Research Article

Hopf Bifurcation and Dynamic Analysis of an Improved Financial System with Two Delays

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The complex chaotic dynamics and multistability of financial system are some important problems in micro- and macroeconomic fields. In this paper, we study the influence of two-delay feedback on the nonlinear dynamics behavior of financial system, considering the linear stability of equilibrium point under the condition of single delay and two delays. The system undergoes Hopf bifurcation near the equilibrium point. The stability and bifurcation directions of Hopf bifurcation are studied by using the normal form method and central manifold theory. The theoretical results are verified by numerical simulation. Furthermore, one feature of the proposed financial chaotic system is that its multistability depends extremely on the memristor initial condition and the system parameters. It is shown that the nonlinear dynamics of financial chaotic system can be significantly changed by changing the values of time delays.

1. Introduction

It is widely recognized that chaos can be obtained in some mathematically simple systems of nonlinear differential equations. With the advent of computers, it is now possible to study the entire parameter space of these systems that result in some desired characteristics of the system. Recently, there has been increasing attention to some unusual examples and application of such systems [1–8]. The financial system is an extremely complex nonlinear dynamical system composed of many elements. The study of the complex nonlinear dynamics behavior of the financial system is an important problem in the fields of micro- and macroeconomy [9]. The uncertain factors bring very important influence to the description of the financial system and make analysis of the financial systems become a very important problem.

Researchers try to explain the core characteristics of economic data: irregular microeconomic fluctuations, unstable macroeconomic fluctuations, irregular growth, and syntax changes [10–13]. However, some inappropriate combination of parameters in the financial system may lead to financial markets in trouble or out of control. Therefore, it is necessary to make a systematic and deep study on the internal syntax characteristics of the complicated financial system. The results will reveal the bifurcation phenomena under different parameter combinations, probe into the causes of the complicated nonlinear dynamics phenomena, and predict and control the complicated financial systems [14, 15]. In addition, multistability is a critical property of nonlinear dynamical systems when coexisting attractors can be obtained for the same parameters, but different initial conditions [16–19]. The flexibility in the systems' performance can be archived without changing parameters.

Wang et al. [20] studied the bifurcation topology and the global complexity of a class of nonlinear financial systems. Ishiyama and Saiki [21] established the macroeconomic growth cycle model and solved the qualitative- and quantitative-related unstable periodic solutions embedded in the chaotic attractor. By using Lyapunov stability theory and Routh–Hurwitz criterion, Zhao et al. [22] studied the global synchronization of the three-dimensional chaotic financial system. Yu et al. [23] used numerical simulation to analyze the Lyapunov exponents and bifurcation diagram of chaotic financial system. Cantore and Levine [24] studied the reparameterized model with evaluation parameters. Gao et al. [25] gave the final bounded estimator set and chaotic synchronization analysis of the financial risks system.

Until now, one of the nonlinear economic and financial dynamic models confirmed by economists comes from a financial system model composed of four subblocks (production, money, security, and labor) [15, 24, 26].

$$\dot{x} = z + (y - a)x, \quad (1a)$$

$$\dot{y} = 1 - by - x^2, \quad (1b)$$

$$\dot{z} = -x - cz, \quad (1c)$$

where x represents the interest rate, y represents the investment demand, z represents the price index, a represents the saving amount, b represents the unit investment cost, and c represents the elasticity of commodity demand; a, b , and c are all normal numbers.

In real life, with the development of economy, there are more and more factors that restrict the development of economy. Some classical chaotic financial systems can not reflect the laws and changes of economic development well. For example, the factors that affect the change of interest rate are related to the average profit rate besides investment demand and price index, and the average profit rate is proportional to the interest rate. Therefore, we construct the following improved chaotic financial system model:

$$\dot{x} = z + (y - a)x, \quad (2a)$$

$$\dot{y} = 1 - by - x^2 - bxy, \quad (2b)$$

$$\dot{z} = -x - cz, \quad (2c)$$

where x is the interest rate, y is the investment demand, z is the price index, a is the saving amount, b is the unit investment cost, and c is the elasticity of commodity demand; a, b , and c are all normal numbers. When the parameters $a = 3$, $b = 0.1$, and $c = 1$ and initial values are at points $(0.1, 2, 0.1)$, system (2a)–(2c) generates chaotic attractors, as shown in Figure 1.

With the development and innovation of financial markets, scholars have found that it will be better to add time delay factor for describing the actual economic markets [27–30]. Chen [31] analyzed the complex nonlinear dynamics, such as periodicity, quasiperiodicity, and chaotic behavior in the delayed feedback of financial systems. Ma

FIGURE 1: Chaotic attractor for the 3D chaotic system ((2a)–(2c)).

and Tu [32] established a class of complex dynamic macroeconomic systems and studied the effect of time delay on savings rate and dynamic financial stability. Holyst and Urbanowicz [33] have shown that the chaotic attractor of the financial model can be stabilized in a periodic track by using Pyragas delayed feedback control. In addition, Ma and Chen [14] added the delayed feedback to the three variables of financial system and gave some results on the existence of Hopf bifurcation and the effect of delayed feedback. Based on political events and other human factors, some scholars have considered the impact of delay and feedback items (see [28, 34]). In practice, financial behaviour is not only affected by a single time delay but often seems to be affected by multiple external shocks. These various external influences embody multiple delays and can be reflected in all variables, i.e., by introducing various delayed feedback items into interest rates x of change of interest rate, they will also have a significant impact on system (2a)–(2c). Therefore, we further consider the double-delay system.

$$\dot{x} = z + (y - a)x + k_1(x(t) - x(t - \tau_1)) + k_2(x(t) - x(t - \tau_2)), \quad (3a)$$

$$\dot{y} = 1 - by - x^2 - bxy, \quad (3b)$$

$$\dot{z} = -x - cz. \quad (3c)$$

where τ_1 and τ_2 are the two time delays and k_1 and k_2 are the feedback control intensities.

