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Research Article

Existence of Positive Solutions for a Class of $(p(x), q(x))$-Laplacian Elliptic Systems with Multiplication of Two Separate Functions

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The paper deals with the study of the existence of weak positive solutions for a new class of the system of elliptic differential equations with respect to the symmetry conditions and the right hand side which has been defined as multiplication of two separate functions by using the sub-supersolutions method (1991 Mathematics Subject Classification: 35J60, 35B30, and 35B40).

1. Introduction

Elliptic systems of differential equations are of crucial importance in modelization and description of a wide variety of phenomena such as fluid dynamics, quantum physics, sound, heat, electrostatics, diffusion, gravitation, chemistry, biology, simulation of airplane, calculator charts, and time prediction. PDEs are equations involving functions of several variables and their derivatives and model multidimensional systems generalizing ODEs (ordinary differential equations), which deal with functions of a single variable and their derivatives (see, for example, [1–15]).

In contrary to ODEs, there is no general result such as the Picard–Lindelöf theorem for PDEs to settle the existence and uniqueness of solutions. Malgrange–Ehrenpreis theorem states that linear partial differential equations with constant coefficients always have at least one solution; another powerful and general result in case of polynomial coefficients is the Cauchy–Kovalevskaya theorem ensuring the existence and uniqueness of a locally analytic solution for PDEs with coefficients that are analytic in the unknown function and its derivatives; otherwise, the existence of solutions is not guaranteed at all for nonanalytic coefficients even if they have derivatives of all orders (see [16]). Given the rich variety of PDEs, there is no general theory of solvability. Instead, research focuses on particular PDEs that are important for applications. It would be desirable when solving a PDE to prove the existence and uniqueness of a regular solution that depends on the initial data given in the problem, but perhaps we are asking too much. A solution with enough smoothness is called a classical solution, but in most cases as for conservation laws, we cannot achieve that much and allow generalized or weak solutions. The point is this: looking for weak solutions allows us to investigate a larger class of candidates, so it is more reasonable to consider as separate the existence and the regularity problems. For various PDEs, this is the best that can be done, and naturally nonlinear equations are more difficult than linear ones. Overall, we know too much about linear PDEs and in best cases, we can express their solutions but too little about nonlinear equations. For linear PDEs, various methods and techniques can be used for separation of variables, method of characteristics, integral transform, change of variables, superposition principle, or even finding a fundamental solution and taking a convolution product to obtain the solution. Variational theory is the most accessible and useful of the methods for nonlinear PDEs, but there are other non-variational techniques of use for nonlinear elliptic and parabolic PDEs such as monotonicity and fixed point methods, semigroup theory, and sub-supersolutions method that played an important role in the study of nonlinear boundary value problems for a long time. Scorza-Dragoni's work in [17] was one of the earliest papers using a pair of ordered solutions of differential inequalities to establish the existence of solution to a given boundary value problem for a nonlinear second-order ordinary differential equation; his work was followed later by Nagumo in [18, 19] which inspired much work on both ordinary and PDEs during the decade of the sixties. Knobloch in [20] introduced the sub-supersolution method to the study of periodic boundary value problems for nonlinear second-order ordinary differential equations using Cesari's method; similar problems and techniques were studied in [21, 22] and still the sub-supersolutions and supersolutions are assumed to be smooth solutions of differential inequalities. Then, the SSM were also used to study Dirichlet and Neumann boundary value problems for semilinear elliptic problems in [23, 24], and even for nonlinear boundary value problems in [25–27] and also for systems of nonlinear ordinary differential equations in [28–30]. The concept of weak sub-supersolutions and supersolutions was first formulated by Hess and Deuel in [31, 32] to obtain existence results for weak solutions of semilinear elliptic Dirichlet problems and was subsequently continued by several authors (see, e.g., [33–43]).

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc (see [44]). Many existence results have been obtained on this kind of problems (see, for example, [44–57]) and in [45] a new class of anisotropic quasilinear elliptic equations with a power-like variable reaction term has been investigated.

