{m_i \hat{\theta}_{i,k-1} + \lambda_i a_{i0} y_{i,k} e_{i,k}}{1 + \lambda_i \mu_i y_{i,k}^2}, \quad (23)$$

where $\lambda_i > 0, \mu_i > 0$, and $0 < m_i < 1$ represent design constants.

The adaptive controller (22) includes estimated parameters $\hat{\theta}_{i,k}$, and the performance of asymptotic tracking with zero error or near zero can be obtained by adjusting the design parameters $\lambda_i > 0, \mu_i > 0$, and $0 < m_i < 1$.

Let $\varphi_{i,k} = \theta_{i,k} - \hat{\theta}_{i,k}$ be the parameter estimation error. The error equation of the closed-loop system can be expressed by substituting (22) into (4) as follows:

$$e_{i,k+1} = (\theta_{i,k} - \hat{\theta}_{i,k}) y_{i,k} + a_{i0} e_{i,k} = \varphi_{i,k} y_{i,k} + a_{i0} e_{i,k}. \quad (24)$$

From (23), we have

$$\hat{\theta}_{i,k} - m_i \hat{\theta}_{i,k-1} = \lambda_i (a_{i0} y_{i,k} e_{i,k} - \mu_i y_{i,k}^2 \hat{\theta}_{i,k}). \quad (25)$$

Thus, as the basis for preparing the stability analysis of the closed-loop system, the relation between the parameter estimation errors at two adjacent instants can be obtained as follows:

$$\begin{aligned}
\varphi_{i,k} &= \theta_{i,k} - \hat{\theta}_{i,k} + m_i \varphi_{i,k-1} - m_i \varphi_{i,k-1} \\
&= m_i \varphi_{i,k-1} + \theta_{i,k} - \hat{\theta}_{i,k} - m_i (\theta_{i,k-1} - \hat{\theta}_{i,k-1}) \\
&= m_i \varphi_{i,k-1} + \theta_{i,k} - m_i \theta_{i,k-1} - (\hat{\theta}_{i,k} - m_i \hat{\theta}_{i,k-1}) \\
&= m_i \varphi_{i,k-1} + \theta_{i,k} - m_i \theta_{i,k-1} - \lambda_i (a_{i0} y_{i,k} e_{i,k} - \mu_i y_{i,k}^2 \hat{\theta}_{i,k}) \\
&= m_i \varphi_{i,k-1} + \theta_{i,k} - m_i \theta_{i,k-1} - \lambda_i a_{i0} y_{i,k} e_{i,k} + \lambda_i \mu_i y_{i,k}^2 (\theta_{i,k} - \varphi_{i,k}) \\
&= m_i \varphi_{i,k-1} + w_{i,k},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
w_{i,k} &= (\theta_{i,k} - m_i \theta_{i,k-1}) - \lambda_i a_{i0} y_{i,k} e_{i,k} \\
&\quad + \lambda_i \mu_i \theta_{i,k} y_{i,k}^2 - \lambda_i \mu_i \varphi_{i,k} y_{i,k}^2.
\end{aligned} \tag{27}$$

For subsequent use, we denote

$$\lambda_{i1} = \frac{m_i^2 \Delta_i - m_i^2 \sqrt{\Delta_i^2 - c_i^2}}{c_i^2}, \tag{28}$$

$$\lambda_{i2} = \frac{m_i^2 \Delta_i + m_i^2 \sqrt{\Delta_i^2 - c_i^2}}{c_i^2}, \tag{29}$$

where $\Delta_i = -a_{i0} c_i + (1 - a_{i0}^2) \mu_i$, $c_i = \mu_i \delta_{i2} - a_{i0}$; $i = 1, 2, \dots, n$. Then, we have the following lemma.

Lemma 1. Let μ_i satisfy

$$\frac{a_{i0}}{\delta_{i2}} < \mu_i < \frac{a_{i0}}{\delta_{i2} - 1 + a_{i0}}, \quad i = 1, 2, \dots, n. \tag{30}$$

Then, $\mu_i, \lambda_{i1}, \lambda_{i2} > 0$, and $\lambda_{i1} < \lambda_{i2}$.

Proof. For the sake of simplicity, the subscript i for all variables is omitted in this proof. For example, a_0 written in this proof actually represents a_{i0} .

We first need to prove that (30) is well defined. It should be noted that $\delta_2 > \delta_2 - 1 + a_0 = 1 + \varepsilon - 1 + a_0 > 0$. Therefore, (30) is well defined and clearly $\mu > 0$.