In this paper, we study the Hopf bifurcation and nonlinear dynamics of an improved financial system with two delays. Firstly, we study the distribution of the roots of the characteristic equations at the equilibrium point. Sufficient conditions for the local stability of the equilibrium point and the existence of Hopf bifurcation are obtained. Secondly, taking two delays as bifurcation parameters and using the canonical form method and the central manifold theorem, we determine the bifurcation direction of the periodic solution and the explicit algorithm for the stability of the bifurcation periodic solution. Under the premise of the existence of local bifurcation, the existence of the bifurcation periodic solution of this system is discussed by using the theory of functional differential equations. Finally, the

correctness of the conclusion is verified by numerical simulation.

2. Existence of Hopf Bifurcation in Financial System

In order to study the influence of time delays on nonlinear dynamic system, the three equilibrium points of the system are obtained as follows:

$$\left(-1, \frac{1+ac}{c}, \frac{1}{c}\right), \quad (4a)$$

$$\left(0, \frac{1}{b}, 0\right), \quad (4b)$$

$$\left(\frac{-b+c-abc}{c}, \frac{1+ac}{c}, \frac{b-c+abc}{c^2}\right). \quad (4c)$$

Here, we only analyze the following equilibrium point:

$$(x_0, y_0, z_0) = \left(0, \frac{1}{b}, 0\right). \quad (5)$$

Linear transformations are given as

$$u_1 = x - x_0, \quad (6a)$$

$$u_2 = y - y_0, \quad (6b)$$

$$u_3 = z - z_0. \quad (6c)$$

System (2a)–(2c) becomes the following equations:

$$\dot{u}_1 = u_3 + \left(u_2 - a + \frac{1}{b}\right)u_1 + k_1(u_1(t) - u_1(t - \tau_1)) \quad (7a)$$

$$+ k_2(u_1(t) - u_1(t - \tau_2)),$$

$$\dot{u}_2 = -u_1 - bu_2 - u_1^2 - bu_1, \quad (7b)$$

$$\dot{u}_3 = -u_1 - cu_3. \quad (7c)$$

The characteristic equation of equations (7a) and (7b) at $(0, 0, 0)$ is

$$\begin{aligned} & b - c + abc + ab\lambda + ac\lambda - \frac{c\lambda}{b} + bc\lambda + a\lambda^2 - \frac{\lambda^2}{b} + b\lambda^2 + c\lambda^2 + \lambda^3 \\ & + e^{-\lambda\tau_1}(-bck_1 - bk_1\lambda - ck_1\lambda - k_1\lambda^2) + e^{-\lambda\tau_2}(-bck_2 - bk_2\lambda - ck_2\lambda - k_2\lambda^2) = 0. \end{aligned} \quad (8)$$

Since the dynamics of differential equations with two delays are very complex, we first discuss the case of $\tau_1 = \tau_2 = 0$, then discuss the case of $\tau_1 > 0, \tau_2 = 0$ with single delay, and finally discuss the case of $\tau_1 > 0, \tau_2 > 0$.

Case 1. $\tau_1 = \tau_2 = 0$.

The characteristic equation (5) becomes

$$b - c + abc + \left(ab + ac - \frac{c}{b} + bc\right)\lambda + \left(a - \frac{1}{b} + b + c\right)\lambda^2 + \lambda^3 = 0. \quad (9)$$

Let

$$p_1 = a - \frac{1}{b} + b + c, \quad (10a)$$

$$p_2 = ab + ac - \frac{c}{b} + bc, \quad (10b)$$

$$p_3 = b - c + abc, \quad (10c)$$

and assume

$$H_1: p_1 > 0, p_3 > 0, p_1 p_2 - p_3 > 0. \quad (11)$$

According to Routh–Hurwitz criterion, if H_1 holds, the equilibrium point $(0, 0, 0)$ of system (7a)–(7c) is locally asymptotically stable.

Case 2. $\tau_1 > 0, \tau_2 = 0$.

The characteristic equation (5) becomes

$$\begin{aligned} & b - c + abc - bck_1 + ab\lambda + ac\lambda - \frac{c\lambda}{b} + bc\lambda - bk_1\lambda - ck_1\lambda + a\lambda^2 - \frac{\lambda^2}{b} \\ & + b\lambda^2 + c\lambda^2 - k_1\lambda^2 + \lambda^3 + e^{-\lambda\tau_1}(bck_1 + bk_1\lambda + ck_1\lambda + k_1\lambda^2) = 0. \end{aligned} \quad (12)$$

If $\lambda = iw$ is a solution of equation (12), then the real part and imaginary part are separated and made equal to zero. We can obtain

$$\begin{aligned} & w^2 + b^2(1 + ac + ck_1(-1 + m) + k_1nw - w^2) \\ & + b(-(a + k_1(-1 + m))w^2 + c(-1 + k_1nw - w^2)) = 0, \end{aligned} \quad (13a)$$

$$\begin{aligned} & -cw + bw(ac + ck_1(-1 + m) + (k_1n - w)w) \\ & + b^2((a + k_1(-1 + m))w + c(-k_1n + w)) = 0, \end{aligned} \quad (13b)$$

where $m = \cos(w\tau_1)$ and $n = \sin(w\tau_1)$.

From formulas (13a) and (13b),

$$\begin{aligned} b^2 w^4 + (1 - 2ab - 2b^2 + a^2 b^2 + b^2 c^2 + 2bk_1 - 2ab^2 k_1) w^2 + b^2 - 2bc + 2ab^2 c \\ + c^2 - 2abc^2 + a^2 b^2 c^2 - 2b^2 ck_1 + 2bc^2 k_1 - 2ab^2 c^2 k_1 = 0. \end{aligned} \quad (14)$$

If all the parameters in system (3a)–(3c) are given, it is easy to calculate the numerical solution of equation (14) by computer. Thus, the following assumptions are given.