In the last few years in [51, 58–60], the regularity and existence of solutions for differential equations with nonstandard $p(x)$-growth conditions have been studied and p-Laplacian elliptic systems with $p(x) = q(x) = p$ (a constant) have been archived. In this work, we study the existence of weak positive solutions for a new class of the system of differential equations with respect to the symmetry conditions by using sub-supersolution method.

2. Preliminaries, Assumptions, and Statement of the Problem

2.1. Plate Problems and Its History. In this paper, we consider the system of differential equations:

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}\left[a(x)f(u)h(v)\right] \text{ in } \Omega, \\ -\Delta_{q(x)}v = \lambda^{q(x)}\left[b(x)g(u)\tau(v)\right] \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega$ and $1 < p(x), q(x) \in C^1(\overline{\Omega})$ are functions with $1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < \infty$, $1 < q^- := \inf_\Omega q(x) \le q^+ := \sup_\Omega q(x)$, and $\Delta_{p(x)}$ is a $p(x)$-Laplacian defined as

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right), \tag{2}$$

and $a, b\colon \overline{\Omega} \longrightarrow \mathbb{R}^+$ are continuous functions, while f, g, h, and τ are monotone functions in $\mathbb{R}^+$ such that

$$\begin{aligned} \lim_{u\longrightarrow+\infty} f(u) &= +\infty, \\ \lim_{u\longrightarrow+\infty} g(u) &= +\infty, \\ \lim_{u\longrightarrow+\infty} h(u) &= +\infty, \\ \lim_{u\longrightarrow+\infty} \tau(u) &= +\infty, \end{aligned} \tag{3}$$

satisfying some natural growth condition at $u = \infty$.

We point out that the extension from p-Laplace operator to $p(x)$-Laplace operator is not trivial, since the $p(x)$-Laplacian has a more complicated structure then the p-Laplace operator, such as it is nonhomogeneous. Moreover, many results and methods for p-Laplacians are not valid for the $p(x)$-Laplacian; for example, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(x))|\nabla u|^{p(x)}\mathrm{d}x}{\int_\Omega (1/p(x))|u|^{p(x)}\mathrm{d}x}, \tag{4}$$

is zero in general, and only under some special conditions, $\lambda_{p(x)}$ is positive (see [53]). Maybe the first eigenvalue and the first eigenfunction of the $p(x)$-Laplacian do not exist, but the fact that the first eigenvalue λ_p is positive and the existence of the first eigenfunction are very important in the study of p-Laplacian problem. There are more difficulties in discussing the existence of solutions of variable exponent problems. In [59], the authors considered the existence of positive weak solutions for the following p-Laplacian problem:

$$\begin{cases} -\Delta_p u = \lambda f(v) \text{ in } \Omega, \\ -\Delta_p u = \lambda g(u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

where the first eigenfunction has been used to construct the subsolution of p-Laplacian problem. Under the condition that for all $M > 0$,

$$\lim_{u\longrightarrow+\infty} \frac{f\left(M(g(u))^{1/p-1}\right)}{u^{p-1}} = 0, \tag{6}$$

the authors gave the existence of positive solutions for problem (5) provided that λ is large enough.

In [48], the existence and nonexistence of positive weak solutions to the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \lambda u^{\alpha} v^{\gamma} \text{ in } \Omega, \\ -\Delta_q u = \lambda u^{\delta} v^{\beta} \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \quad (7)$$

has been considered where the first eigenfunction has been used to construct the subsolution of problem (7) and the following results were obtained:

(i) If $\alpha, \beta \geq 0, \gamma, \delta > 0, \theta = (p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$, then problem (7) has a positive weak solution for each $\lambda > 0$.

(ii) If $\theta = 0$ and $p\gamma = q(p-1-\alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (7) has no nontrivial nonnegative weak solution. For further generalizations of system (7), we refer to [49, 50].