To prove that λ_1 and λ_2 are also well defined, we need to first prove that $\Delta^2 - c^2 > 0$. To do this, let us consider the following equality:

$$\begin{aligned}
\Delta - c &= -a_0 c + (1 - a_0^2) \mu - c = -(1 + a_0) c + (1 - a_0^2) \mu \\
&= -(1 + a_0) [(\delta_2 - 1 + a_0) \mu - a_0].
\end{aligned} \tag{31}$$

Since $\mu < a_0 / (\delta_2 - 1 + a_0)$, we have $\Delta > c$. Considering that $\mu > a_0 / \delta_2$, we obtain $c = \mu \delta_2 - a_0 > 0$. Thus, we further have $\Delta > 0$, and hence $\Delta^2 - c^2 > 0$. Therefore, both λ_1 and λ_2 are well defined. It is clear that $\lambda_1, \lambda_2 > 0$ and $\lambda_1 < \lambda_2$. $\square$

We set $T_i = \min\{\eta_i, (d_i - \eta_i)\}$, $i = 1, 2, \dots, n$; $T_{\min} = \min_{1 \leq i \leq n} \{T_i / G_i\}$. We require the following lemma to summarize the performance of our proposed control strategy.

Lemma 2 (see [43], Lemma 6). Let $x, y \in R$. Then,

$$2xy \leq \varepsilon x^2 + \frac{1}{\varepsilon} y^2, \tag{32}$$

for any $\varepsilon > 0$.

Theorem 2. Consider an MIMO nonaffine nonlinear system (1) satisfying Assumptions 1 and 2. If the adaptive tracking controller is designed as (22) with update law (23), the sampling period $T_s < T_{\min}$, and controller parameters $\mu_i \in (a_{i0} / \delta_{i1}, a_{i0} / (\delta_{i2} - 1 + a_{i0}))$, $\lambda_i \in (\lambda_{i1}, \lambda_{i2})$ ($i = 1, 2, \dots, n$), then the following properties are guaranteed:

- (1) The tracking errors $e_{i,k}$ ($i = 1, 2, \dots, n$) and each parameter estimation error $\varphi_{i,k}$ are both bounded.
- (2) The tracking errors $e_{i,k}$ ($i = 1, 2, \dots, n$) converge to the neighborhoods of the origin.

Proof. First, we note that from (20), $\varepsilon_i < 1$ when the sampling period $T_s < T_{\min}$. Thus, $\delta_{i1} = 1 - \varepsilon_i > 0$. Since $\mu_i > a_{i0} / \delta_{i1}$, it is easy to see that $\mu_i > a_{i0} / \delta_{i2}$. From Lemma 1, there exist real numbers λ_i such that $\lambda_i \in (\lambda_{i1}, \lambda_{i2})$ for $i = 1, 2, \dots, n$. Now, consider the Lyapunov function given by

$$V_{i,k} = e_{i,k+1}^2 + \varphi_{i,k}^2. \tag{33}$$

Part 1. Calculating the difference of the Lyapunov function (33).

By utilizing (24), we can derive the first difference of the Lyapunov function (33) as

$$\begin{aligned}
\Delta V_{i,k} &= V_{i,k} - V_{i,k-1} \\
&= e_{i,k+1}^2 - e_{i,k}^2 + \varphi_{i,k}^2 - \varphi_{i,k-1}^2 \\
&= (\varphi_{i,k} y_{i,k} + a_{i0} e_{i,k})^2 - e_{i,k}^2 + (\varphi_{i,k}^2 - \varphi_{i,k-1}^2) \\
&= a_{i0}^2 e_{i,k}^2 + \varphi_{i,k}^2 y_{i,k}^2 + 2a_{i0} \varphi_{i,k} y_{i,k} e_{i,k} - e_{i,k}^2 + (\varphi_{i,k}^2 - \varphi_{i,k-1}^2).
\end{aligned} \tag{34}$$

Using (26), the last term $-\varphi_{i,k-1}^2$ in (34) can be written as

$$-\varphi_{i,k-1}^2 = -\left(\frac{\varphi_{i,k} - w_{i,k}}{m_i}\right)^2 = -\frac{1}{m_i^2} \varphi_{i,k}^2 - \frac{w_{i,k}^2}{m_i^2} + \frac{2\varphi_{i,k} w_{i,k}}{m_i^2}. \tag{35}$$