Suppose H_2 that equation (14) has at least one positive real root.

If H_2 is assumed to be true and equation (14) has two positive real roots ω_k ($k = 1, 2$), we have

$$\tau_1(k, j) = \frac{1}{\omega_k} [\arccos(p) + 2j\pi], \quad Q \geq 0, \quad (15a)$$

$$\tau_1(k, j) = \frac{1}{\omega_k} [2\pi - \arccos(p) + 2j\pi], \quad Q \leq 0, \quad k = 1, 2, \quad j = 0, 1, \dots, \quad (15b)$$

where

$$P = \frac{bc - c^2 + abc^2 - bc^2 k_1 - w^2 + abw^2 - bk_1 w^2}{bk_1 (c^2 + w^2)}, \quad (16a)$$

$$Q = \frac{w - c^2 w - w^3}{k_1 (c^2 + w^2)}. \quad (16b)$$

$\pm i\omega_k$ is a pair of pure virtual root under $\tau_1 = \tau_1(k, j)$ of equation (12). Let us take

$$\tau_1^0 = \min\{\tau_1(k, j), k = 1, 2; j = 0, 1, \dots\}. \quad (17)$$

Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ be the virtual root of equation (12) near $\tau_1 = \tau_1(k, j)$. By differential degeneracy of equation (12) with respect to τ , we can obtain

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1} \right]^{-1} &= \frac{e^{\lambda\tau_1} [ab^2 - c + abc + b^2 c - (b^2 - bc)k_1 + 2(ab + b^2 + bc - 1)\lambda]}{bk_1 \lambda (b + \lambda) (c + \lambda)} \\ &+ \frac{3b\lambda^2 + b^2 k_1 + bck_1 - b^2 ck_1 \tau_1 - b^2 k_1 \lambda \tau_1 - bck_1 \lambda^2 \tau_1}{bk_1 \lambda (b + \lambda) (c + \lambda)}. \end{aligned} \quad (18)$$

When $\lambda = i\omega$ is substituted into equation (18), we have

$$\left[\frac{d(\operatorname{Re}\lambda)}{d\tau_1} \right]^{-1} = \frac{f(\omega_k)}{\Lambda}. \quad (19)$$

Among them,

$$\begin{aligned} f(\omega_k) &= a^2 b^2 c^2 + b^4 c^2 + 2b^3 k_1 - 2ab^4 k_1 - 2b^2 ck_1 + 2bc^2 k_1 \\ &+ (2 - 4ab - 4b^2 + 2a^2 b^2 + 2b^4 + 2b^2 c^2 \\ &+ 4bk_1 - 4ab^2 k_1) \omega k^2 \\ &+ 2b^2 - 2ab^3 - 2b^4 + a^2 b^4 - 2bc + 2ab^2 c \\ &+ c^2 - 2abc^2 - 2ab^2 c^2 k_1 + 3b^2 \omega k^4, \end{aligned} \quad (20a)$$

$$\Lambda = b^2 k l^2 (b^2 + \omega k^2) (c^2 + \omega k^2). \quad (20b)$$

Hypothesis 1. H_3 :

$$\left[\frac{d(\operatorname{Re}\lambda)}{d\tau_1} \right]^{-1} \neq 0. \quad (21)$$

Therefore, there are the following theorems.

Theorem 1. If (H_1) , (H_2) , and (H_3) are assumed to be true, then the equilibrium $(0, (1/b), 0)$ is locally stable for $\tau_1 \in (0, \tau_1^0)$. When $\tau_1 > \tau_1^0$ equilibrium $(0, (1/b), 0)$ is unstable and Hopf bifurcation occurs in system (3a)–(3c) at $\tau_1 = \tau_1^0$.

Case 3. $\tau_1 > 0, \tau_2 > 0$.

The corresponding characteristic equation of system (3a)–(3c) is (8). Now, let delay $\tau_1 \in (0, \tau_1^0)$, and τ_2 is taken as a parameter. Assuming $\lambda = i\sigma$ is the characteristic root under the two delays in the characteristic equation (5), then

$$\begin{aligned}
& \lambda^3 + \left(a - \frac{1}{b} + b + c - k_1 - k_2\right) \lambda^2 + \left(ab + ac - \frac{c}{b} + bc - bk_1 - ck_1 - bk_2 - ck_2\right) \lambda \\
& + b - c + abc - bck_1 - bck_2 + (\cos(\sigma\tau_1) - \sin(\sigma\tau_1))(bck_1 + bk_1\lambda + ck_1\lambda + k_1\lambda^2) \\
& + (\cos(\sigma\tau_2) - \sin(\sigma\tau_2))(bck_2 + bk_2\lambda + ck_2\lambda + k_2\lambda^2) = 0.
\end{aligned} \tag{22}$$

From (22),

$$\cos(\sigma\tau_2) = -\frac{1}{bk_2(c^2 + \sigma^2)} \begin{pmatrix} bc - c^2 + abc^2 - bc^2k_1 - bc^2k_2 + bc^2k_1 \cos(\sigma\tau_1) \\ -\sigma^2 + ab\sigma^2 - bk_1\sigma^2 - bk_2\sigma^2 + bk_1 \cos(\sigma\tau_1)\sigma^2 \end{pmatrix}, \tag{23a}$$

$$\sin(\sigma\tau_2) = -\frac{c^2k_1 \sin(\sigma\tau_1) + \sigma - c^2\sigma + k_1 \sin(\sigma\tau_1)\sigma^2 - \sigma^3}{k_2(c^2 + \sigma^2)}. \tag{23b}$$