As already discussed before, on the $p(x)$-Laplacian problems, maybe the first eigenvalue and the first eigenfunction of the $p(x)$-Laplacian do not exist even if the first eigenfunction of the $p(x)$-Laplacian exists. Because of the nonhomogeneous property of the $p(x)$-Laplacian, the first eigenfunction cannot be used to construct the subsolutions of $p(x)$-Laplacian problems. Moreover, in [47, 61], the authors studied the existence of solutions for problem (5), where some symmetry conditions are imposed. Then, in [46], the existence of positive solutions of the system was investigated:

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)} f(v) \text{ in } \Omega, \\ -\Delta_{p(x)} u = \lambda^{p(x)} g(u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \Omega, \end{cases} \quad (8)$$

without any symmetry conditions. Motivated by the ideas introduced in [47], the authors proved the existence of a positive solution when λ is large enough and satisfies condition (6) and they did not assume any symmetric condition and did not assume any sign condition on $f(0)$ and $g(0)$. Also the authors proved the existence of positive solutions with multiparameters; in this paper, we extend this given system of differential equations, where we establish the existence of a positive solution for a new class of this system with respect to the symmetry conditions by constructing a positive subsolution and supersolution and $p, q \in C^1(\overline{\Omega})$ are functions, $\lambda, \lambda_1, \lambda_2, \mu_1$, and μ_2 are positive parameters, and $\Omega \subset \mathbb{R}^N$ is a bounded domain and we did not assume any sign condition on $f(0)$, $g(0)$, $h(0)$, and $\tau(0)$.

2.2. Preliminary Results. In order to discuss problem (1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly, we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [54]).

Let us define

$$L^{p(x)}(\Omega) = \left\{ \begin{array}{c} u\colon u \text{ is a measurable real} - \text{valued function such that} \\ \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \end{array} \right\}. \quad (9)$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$\begin{aligned} |u(x)|_{p(x)} &= \inf\left\{ \lambda > 0\colon \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\}, \\ W^{1,p(x)}(\Omega) &= \left\{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \right\}, \end{aligned} \quad (10)$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega). \quad (11)$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 1 (see [59]). *The spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Throughout the paper, we will assume that

$(H1)$ $p, q \in C^1(\overline{\Omega})$ and $1 < p^- \leq p^+, 1 < q^- \leq q^+$

$(H2)$ $f, g, h, \tau\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ are C^1 monotone functions such that

$$\begin{aligned} &\lim_{u \longrightarrow +\infty} f(u) = +\infty, \\ &\lim_{u \longrightarrow +\infty} g(u) = +\infty; \\ &\lim_{u \longrightarrow +\infty} h(u) = +\infty, \\ &\lim_{u \longrightarrow +\infty} \tau(u) = +\infty. \end{aligned} \quad (12)$$

$(H3)$ $\exists r > 0$ such that

$$\lim_{u \longrightarrow +\infty} \frac{f(u) h(c u^{r/q^- - 1})}{u^{p^- - 1}} = 0, \quad (13)$$

for all $c > 0$,

$$\lim_{u \longrightarrow +\infty} \frac{g(u) \tau(k u^{r/q^- - 1})}{u^{r}} = 0, \quad (14)$$

for all $k > 0$.

$(H4)\, a, b$: $\overline{\Omega} \longrightarrow \mathbb{R}^+$ are continuous functions, such that

$$\begin{aligned} a_1 &= \min_{x\in\overline{\Omega}} a(x), \\ b_1 &= \min_{x\in\overline{\Omega}} b(x), \\ a_2 &= \max_{x\in\overline{\Omega}} a(x), \\ b_2 &= \max_{x\in\overline{\Omega}} b(x). \end{aligned} \tag{15}$$

We define

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega). \tag{16}$$

Then, L: $W_0^{1,p(x)}(\Omega) \longrightarrow (W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded, and strictly monotone operator, and it is a homeomorphism (see [61], Theorem 3.1).

