Research Article

Evolutionary Dynamics of Stochastic SEIR Models with Migration and Human Awareness in Complex Networks

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In this paper, a stochastic SEIR (Susceptible-Exposed-Infected-Removed) epidemic dynamic model with migration and human awareness in complex networks is constructed. The awareness is described by an exponential function. The existence of global positive solutions for the stochastic system in complex networks is obtained. The sufficient conditions are presented for the extinction and persistence of the disease. Under the conditions of disease persistence, the distance between the stochastic solution and the local disease equilibrium of the corresponding deterministic system is estimated in the time sense. Some numerical experiments are also presented to illustrate the theoretical results. Although the awareness introduced in the model cannot affect the extinction of the disease, the scale of the disease will eventually decrease as human awareness increases.

1. Introduction

Infectious diseases have been threatening human health and affecting people's life. Many researchers use mathematical models to study the spread and control strategies of infectious diseases. Some current compartment models were proposed by Kermack and McKendrick (see [1–3]). Since then, many researchers have proposed other models of infectious diseases based on these models.

It is usually assumed that the probability of contact between all different individuals is consistent in early classical compartment models. However, there is obvious heterogeneity in human behavior and interpersonal communication in society because there might exist some members who could spread the disease to many other members. Therefore, it is more suitable that the disease transmission is modeled over complex networks.

Furthermore, most catenary systems have scale-free properties of several orders of magnitude. For example, Pastor-Satorras and Vespignani studied SIS (Susceptible-Infected-Susceptible) models on scale-free networks by mean-field approximation (see [4]). In order to comprehend the spread tendency of infectious diseases, more and more

researchers concerned about the stability and asymptotic behavior of epidemic models on complex networks. For example, Wei et al. researched the global stability of the endemic equilibrium for a network-based SIS epidemic model with the nonmonotone incidence rate, a SIS epidemic model with feedback mechanism on networks and a SIQRS epidemic model on complex networks in papers [5–7], respectively. Li found a threshold value for the transmission rate of a network-based SIS epidemic model with the nonmonotone incidence rate (see [8]). Yang et al. obtained the basic reproduction numbers of the SIS model and SIR model with infection age to investigate the disease transmission on complex networks (see [9, 10]). These results provide good theoretical help for the prevention and control of infectious diseases.

In addition, there are some diseases with incubation period. Humans can be divided into susceptible, infectious, and recovered population and undiagnosed infectious population. Undiagnosed infectious population are individuals with the disease who pass them on to susceptible population, such as measles and pertussis. The impact of undiagnosed populations on the spread of infectious diseases is also important. The susceptible-

exposed-infective-recovered (SEIR) model is suited to model this kind of diseases in the incubation period. So, some SEIR epidemic models on the scale-free networks are presented. Using the analytical method, the threshold values on the stability of disease-free equilibrium and the persistence of the disease have been given in [11]. By researching the SEIR epidemic models on complex network, Zhu et al. found that epidemic propagation depends equally on the infection rate and the time of latent and infected periods, especially on the nature of the contacts among individuals (see [12]). Liu and Li discussed a new epidemic SEIR model with discrete delay on complex network in paper [13]. Moreover, there are many researchers discussed rumor spreading problems by SIER propagation model on heterogeneous network (see [14, 15]).

In real communities, susceptible people will be vigilant in the presence of infectious diseases, thus reducing the probability of contact with suspected infected people. As the number of patients increases, susceptible people become more vigilant. As a result, susceptible people will raise awareness of staying away from disease. The awareness reveals “washing hands,” “staying at home,” and “increasing the distance between people.” The improvement of this awareness does not only lower the incidence of the disease but, in some cases, can also prevent the outbreak of the disease. Funk et al. investigated how the spread of awareness prompted by a first-hand contact with the disease affects the spread of the disease (see [16]). Agaba et al. provided information about potential spread of disease in a population (see [17]). Recently, there are also some research results about epidemic models considering awareness progress on complex networks. Yuan et al. considered an epidemic disease model about the effect of awareness programs on complex networks and obtained the basic reproduction number in [18]. Wu and Fu studied a discrete-time SIS model with awareness interactions on degree-uncorrelated networks. Their findings explored the effects of various types of awareness on epidemic spreading and addressed their roles in the epidemic control (see [19]). She et al. presented a theoretical analysis of the SIS epidemic spread from the perspective of bipartite network and risk aversion (see [20]).

In real life, biological populations are inevitably affected by external uncertainties, such as absolute humidity, temperature, and precipitation. In order to be more practical, it is necessary to introduce the randomness into the deterministic model. The importance of stochasticity in the epidemic dynamics has been acknowledged for a long time. There are many research results in stochastic epidemic models. For instance, Lahrouz and Omari researched a stochastic SIRS epidemic model with general incidence rate in a population of varying size and obtained sufficient conditions for the extinction and the existence of a unique stationary distribution (see [21]). Li et al. studied a stochastic SIRS epidemic model with nonlinear incidence rate and varying population size and obtained sufficient conditions for extinction and persistence of the disease (see [22]). Ji and

Jiang established a threshold condition under stochastic perturbation for persistence or extinction of the disease and global dynamical behavior for stochastic SIR epidemic models (see [23, 24]).

Especially, there are also some results in stochastic SEIR epidemic model. For example, Zhang and Wang established stochastic SEIR models with jumps, which are used to describe the wide spread of the infectious diseases due to the medical negligence (see [25]). Witbooi proved that there is almost sure exponential stability of the disease-free equilibrium for an SEIR epidemic model with independent stochastic perturbations (see [26]). Liu et al. researched the asymptotic behaviors of the disease-free equilibrium and the endemic equilibrium for a stochastic delayed SEIR epidemic model with nonlinear incidence (see [27]). Liu et al. proposed a two-group stochastic SEIR model with infinite delay and obtained the sufficient conditions for asymptotic stability of endemic equilibrium in [28]. Han et al. researched the dynamics behaviors of a nonlinear stochastic SEIR epidemic system with varying population size in [29]. In recent years, there are several results for some stochastic epidemic models over complex networks, for example, Krause et al. established a stochastic epidemic metapopulation model and solved the stochastic model numerically. Furthermore, authors designed some control approaches to control the spread of a stochastic SIS epidemic over spatial networks (see [30]). Cao and Jin studied the epidemic threshold and ergodicity of an SIS model in switched networks (see [31]). Nevertheless, the research results about stochastic SEIR epidemic models over complex networks are seldom.

