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Research Article

Stability Analysis of a Reaction-Diffusion Heroin Epidemic Model

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A reaction-diffusion (R-D) heroin epidemic model with relapse and permanent immunization is formulated. We use the basic reproduction number R_0 to determine the global dynamics of the models. For both the ordinary differential equation (ODE) model and the R-D model, it is shown that the drug-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and if $R_0 > 1$, the drug-addiction equilibrium is globally asymptotically stable. Some numerical simulations are also carried out to illustrate our analytical results.

1. Introduction

Heroin comes from opioids, commonly known as "opium," derived from the poppy plant. Pure heroin is a white powdery substance or a white crystalline powder. It is well known that long-term consumption and injection of heroin can cause personality disintegration, psychological metamorphosis, and lifespan to be reduced. There are more and more registered drug users, and this number is growing continuously [1, 2]. Heroin abuse and independence has been bringing tremendous pressures on the social and public health systems due to its prevalence all over the world [3, 4].

Since the spread of heroin is as contagious as infectious diseases, it is a new trend to study heroin transmission from the perspective of infectious disease dynamics [5–16]. In 2007, White and Comiskey established an ODE model for heroin infectious diseases [14]. They studied the dynamics using a threshold R_0 and showed that prevention is better than treatment. In 2009, this model was reconsidered by Mulone and Straughan [11], and the stability of the positive equilibrium point of the model is obtained by the authors by using the eigenvalue equation and Poincare–Bendixson theory. In 2011, Wang et al. [15] used the bilinear law incidence function instead of standard incidence, and they also analyzed the dynamic behavior of the heroin model. In order to better study the dynamics of heroin infectious diseases, many other different epidemic models have been formulated and studied in various ways [5–10, 12, 13, 16].

As known to all that many people who are detoxifying are not really willing to go, and lots of people still keep in touch with friends who use drugs after they have a successful drug treatment, thus they have a high probability of reusing drugs. Hence, some scholars considered relapse in the drug model and analyzed its stability [12, 17]. However, there are some drug users who experienced the harmful effect of drugs and realized the happiness of being an ordinary person after they have a successful drug treatment, such persons will be far away from drugs, so we are optimistic that they will continue this good habit for a lifetime. In addition, there are some people who have been well educated since childhood and have a healthy living environment and strong willpower. So, they do not take drugs from the start to the finish. We call these two types of people permanently immunized against drugs.

In recent years, it has been well recognized that spatial diffusion and environmental heterogeneity have important effects on the persistence and extinction of contagious diseases [18–38]. In the case of the heroin epidemic model, the spatial distribution of the susceptible or infected person is uneven, and the density may change at any time and place, thus it is more reasonable to use the R-D equations to describe the spread of drug abusers. Moreover, to the best of

our knowledge, the heroin infectious disease models in which population density depends on both time and space variables are rarely studied.

In this work, we first formulate an R-D heroin model with relapse and permanent immunization, and then study its global dynamics. We organize this paper as follows: in Section 2, the model is derived and the positive property of the solutions for the model is proved. In Section 3, the model without diffusion is analyzed, and we obtain the basic reproduction number and show the stability of all the equilibria. In Section 4, the stability of the R-D model is obtained. In Section 5, we illustrate our results by some numerical simulations. Finally, we finish this paper with a concluding discussion.

2. The Model

2.1. Model Formulation. We divide the total population into five compartments: S, U_1, U_2, Q, and R. Here, S represents the number of susceptible individuals who have never used heroin; U_1 represents the number of heroin users; U_2 represents the number of heroin users undergoing treatment; Q represents the number of people who have used drugs and are not taking drugs at this stage, but may be taking drugs in the future; and R represents the number of people who never use drugs or these successful detoxification people do not take drugs anymore. We assume that drug users are not able to heal themselves through self-control, and if they want to abstain from drugs, they have to go through treatment. We also assume that not all people can be cured completely. If the drug users are successfully cured, individuals in compartment U_2 will enter into the compartment of Q. If the treatment is terminated or failure, the people who failed the treatment will still take drugs. Some of these successful detoxification people will redrug because they cannot resist the temptation of drugs, and some will never take drugs because they know the harm of drug abuse. Thus, the total population is given by