Therefore, the equation about σ can be obtained:

$$\begin{aligned}
& b^2 - 2bc + 2ab^2c + c^2 - 2abc^2 + a^2b^2c^2 - 2b^2ck_1 + 2bc^2k_1 - 2ab^2c^2k_1 + c^2b^2k_1 - 2b^2ck_2 \\
& + 2bc^2k_2 - 2ab^2c^2k_2 + 2b^2c^2k_1k_2 + 2b^2ck_1 \cos(\sigma\tau_1) - 2bc^2k_1 \cos(\sigma\tau_1) \\
& + 2ab^2c^2k_1 \cos(\sigma\tau_1) - 2b^2c^2k_1^2 \cos(\sigma\tau_1) - 2b^2c^2k_1k_2 \cos(\sigma\tau_1) + b^2c^2k_1^2 \\
& + 2b^2k_1 \sin(\sigma\tau_1)(1 - c^2)\sigma + (1 - 2ab - 2b^2 + a^2b^2 + b^2c^2 + 2bk_1 - 2ab^2k_1 \\
& + b^2k_1^2 + 2bk_2 - 2ab^2k_2 + 2b^2k_1k_2 - 2bk_1 \cos(\sigma\tau_1) + 2ab^2k_1 \cos(\sigma\tau_1) \\
& - 2b^2k_1^2 \cos(\sigma\tau_1) - 2b^2k_1k_2 \cos(\sigma\tau_1) + b^2k_1^2)\sigma^2 - 2b^2k_1 \sin(\sigma\tau_1)\sigma^3 + b^2\sigma^4 = 0.
\end{aligned} \tag{24}$$

Obviously, equation (24) has at most a positive real root of N ($N \leq 4$), which is denoted as σ_h ($h = 1, 2, \dots, N$). Similarly, the following can be obtained:

$$\tau_2(h, j) = \frac{1}{\sigma_h} [\arccos(\bar{p}) + 2j\pi], \quad \bar{Q} \geq 0, \tag{25a}$$

$$\tau_2(h, j) = \frac{1}{\sigma_h} [2\pi - \arccos(\bar{p}) + 2j\pi], \quad \bar{Q} < 0, \tag{25b}$$

where

$$\bar{p} = -\frac{1}{bk_2(c^2 + \sigma^2)} \begin{pmatrix} bc - c^2 + abc^2 - bc^2k_1 - bc^2k_2 \\ +bc^2k_1 \cos(\sigma\tau_1) - \sigma^2 + ab\sigma^2 \\ -bk_1\sigma^2 - bk_2\sigma^2 + bk_1 \cos(\sigma\tau_1)\sigma \end{pmatrix}, \tag{26a}$$

$$\begin{aligned}
\bar{Q} &= -\frac{c^2k_1 \sin(\sigma\tau_1) + \sigma - c^2\sigma + k_1 \sin(\sigma\tau_1)\sigma^2 - \sigma^3}{k_2(c^2 + \sigma^2)}, \\
&h = 1, 2, \dots, N; j = 0, 1, \dots
\end{aligned} \tag{26b}$$

Then, there is a point column $\tau_2 = \tau_2(h, j)$ satisfying equation (26a) and (26b),

$$\tau_2^0 = \min\{\tau_2(h, j)\}, \quad h = 1, 2, \dots, N; j = 0, 1, \dots \tag{27}$$

Correspondingly, $h = h_0 \in \{1, 2, \dots, N\}$ and $\sigma = \sigma_{h_0}$.

When $\tau_1 \in (0, \tau_1^0)$ and $\tau_2 \in \tau_2^0$, equation (8) has a pair of pure virtual root.

Hypothesis 2

$$H_4: \left[\frac{d(\text{Re}\lambda)}{d\tau_2} \right]^{-1} \neq 0. \tag{28}$$

In this way, by using the general Hopf bifurcation theorem [35–40] for functional differential equations, the results on the stability and bifurcation of system (3a)–(3c) are obtained.

Theorem 2. Assuming (H_4) holds and $\tau_1 \in (0, \tau_1(2, 0))$ holds, the equilibrium $(0, (1/b), 0)$ of system (3a)–(3c) is asymptotically stable at $\tau_2 \in (0, \tau_2^0)$. When $\tau_2 \in \tau_2^0$ occurs, Hopf bifurcation occurs in system (3a)–(3c).

3. Period Solution and Stability of Hopf Bifurcation

In this section, we study the relevant properties of Hopf bifurcation in financial system (3a)–(3c) under the condition of delays $\tau_1 > 0$, $\tau_2 > 0$.

Using the ideas of Hassard et al. [36], the exact expression of Hopf bifurcation property of system (3a)–(3c) is considered by using central manifold theorem. Here, we consider the Hopf bifurcation of system (4a)–(4c) at the

equilibrium point $(0, 0, 0)$ for $\tau_2 = \tau_2^0$. The financial system can be converted to the following equation:

$$\dot{u}(t) = L_u(u_t) + f(\mu, u_t), \quad (29)$$

where

$$u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3, \quad (30)$$

$$L_\mu: C \longrightarrow R^3, f: R \times C \longrightarrow R^3, u_t(\theta) = u(t + \theta) \in C, \quad (31a)$$

$$L_\mu(\phi) = A(0) + B_1(-\tau_1) + B_2(-(\tau_2^0 + \mu)). \quad (31b)$$

According to systems (3a)–(3c) and (7a)–(7c), it can be seen that

$$A = \begin{pmatrix} \frac{1}{(b-a)} + k_1 = k_2 & 0 & 1 \\ -1 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}, \quad (32a)$$

$$B_1 = \begin{pmatrix} -k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (32b)$$

$$B_2 = \begin{pmatrix} -k_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (32c)$$