Inspired by the abovementioned factors, we will establish a stochastic SEIR models with human awareness in complex networks in this paper, which consider the effects of undiagnosed populations, human awareness, and external uncertainties on the spread of infectious diseases. The rest of this paper is as follows. The stochastic SEIR models with migration and human awareness in complex networks are established in Section 2. In Section 3, we discuss the existence of global positive solutions for the stochastic system in complex networks. In Sections 4 and 5, we study the persistence and extinction of the disease. In Section 6, we introduce numerical simulations to illustrate the main results. Finally, we close the paper with the summary of main results.

2. The Model with Migration and Human Awareness

In this section, we will establish a stochastic SEIR model on complex networks. This model will reflect the impact of susceptible, infected, recovered, and undiagnosed populations on the spread of infectious diseases. Furthermore, the effects of migration and human awareness on the spread of disease will also be considered on complex networks, and suppose that the network is inaccessible to people who have been diagnosed with the disease. The basic SEIR model on complex networks is as follows:

$$\begin{cases} dS_k(t) = [(1 - \alpha(t))\Delta - \lambda k S_k(t)\Theta(t) - b S_k(t)]dt, \\ dE_k(t) = [\alpha(t)\Delta + \lambda k S_k(t)\Theta(t) - (\mu + b + \gamma_1)E_k(t)]dt, \\ dI_k(t) = [\mu E_k(t) - (\gamma_2 + b)I_k(t)]dt, \\ dR_k(t) = (\gamma_1 E_k(t) + \gamma_2 I_k(t) - b R_k(t))dt, \end{cases} \quad (1)$$

where

$$\begin{aligned} \Theta(t) &= \frac{1}{\langle k \rangle} \sum_{k=1}^n k p_k (E_k(t) + I_k(t)), \\ \langle k \rangle &= \sum_{k=1}^n k p_k, \\ \lim_{t \rightarrow +\infty} \alpha(t) &= \alpha, \end{aligned} \quad (2)$$

and S_k , E_k , I_k , and R_k denote the number of the population that are susceptible, undiagnosed infectious (also known as “exposed,” which have the disease and can pass it on to a susceptible person), infectious and recovered with immunity at node k , respectively. The parameter λ is the probability of exposure each time a susceptible person comes into contact with a pathogen carrier. b is for the death and emigration of the susceptible, exposed, infectious, and recovered. Δ is a growing and migrating population of susceptible and undiagnosed infectious individuals. $\Theta(t)$ is a probability of a randomly selected link pointing to an undiagnosed or infected individual. $\alpha(t) \in [0, 1]$ is the proportion of these undiagnosed infectious individuals over time t , and $\alpha_m \leq \alpha(t) \leq \alpha_M$. μ is the conversion rate from E_k to I_k . γ_1 and γ_2 represent the response rate from E_k and I_k to R_k , respectively. In addition, all parameters of model (1) are assumed to be positive constants.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets), and $\mathbb{R}_+^{4n} = \{x \in \mathbb{R}^{4n}: x_i > 0, i = 1, 2, \dots, 4n\}$.

Consider that the awareness of the population increases with the size of the infected population during an epidemic, and the rate of increase in awareness gradually decreases. Therefore, this awareness is described by $e^{-c\Theta(t)}$. Supposing that $\Delta = b$, then we obtain the following model:

$$\begin{cases} dS_k(t) = [(1 - \alpha(t))b - \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - b S_k(t)]dt, \\ dE_k(t) = [\alpha(t)b + \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - (\mu + b + \gamma_1)E_k(t)]dt, \\ dI_k(t) = [\mu E_k(t) - (\gamma_2 + b)I_k(t)]dt, \\ dR_k(t) = (\gamma_1 E_k(t) + \gamma_2 I_k(t) - b R_k(t))dt. \end{cases} \quad (3)$$

Assume that the transmission rate λ and the conversion rate μ in system (3) are perturbed by some random environmental effects; then, we have

$$\begin{aligned} \lambda &= \bar{\lambda} + \sigma_1 \xi_1(t), \\ \mu &= \bar{\mu} + \sigma_2 \xi_2(t), \end{aligned} \quad (4)$$

where $\xi_1(t)$ and $\xi_2(t)$ are the white noise with mean zero and variance one. $\bar{\lambda}$ and $\bar{\mu}$ represents the average of λ and μ in the stochastic setting, respectively. $\sigma_1, \sigma_2 > 0$ denotes the white noise intensity. Rewrite $\bar{\lambda}$ and $\bar{\mu}$ as λ and μ for convenience. Noting that $\xi_i(t)dt = dB_i(t)$, $i = 1, 2$, where $B_i(t)$, $i = 1, 2$ represents the standard Brownian motion. Then, we obtain the following stochastic epidemic SEIR model:

$$\begin{cases} dS_k(t) = [(1 - \alpha(t))b - \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - b S_k(t)]dt - \sigma_1 k S_k(t)\Theta(t)e^{-c\Theta(t)}dB_1, \\ dE_k(t) = [b\alpha(t) + \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - (\mu + b + \gamma_1)E_k(t)]dt + \sigma_1 k S_k(t)\Theta(t)e^{-c\Theta(t)}dB_1 - \sigma_2 E_k(t)dB_2, \\ dI_k(t) = [\mu E_k(t) - (\gamma_2 + b)I_k(t)]dt + \sigma_2 E_k(t)dB_2, \\ dR_k(t) = (\gamma_1 E_k(t) + \gamma_2 I_k(t) - b R_k(t))dt. \end{cases} \quad (5)$$

Denoting $N_k(t)$ is the size of total population with k - nodes, let $N_k(t) = S_k(t) + E_k(t) + I_k(t) + R_k(t)$ and $N_k(0) = 1$.