$$N = S + U_1 + U_2 + Q + R. \tag{1}$$

We give the transfer diagram of the model in Figure 1. According to Figure 1, we obtain the following R-D heroin epidemic model with relapse and permanent immunization:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S + \Lambda - \beta_1 S U_1 - (\mu + \alpha_1) S, & l \in \Omega,\ t > 0, \\ \dfrac{\partial U_1}{\partial t} = d_2 \Delta U_1 + \beta_1 S U_1 + k_2 Q + \beta_2 U_2 - (\mu + \delta_1 + k) U_1, & l \in \Omega,\ t > 0, \\ \dfrac{\partial U_2}{\partial t} = d_3 \Delta U_2 + k U_1 - (\mu + \delta_2 + k_1 + \beta_2) U_2, & l \in \Omega,\ t > 0, \\ \dfrac{\partial Q}{\partial t} = d_4 \Delta Q + k_1 U_2 - (\mu + k_2 + \alpha_2) Q, & l \in \Omega,\ t > 0, \\ \dfrac{\partial R}{\partial t} = \alpha_1 S + \alpha_2 Q - \mu R l \in \Omega, & t > 0, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial U_1}{\partial n} = \dfrac{\partial U_2}{\partial n} = \dfrac{\partial Q}{\partial n} = \dfrac{\partial R}{\partial n} = 0, & l \in \Omega,\ t > 0, \\ S(l, 0) = S_0(l) > 0, \\ U_1(l, 0) = U_{10}(l) > 0, \\ U_2(l, 0) = U_{20}(l) > 0, \\ Q(l, 0) = Q_0(l) > 0, \\ R(l, 0) = R_0(l) > 0, \\ l \in \Omega. \end{cases} \tag{2}$$

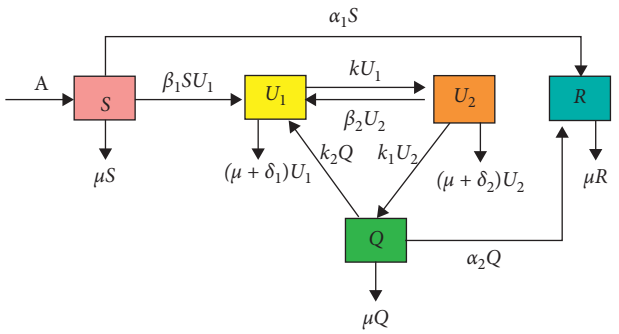


Figure 1: Transfer diagram of the heroin epidemic model with relapse and permanent immunization.

We assume that the bounded domain $\Omega \subset \mathbb{R}^5$ has a smooth boundary $\partial\Omega$. As shown in (2), the homogeneous Neumann boundary conditions indicate that the population movements will not cross the border. Δ is the usual Laplacian operator on $\mathbb{R}^5$. $d_i > 0$ $(i = 1, 2, 3, 4)$ are the diffusion coefficients. As mentioned above, R is a compartment of complete rehabilitation. The individuals in R are permanently immunized, and we can see that the R equation is uncoupled with the other equations of (2). Hence, the diffusion of R is not considered. We assume that all parameters in the model are positive constants, and the meaning of the parameters is described in Table 1.

To investigate the global dynamic behavior of system (2), we first study its ODE counterpart version as follows:

$$\begin{cases} \dot{S} = \Lambda - \beta_1 S U_1 - (\mu + \alpha_1)S, \\ \dot{U}_1 = \beta_1 S U_1 + k_2 Q + \beta_2 U_2 - (\mu + \delta_1 + k)U_1, \\ \dot{U}_2 = kU_1 - (\mu + \delta_2 + k_1 + \beta_2)U_2, \\ \dot{Q} = k_1 U_2 - (\mu + k_2 + \alpha_2)Q, \\ \dot{R} = \alpha_1 S + \alpha_2 Q - \mu R. \end{cases} \tag{3}$$

2.2. The Basic Properties of Model (3)

Lemma 1. *If the initial values $S(0), U_1(0), U_2(0)$, $Q(0)$, and $R(0)$ are positive, then model (3) has positive solutions $S(t)$, $U_1(t)$, $U_2(t)$, $Q(t)$, and $R(t)$ for all $t > 0$.*

Since the proof of the above lemma is direct, we omit it.