Let

$$\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))^T. \quad (33)$$

Then,

$$f(\mu, u_t) = (\mu + \tau_k) \begin{pmatrix} \varphi_1(0)\varphi_2(0) \\ -\varphi_1^2(0) \\ 0 \end{pmatrix}. \quad (34)$$

When the equilibrium point $(0, 0, 0)$ of system (29) passes Hopf bifurcation at $\mu = 0$, the characteristic equation has a pair of pure virtual root $i\sigma_h$ and $-i\sigma_h$. According to Ritz representation theorem, there is a matrix function of bounded variation

$$L_\mu(\phi) = \int_{-\tau_1}^0 d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C. \quad (35)$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} A + B_1 + B_2, & \theta = 0, \\ B_1 + B_2, & \theta \in (-\tau_2, 0), \\ B_1, & \theta \in (-\tau_1, -\tau_2), \\ 0, & \theta \in -\tau_1. \end{cases} \quad (36)$$

For $\phi \in C([-\tau_1, 0], R^3)$, we define

$$A(\mu)\phi = \frac{d\phi(\theta)}{d\theta}, \quad \theta \in [-\tau_1, 0], \quad (37a)$$

$$A(\mu)\phi = \int_{-\tau_1}^0 d\eta(\xi, \mu)\phi\xi, \quad \theta = 0, \quad (37b)$$

$$R(\mu)\phi = 0, \quad \theta \in [-\tau_1, 0], \quad (38a)$$

$$R(\mu)\phi = f(\mu, \phi), \quad \theta = 0. \quad (38b)$$

To simplify, equation (29) can be written in the following form:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \quad (39)$$

in which

$$u_t = u(t + \theta), \quad \theta \in [-\tau_1, 0]. \quad (40)$$

The adjoint operator A^* that defines A for $\psi \in C^1([0, \tau_1], (R^3)^*)$ is as follows:

$$A^*\psi(s) = -\frac{d\psi(s)}{ds}, \quad s \in (0, \tau_1), \quad (41a)$$

$$A^*\psi(s) = \int_{-\tau_1}^0 d\eta^T(t, 0)\psi(-t), \quad s = 0. \quad (41b)$$

In addition, we define a bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-\tau_1}^0 \int_{\varsigma=0}^\theta \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \quad (42a)$$

$$\eta(\theta) = \eta(\theta, 0). \quad (42b)$$

According to the above analysis, σ_h and $-i\sigma_{h_0}$ are the eigenvalues of $A(0)$ and $A^*(0)$. Let $q(\theta)$ be the eigenvector corresponding to the eigenvalue $i\sigma_{h_0}$ of $A(0)$ and $q^*(\theta)$ be the eigenvector corresponding to the eigenvalue $-i\sigma_{h_0}$ of $A^*(0)$. There are

$$A(0)q(\theta) = i\sigma_{h_0}q(\theta), \quad (43a)$$

$$A^*(0)q^*(\theta) = -i\sigma_{h_0}q^*(\theta). \quad (43b)$$

Through simple calculation, we can get

$$q(0) = (1, \alpha, \beta)^T = \left(1, \frac{1}{-b - \sigma_{h_0}i}, \frac{1}{-c - \sigma_{h_0}i}\right)^T, \quad (44a)$$

$$\begin{aligned} q^*(s) &= D(1, \alpha, \beta)e^{is\sigma_{h_0}} = D\left(1, 0, \frac{1}{-c - \sigma_{h_0}i}\right)e^{is\sigma_{h_0}} \\ &= \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*)(1, \alpha, \beta)^T - \int_{-\tau_1}^0 \int_{\xi=0}^\theta \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*) \\ &\quad \cdot e^{-i(\xi-\theta)\sigma_{h_0}}d\eta(\theta)(1, \alpha, \beta)^T e^{is\sigma_{h_0}}d\xi \\ &= \bar{D}\{1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* - k_1\tau_1 e^{-i\sigma_{h_0}} - k_2\tau_2^0 e^{-i\sigma_{h_0}}\}, \end{aligned} \quad (44b)$$

where D is a constant, making $\langle q^*(s), q(\theta) \rangle = 1$ valid.

Therefore, we have

$$D = \frac{1}{1 + \alpha \bar{\alpha}^* + \beta \bar{\beta}^* - k_1 \tau_1 e^{i\sigma_{h_0}} - k_2 \tau_2^0 e^{i\sigma_{h_0}}}. \quad (45)$$

Using the same notation of Ruan et al. [37], we can calculate the center popularity of $\mu = 0$. Let μ_1 be the solution of equation (29) when $\mu = 0$, defining

$$z(t) = \langle q^*, u_t \rangle, \quad (46a)$$

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + W_{30}(\theta) \frac{z^3}{6} + \dots, \quad (46b)$$

where z and $\bar{z}$ are the local coordinates of the central epidemic. When $\mu = 0$,

$$\begin{aligned} \bar{z}(t) &= i\sigma_k \tau_k z + \langle q^*(\theta), f(0, W(z(t), \bar{z}(t), \theta) + 2R_e\{z(t)q(\theta)\}) \rangle \\ &= i\sigma_{h_0} z + \bar{q}^*(0) (f(0, W(z(t), \bar{z}(t), 0)) + 2R_e\{z(t)q(0)\}). \end{aligned} \quad (47)$$

Make

$$f(0, W(z(t), \bar{z}(t), 0) + 2R_e\{z(t)q(0)\}) = f_0(z, \bar{z}). \quad (48)$$

Then,

$$\dot{z}(t) = i\sigma_{h_0} z + \bar{q}^*(0) f_0(z, \bar{z}) = i\sigma_{h_0} z + g(z, \bar{z}), \quad (49a)$$

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{21}(\theta) \frac{z^2 \bar{z}}{2} + \dots. \quad (49b)$$