Now, substituting (27) into $\varphi_{i,k} w_{i,k}$ in the last term of (35) and employing $y_{i,k} = e_{i,k} + y_{ir}$, we then have

$$\begin{aligned}
\varphi_{i,k} w_{i,k} &= \varphi_{i,k} \left[(\theta_{i,k} - m_i \theta_{i,k-1}) - \lambda_i a_{i0} y_{i,k} e_{i,k} + \lambda_i \mu_i \theta_{i,k} y_{i,k}^2 - \lambda_i \mu_i \varphi_{i,k} y_{i,k}^2 \right] \\
&= (\theta_{i,k} - m_i \theta_{i,k-1}) \varphi_{i,k} - \lambda_i a_{i0} \varphi_{i,k} y_{i,k} e_{i,k} + \lambda_i \mu_i \theta_{i,k} y_{i,k}^2 \varphi_{i,k} - \lambda_i \mu_i \varphi_{i,k}^2 y_{i,k}^2 \\
&= (\theta_{i,k} - m_i \theta_{i,k-1}) \varphi_{i,k} - \lambda_i a_{i0} \varphi_{i,k} y_{i,k} e_{i,k} + \lambda_i \mu_i \theta_{i,k} \varphi_{i,k} y_{i,k} (e_{i,k} + y_{ir}) - \lambda_i \mu_i \varphi_{i,k}^2 y_{i,k}^2 \\
&= -\lambda_i \mu_i \varphi_{i,k}^2 y_{i,k}^2 + (\theta_{i,k} - m_i \theta_{i,k-1}) \varphi_{i,k} + \lambda_i \varphi_{i,k} y_{i,k} e_{i,k} (-a_{i0} + \mu_i \theta_{i,k}) + \lambda_i \mu_i \theta_{i,k} \varphi_{i,k} y_{i,k} y_{ir}.
\end{aligned} \tag{36}$$

Substituting (36) into (35) and then into (34) yields

$$\begin{aligned}
\Delta V_{i,k} &= (-1 + a_{i0}^2) e_{i,k}^2 + \varphi_{i,k}^2 y_{i,k}^2 + 2a_{i0} \varphi_{i,k} y_{i,k} e_{i,k} + \varphi_{i,k}^2 - \frac{1}{m_i^2} \varphi_{i,k}^2 - \frac{w_{i,k}^2}{m_i^2} \\
&\quad + \frac{2}{m_i^2} \left[-\lambda_i \mu_i \varphi_{i,k}^2 y_{i,k}^2 + (\theta_{i,k} - m_i \theta_{i,k-1}) \varphi_{i,k} + \lambda_i \varphi_{i,k} y_{i,k} e_{i,k} (-a_{i0} + \mu_i \theta_{i,k}) + \lambda_i \mu_i \theta_{i,k} \varphi_{i,k} y_{i,k} y_{ir} \right] \\
&= (-1 + a_{i0}^2) e_{i,k}^2 + \left(-\frac{1}{m_i^2} + 1 \right) \varphi_{i,k}^2 - \frac{1}{m_i^2} w_{i,k}^2 + \varphi_{i,k}^2 y_{i,k}^2 \left(-\frac{2\lambda_i \mu_i}{m_i^2} + 1 \right) \\
&\quad + 2\varphi_{i,k} y_{i,k} e_{i,k} \left[a_{i0} + \frac{1}{m_i^2} (-\lambda_i a_{i0} + \lambda_i \mu_i \theta_{i,k}) \right] + \frac{2}{m_i^2} \lambda_i \mu_i \theta_{i,k} \varphi_{i,k} y_{i,k} y_{ir} + \frac{2}{m_i^2} (\theta_{i,k} - m_i \theta_{i,k-1}) \varphi_{i,k}.
\end{aligned} \tag{37}$$

Denote

$$\begin{aligned}
S_{i1} &= a_{i0} + \frac{1}{m_i^2} (-\lambda_i a_{i0} + \lambda_i \mu_i \theta_{i,k}), \\
S_{i2} &= \frac{1}{m_i^2} \lambda_i \mu_i \theta_{i,k} y_{ir}, \\
S_{i3} &= \frac{1}{m_i^2} (\theta_{i,k} - m_i \theta_{i,k-1}).
\end{aligned} \tag{38}$$

Then, (37) can be rewritten as

$$\begin{aligned}
\Delta V_{i,k} &= (-1 + a_{i0}^2) e_{i,k}^2 + \left(-\frac{1}{m_i^2} + 1 \right) \varphi_{i,k}^2 - \frac{1}{m_i^2} w_{i,k}^2 \\
&\quad + \varphi_{i,k}^2 y_{i,k}^2 \left(-\frac{2\lambda_i \mu_i}{m_i^2} + 1 \right) + 2S_{i1} \varphi_{i,k} y_{i,k} e_{i,k} \\
&\quad + 2S_{i2} \varphi_{i,k} y_{i,k} + 2S_{i3} \varphi_{i,k}.
\end{aligned} \tag{39}$$

Part 2. Enlarging the difference $\Delta V_{i,k}$ denoted by (39).