Lemma 2. *All feasible solutions of the model (3) are bounded and enter the following region:*

$$\Omega = \left\{ (S(t), U_1(t), U_2(t), Q(t), R(t)) \in R_+^5 \,\middle|\, 0 \le S + U_1 + U_2 + Q + R \le \frac{\Lambda}{\mu} \right\}. \tag{4}$$

Proof. If (S, U_1, U_2, Q, R) is a solution of model (3) with nonnegative initial conditions, adding the five equations yields

$$\begin{aligned} \dot{S} + \dot{U}_1 + \dot{U}_2 + \dot{Q} + \dot{R} &= \Lambda - \mu S - \mu U_1 - \mu U_2 - \mu Q - \mu R - \delta_1 U_1 - \delta_2 U_2 \\ &= \Lambda - \mu(S + U_1 + U_2 + Q + R) - (\delta_1 U_1 + \delta_2 U_2) \\ &\le \Lambda - \mu(S + U_1 + U_2 + Q + R) \\ &= \Lambda - \mu N(t), \end{aligned} \tag{5}$$

where

$$N(t) = S(t) + U_1(t) + U_2(t) + Q(t) + R(t), \tag{6}$$

which indicates that

$$0 \le N(t) \le \frac{\Lambda}{\mu} + N(0)e^{-\mu t}, \tag{7}$$

where $N(0)$ is the initial value. Thus, $0 \le N(t) \le (\Lambda/\mu)$, as $t \longrightarrow \infty$. This completes the proof. □

3. Global Dynamics of the ODE Model (3)

3.1. The Basic Reproduction Number and Stability of Drug-Free Equilibrium. It is easy to get the drug-free equilibrium of system (3):

$$E_0 = \left(\frac{\Lambda}{\mu + \alpha_1}, 0, 0, 0, \frac{\alpha_1 \Lambda}{\mu(\mu + \alpha_1)} \right). \tag{8}$$

We now use the next-generation matrix method formulated in [41] to derive the basic reproduction number R_0 of model (3).

Let $l = (U_1, U_2, Q, S, R)^T$, then system (3) can be written as

$$\frac{dl}{dt} = P(l) - W(l), \tag{9}$$

where

Table 1: Description of parameters.

Parameter	Description	Data estimated	Data sources
$S(l,t)$	Number of susceptible people at location l and time t		
$U_1(l,t)$	Number of heroin users at location l and time t		
$U_2(l,t)$	Number of heroin users undergoing treatment at location l and time t		
$Q(l,t)$	Number of people who have used drugs at location l and time t		
$R(l,t)$	Number of people who never use drugs at location l and time t		
Λ	Recruitment rate of the population	1	[39]
μ	Natural death rate	0.02	[40]
β_1	Addition rate from S to abusers	Variable	
β_2	The proportion of failure treatment	0.0011	[40]
δ_1	The heroin-related death rate of U_1	0.01	Estimate
δ_2	The heroin-related death rate of being treated	0.005	Estimate
k	Progression rate to U_2 from U_1	0.0095	[40]
k_1	The proportion of successful treatment	Variable	
k_2	Addition rate from Q to abusers	Variable	
α_1	The permanent withdrawal rate from S to R	Variable	Estimate
α_2	The permanent withdrawal rate from Q to R	0.0001	Estimate

$$P(l) = \begin{pmatrix} \beta_1 S U_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
$$W(l) = \begin{pmatrix} (\mu+\delta_1+k)U_1 - k_2 Q - \beta_2 U_2 \\ (\mu+\delta_2+k_1+\beta_2)U_2 - kU_1 \\ (\mu+k_2+\alpha_2)Q - k_1 U_2 \\ (\mu+\alpha_1)S + \beta_1 S U_1 - \Lambda \\ \mu R - \alpha_1 S - \alpha_2 Q \end{pmatrix}. \tag{10}$$