3. Existence of the Global and Positive Solution

To investigate the dynamical behavior of system (5), the first concern is whether the solution is global. Moreover, for an epidemic dynamics model, whether the value is positive is also required. The following theorem shows that the solution of system (5) is global and positive.

Theorem 1. *For any initial value $(S_k(0), E_k(0), I_k(0), R_k(0)) \in \mathbb{R}_+^4$, $k = 1, \dots, n$ and $N_k(0) = 1$, there is a unique*

solution $(S_k(t), E_k(t), I_k(t), R_k(t))$, $k = 1, \dots, n$ of system (5) on $t \geq 0$, and the solution will remain in $\mathbb{R}_+^4$ with probability 1.

Proof. Define the stopping time τ^+ by

$$\begin{aligned} \tau^+ &= \inf\{t \in [0, +\infty): S_k \leq 0 \text{ or } E_k \leq 0 \text{ or } I_k \leq 0 \text{ or } R_k \leq 0, k \\ &= 1, \dots, n\}. \end{aligned} \quad (6)$$

To show this solution is positive globally in the first octant, we only need to show that $\tau^+ = +\infty$ a.s. Assume that there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(\tau^+ < T) > \varepsilon$. Let $\tilde{\Omega} = \{\omega \in \Omega: \tau^+ < T\}$, where ω represents a sample point of the sample space Ω . Define a C^2 - function $V: \mathbb{R}_+^{4n} \rightarrow \mathbb{R}$ as follows:

$$V(X) = \sum_{k=1}^n [\ln S_k + \ln E_k + \ln I_k + \ln R_k]. \quad (7)$$

Applying Itô's formula, we obtain

$$\begin{aligned} dV(X) &= \sum_{k=1}^n \left\{ \left[\frac{(1-\alpha(t))b}{S_k} - \lambda k \Theta e^{-c\Theta} - b - \frac{1}{2} \sigma_1^2 (k \Theta e^{-c\Theta})^2 \right] dt + \left[\frac{\alpha(t)b}{E_k} + \frac{\lambda k S_k \Theta e^{-c\Theta}}{E_k} - (\mu + b + \gamma_1) - \frac{1}{2} \sigma_1^2 \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right)^2 - \frac{1}{2} \sigma_2^2 \right] dt \right. \\ &\quad \left. + \left[\frac{\mu E_k}{I_k} - (\gamma_2 + b) - \frac{1}{2} \sigma_2^2 \left(\frac{E_k}{I_k} \right)^2 \right] dt + \left[\frac{\mu E_k}{R_k} - \frac{\gamma_1 E_k}{R_k} - b \right] dt - \sigma_1 k \Theta e^{-c\Theta} dB_1 + \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1 - \sigma_2 dB_2 + \sigma_2 \frac{E_k}{I_k} dB_2 \right\}, \\ &\geq \sum_{k=1}^n \left\{ K(S_k, E_k, I_k, R_k) dt - \sigma_1 k \Theta e^{-c\Theta} dB_1 + \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1 - \sigma_2 dB_2 + \sigma_2 \frac{E_k}{I_k} dB_2 \right\}, \end{aligned} \quad (8)$$

where

$$K(S_k, E_k, I_k, R_k) = -\lambda k \Theta e^{-c\Theta} - \frac{1}{2} \sigma_1^2 (k \Theta e^{-c\Theta})^2 - \frac{1}{2} \sigma_1^2 \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right)^2 - \frac{1}{2} \sigma_2^2 - \frac{1}{2} \sigma_2^2 \left(\frac{E_k}{I_k} \right)^2 - \frac{\gamma_1 E_k}{R_k} - (\mu + \gamma_1 + \gamma_2 + 4b). \quad (9)$$

Integrating both sides of (8) from 0 to $\tau^+ \wedge t$, we obtain

$$V(\tau^+ \wedge t) \geq V(X_0) + \sum_{k=1}^n \left[\int_0^{\tau^+ \wedge t} K(S_k(s), E_k(s), I_k(s), R_k(s)) ds - \int_0^{\tau^+ \wedge t} \left(1 - \frac{S_k}{E_k} \right) \sigma_1 k \Theta e^{-c\Theta} dB_1 - \int_0^{\tau^+ \wedge t} \sigma_2 \left(1 - \frac{E_k}{I_k} \right) dB_2 \right]. \quad (10)$$

Let $t \rightarrow +\infty$ in both sides of the above inequality, and we obtain from the definition of τ^+ that

$$-\infty > -\infty, \quad (11)$$

which contradicts our assumption, and the proof of Theorem 1 is complete. $\square$

Lemma 1. For any initial value $X_0 = (S_1(0), E_1(0), I_1(0), R_1(0), \dots, S_n(0), E_n(0), I_n(0), R_n(0))$ in $\mathbb{R}_+^{4n}$ and $N_k(0) = S_k(0) + E_k(0) + I_k(0) + R_k(0) = 1, k = 1, 2, \dots, n$, the solution of system (5) has the property that

$$N_k(t) = S_k(t) + E_k(t) + I_k(t) + R_k(t) = 1, \quad (12)$$

for all $\omega \in \Omega$.

Proof. The proof is straightforward. Summing up the four equations in (5), we can obtain

$$\frac{dN_k}{dt} = 0. \quad (13)$$

By Theorem 1, we obtain the next assertion that

$$0 \leq S_k(t), E_k(t), I_k(t), R_k(t) \leq 1. \quad (14) \quad \square$$

4. Persistence of the Disease

Now let us consider the persistence of disease, which corresponds to the stochastic strong persistence in the mean defined by Zhao et al. [32]. The following theorem shows that the disease will be almost surely persistent in the time mean sense.