For further analysis, we derive the ranges of S_{i1} , S_{i2} , and S_{i3} shown in (38). Let S_{ij1} and S_{ij2} denote the minimum and maximum values of S_{ij} ($j = 1, 2, 3$), respectively. Based on the expressions of S_{i1} and S_{i3} , we have

$$\begin{aligned}
S_{i11} &= a_{i0} + \frac{1}{m_i^2} (\lambda_i \mu_i \delta_{i1} - \lambda_i a_{i0}), \\
S_{i12} &= a_{i0} + \frac{1}{m_i^2} (\lambda_i \mu_i \delta_{i2} - \lambda_i a_{i0}), \\
S_{i31} &= \frac{1}{m_i^2} (\delta_{i1} - m_i \delta_{i2}), \\
S_{i32} &= \frac{1}{m_i^2} (\delta_{i2} - m_i \delta_{i1}).
\end{aligned} \tag{40}$$

We denote $S_{i1m} = \max\{|S_{i11}|, |S_{i12}|\}$ and $S_{i3m} = \max\{|S_{i31}|, |S_{i32}|\}$. Then, $|S_{i1}| \leq S_{i1m}$ and $|S_{i3}| \leq S_{i3m}$. From (20), $|S_{i2}| \leq S_{i2m} = 1/m_i^2 \lambda_i \mu_i (1 + \varepsilon_i) \cdot |y_{ir}|$.

Now, employing Lemma 2, we obtain

$$2S_{i1} \varphi_{i,k} y_{i,k} e_{i,k} \leq 2S_{i1m} |\varphi_{i,k} y_{i,k}| \cdot |e_{i,k}| \leq \frac{S_{i1m}}{\gamma_{i1}} \varphi_{i,k}^2 y_{i,k}^2 + \gamma_{i1} S_{i1m} e_{i,k}^2, \tag{41}$$

$$\begin{aligned}
2S_{i2} \varphi_{i,k} y_{i,k} &\leq \frac{S_{i2m}}{\gamma_{i2}} \varphi_{i,k}^2 y_{i,k}^2 + \gamma_{i2} S_{i2m}, \\
2S_{i3} \varphi_{i,k} &\leq \frac{S_{i3m}}{\gamma_{i3}} \varphi_{i,k}^2 + \gamma_{i3} S_{i3m},
\end{aligned} \tag{42}$$

provided that γ_{i1} , γ_{i2} , and $\gamma_{i3} > 0$. Substituting these three inequalities into (39) yields

$$\begin{aligned}
\Delta V_{i,k} &\leq (-1 + a_{i0}^2)e_{i,k}^2 + \left(-\frac{1}{m_i^2} + 1\right)\varphi_{i,k}^2 - \frac{1}{m_i^2}w_{i,k}^2 + \varphi_{i,k}^2\gamma_{i,k}^2 \left(-\frac{2\lambda_i\mu_i}{m_i^2} + 1\right) + \frac{S_{i1m}}{\gamma_{i1}}\varphi_{i,k}^2\gamma_{i,k}^2 + \gamma_{i1}S_{i1m}e_{i,k}^2 + \frac{S_{i2m}}{\gamma_{i2}}\varphi_{i,k}^2\gamma_{i,k}^2 \\
&\quad + \gamma_{i2}S_{i2m} + \frac{S_{i3m}}{\gamma_{i3}}\varphi_{i,k}^2 + \gamma_{i3}S_{i3m} \\
&\leq (-1 + a_{i0}^2 + \gamma_{i1}S_{i1m})e_{i,k}^2 + \left(-\frac{2\lambda_i\mu_i}{m_i^2} + 1 + \frac{S_{i1m}}{\gamma_{i1}} + \frac{S_{i2m}}{\gamma_{i2}}\right)\varphi_{i,k}^2\gamma_{i,k}^2 + \left(-\frac{1}{m_i^2} + 1 + \frac{S_{i3m}}{\gamma_{i3}}\right)\varphi_{i,k}^2 + \gamma_{i2}S_{i2m} + \gamma_{i3}S_{i3m}.
\end{aligned} \tag{43}$$

Denote

$$\begin{aligned}
N_{i1} &= 1 - a_{i0}^2 - \gamma_{i1}S_{i1m}, \\
N_{i2} &= \frac{2\lambda_i\mu_i}{m_i^2} - 1 - \frac{S_{i1m}}{\gamma_{i1}} - \frac{S_{i2m}}{\gamma_{i2}}, \\
N_{i3} &= \frac{1}{m_i^2} - 1 - \frac{S_{i3m}}{\gamma_{i3}}, \\
\delta_{im} &= \gamma_{i2}S_{i2m} + \gamma_{i3}S_{i3m}.
\end{aligned} \tag{44}$$