The Jacobian matrices of $P(l)$ and $W(l)$ at the drug-free equilibrium E_0 are as follows:

$$DP(E_0) = \begin{pmatrix} P_{3\times3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$DW(E_0) = \begin{pmatrix} & W_{3\times3} & & 0 & 0 \\ \beta_1 \dfrac{\Lambda}{\mu+\alpha_1} & 0 & 0 & \mu+\alpha_1 & 0 \\ 0 & 0 & -\alpha_2 & -\alpha_1 & \mu \end{pmatrix}, \tag{11}$$

where

$$P_{3\times3} = \begin{pmatrix} \beta_1 \dfrac{\Lambda}{\mu+\alpha_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$W_{3\times3} = \begin{pmatrix} \mu+\delta_1+k & -\beta_2 & -k_2 \\ -k & \mu+\delta_2+k_1+\beta_2 & 0 \\ 0 & -k_1 & \mu+k_2+\alpha_2 \end{pmatrix}. \tag{12}$$

Then, the basic reproduction number R_0 is

$$R_0 = \rho\left(PW^{-1}\right) = \frac{\beta_1 \Lambda (\mu+\delta_2+k_1+\beta_2)(\mu+k_2+\alpha_2)}{(\mu+\alpha_1)[(\mu+\delta_1+k)(\mu+\delta_2+k_1+\beta_2)(\mu+k_2+\alpha_2) - k\beta_2(\mu+k_2+\alpha_2) - kk_1k_2]}. \tag{13}$$

The following result on the local stability of E_0 can be obtained directly by Theorem 2 in [41], and we thus omit its proof.

Theorem 1. *The drug-free equilibrium E_0 is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$.*

Theorem 2. *If $R_0 \le 1$, the drug-free equilibrium E_0 of system (3) is globally asymptotically stable.*

Proof. Inspired by [28], we introduce the following Lyapunov function:

$$V = S - S_0 - S_0 \ln \frac{S}{S_0} + U_1 + U_2 + Q. \tag{14}$$

It follows that the derivative of V is

$$\begin{aligned}
\dot{V} &= \dot{S} + \dot{U}_1 + \dot{U}_2 + \dot{Q} - \frac{S_0}{S}\dot{S} \\
&= S_0(\mu + \alpha_1) - S(\mu + \alpha_1) - (\mu + \delta_2)U_2 - (\mu + \alpha_2)Q - \frac{S_0}{S}S_0(\mu + \alpha_1) - \beta_1 S_0 U_1 + (\mu + \alpha_1)S_0 \\
&= -(\mu + \alpha_1)S_0\left(\frac{S_0}{S} + \frac{S}{S_0} - 2\right) - (\mu + \delta_2)U_2 - (\mu + \alpha_2)Q - (\mu + \delta_1)\left[1 - \frac{\beta_1\Lambda}{(\mu + \alpha_1)(\mu + \delta_1)}\right]U_1 \\
&\le -(\mu + \alpha_1)S_0\left(\frac{S_0}{S} + \frac{S}{S_0} - 2\right) - (\mu + \delta_2)U_2 - (\mu + \alpha_2)Q - (\mu + \delta_1)(1 - R_0)\Big]U_1.
\end{aligned} \tag{15}$$

Here, we sued the equalities $\Lambda = S_0(\mu + \alpha_1)$. Since $(S_0/S) + (S/S_0) - 2 \ge 0$, if $R_0 \le 1$, we have $\dot{V} \le 0$. Clearly, $\dot{V} \le 0$ if and only if $S = S_0$ and $U_1 = U_2 = Q = 0$. Substituting $S = S_0$ and $U_1 = U_2 = Q = 0$ into system (3), we get $R \longrightarrow (\alpha_1\Lambda/\mu(\mu + \alpha_1))$ as $t \longrightarrow \infty$. According to LaSalle's invariance principle [42], we obtain that the drug-free equilibrium E_0 of system (3) is globally asymptotically stable in Ω if $R_0 \le 1$. This completes the proof. □

3.2. Existence and Stability of the Drug-Addiction Equilibrium. Let the right side of model (3) be equal to zero. Then we get

$$\begin{cases}
\Lambda - \beta_1 S^* U_1^* - (\mu + \alpha_1)S^* = 0, \\
\beta_1 S^* U_1^* + k_2 Q^* + \beta_2 U_2^* - (\mu + \delta_1 + k)U_1^* = 0, \\
kU_1^* - (\mu + \delta_2 + k_1 + \beta_2)U_2^* = 0, \\
k_1 U_2^* - (\mu + k_2 + \alpha_2)Q^* = 0, \\
\alpha_1 S^* + \alpha_2 Q^* - \mu R^* = 0.
\end{cases} \tag{16}$$