Because

$$q(\theta) = (1, \alpha, \beta)^T e^{i\theta \sigma_{h_0}}, \quad (50a)$$

$$\begin{aligned} u_t(\theta) &= (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta)) \\ &= W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta), \end{aligned} \quad (50b)$$

we can obtain

$$u_{1t}(0) = z + \bar{z} + W_{20}^{(1)} \frac{z^2}{2} + W_{11}^{(1)} z \bar{z} + W_{02}^{(1)} \frac{\bar{z}^2}{2} + \dots, \quad (51a)$$

$$u_{2t}(0) = \alpha z + \bar{\alpha} \bar{z} + W_{20}^{(2)} \frac{z^2}{2} + W_{11}^{(2)} z \bar{z} + W_{02}^{(2)} \frac{\bar{z}^2}{2} + \dots, \quad (51b)$$

$$u_{3t}(0) = \beta z + \bar{\beta} \bar{z} + W_{20}^{(3)} \frac{z^2}{2} + W_{11}^{(3)} z \bar{z} + W_{02}^{(3)} \frac{\bar{z}^2}{2} + \dots. \quad (51c)$$

Based on the above formula (49b), it can be seen that

$$g(z, \bar{z}) = q^*(0) f_0(z, \bar{z})$$

$$\begin{aligned} &= \bar{D} \begin{pmatrix} u_{1t}(0) \\ -u_{1t}^2(0) - bu_{1t}(0)u_{2t}(0) \\ 0 \end{pmatrix} \\ &= D \left(z + \bar{z} + W_{20}^{(1)} \frac{z^2}{2} + W_{11}^{(1)} z \bar{z} + W_{02}^{(1)} \frac{\bar{z}^2}{2} + \dots \right) \\ &= \left(\alpha z + \bar{\alpha} \bar{z} + W_{20}^{(2)} \frac{z^2}{2} + W_{11}^{(2)} z \bar{z} + W_{02}^{(2)} \frac{\bar{z}^2}{2} + \dots \right). \end{aligned} \quad (52)$$

By using the method of comparison coefficient, we obtain

$$g_{20} = 2\alpha \bar{D}, \quad (53a)$$

$$g_{21} = 2\bar{D} \left[\frac{1}{2} W_{20}^{(1)}(0) \bar{\alpha} + \frac{1}{2} W_{20}^{(1)}(0) + W_{11}^{(2)}(0) + W_{11}^{(1)}(0) \alpha \right], \quad (53b)$$

$$g_{02} = 2\bar{\alpha} \bar{D}, \quad (53c)$$

$$g_{11} = (\alpha + \bar{\alpha}) \bar{D}. \quad (53d)$$

In order to calculate $W_{20}(\theta)$ and $W_{11}(\theta)$, we use

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} A(0)W - 2R_e\{\bar{q}^*(0)f_0q(\theta)\}, & \theta \in [-\tau_1, 0], \\ A(0)W - 2R_e\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0, \end{cases} \end{aligned} \quad (54)$$

Make

$$H(z, \bar{z}, \theta) = \begin{cases} 2R_e\{\bar{q}^*(0)f_0q(\theta)\}, & \theta \in [-\tau_1, 0], \\ 2R_e\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0, \end{cases} \quad (55)$$

We rewrite (54):

$$\dot{W} = A(0)W + H(z, \bar{z}, \theta), \quad (56)$$

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{\bar{z}^2}{2} + H_{11}(\theta) z \bar{z} + H_{02}(\theta) \frac{z^2}{2} + \dots. \quad (57)$$

Using (54) and (55), one can obtain

$$(A(0) - 2i\sigma_{h_0})W_{20}(\theta) = -H_{20}, \quad (58a)$$

$$A(0)W_{11}(\theta) = -H_{11}(\theta), \quad (58b)$$

$$\begin{aligned} \theta \in [-\tau_1, 0], \quad H(z, \bar{z}, \theta) &= -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\ &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \end{aligned} \quad (59a)$$

The combination formula (57) is obtained:

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad (60a)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (60b)$$

Using equations (58a) and (58b) and (60a) and (60b), it is easy to obtain

$$W_{20} = 2i\sigma_{h_0}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \quad (61a)$$

$$W_{20} = \frac{ig_{20}}{\sigma_{h_0}}q(0)e^{i\theta\sigma_{h_0}} + \frac{i\bar{g}_{02}}{3\sigma_{h_0}}\bar{q}(0)e^{-i\theta\sigma_{h_0}} + Ee^{2i\theta\sigma_{h_0}}, \quad (61b)$$

$$W_{11} = \frac{ig_{11}}{\sigma_{h_0}}q(0)e^{i\theta\sigma_{h_0}} + \frac{i\bar{g}_{11}}{\sigma_{h_0}}\bar{q}(0)e^{-i\theta\sigma_{h_0}} + E_2. \quad (62)$$

Make

$$E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3, \quad (63a)$$

$$E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3. \quad (63b)$$

Combine formulas (58a) and (58b) again to obtain

$$\dot{W}_{20}(\theta) = \int_{-\tau_1}^0 d\eta(\theta)W_{20}(\theta) = 2i\theta\sigma_{h_0}W_{20}(\theta) - H_{20}(\theta), \quad (64a)$$

$$\dot{W}_{11}(\theta) = \int_{-\tau_1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(\theta), \quad (64b)$$

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2(\alpha, -1 - b\alpha, 0), \quad (65a)$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + (\alpha + \bar{\alpha}, -2 - b(\bar{\alpha} + \alpha, 0))^T, \quad (65b)$$

$$\left(i\sigma_{h_0} - \int_{-\tau_1}^0 e^{i\theta\sigma_{h_0}}d\eta(\theta)\right)q(0) = 0, \quad (65c)$$

$$\left(-i\sigma_{h_0} - \int_{-\tau_1}^0 e^{-i\theta\sigma_{h_0}}d\eta(\theta)\right)\bar{q}(0) = 0. \quad (65d)$$