Define A: $W_0^{1,p(x)}(\Omega) \longrightarrow (W_0^{1,p(x)}(\Omega))^*$ as for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$,

$$\langle A(u), \varphi\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + h(x,u)\varphi\right) dx, \tag{17}$$

where $h(x,u)$ is continuous on $\overline{\Omega} \times \mathbb{R}$ and $h(x)$ is increasing. It is easy to check that A is a continuous bounded mapping. Copying the proof of [44], we have the following lemma:

Lemma 1 (see [45]) (comparison principle). *Let $u, v \in W_0^{1,p(x)}(\Omega)$ satisfy*

$$\begin{aligned} Au - Av \geq 0 \text{ in} \left(W_0^{1,p(x)}(\Omega)\right)^*, \\ \varphi(x) = \min\{u(x) - v(x), 0\}. \end{aligned} \tag{18}$$

If

$$\varphi(x) \in W_0^{1,p(x)}(\Omega), \quad (\text{i.e}\, u \geq v \,\text{on}\, \partial\Omega), \tag{19}$$

then $u \geq v$ a.e. in Ω.

Definition 1. Let $(u,v) \in (W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega))$; the couple (u,v) is said to be a weak solution of (1) if it satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \lambda^{p(x)} [a(x) f(u) h(v)] \varphi \, dx, \\ \displaystyle\int_\Omega |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \varphi \, dx = \int_\Omega \lambda^{q(x)} [b(x) g(u) \tau(v)] \psi \, dx, \end{cases} \tag{20}$$

for all $(\varphi, \psi) \in (W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega))$ *with* $(\varphi,\psi) \geq 0$.

Here, and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to denote the distance of Ω. Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\varepsilon = \{x \in \Omega\colon d(x,\partial\Omega) < \varepsilon\}$.

Since $\partial\Omega$ is C^2 regularly, there exists a constant $\delta \in (0,1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$ and $|\nabla d(x)| = 1$.

Denote

$$\begin{aligned} v_1(x) &= \begin{cases} \gamma d(x), d(x) < \delta, \\ \displaystyle\gamma\delta + \int_\delta^{d(x)} \gamma \left(\frac{2\delta - t}{\delta}\right)^{2/p^- - 1} (a_2)^{2/p^- - 1} dt, \\ \delta \leq d(x) < 2\delta, \\ \displaystyle\gamma\delta + \int_\delta^{2\delta} \gamma \left(\frac{2\delta - t}{\delta}\right)^{2/p^- - 1} (a_2)^{2/p^- - 1} dt, \\ 2\delta \leq d(x). \end{cases} \\ v_2(x) &= \begin{cases} \gamma d(x), d(x) < \delta, \\ \displaystyle\gamma\delta + \int_\delta^{d(x)} \gamma \left(\frac{2\delta - t}{\delta}\right)^{2/p^- - 1} (b_2)^{2/p^- - 1} dt, \\ \delta \leq d(x) < 2\delta, \\ \displaystyle\gamma\delta + \int_\delta^{2\delta} \gamma \left(\frac{2\delta - t}{\delta}\right)^{2/p^- - 1} (b_2)^{2/p^- - 1} dt, \\ 2\delta \leq d(x). \end{cases} \end{aligned} \tag{21}$$

Obviously, $0 \leq v_1(x), v_2(x) \in C^1(\overline{\Omega})$. Considering

$$\begin{cases} -\Delta_{p(x)} \omega(x) = \eta \text{ in } \Omega, \\ \omega = \text{on } \partial\Omega, \end{cases} \tag{22}$$

we have the following result

Lemma 2 (Lemma 2.1 in [52]). *If positive parameter η is large enough and ω is the unique solution of (22), then we have*

(i) For any $\theta \in (0,1)$, there exists a positive constant C_1 such that

$$C_1 \eta^{1/p^+ - 1 + \theta} \leq \max_{x\in\overline{\Omega}} \omega(x), \tag{23}$$