Then, we have

$$\Delta V_{i,k} \leq -N_{i1}e_{i,k}^2 - N_{i2}\varphi_{i,k}^2\gamma_{i,k}^2 - N_{i3}\varphi_{i,k}^2 + \delta_{im}, \tag{45}$$

this is,

$$\begin{aligned}
\Delta V_{i,k} &= e_{i,k+1}^2 + \varphi_{i,k}^2 - e_{i,k}^2 - \varphi_{i,k-1}^2 \leq -N_{i1}e_{i,k}^2 \\
&\quad - N_{i2}\varphi_{i,k}^2\gamma_{i,k}^2 - N_{i3}\varphi_{i,k}^2 + \delta_{im}.
\end{aligned} \tag{46}$$

Part 3. Giving the properties of N_{i1} , N_{i2} , and N_{i3} in (46).

Based on the bounds of the characteristic parameters provided by (20), we can prove that there exist γ_{i1} , γ_{i2} , and $\gamma_{i3} > 0$, such that $N_{i1} \in (0, 1)$; N_{i2} and $N_{i3} > 0$. To prove these, it suffices to show that

$$2\lambda_i\mu_i - m_i^2 > 0, \quad \frac{m_i^2 S_{i1m}}{2\lambda_i\mu_i - m_i^2} < \frac{1 - a_{i0}^2}{S_{i1m}}. \tag{47}$$

Indeed, if (47) holds, then there exist γ_{i1} that satisfy

$$0 < \frac{m_i^2 S_{i1m}}{2\lambda_i\mu_i - m_i^2} < \gamma_{i1} < \frac{1 - a_{i0}^2}{S_{i1m}}. \tag{48}$$

Further, we have $N_{i1} = 1 - a_{i0}^2 - \gamma_{i1}S_{i1m} \in (0, 1)$, and $2\lambda_i\mu_i\gamma_{i1} - m_i^2\gamma_{i1} - m_i^2S_{i1m} > 0$. Thus, there exists a positive number

$$\gamma_{i2} > \frac{m_i^2\gamma_{i1}S_{i2m}}{2\lambda_i\mu_i\gamma_{i1} - m_i^2\gamma_{i1} - m_i^2S_{i1m}}. \tag{49}$$

Note that

$$N_{i2} = \frac{2\lambda_i\mu_i\gamma_{i1} - m_i^2\gamma_{i1} - m_i^2S_{i1m}}{m_i^2\gamma_{i1}} - \frac{S_{i2m}}{\gamma_{i2}}. \tag{50}$$

Consequently, we can obtain $N_{i2} > 0$ for any γ_{i1} and γ_{i2} that satisfy (48) and (49), respectively.

In what follows, we shall prove that (47) holds indeed.

For simplicity, the subscript i of variables is omitted until the end of the proof of (47), except for S_{i11} and S_{i12} . For example, we use the notation μ instead of μ_i .

(i) First, we prove the first formula in (47).

Since

$$\begin{aligned}
\{2\mu\Delta - c^2\}^2 - 4\mu^2[\Delta^2 - c^2] &= 4\mu^2\Delta^2 + c^4 - 4\mu\Delta c^2 - 4\mu^2\Delta^2 + 4\mu^2c^2 \\
&= c^4 - 4\mu\Delta c^2 + 4\mu^2c^2 = c^4 - 4\mu[-a_0c + (1 - a_0^2)\mu]c^2 + 4\mu^2c^2 \\
&= c^4 + 4\mu a_0c^3 - 4\mu^2c^2 + 4\mu^2a_0^2c^2 + 4\mu^2c^2 \\
&> 0.
\end{aligned} \tag{51}$$

We have $2\mu\Delta - c^2 > 2\mu\sqrt{\Delta^2 - c^2}$. Consequently, we obtain $2\mu((\Delta - \sqrt{\Delta^2 - c^2})/c^2) > 1$. Multiplying both sides of the above inequality by m^2 yields $(2\mu(m^2\Delta - m^2\sqrt{\Delta^2 - c^2})/c^2) > m^2$. That is, $2\lambda_1\mu - m^2 > 0$. Thus, $2\lambda\mu - m^2 > 0$.

(ii) Next, we prove the second formula in (47).