Substituting equations (62) and (65a)–(65d) into (64a) and (64b), there is

$$\left(2i\sigma_{h_0}I - \int_{-\tau_1}^0 e^{i\theta\sigma_{h_0}}d\eta(\theta)\right)E_1 = 2(\alpha, -1 - b\alpha, 0)^T. \quad (66)$$

Therefore,

$$\begin{pmatrix} 2i\sigma_{h_0} + \Pi & 0 & -1 \\ 1 & 2i\sigma_{h_0} + b & 0 \\ 1 & 0 & 2i\sigma_{h_0} + c \end{pmatrix} E_1 = 2 \begin{pmatrix} \alpha \\ -1 - b\alpha \\ 0 \end{pmatrix}, \quad (67)$$

where

$$\Pi = a - \frac{1}{b} - k_1 - k_2 + k_2 e^{-2i\sigma_{h_0}\tau_2^0}, \quad (68a)$$

$$f(\tau_1, \tau_2^0) = k_1 e^{-2i\sigma_{h_0}\tau_1} + k_2 e^{-2i\sigma_{h_0}\tau_2^0}. \quad (68b)$$

Let

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \quad (69)$$

in which

$$\Delta_{11} = \begin{vmatrix} 2\alpha & 0 & -1 \\ -2 - 2b\alpha & 2i\sigma_{h_0} & 0 \\ 0 & 0 & 2i\sigma_{h_0} + c \end{vmatrix}, \quad (70a)$$

$$\Delta_{12} = \begin{vmatrix} 2i\sigma_{h_0} + \Pi & 2\alpha & -1 \\ 1 & -2 - 2b\alpha & 0 \\ 1 & 0 & 2i\sigma_{h_0} + c \end{vmatrix}, \quad (70b)$$

$$\Delta_{13} = \begin{vmatrix} 2i\sigma_{h_0} + \Pi & 0 & 2\alpha \\ 1 & 2i\sigma_{h_0} + b & -b - 2b\alpha \\ 1 & 0 & 0 \end{vmatrix}, \quad (70c)$$

$$\Delta_1 = \begin{vmatrix} \frac{2i\sigma_{h_0} + a - 1}{b - k_1 - k_2 + f(\tau_1, \tau_2^0)} & 0 & -1 \\ 1 & 2i\sigma_{h_0} + b & 0 \\ 1 & 0 & 2i\sigma_{h_0} + c \end{vmatrix}. \quad (70d)$$

Similarly, we can also have

$$\begin{pmatrix} \frac{a-1}{b-k_1-k_2} & 0 & -1 \\ 1 & b & 0 \\ 1 & b & c \end{pmatrix} E_2 = \begin{pmatrix} \alpha + \bar{\alpha} \\ -2 - b(\bar{\alpha} + \alpha) \\ 0 \end{pmatrix}. \quad (71)$$

Let

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, \quad (72)$$

where

$$\Delta_{21} = \begin{vmatrix} \alpha + \bar{\alpha} & 0 & -1 \\ -2 - b(\bar{\alpha} + \alpha) & b & 0 \\ 1 & 0 & c \end{vmatrix}, \quad (73a)$$

$$\Delta_{22} = \begin{vmatrix} \frac{a-1}{b-k_1-k_2} & \alpha + \bar{\alpha} & 0 \\ 1 & -2 - b(\bar{\alpha} + \alpha) & 0 \\ 1 & 0 & c \end{vmatrix}, \quad (73b)$$

$$\Delta_{23} = \begin{vmatrix} \frac{a-1}{b-k_1-k_2} & 0 & \alpha + \bar{\alpha} \\ 1 & b & -2 - b(\bar{\alpha} + \alpha) \\ 1 & 0 & 0 \end{vmatrix}, \quad (73c)$$

FIGURE 2: The phase diagram (a) and the time series diagram (b) of system (3a)–(3c) with initial value $(0.1, 2, 0.1)$ for $\tau_1 = 1.5 > \tau_1^0$ and $\tau_2 = 0$.

$$\Delta_2 = \begin{vmatrix} a - 1/b - k_1 - k_2 & 0 & -1 \\ 1 & b & 0 \\ 1 & 0 & c \end{vmatrix}. \quad (73d)$$

Furthermore, g_{ij} can be determined by the coefficients and time delays of system (7a)–(7c). Thus, the following values can be calculated using the method in [37]:

$$C_1(0) = \frac{i}{2\sigma_{h_0}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \quad (74a)$$

$$\mu_2 = \frac{\text{Re}\{C_1(0)\}}{e\{d\lambda/d\tau_2^0\}}, \quad (74b)$$

$$T_2 = -\frac{\text{Im}\{C_1(0) + \mu_2 \text{Im}\{d\lambda/d\tau_2^0\}\}}{\sigma_{h_0}}, \quad (74c)$$

$$\beta_2 = 2R_e\{C_1(0)\}. \quad (74d)$$

Formulas (74a)–(74d) determine the critical point above the central flow τ_2^0 . Now, the properties of periodic solutions of system (3a)–(3c) could be obtained. Therefore, we obtain the following theorem.

Theorem 3. For system (3a)–(3c), when $\tau_1 \in (0, \tau_1^0)$,

- (1) The symbol μ_2 determines the orientation of Hopf bifurcation. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical). If $\tau_2 > \tau_2^0$ ($\tau_2 < \tau_2^0$), the bifurcation period solution exists.
- (2) The sign of β_2 determines the stability of the bifurcation period solution: if $\beta_2 < 0$ ($\beta_2 > 0$), the period solution is stable (unstable).

- (3) The symbol T_2 determines the period of bifurcation period solution. If $T_2 > 0$ ($T_2 < 0$), the period solution is increased (decreased).