(ii) and, there exists a positive constant C_2 such that

$$\max_{x\in\overline{\Omega}} \omega(x) \leq C_2 \eta^{1/p^- - 1}. \tag{24}$$

3. Main Result

In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

Theorem 1. *Assume that the conditions $(H1)-(H4)$ are satisfied. Then problem (1) has a positive solution when λ is large enough.*

Proof. We shall establish Theorem 1 by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (1) such that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$, that is, (ϕ_1, ϕ_2) and (z_1, z_2) satisfy

$$\begin{cases} \int_\Omega |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 \cdot \nabla \varphi \, dx \le \int_\Omega \lambda^{p(x)} [a(x) f(\phi_1) h(\phi_2)] \varphi \, dx, \\ \int_\Omega |\nabla \phi_2|^{q(x)-2} \nabla \phi_2 \cdot \nabla \psi \, dx \le \int_\Omega \lambda^{q(x)} [b(x) g(\phi_1) \tau(\phi_2)] \psi \, dx, \\ \int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi \, dx \ge \int_\Omega \lambda^{p(x)} [a(x) f(z_1) h(z_2)] \varphi \, dx, \\ \int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi \, dx \ge \int_\Omega \lambda^{q(x)} [b(x) g(z_1) \tau(z_2)] \psi \, dx, \end{cases} \quad (25)$$

for all $(\varphi, \psi) \in (W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega))$ with $(\varphi, \psi) \ge 0$. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [52]), problem (1) has a positive solution.

Step 1. We will construct a subsolution of (1). Let $\sigma \in (0, \delta)$ be small enough. Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{2/p^- - 1} (a_1)^{2/p^- - 1} dt, \\ \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{2/p^- - 1} (a_1)^{2/p^- - 1} dt, \\ 2\delta \le d(x), \end{cases}$$
$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{2/p^- - 1} (b_1)^{2/q^- - 1} dt, \quad \sigma \le d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma} \right)^{2/p^- - 1} (b_1)^{2/q^- - 1} dt, \quad 2\delta \le d(x). \end{cases} \quad (26)$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\overline{\Omega})$.

Denote

$$\alpha = \min\left\{ \frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, \frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, 1 \right\},$$
$$\zeta = \begin{cases} \min\left[\left(\frac{-\alpha}{a_1 f(0) h(0)} \right)^{1/p^+}; \left(\frac{-\alpha}{a_1 f(0) h(0)} \right)^{1/p^-}; \left(\frac{-\alpha}{b_1 g(0) \tau(0)} \right)^{1/q^+}; \left(\frac{-\alpha}{b_1 g(0) \tau(0)} \right)^{1/q^-} \right], & \text{if } f(0)h(0) < 0,\ g(0)\tau(0) < 0, \\ \min\left[\left(\frac{-\alpha}{a_1 f(0) h(0)} \right)^{1/p^+}; \left(\frac{-\alpha}{a_1 f(0) h(0)} \right)^{1/p^-} \right], & \text{if } f(0)h(0) < 0,\ g(0)\tau(0) > 0, \\ \min\left[\left(\frac{-\alpha}{b_1 g(0) \tau(0)} \right)^{1/q^+}; \left(\frac{-\alpha}{b_1 g(0) \tau(0)} \right)^{1/q^-} \right], & \text{if } f(0)h(0) > 0,\ g(0)\tau(0) < 0, \\ 1, & \text{if } f(0)h(0) > 0,\ g(0)\tau(0) > 0. \end{cases} \quad (27)$$

By some simple computations, we can obtain

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k\left(e^{kd(x)}\right)^{p(x)-1} \times \left[(p(x)-1) + \left(d(x) + \dfrac{\ln k}{k}\right)\nabla p \nabla d + \dfrac{\Delta d}{k}\right], \ d(x) < \sigma, \\ \dfrac{1}{2\delta - \sigma} \dfrac{2(p(x)-1)}{p^- - 1} - \left(\dfrac{2\delta - d}{2\delta - \sigma}\right) \times \left[\left(\ln ke^{k\sigma}\right) \times \left(\dfrac{2\delta - d}{2\delta - \sigma}\right)^{2/p^- - 1} \nabla p \nabla d + \Delta d\right] \\ \times \left(Ke^{k\sigma}\right)^{p(x)-1} \left(\frac{2\delta - d}{2\delta - \sigma}\right)^{2(p(x)-1)/p^- - 1 - 1} (a_1), \quad \sigma \le d(x) < 2\delta, \\ 0, \quad 2\delta \le d(x), \end{cases}$$