Note that $\delta_1 = 1 - \varepsilon > 0$ when the sampling period $T_s < T_{\min}$. Since $\mu > a_0/\delta_1$, we have $\mu\delta_1 > a_0$ and $\mu\delta_1 - a_0 > 0$. Therefore, $S_{i11} > 0$. Thus, we obtain $S_{1m} = S_{i12} = a_0 + 1/m^2(\lambda\mu\delta_2 - \lambda a_0)$. Then, from $c = \mu\delta_2 - a_0$, we have

$$\begin{aligned}
m^2S_{1m} - (1 - a_0^2)(2\lambda\mu - m^2) &= \frac{1}{m^2}[c\lambda + m^2a_0]^2 \\
&\quad - (1 - a_0^2)(2\lambda\mu - m^2) \\
&= \frac{1}{m^2}[c^2\lambda^2 - 2[-m^2a_0c \\
&\quad + m^2(1 - a_0^2)\mu]\lambda + m^4].
\end{aligned} \tag{52}$$

From $\lambda \in (\lambda_1, \lambda_2)$, we have $c^2\lambda^2 - 2[-m^2a_0c + m^2(1 - a_0^2)\mu]\lambda + m^4 < 0$, i.e.,

$$m^2 S_{1m}^2 < (1 - a_0^2)(2\lambda\mu - m^2). \quad (53)$$

Therefore, we have $m^2 S_{1m}/(2\lambda\mu - m^2) < (1 - a_0^2)/S_{1m}$. Thus, the second formula in (47) is proved.

Now, we notice that $N_{i3} > 0$ can be guaranteed by taking positive numbers $\gamma_{i3} > m_i^2 S_{i3m}/(1 - m_i^2)$.

Part 4. Proving the boundedness for the tracking errors $e_{i,k}$ and parameter estimation error $\varphi_{i,k}$ and convergence for tracking errors $e_{i,k}$.

From the above results and (46), we can obtain the following result:

$$\begin{aligned} e_{i,k+1}^2 + (1 + N_{i3})\varphi_{i,k}^2 &\leq (1 - N_{i1})e_{i,k}^2 + \varphi_{i,k-1}^2 - N_{i2}\varphi_{i,k}^2\gamma_{i,k}^2 + \delta_{im} \\ &\leq (1 - N_{i1})e_{i,k}^2 + \varphi_{i,k-1}^2 + \delta_{im}. \end{aligned} \quad (54)$$

Denote $N_i = \max\{1/(1 + N_{i3}), 1 - N_{i1}\}$; then, $N_i \in (0, 1)$. Thus, we have

$$e_{i,k+1}^2 + (1 + N_{i3})\varphi_{i,k}^2 \leq (1 - N_{i1})e_{i,k}^2 + \frac{1}{1 + N_{i3}}[(1 + N_{i3})\varphi_{i,k-1}^2] + \delta_{im} \leq N_i[e_{i,k}^2 + (1 + N_{i3})\varphi_{i,k-1}^2] + \delta_{im}. \quad (55)$$

In other words, we obtained a recursion relation that can yield

$$\begin{aligned} e_{i,k+1}^2 + (1 + N_{i3})\varphi_{i,k}^2 &\leq N_i^2[e_{i,k-1}^2 + (1 + N_{i3})\varphi_{i,k-2}^2] + N_i\delta_{im} + \delta_{im} \leq \dots \\ &\leq N_i^k[e_{i,1}^2 + (1 + N_{i3})\varphi_{i,0}^2] + N_i^{k-1}\delta_{im} + N_i^{k-2}\delta_{im} + \dots + N_i^2\delta_{im} + N_i\delta_{im} + \delta_{im} \\ &= N_i^k[e_{i,1}^2 + (1 + N_{i3})\varphi_{i,0}^2] + \frac{1 - N_i^k}{1 - N_i}\delta_{im} \\ &< e_{i,1}^2 + (1 + N_{i3})\varphi_{i,0}^2 + \frac{1}{1 - N_i}\delta_{im}. \end{aligned} \quad (56)$$

From this inequality, both $e_{i,k}$ and $\varphi_{i,k}$ are bounded, and we can obtain

$$e_{i,k+1}^2 \leq N_i^k[e_{i,1}^2 + (1 + N_{i3})\varphi_{i,0}^2] + \frac{1 - N_i^k}{1 - N_i}\delta_{im}, \quad (57)$$

which implies $\limsup_{k \rightarrow +\infty} e_{i,k+1}^2 \leq \delta_{im}/(1 - N_i)$. Thus, $\limsup_{k \rightarrow +\infty} |e_{i,k}| \leq \sqrt{\delta_{im}/(1 - N_i)}$. By the definition of the supremum limit, there exist integers $M_i > 0$ for all sufficiently small $\bar{\epsilon}_i > 0$ such that

$$|e_{i,k}| \leq \rho_i = \max\left\{|e_{i,1}|, \dots, |e_{i,M_i}|, \bar{\epsilon}_i + \sqrt{\frac{\delta_{im}}{1 - N_i}}\right\}. \quad (58)$$