Lemma 2. For any initial value $(S_k(0), E_k(0), I_k(0), R_k(0)) \in \{(S_k, E_k, I_k, R_k) \in \mathbb{R}_+^4 \mid 0 \leq S_k, E_k, I_k, R_k \leq 1\}, k = 1, \dots, n$ and $N_k(0) = 1, \exists T > 1$, when $t > T$, the solution $E_k(t)$ of system (5) has the property that

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t E_k(s) ds \geq \alpha_m b. \quad (15)$$

Proof. From the second equation of system (5), we obtain

$$dE_k(t) \geq [b\alpha_m - (\mu + b + \gamma_1)E_k(t)]dt + \sigma_1 k S_k(t) e^{-c\Theta(t)} dB_1 - \sigma_2 E_k(t) dB_2. \quad (16)$$

Considering the actual situation of parameter μ, b , and γ_1 , integrating the above inequality via t along $[0, t]$, and then dividing both sides by t , we have

$$\frac{E_k(t) - E_k(0)}{t} \geq b\alpha_m - (\mu + b + \gamma_1) \frac{1}{t} \int_0^t E_k ds + \frac{1}{t} \int_0^t \sigma_1 k S_k \Theta e^{-c\Theta} dB_1(s) - \frac{1}{t} \int_0^t \sigma_2 E_k dB_2(s). \quad (17)$$

From the strong law of large numbers for local martingales, we obtain

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t E_k(s) ds \geq \alpha_m b. \quad (18)$$

The proof of the lemma is complete. $\square$

Theorem 2. For any initial value $(S_k(0), E_k(0), I_k(0), R_k(0)) \in \{(S_k, E_k, I_k, R_k) \in \mathbb{R}_+^4 \mid 0 \leq S_k, E_k, I_k, R_k \leq 1\}, k = 1, \dots, n$ and $N_k(0) = 1$, if $\alpha > 0$, then the solution $(S_k(t), E_k(t), I_k(t), R_k(t)), k = 1, \dots, n$ of system (5) has the property that

$$\begin{aligned} \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t S_k(s) ds &\geq \frac{(1 - \alpha_M)b}{\lambda k + b}, \quad \text{a.s.,} \\ \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t E_k(s) ds &\geq \frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q\left(\frac{1}{\alpha_m b}\right) - \alpha_M b - \frac{\lambda k}{\alpha_m b}, \quad \text{a.s.,} \\ \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t I_k(s) ds &\geq \frac{\mu}{\gamma_2 + b} \left[\frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q\left(\frac{1}{\alpha_m b}\right) - \alpha_M b - \frac{\lambda k}{\alpha_m b} \right], \quad \text{a.s.,} \\ \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t R_k(s) ds &\geq \frac{\mu\gamma_2 + \gamma_1\gamma_2 + \gamma_1 b}{\gamma_2 b + b^2} \left[\frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q\left(\frac{1}{\alpha_m b}\right) - \alpha_M b - \frac{\lambda k}{\alpha_m b} \right], \quad \text{a.s.,} \end{aligned} \quad (19)$$

where $Q(x) = -(1/2)\sigma_1^2 k^2 x^2 + \lambda k x + b\alpha_m - (\mu + b + \gamma_1) - (1/2)\sigma_2^2$.

Proof. Let $(S_k(t), E_k(t), I_k(t), R_k(t))$ be a solution of system (5) with any initial value $(S_k(0), E_k(0), I_k(0), R_k(0)) \in \mathbb{R}_+^4$

and $N_k(0) = 1$. By Theorem 1, it holds that $0 < \lim_{t \rightarrow +\infty} \Theta \leq 1$ for all $\omega \in \Omega$. This means that there exists a $T^\omega = T^\omega(\varepsilon) > 0$ such that $\Theta(\omega, t) \leq 1 + \varepsilon$ for all $t > T^\omega$, any $\omega \in \Omega$ and any sufficiently small $\varepsilon > 0$. From the first equation of system (5), we derive that, for all $t > T^\omega$,

$$dS_k(t) \geq [(1 - \alpha_M)b - \lambda k S_k(1 + \varepsilon) - b S_k]dt - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1 = [(1 - \alpha_M)b - (\lambda k(1 + \varepsilon) + b)S_k]dt - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1, \quad (20)$$

where ω is omitted. For convenience and without losing generality, we assume that the above inequality holds for all

$t > 0$. Integrating the above inequality and then dividing both sides by t , we obtain

$$\frac{S_k(t) - S_k(0)}{t} \geq (1 - \alpha_M)b - (\lambda k(1 + \varepsilon) + b) \int_0^t S_k(s) ds - \frac{1}{t} \int_0^t \sigma_1 k S_k \Theta e^{-c\Theta} dB_1(s). \quad (21)$$

From $\lim_{t \rightarrow +\infty} S_k(t) \leq 1$ for all $\omega \in \Omega$ and strong law of large numbers for local martingales, we obtain

$$\lim_{t \rightarrow +\infty} \left(\frac{S_k(t) - S_k(0)}{t} + \frac{1}{t} \int_0^t \sigma_1 k S_k \Theta e^{-c\Theta} dB_1(s) \right) = 0, \quad \text{a.s.}, \quad (22)$$

together with (21), which implies

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t S_k(s) ds \geq \frac{(1 - \alpha_M)b}{\lambda k(1 + \varepsilon) + b}, \quad \text{a.s.} \quad (23)$$

Because of the arbitrariness of ε , we have

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t S_k(s) ds \geq \frac{(1 - \alpha_M)b}{\lambda k + b}, \quad \text{a.s.} \quad (24)$$

This is the required first assertion.

Then, we will prove the second assertion.