$$-\Delta_{q(x)}\phi_2 = \begin{cases} -k\left(e^{kd(x)}\right)^{q(x)-1} \times \left[\begin{array}{c}(q(x)-1) \\ +\left(d(x) + \dfrac{\ln k}{k}\right)\nabla q \nabla d + \dfrac{\Delta d}{k}\end{array}\right], \quad d(x) < \sigma, \\ \left\{\dfrac{1}{2\delta - \sigma} \dfrac{2(q(x)-1)}{q^- - 1} - \left(\dfrac{2\delta - d}{2\delta - \sigma}\right) \times \left[\left(\ln ke^{k\sigma}\right) \times \left(\dfrac{2\delta - d}{2\delta - \sigma}\right)^{2/q^- - 1} \nabla q \nabla d + \Delta d\right]\right\} \\ \times \left(Ke^{k\sigma}\right)^{q(x)-1} \left(\frac{2\delta - d}{2\delta - \sigma}\right)^{(2(q(x)-1)/q^- - 1) - 1} (b_1), \quad \sigma \le d(x) < 2\delta, \\ 0, \quad 2\delta \le d(x). \end{cases} \tag{28}$$

From $(H2)$, there exists a positive constant $M > 1$ such that

$$\begin{aligned} f(M-1)h(M-1) &\ge 1, \\ g(M-1)\tau(M-1) &\ge 1. \end{aligned} \tag{29}$$

Let $\sigma = (1/k)\ln M$, then

$$\sigma k = \ln M. \tag{30}$$

If k is sufficiently large, from (30), we have

$$-\Delta_{p(x)}\phi_1 \le -k^{p(x)}\alpha, \quad d(x) < \sigma. \tag{31}$$

Let $\lambda = \zeta k$. We claim that

$$-k^{p(x)}\alpha \le a_1 f(0)h(0)\lambda^{p(x)}, \quad \forall x \in \Omega, \tag{32}$$

Indeed, by definition of λ, the last inequality is obvious when $f(0)h(0) > 0$.

When $f(0)h(0) < 0$, we can notice that

$$\frac{\lambda}{k} \le \left(\frac{-\alpha}{a_1 f(0)h(0)}\right)^{1/p(x)}, \quad \forall x \in \Omega, \tag{33}$$

Then, we have

$$-\Delta_{p(x)}\phi_1 \le -k^{p(x)}\alpha \le \lambda^{p(x)}\left(a_1 f(\phi_1)h(\phi_2)\right), \quad d(x) < \sigma. \tag{34}$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, there exists a positive constant C_3 such that

$$\begin{cases} -\Delta_{p(x)}\phi_1 \le \left(Ke^{k\sigma}\right)^{p(x)-1} \times \left(\frac{2\delta - d}{2\delta - \sigma}\right)^{(2(p(x)-1)/p^- - 1) - 1} a_1 \\ \times \left|\dfrac{1}{2\delta - \sigma} \dfrac{2(p(x)-1)}{p^- - 1} - \left(\dfrac{2\delta - d}{2\delta - \sigma}\right) \times \left[\left(\ln ke^{k\sigma}\right) \times \left(\dfrac{2\delta - d}{2\delta - \sigma}\right)^{2/p^- - 1} \nabla p \nabla d + \Delta d\right]\right|, \\ \le C_3\left(Ke^{k\sigma}\right)^{p(x)-1} a_1 \ln k, \\ \sigma \le d(x) < 2\delta. \end{cases} \tag{35}$$