Hence, Theorem 2 is proved. $\square$

Remark 2. If $f_i(x, u)$ is bounded, that is, there exists a constant $L_i > 0$ such that $|f_i(x, u)| \leq L_i$, then the tracking error $e_i(t) = x_i(t) - x_{ir}$ of system (1) converges to the neighborhood of the origin under the conditions in Theorem 2. In fact, for any $t \in [kT_s, (k+1)T_s]$, integrating (1) from kT_s to t and noting $\dot{e}_i(t) = \dot{x}_i(t)$, one can easily obtain $|e_i(t)| \leq |e_i(kT_s)| + T_s L_i \leq \rho_i + T_s L_i$, where ρ_i is defined in (58).

4. Numerical Example

To verify the effectiveness of the developed control method, consider a nonaffine double-pendulum system. The dynamic equations of motion are given as [44]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \end{bmatrix}, \quad (59)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t),$$

with $x(0) = [0.2, 0.1, 0.2, 0.1]^T$. $x = [x_1, x_2, x_3, x_4]^T$ is the vector of measurable states, $u = [u_1 u_2]^T$ is the vector control inputs, and

$$\begin{aligned} f_1(x, u) &= f_{11}(x)M_1(x, u_1) + f_{12}(x)M_2(x, u_2) + f_{13}(x), \\ f_2(x, u) &= f_{21}(x)M_1(x, u_1) + f_{22}(x)M_2(x, u_2) + f_{23}(x). \end{aligned} \quad (60)$$

Here,

$$\begin{aligned}
f_{11}(x_1, x_3) &= \frac{12}{l_1^2 [4m_1 + 12m_2 - 9m_2 \cos^2(x_3 - x_1)]}, \\
f_{12}(x_1, x_3) &= \frac{12l_2 + 18l_1 \cos(x_3 - x_1)}{l_1^2 l_2 [9m_2 \cos^2(x_3 - x_1) - 4m_1 - 12m_2]}, \\
f_{21}(x_1, x_3) &= \frac{18 \cos(x_3 - x_1)}{l_1 l_2 [9m_2 \cos^2(x_3 - x_1) - 4m_1 - 12m_2]}, \\
f_{22}(x_1, x_3) &= \frac{12m_1 + 36m_2}{l_2^2 m_2 [4m_1 + 12m_2 - 9m_2 \cos^2(x_3 - x_1)]} + \frac{18l_2 \cos(x_3 - x_1)}{l_1 l_2^2 [4m_1 + 12m_2 - 9m_2 \cos^2(x_3 - x_1)]}, \\
f_{13}(x_1, x_2, x_3, x_4) &= \frac{9gm_2 \sin(2x_3 - x_1)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} - \frac{9l_1 m_2 x_2^2 \sin(2x_3 - 2x_1)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} \\
&\quad + \frac{12l_2 m_2 x_4^2 \sin(x_3 - x_1)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} - \frac{(15gm_2 + 12gm_1) \sin(x_1)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]}, \\
f_{23}(x_1, x_2, x_3, x_4) &= \frac{12l_1^2 x_2^2 \sin(x_3 - x_1)(m_1 + 3m_2)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} - \frac{9l_2 m_2 x_4^2 \sin(2x_3 - 2x_1)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} \\
&\quad - \frac{9g \sin(x_3 - 2x_1)(m_1 + 2m_2)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]} + \frac{3g \sin(x_3)(m_1 + 6m_2)}{l_1 [15m_2 + 8m_1 - 9m_2 \cos(2x_3 - 2x_1)]}, \\
M_1(u_1) &= \tanh(u_1) + 0.35u_1, \\
M_2(u_2) &= \tanh(u_2) + 0.35u_2,
\end{aligned} \tag{61}$$

where m_1 and m_2 are the mass of pendulums 1 and 2, respectively, and the meaning of other symbols in (61) can be found in [44].

Herein, the control objective is to use our adaptive controller such that the states x_1 and x_3 are adjusted to 0 from the initial state. In this example, the parameter values of the nonaffine double-pendulum system are $m_1 = m_2 = 1$ kg, $l_1 = l_2 = 1$ m, and $g = 9.81$ m/s². The initial values of the parameters $\hat{\theta}_{i,k}$ are chosen to be $\hat{\theta}_{i,0} = 1$ for each $i = 1, 2$. The parameters of the controller are selected as $a_{i0} = 0.01T_s$, $\lambda_i = 1000$, $\mu_i = 0.001$, and $m_i = 0.1T_s$ for $i = 1, 2$. Meanwhile, we set $\eta_i = 0.0001$, $d_i = 0.0002$ for $i = 1, 2$, and the sampling period $T_s = 0.001$.