By Itô's formula, it can be seen from the second equation of system (5) that

$$\begin{aligned} d \ln E_k(t) &\geq \left[b\alpha_m + \frac{\lambda k S_k \Theta e^{-c\Theta}}{E_k} - (\mu + b + \gamma_1) - \frac{1}{2} \sigma_1^2 k^2 \left(\frac{\sigma_1 S_k \Theta e^{-c\Theta}}{E_k} \right)^2 - \frac{1}{2} \sigma_2^2 \right] dt + \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(t) - \sigma_2 dB_2(t) \\ &= Q \left(\frac{S_k \Theta e^{-c\Theta}}{E_k} \right) dt + \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(t) - \sigma_2 dB_2(t). \end{aligned} \quad (25)$$

Integrating both sides yields

$$\ln E_k(t) \geq \ln E_k(0) + \int_0^t Q \left(\frac{S_k \Theta e^{-c\Theta}}{E_k} \right) ds + \int_0^t \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(s) - \int_0^t \sigma_2 dB_2(s). \quad (26)$$

According to Lemma 2, supposing that $E_k \geq \alpha_m b$, then we compute

$$Q \left(\frac{S_k \Theta e^{-c\Theta}}{E_k} \right) - Q \left(\frac{1}{\alpha_m b} \right) = \frac{1}{2} \sigma_1^2 k^2 \left[\left(\frac{1}{\alpha_m b} \right)^2 - \left(\frac{S_k \Theta e^{-c\Theta}}{E_k} \right)^2 \right] - \lambda k \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right). \quad (27)$$

By Lemma 1 and Theorem 1, we assume that $S_k(t) \leq 1, E_k(t) \leq 1, I_k(t) \leq 1$ for all $t > 0$, which yields from (27) that

$$\begin{aligned} Q \left(\frac{S_k \Theta e^{-c\Theta}}{E_k} \right) &= Q \left(\frac{1}{\alpha_m b} \right) - \left[\lambda k - \frac{1}{2} \sigma_1^2 k^2 \left(\frac{1}{\alpha_m b} + \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) \right] \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) \\ &\geq Q \left(\frac{1}{\alpha_m b} \right) - \left(\lambda k - \frac{1}{2} \sigma_1^2 k^2 \right) \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right). \end{aligned} \quad (28)$$

Substituting this inequality into (26), we obtain

$$\ln E_k(t) \geq \ln E_k(0) + Q \left(\frac{1}{\alpha_m b} \right) t - \left(\lambda k - \frac{1}{2} \sigma_1^2 k^2 \right) \int_0^t \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) ds + \int_0^t \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(s) - \int_0^t \sigma_2 dB_2(s). \quad (29)$$

From the first equation of system (5), we have

$$\begin{aligned}
 dS_k(t) &\geq \left[(1 - \alpha(t))b - \frac{\lambda k S_k \Theta e^{-c\Theta}}{E_k} - b S_k \right] dt - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1 = \left[(1 - \alpha(t))b - \frac{\lambda k}{\alpha_m b} + \lambda k \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) - b S_k \right] dt \\
 &\quad - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1 \\
 &\geq \left[(1 - \alpha_M)b - \frac{\lambda k}{\alpha_m b} + \lambda k \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) - b - E_k(t) \right] dt - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1 \\
 &= \left[-\alpha_M b - \frac{\lambda k}{\alpha_m b} + \lambda k \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) b - E_k(t) \right] dt - \sigma_1 k S_k \Theta e^{-c\Theta} dB_1.
 \end{aligned} \tag{30}$$

Then, we obtain

$$\lambda k \int_0^t \left(\frac{1}{\alpha_m b} - \frac{S_k \Theta e^{-c\Theta}}{E_k} \right) ds \leq S_k(t) - S_k(0) + \frac{\lambda k}{\alpha_m b} t + \alpha_M b t + \int_0^t E_k(s) ds - \int_0^t \sigma_1 k S_k \Theta e^{-c\Theta} dB_1(s). \tag{31}$$

Substituting inequality (31) into (29), we obtain

$$\ln E_k(t) \geq Q \left(\frac{1}{\alpha_m b} \right) t - \frac{2\lambda k - \sigma_1^2 k^2}{2\lambda k} \left(\alpha_M b t + \frac{\lambda k}{\alpha_m b} t + \int_0^t E_k(s) ds \right) + Z(t), \tag{32}$$

where

$$Z(t) = \ln E_k(0) - \frac{2\lambda k - \sigma_1^2 k^2}{2\lambda k} \left(S_k(t) - S_k(0) - \int_0^t \sigma_1 k S_k \Theta e^{-c\Theta} dB_1(s) \right) + \int_0^t \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(s) - \int_0^t \sigma_2 dB_2(s). \tag{33}$$

By Theorem 1, Lemma 1 and the strong law of large numbers for local martingales, we have

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \frac{Z(t)}{t} &= 0, \text{ a.s.}, \\
 \limsup_{t \rightarrow +\infty} \frac{\ln E_k(t)}{t} &\leq 0.
 \end{aligned} \tag{34}$$

Adding with (32), they can yield the following inequation:

$$\frac{I_k(t) - I_k(0)}{t} = \mu \frac{1}{t} \int_0^t E_k(s) ds - (\gamma_2 + b) \frac{1}{t} \int_0^t I_k(s) ds + \frac{1}{t} \int_0^t \sigma_2 E_k(s) dB_2(s). \tag{36}$$

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t E_k(s) ds \geq \frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q \left(\frac{1}{\alpha_m b} \right) - \alpha_M b - \frac{\lambda k}{\alpha_m b} \quad \text{a.s.} \tag{35}$$

This is the required second assertion.

Then, we shall prove the third assertion. Integrating the third equation of system (5) and dividing both sides by t , we obtain

Adding with (35), they can yield the following inequation:

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t I_k(s) ds \geq \frac{\mu}{\gamma_2 + b} \left[\frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q\left(\frac{1}{\alpha_m b}\right) - \alpha_M b - \frac{\lambda k}{\alpha_m b} \right], \quad \text{a.s.} \quad (37)$$

This is the required third assertion. Finally, similar to the proof of the third assertion, we can obtain

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t R_k(s) ds \geq \frac{\mu\gamma_2 + \gamma_1\gamma_2 + \gamma_1 b}{\gamma_2 b + b^2} \left[\frac{2\lambda k}{2\lambda k - \sigma_1^2 k^2} Q\left(\frac{1}{\alpha_m b}\right) - \alpha_M b - \frac{\lambda k}{\alpha_m b} \right], \quad \text{a.s.} \quad (38)$$

This completes the proof of Theorem 2.