If k is sufficiently large, we have

$$C_3\left(ke^{k\sigma}\right)^{p(x)-1}(a_1)\ln k = C_3(kM)^{p(x)-1} a_1 \ln k \le \lambda^{p(x)} a_1. \tag{36}$$

Then,

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)} a_1,\quad \sigma \le d(x) < 2\delta. \tag{37}$$

Since $\phi_1(x), \phi_2(x)$, and f, h are monotone, when λ is large enough, we have

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)}\left(a_1 f(\phi_1) h(\phi_2)\right),\quad \sigma \le d(x) < 2\delta, \tag{38}$$

$$-\Delta_{p(x)}\phi_1 = 0 \le \lambda^{p(x)} a_1 \le \lambda^{p(x)}\left(a_1 f(\phi_1) h(\phi_2)\right),\quad 2\delta \le d(x). \tag{39}$$

Combining (34), (38), and (39), we can deduce that

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)}\left(a(x) f(\phi_1) h(\phi_2)\right), \quad \text{a.e. on } \Omega. \tag{40}$$

Similarly,

$$-\Delta_{q(x)}\phi_2 \le \lambda^{q(x)}\left(b(x) g(\phi_1) \tau(\phi_2)\right), \quad \text{a.e. on } \Omega. \tag{41}$$

From (40) and (41), we can see that (ϕ_1, ϕ_2) is a subsolution of problem (1).

Step 2. We will construct a supersolution of problem (1); we consider

$$\begin{cases} -\Delta_{p(x)} z_1 = \lambda^{p+} a_2 \mu \text{ in } \Omega, \\ -\Delta_{q(x)} z_2 = \lambda^{q+} b_2 \beta^r \text{ in } \Omega, \\ z_1 = z_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{42}$$

where $r > 0$ is the positive number that verifies $(H3)$ and $\beta = \max_{x\in\overline{\Omega}} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution of problem (1).

For $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \ge 0$, it is easy to see that

$$\int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi \, dx = \int_\Omega \lambda^{q+} b_2 \beta^r \psi \, dx. \tag{43}$$

By $(H4)$, for a μ large enough, using Lemma 2, we have

$$\beta^r \ge g(\beta) \tau\left(C_2 \left(\lambda^{q^+} b_2 \beta^r\right)^{1/q^- - 1}\right). \tag{44}$$

Hence,

$$\begin{aligned}\int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi \, dx &\ge \int_\Omega \lambda^{q^+} b_2 g(\max z_1) \tau(\max z_2) \psi \, dx, \\ &\ge \int_\Omega \lambda^{q(x)} b(x) g(z_1) \tau(z_2) \psi \, dx.\end{aligned} \tag{45}$$

Also, for $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \ge 0$, it is easy to see that

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi \, dx = \int_\Omega \lambda^{p+} a_2 \mu \varphi \, dx. \tag{46}$$

By $(H3)$ and Lemma 2, when μ is sufficiently large, we have

$$a_2 \lambda^{p^+} \mu \ge \left[\frac{1}{C_2}\beta\right]^{p^- - 1} \ge a_2 \lambda^{p^+} f(\beta) h\left(C_2\left(\lambda^{q^+} b_2 \beta^r\right)^{1/q^- - 1}\right). \tag{47}$$

Then,

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi \, dx \ge \int_\Omega \lambda^{p(x)} a(x) f(z_1) h(z_2) \varphi \, dx. \tag{48}$$

According to (45) and (48), we can conclude that (z_1, z_2) is a supersolution of problem (1). It only remains to prove that $\phi_1 \le z_1$ and $\phi_2 \le z_2$.