Figures 1–4 depict the simulation results. As shown in the figures, the system can effectively track the expected outputs, thereby confirming the effectiveness of our proposed control approach.

To demonstrate the effectiveness of the proposed control strategy, we use the fuzzy adaptive control method in [45] instead of our proposed adaptive controller. The control strategy proposed in [45] can be described as follows.

Control laws are designed as follows:

$$u_i = u_{di} + u_{fi} + u_{hi}, \quad i = 1, 2, \tag{62}$$

where $u_{di} = K_i^T \bar{e}_i$, $\bar{e}_i = [e_i, \dot{e}_i]^T$, e_i is the tracking error of x_i , $K_i = [k_{i2}, k_{i1}]^T$, and all roots of the polynomial $h(s) = s^2 + k_{i1}s + k_{i2}$ are in the open left-half plane; the fuzzy control terms $u_{fi} = \theta_i^T \xi_i(x)$, in which $\xi_i(x)$ is fuzzy basis function, and we use the following adaptation laws to adjust the unknown parameter vectors of fuzzy systems:

$$\dot{\theta}_i = -\eta_i \bar{e}_i^T P_i B \xi_i(x) - \sigma_i \eta_i \|\bar{e}_i\| \theta_i, \tag{63}$$

where $\eta_i, \sigma_i > 0$ are adaptive parameter rates;

$$u_{hi} = -\frac{1}{r_i} B^T P_i e_i, \quad i = 1, 2, \tag{64}$$

where $r_i > 0$ is a design parameter, $B = [0, 1]^T$, and P_i is a symmetric positive definite matrix satisfying the Riccati equation

$$A_i^T P_i + P_i A_i + Q_i - \left(\frac{2}{r_i} - \frac{1}{\rho_i^2} \right) P_i B B^T P_i = 0, \tag{65}$$

where $Q_i > 0$ and $A_i = \begin{bmatrix} 0 & 1 \\ -k_{i2} & -k_{i1} \end{bmatrix}$.

In the simulation, the design parameters are $k_{i1} = 2$,

$$k_{i2} = 1, \quad Q_i = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad r_i = 0.001, \quad \text{and} \quad \rho_i = 0.1, \quad \text{for}$$

FIGURE 1: Trajectories of the first output x_1 for our controller. FIGURE 2: Trajectories of the second output x_3 for our controller. FIGURE 3: Control input u_1 for our controller. FIGURE 4: Control input u_2 for our controller. FIGURE 5: Trajectories of the first output x_1 for the fuzzy controller [45]. FIGURE 6: Trajectories of the second output x_3 for the fuzzy controller [45].

$i = 1, 2$. The adaptive parameter rates are chosen as $\eta_i = 0.1$ and $\sigma_i = 10$, for $i = 1, 2$. With respect to fuzzy logic systems, Mamdani fuzzy models are employed to obtain the fuzzy

control terms u_{f1} and u_{f2} , and we define five Gaussian membership functions for each variable x_i , $i = 1, 2, 3, 4$, distributed uniformly on interval $[-0.8, 0.8]$. Then, there are

FIGURE 7: Control input u_1 for the fuzzy controller [45].

FIGURE 8: Control input u_2 for the fuzzy controller [45].

$2 \times 5 \times 5 \times 5 \times 5 = 1250$ rules that are used in the simulation. The initial state of each variable is the same as in the above simulation.

The results of this simulation are illustrated in Figures 5–8. The figures indicate that the adaptive fuzzy controller can also achieve better performance. However, it should be emphasized that this is the result of more than 2 min of CPU time running on a computer (Lenovo workstation P720 with dual Intel® Xeon® Gold 6136 Processor), whereas the results shown in Figures 1–4 only require 7 s of CPU time on the same computer.

There are 1250 parameters that need to be updated online for the scheme in [45], whereas only 2 parameters are required in our scheme. The fewer parameters used in our scheme help to make the proposed algorithm more efficient.

5. Conclusions

In this study, the control method for a class of uncertain nonaffine systems is studied for the first time by using a first-order characteristic model. Specifically, we first transformed the nonaffine uncertain system into a linear first-order characteristic model with bounded time-varying parameters. Based on this model, we designed an adaptive controller with

a simple structure and proved that the tracking errors converge to the neighborhoods of the origin. The proposed approach can avoid the controller singularity because there is no inverse operator construction in the controller. As an illustration of the effectiveness of the proposed method, a numerical example is provided.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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