Recall that $\lim_{t \rightarrow +\infty} \alpha(t) = \alpha$, which means that we have $|\alpha(t) - \alpha| < \varepsilon$ when $t \geq T$, $\exists T > 0$, for any arbitrarily small

number $\varepsilon > 0$. Then, we consider the following simplified version of system (5):

$$\begin{cases} dS_k(t) = [(1 - \alpha)b - \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - bS_k(t)]dt - \sigma_1 k S_k(t)\Theta(t)e^{-c\Theta(t)} dB_1, \\ dE_k(t) = [b\alpha + \lambda k S_k(t)\Theta(t)e^{-c\Theta(t)} - (\mu + b + \gamma_1)E_k(t)]dt + \sigma_1 k S_k(t)\Theta(t)e^{-c\Theta(t)} dB_1 - \sigma_2 E_k(t) dB_2, \\ dI_k(t) = [\mu E_k(t) - (\gamma_1 + b)I_k(t)]dt + \sigma_2 E_k(t) dB_2, \end{cases} \quad (39)$$

where $\gamma_1 = \gamma_2$. $\square$

Theorem 3. Suppose that $(S_k(t), E_k(t), I_k(t)), k = 1, \dots, n$ is the solution of system (39) for any initial value $(S_k(0), E_k(0),$

$I_k(0)) \in \{(S_k, E_k, I_k) \in \mathbb{R}_+^3 \mid 0 \leq S_k, E_k, I_k \leq 1\}, k = 1, \dots, n$ and $N_k(0) = 1$. If $\Theta \leq \Theta^*$ and $\Theta^* e^{-c\Theta^*} \leq \Theta e^{-c\Theta}$, then we have

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E} \left[\int_0^t \left((\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} + \frac{1}{\langle k \rangle} \sum_{k=1}^n k p_k \left[(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} \right] \right) ds \right] \leq \eta, \quad (40)$$

where $(S_k^*, E_k^*, I_k^*), k = 1, \dots, n$ is the unique endemic equilibrium E^* of the corresponding deterministic equation of system (39), and

$$\eta = \frac{1}{\langle k \rangle} \sum_{k=1}^n k p_k \left[\frac{1}{2} \sigma_1^2 (k \Theta^*)^2 + \sigma_2^2 \Theta^{*2} \left(\frac{1}{\alpha b} \right)^2 + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} \right]. \quad (41)$$

Proof. For any arbitrarily small number $\varepsilon > 0$, there exist a $T > 0$, when $t \geq T$, we have $\alpha(t) - \alpha < \varepsilon$. The corresponding deterministic version of system (39) has a unique endemic equilibrium E^* satisfying

$$\begin{cases} (1 - \alpha)b - \lambda k S_k^* \Theta^* e^{-c\Theta^*} - bS_k^* = 0, \\ b\alpha + \lambda k S_k^* \Theta^* e^{-c\Theta^*} - (\mu + b + \gamma_1)E^* = 0, \\ \mu E_k^* - (\gamma_1 + b)I_k^* = 0. \end{cases} \quad (42)$$

Define C^2 functions as follows:

$$V(t) = V_1(t) + V_2(t),$$

$$V_1 = \Theta - \Theta^* - \Theta^* \ln\left(\frac{\Theta}{\Theta^*}\right), \quad (43)$$

$$V_2 = \frac{1}{\langle k \rangle} \sum_{k=1}^n k p_k \left[S_k - S_k^* - S_k^* \ln\left(\frac{S_k}{S_k^*}\right) \right].$$

Applying Itô's formula, we have

$$\begin{aligned}
 LV_1 &= \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{\Theta - \Theta^*}{\Theta} (\lambda k S_k \Theta e^{-c\Theta} - \lambda k S_k^* \Theta^* e^{-c\Theta^*} - (\gamma_1 + b)(E_k + I_k - E_k^* - I_k^*)) + \frac{1}{2} \sigma_1^2 (k S_k \Theta^* e^{-c\Theta})^2 + \sigma_2^2 \Theta^{*2} \left(\frac{E_k}{\Theta} \right)^2 \right] \\
 &= -(\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} + \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{\Theta - \Theta^*}{\Theta} (\lambda k S_k \Theta e^{-c\Theta} - \lambda k S_k^* \Theta^* e^{-c\Theta^*}) + \frac{1}{2} \sigma_1^2 (k S_k \Theta^* e^{-c\Theta})^2 + \sigma_2^2 \Theta^{*2} \left(\frac{E_k}{\Theta} \right)^2 \right] \\
 &\leq -(\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} + \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{\Theta - \Theta^*}{\Theta} (\lambda k S_k \Theta e^{-c\Theta} - \lambda k S_k^* \Theta^* e^{-c\Theta^*}) + \frac{1}{2} \sigma_1^2 (k \Theta^*)^2 + \sigma_2^2 \Theta^{*2} \left(\frac{1}{ab} \right)^2 \right],
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 LV_2 &= \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{S_k - S_k^*}{S_k} (\lambda k S_k^* \Theta^* e^{-c\Theta^*} + b S_k^* - \lambda k S_k \Theta e^{-c\Theta} - b S_k) + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} (\Theta e^{-c\Theta})^2 \right] \\
 &= \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{S_k - S_k^*}{S_k} (b S_k^* - b S_k + \lambda k S_k^* \Theta^* e^{-c\Theta^*} - \lambda k S_k \Theta e^{-c\Theta} - \lambda k S_k \Theta^* e^{-c\Theta^*} + \lambda k S_k \Theta^* e^{-c\Theta}) + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} (\Theta e^{-c\Theta})^2 \right] \\
 &= \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[-(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} + \frac{S_k - S_k^*}{S_k} (-\lambda k S_k \Theta e^{-c\Theta} + \lambda k S_k \Theta^* e^{-c\Theta^*}) \right] + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} (\Theta e^{-c\Theta})^2 \\
 &\leq \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[-(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} + \frac{S_k - S_k^*}{S_k} (-\lambda k S_k \Theta e^{-c\Theta} + \lambda k S_k \Theta^* e^{-c\Theta^*}) + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} \right].
 \end{aligned} \tag{45}$$