In the definition of $v_1(x)$, let

$$\gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_1(x) + \max_{\overline{\Omega}} |\nabla \phi_1|(x)\right). \tag{49}$$

We claim that

$$\phi_1(x) \le v_1(x), \quad \forall x \in \Omega. \tag{50}$$

From the definition of v_1, it is easy to see that

$$\begin{aligned} \phi_1(x) &\le 2 \max_{\overline{\Omega}} \phi_1(x) \le v_1(x), \quad \text{when } d(x) = \delta, \\ \phi_1(x) &\le 2 \max_{\overline{\Omega}} \phi_1(x) \le v_1(x), \quad \text{when } d(x) \ge \delta, \\ \phi_1(x) &\le v_1(x), \quad \text{when } d(x) < \delta. \end{aligned} \tag{51}$$

Since $v_1 - \phi_1 \in C^1(\overline{\partial\Omega_\delta})$, there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1(x_0) - \phi_1(x_0) = \min_{x_0 \in \overline{\partial\Omega_\delta}} \left(v_1(x_0) - \phi_1(x_0)\right). \tag{52}$$

If $v_1(x_0) - \phi_1(x_0) < 0$, it is easy to see that $0 < d(x) < \delta$ and then

$$\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0. \tag{53}$$

From the definition of v_1, we have

$$\begin{aligned}|\nabla v_1(x_0)| = \gamma &= \frac{2}{\delta}\left(\max_{\overline{\Omega}} \phi_1(x_0) + \max_{\overline{\Omega}} |\nabla \phi_1|(x_0)\right) \\ &> |\nabla \phi_1|(x_0).\end{aligned} \tag{54}$$

It is a contradiction to

$$\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0. \tag{55}$$

Thus, (50) is valid.

Obviously, there exists a positive constant C_3 such that $\gamma \le C_3 \lambda$.

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 2, there exists a positive constant C_4 such that

$$-\Delta_{p(x)} v_1(x) \le C_* \gamma^{p(x)-1+\theta} \le C_4 \lambda^{p(x)-1+\theta}, \quad \text{a.e, in } \Omega, \text{ where } \theta \in (0,1). \tag{56}$$

When $\eta \geq \lambda^{p^+}$ is large enough, we have $-\Delta_{p(x)} v_1(x) \leq \eta$. According to the comparison principle, we have

$$v_1(x) \leq \omega(x). \quad (57)$$

From (50) and (57), when $\eta \geq \lambda^{p^+}$ and $\lambda \geq 1$ are sufficiently large, we have, for all $x \in \Omega$,

$$\phi_1(x) \leq v_1(x) \leq \omega(x). \quad (58)$$

According to the comparison principle, when μ is large enough, we have, for all $x \in \Omega$,

$$v_1(x) \leq \omega(x) \leq z_1(x). \quad (59)$$

Combining the definition of $v_1(x)$ and (58), it is easy to see that, for all $x \in \Omega$,

$$\phi_1(x) \leq v_1(x) \leq \omega(x) \leq z_1(x). \quad (60)$$

When $\mu \geq 1$ and λ is large enough, from Lemma 2, we can see that β is large enough, and then $\lambda^{q+} b_2 \beta^r$ is a large enough. Similarly, we have $\phi_2 \leq z_2$. This completes the proof.

4. Conclusion

Validity of the comparison principle and of the SSM for local and nonlocal problems as the stationary and evolutionary Kirchhoff Equation was an important subject in the last few years (see, for example, [44, 53, 58, 62–66]), where the authors showed by giving different counterexamples that the simple assumption M increasing somewhere is enough to make the comparison principle and SSM hold false contradiction and clear up some results in the literature. Moreover, the two conditions that M is nonincreasing and H is increasing turn out to be necessary and sufficient, at least for the validity of the comparison principle. It is worth to note that in [45, 67], C. O. Alves and F. J. S. A. Correa developed a new SSM for problem (1) to deal with the increasing M case. The result is obtained by using a kind of Minty–Browder theorem for a suitable pseudomonotone operator, but instead of constructing a subsolution, the authors assumed the existence of a whole family of functions which satisfy a stronger condition than just being subsolutions; the inconvenience is that these stronger conditions restrict the possible right hand sides in (1). Another SSM for nonlocal problems is obtained in [45] for a problem involving a nonlocal term with a Lebesgue norm, instead of the Sobolev norm appearing in (1).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally in this article. All authors read and approved the final manuscript.

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