4. Numerical Results and Analysis

In this section, we will give numerical simulations on the theoretical results of Hopf bifurcation with two delays.

Given the parameters $a = 3, b = 0.51, c = 1.0, k_1 = 1$, and $k_2 = 5$, it is verified that at that time $\tau_1 = \tau_2 = 0$, the parameters satisfy the assumption H_1 that the equilibrium point $(0, (1/b), 0)$ of system (3a)–(3c) without time delay is asymptotically stable.

- (1) $\tau_1 > 0, \tau_2 = 0$

If equation (8) has a pair of pure virtual root $i\omega$ and $-i\omega$, there are two positive roots $w_1 = 0.20211$ and $w_2 = 1.39915$. Substituting w_1 and w_2 into equations (15a) and (15b), we can obtain

$$\tau_1(1, j) = 15.5044 + 31.0872j, \quad (75a)$$

$$\tau_1(2, j) = 1.39921 + 4.49073j, \quad (j = 0, 1, 2, \dots). \quad (75b)$$

That is, $\tau_1^0 = 1.39921$, and we have the following result for $\tau_1 = \tau_1^0$:

$$H_3 : \frac{d(R_e\lambda)}{d\tau_1} = 0.23975 > 0. \quad (76)$$

According to Theorem 1, equilibrium point $(0, (1/b), 0)$ of system (3a)–(3c) is asymptotically

FIGURE 3: Phase diagram (a) and time sequence diagram (b) of system (3a)–(3c) with initial value $(0.1, 2, 0.1)$ for $\tau_1 = 0.45 < \tau_1^0$ and $\tau_2 = 0.1 < \tau_2^0$.

FIGURE 4: Phase space (a) and time series (b) of system (3a)–(3c) with initial value $(0.1, 2, 0.1)$ for $\tau_1 = 0.45 < \tau_1^0$ and $\tau_2 = 0.1 < \tau_2^0$.

stable for $\tau_1 \in (0, \tau_1^0)$. Hopf bifurcation occurs when $\tau_1 = \tau_1^0$, $\tau_2 = 0$, and a stable period solution is obtained. The time series diagram and phase diagram are shown in Figure 2.

(2) $\tau_1 > 0$, $\tau_2 > 0$

Let $\tau_1 = 0.45 < \tau_1^0$ and consider $\tau_2 > 0$. Suppose that equation (22) has a pair of pure virtual root $i\sigma$ and $-i\sigma$. Then, according to equation (24), it is found that $N = 1$. There is $\sigma_1 = 3.05932$. Substituting σ_1 into equations (25a) and (25b), we can obtain

$$\tau_2(2, j) = 0.11917 + 2.05378j, \quad (j = 0, 1, 2, \dots). \quad (77)$$

That is, $\tau_2^0 = 0.11917$ and

$$\frac{d(R_e \lambda)}{d\tau_2} = 0.227516 > 0. \quad (78)$$

According to Theorem 2, we know when $\tau_2 \in (0, \tau_2^0)$, the equilibrium point $(0, (1/b), 0)$ is asymptotically stable (as shown in Figure 3: $\tau_1 = 0.45 < \tau_1^0$ and $\tau_2 = 0.1 < \tau_2^0$).

When $\tau_1 = 0.45$ and $\tau_2 = \tau_2^0$, Hopf bifurcation occurs and we can have

FIGURE 5: Bifurcation diagram of system (3a)–(3c) with initial value $(0.1, 2, 0.1)$ and $\tau_1 = 0.45 < \tau_1^0$ and $\tau_2 \in [1, 4]$.

FIGURE 6: Time series of system (3a)–(3c) with $\tau_1 = 0.45 < \tau_1^0$ and $\tau_2 = 3.8$: (a) initial value $(0.01, 1, 0.01)$ and periodic orbit; (b) initial value $(0.1, 0, 0)$ and means chaos.

$$C_1(0) = -2.29734 + 4.10025i, \quad (79a)$$

$$\mu_2 = 0.92535 > 0, \quad (79b)$$

$$\beta_2 = -4.59468 < 0. \quad (79c)$$

Therefore, the bifurcation direction is $\tau_2 > \tau_2^0$, and the period solution is stable, as shown in Figure 4: $\tau_1 = 0.45$ and $\tau_2 = 0.15$. In addition, the complex dynamics of system (3a)–(3c) could be shown from the bifurcation diagram in Figure 5.

5. Multistability in the Improved Financial System (3a)–(3c) with Two Delays

When $\tau_1 = 0.45$ and $\tau_2 = 3.8$, we will find the multistability in the systems without changing parameters. Given the parameters $a = 3$, $b = 0.51$, $c = 1.0$, $k_1 = 1$, and $k_2 = 5$, we have obtained the periodic attractor with initial value $(0.01, 1, 0.01)$ in Figure 6(a). However, we also obtain chaos for same parameters' values but different initial values $(0.1, 0, 0)$ in Figure 6(b). Therefore, when different initial conditions are taken, the coexisting and different attractors

are exhibited. We know multistability is a critical property of nonlinear dynamical systems [41–44]. Since the crisis of the financial system is subject to various factors, the nature of the multi-steady state plays an important role in making correct decisions for government workers.

6. Conclusion

Time delay is a very sensitive factor in financial systems with multistability. Financial systems with multiple time delays have richer dynamic characteristics than those with single time delay. Two-delay feedback can effectively control the unstable behavior of financial markets. In this paper, Hopf bifurcation of an improved financial model with two time delays is studied in detail. The existence of the bifurcation period solution of this system is discussed by using the theory of functional differential equations. Complexity of the proposed financial chaotic system is studied from the bifurcation diagram that its multistability depends extremely on the memristor initial condition and the system parameters. In summary, time delay is one of the effective methods to control the stability of the financial market, so it can provide a theoretical reference for relevant departments to regulate economic behavior.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this research work.

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