Combining (44) with (45), this implies

$$\begin{aligned}
 LV &\leq -(\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} - \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} \right] + \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \\
 &\quad \cdot \left[\frac{\lambda k}{\Theta} (\Theta S_k \Theta^* (e^{-c\Theta^*} - e^{-c\Theta}) + S_k^* (\Theta^* e^{-c\Theta^*} - \Theta e^{-c\Theta}) (\Theta^* - \Theta)) + \frac{1}{2} \sigma_1^2 (k \Theta^*)^2 + \sigma_2^2 \Theta^{*2} \left(\frac{1}{ab} \right)^2 + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} \right] \\
 &\leq -(\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} - \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} \right] \\
 &\quad + \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[\frac{1}{2} \sigma_1^2 (k \Theta^*)^2 + \sigma_2^2 \Theta^{*2} \left(\frac{1}{ab} \right)^2 + \frac{1}{2} \sigma_1^2 k^2 S_k^{*2} \right].
 \end{aligned} \tag{46}$$

Based on Theorem 6 in [33], it can be obtain that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E} \left[\int_0^t \left((\gamma_1 + b) \frac{(\Theta - \Theta^*)^2}{\Theta} + \frac{1}{\langle k \rangle} \sum_{k=1}^n kp_k \left[(b + \lambda k \Theta^* e^{-c\Theta^*}) \frac{(S_k - S_k^*)^2}{S_k} \right] \right) ds \right] \leq \eta. \tag{47}$$

The proof of the theorem is complete. $\square$

Remark 1. From this result, the distance between the stochastic solution and the local disease equilibrium of the

corresponding deterministic system can be estimated in the time sense under the conditions of disease persistence. η is related to the strength of the two white noise environment. Θ corresponds to the size of the infected person. When an

endemic disease is formed, the scale of the final disease will fluctuate in the vicinity of the equilibrium point of the determining system under the influence of the environment. The greater the noise intensity, the more violent the fluctuation.

5. Extinction of a Disease

Finally, let us consider the case of $\alpha = 0$, which assumes that no undiagnosed infection can enter people. As people migrate, the disease will persist as long as there are undiagnosed infections. $\alpha = 0$ will be an ideal case before migration is controlled. In this case, system (5) can be written as follows:

$$\begin{cases} dS_k(t) = [b - \lambda k S_k(t) \Theta(t) e^{-c\Theta(t)} - b S_k(t)] dt - \sigma_1 k S_k(t) \Theta(t) e^{-c\Theta(t)} dB_1, \\ dE_k(t) = [\lambda k S_k(t) \Theta(t) e^{-c\Theta(t)} - (\mu + b + \gamma_1) E_k(t)] dt + \sigma_1 k S_k(t) \Theta(t) e^{-c\Theta(t)} dB_1 - \sigma_2 E_k(t) dB_2, \\ dI_k(t) = [\mu E_k(t) - (\gamma_2 + b) I_k(t)] dt + \sigma_2 E_k(t) dB_2, \\ dR_k(t) = (\gamma_1 E_k(t) + \gamma_2 I_k(t) - b R_k(t)) dt. \end{cases} \quad (48)$$

At this point, we can discuss the conditions for disease extinction in system (48). The following theorem will show the conditions for disease extinction.

Lemma 3 (see [34, 35]). *Let H be a given separable real-valued Hilbert space, $\{B(t): t \in \mathbb{R}_+\}$ be a canonical cylindrical Wiener process defined on H , $\delta > 1$, and $\{\tau_h\}$ and $\{\nu_h\}$ be two sequences of positive numbers. Then, for almost all $\omega \in \Omega$, there exists a random integer $h_0(\omega)$ such that, for all $h \geq h_0$,*

$$\int_0^t f(s) dB(s) \leq \frac{1}{2} \nu_h \int_0^t [f(s)]^2 ds + \frac{\delta \ln h}{\nu_h}, \quad 0 \leq t \leq \tau_h. \quad (49)$$

Theorem 4. *For any initial value $(S_k(0), E_k(0), I_k(0), R_k(0)) \in \{(S_k, E_k, I_k, R_k) \in \mathbb{R}_+^4 \mid 0 \leq S_k, E_k, I_k, R_k \leq 1\}$, $k = 1, \dots, n$, and $N_k(0) = 1$, the solution $E_k(t)$ of system (48) has the property that*

$$\limsup_{t \rightarrow +\infty} \frac{\ln E_k(t)}{t} \leq \frac{\lambda^2}{2\sigma_1^2} - (\mu + b + \gamma_1) - \frac{1}{2}\sigma_2^2, \quad \text{a.s.} \quad (50)$$

If $\lambda^2 - (2\mu + 2b + 2\gamma_1 + \sigma_2^2)\sigma_1^2 < 0$, then $E_k(t)$ tends to zero exponentially almost surely.

Proof. By using Itô's formula, we have

$$\ln E_k(t) = \ln E_k(0) + \int_0^t \left[-\frac{1}{2} \sigma_1^2 \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right)^2 + \lambda \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right) - (\mu + b + \gamma_1) - \frac{1}{2} \sigma_2^2 \right] ds + \int_0^t \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(s) - \int_0^t \sigma_2 dB_2(s). \quad (51)$$

Choosing $\delta > 1$, $\nu_h > 0$, and $\tau_h = h$ in Lemma 3, then for almost all $\omega \in \Omega$, there exists a random integer $h_0(\omega)$ such that, for all $h \geq h_0$,

$$\int_0^t \frac{\sigma_1 k S_k \Theta e^{-c\Theta}}{E_k} dB_1(s) \leq \frac{1}{2} \nu_h \int_0^t \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right)^2 ds + \frac{\delta \ln h}{\nu_h}, \quad 0 \leq t \leq h. \quad (52)$$

Substituting (52) into (51), we obtain that

$$\begin{aligned} \ln E_k(t) &\leq \ln E_k(0) \\ &+ \int_0^t \left[-\frac{1}{2} (1 - \nu_k) \sigma_1^2 \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right)^2 + \lambda \left(\frac{k S_k \Theta e^{-c\Theta}}{E_k} \right) - (\mu + b + \gamma_1) - \frac{1}{2} \sigma_2^2 \right] ds \\ &+ \frac{\delta \ln h}{\nu_h} - \int_0^t \sigma_2 dB_2(s). \end{aligned} \quad (53)$$

FIGURE 1: Under the initial conditions given in Example 1, the time evolution of the average density of each state in a complex network.

Noting that

$$-\frac{1}{2}(1-\nu_k)\sigma_1^2\left(\frac{kS_k\Theta e^{-c\Theta}}{E_k}\right)^2 + \lambda\left(\frac{kS_k\Theta e^{-c\Theta}}{E_k}\right) \leq \frac{\lambda^2}{2(1-\nu_h)\sigma_1^2}, \quad (54)$$

it follows that, for all $t \in [0, h]$ and $h > h_0$,

$$\ln E_k(t) \leq \ln E_k(0) + \int_0^t \left[\frac{\lambda^2}{2(1-\nu_h)\sigma_1^2} - (\mu + b + \gamma_1) - \frac{1}{2}\sigma_2^2 \right] ds + \frac{\delta \ln h}{\nu_h} - \int_0^t \sigma_2 dB_2(s), \quad (55)$$

dividing both sides by t ; let $h \rightarrow +\infty$ and $t \rightarrow \infty$, and applying the strong law of large numbers, we have

$$\limsup_{t \rightarrow +\infty} \frac{\ln E_k(t)}{t} \leq \frac{\lambda^2}{2(1-\nu_h)\sigma_1^2} - (\mu + b + \gamma_1) - \frac{1}{2}\sigma_2^2, \quad \text{a.s.} \quad (56)$$

Finally, by sending $\nu_h \rightarrow 0$, we obtain the desired inequality of Theorem 4. $\square$

6. Numerical Simulations

In this section, we will assume that all parameters are given in appropriate units.

Example 1. Consider system (48) with initial condition $S_k(0) = 0.7$, $E_k(0) = 0.2$, $I_k(0) = 0.1$, and $R_k(0) = 0$ under different system parameters:

FIGURE 2: Under the initial conditions given in Example 2, the time evolution of each state mean in the complex network in the stochastic model and the deterministic model, respectively.

$$\begin{aligned}
 \alpha(t) &= 0, \\
 \mu &= 0.3, \\
 b &= 0.05, \\
 \lambda &= 0.1, \\
 \gamma_2 &= 0.2, \\
 c &= 1, \\
 \gamma_1 &= 0.25, \\
 n &= 50, \\
 \sigma_1 &= 0.2, \\
 \sigma_2 &= 0.2.
 \end{aligned} \tag{57}$$

Based on Theorem 4, we can conclude that the solution of (48) obeys

$$\limsup_{t \rightarrow +\infty} \frac{\ln E_k(t)}{t} \leq -0.0396, \quad \text{a.s.} \tag{58}$$

The computer simulation in Figure 1 supports this result clearly, illustrating the extinction of the disease.

Example 2. Consider system (48) with initial condition $S_k(0) = 0.6$, $E_k(0) = 0.2$, $I_k(0) = 0.2$, and $R_k(0) = 0$ under different system parameters:

FIGURE 3: Under the initial conditions given in Example 3, the time evolution of each state mean in the stochastic model with $c = 1, 5, 15$, respectively.

$$\begin{aligned}
 \alpha(t) &= 0.3, \\
 \mu &= 0.3, \\
 b &= 0.05, \\
 \lambda &= 0.1, \\
 \gamma_2 &= 0.2, \\
 c &= 1, \\
 \gamma_1 &= 0.25, \\
 n &= 50, \\
 \sigma_1 &= 0.25, \\
 \sigma_2 &= 0.2.
 \end{aligned} \tag{59}$$

Figure 2 shows the time plots of each state mean in the complex network in the stochastic model and the deterministic model, respectively.

Example 3. Consider system (48) with initial condition $S_k(0) = 0.6, E_k(0) = 0.2, I_k(0) = 0.2$, and $R_k(0) = 0$ under different system parameters:

$$\begin{aligned}
 \alpha(t) &= 0.3, \\
 \mu &= 0.1, \\
 b &= 0.1, \\
 \lambda &= 0.2,
 \end{aligned}$$

$$\begin{aligned}
 \gamma_2 &= 0.2, \\
 \gamma_1 &= 0.2, \\
 n &= 50, \\
 \sigma_1 &= 0.1, \\
 \sigma_2 &= 0.1.
 \end{aligned}
 \tag{60}$$

Figure 3 shows the time plots of each state mean in the stochastic model with different values of c , respectively. The results show that as the value of c increases, the scale of the disease decreases.

7. Conclusion

When infectious diseases are prevalent, healthy people, especially susceptible people, will be alert to the disease and thus reduce their contact with such people with disease or suspected disease. We take this property into account in the model. Moreover, in the real world, there are various random factors that have a significant impact on the rate of infection and the rate of diagnosis of patients with the disease, so it is necessary to consider this randomness in the model. In this paper, we have considered a stochastic SEIR epidemic model with the external variability in the transmission rate λ and the conversion rate μ . Combining analytical results with numerical simulations, we discussed the effects of these environmental noises on the transmission dynamics of epidemics. When outsiders enter the network, if they have undiagnosed infections, the disease persists. If there is no infected person in the alien population, then when $\lambda^2 - (2\mu + 2b + 2\gamma_1 + \sigma_2^2)\sigma_1^2 < 0$, the disease dies out almost surely. Through numerical simulation, we found that the awareness introduced in the model cannot affect the extinction of the disease, but the scale of the disease will eventually decrease as the increase of human awareness, which may contribute to the control of the disease.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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