Then, the unique drug-addiction equilibrium $E^* = (S^*, U_1^*, U_2^*, Q^*, R^*)$ exists if $R_0 > 1$, where

$$\begin{aligned}
S^* &= \frac{\Lambda}{\beta_1 U_1^* + \mu + \alpha_1}, \\
U_1^* &= \frac{(\mu + \alpha_1)(R_0 - 1)}{\beta_1}, \\
U_2^* &= \frac{kU_1^*}{\mu + \delta_2 + k_1 + \beta_2}, \\
Q^* &= \frac{k_1 k U_1^*}{(\mu + k_2 + \alpha_2)(\mu + \delta_2 + k_1 + \beta_2)}, \\
R^* &= \frac{\alpha_1\Lambda}{\mu(\beta_1 U_1^* + \mu + \alpha_1)} + \frac{\alpha_2 k_1 k U_1^*}{\mu(\mu + k_2 + \alpha_2)(\mu + \delta_2 + k_1 + \beta_2)}.
\end{aligned} \tag{17}$$

Theorem 3. *If $R_0 > 1$, then the unique drug-addiction equilibrium E^* of system (3) is global asymptotically stable.*

Proof. If $R_0 > 1$, then there exists a unique drug-addiction equilibrium E^*. We now introduce a Lyapunov function V as follows:

$$V = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + A\left(U_1 - U_1^* - U_1^* \ln \frac{U_1}{U_1^*}\right) + B\left(U_2 - U_2^* - U_2^* \ln \frac{U_2}{U_2^*}\right) + C\left(Q - Q^* - Q^* \ln \frac{Q}{Q^*}\right), \tag{18}$$

where A, B, and C are the positive constants to be determined later. It follows that the derivative of V is

$$\begin{aligned}
\dot{V} &= \dot{S}\left(1 - \frac{S^*}{S}\right) + A\dot{U}_1\left(1 - \frac{U_1^*}{U_1}\right) + B\dot{U}_2\left(1 - \frac{U_2^*}{U_2}\right) + C\dot{Q}\left(1 - \frac{Q^*}{Q}\right) \\
&= \left(1 - \frac{S^*}{S}\right)[\Lambda - \beta_1 S U_1 - (\mu + \alpha_1)S] + A\left(1 - \frac{U_1^*}{U_1}\right)[\beta_1 S U_1 + k_2 Q + \beta_2 U_2 - (\mu + \delta_1 + k)U_1] \\
&\quad + B\left(1 - \frac{U_2^*}{U_2}\right)[kU_1 - (\mu + \delta_2 + k_1 + \beta_2)U_2] + C\left(1 - \frac{Q^*}{Q}\right)[k_1 U_2 - (\mu + k_2 + \alpha_2)Q] \\
&= \left(1 - \frac{S^*}{S}\right)[\beta_1 S^* U_1^* + (\mu + \alpha_1)S^* - \beta_1 S U_1 - (\mu + \alpha_1)S] \\
&\quad + A\left(1 - \frac{U_1^*}{U_1}\right)\left(\beta_1 S U_1 + k_2 Q + \beta_2 U_2 - \frac{\beta_1 S^* U_1^* + k_2 Q^* + \beta_2 U_2^*}{U_1^*} U_1\right) \\
&\quad + B\left(1 - \frac{U_2^*}{U_2}\right)\left(kU_1 - \frac{kU_1^*}{U_2^*} U_2\right) + C\left(1 - \frac{Q^*}{Q}\right)\left(k_1 U_2 - \frac{k_1 U_2^*}{Q^*} Q\right).
\end{aligned} \tag{19}$$

Let $(S/S^*) = x$, $(U_1/U_1^*) = y$, $(U_2/U_2^*) = z$, and $(Q/Q^*) = u$. Then, we have

$$\begin{aligned}
\dot{V} &= \left(1 - \frac{1}{x}\right)[\beta_1 S^* U_1^* + (\mu + \alpha_1)S^* - \beta_1 S^* U_1^* xy - (\mu + \alpha_1)S^* x] \\
&\quad + A\left(1 - \frac{1}{y}\right)(\beta_1 S^* U_1^* xy + k_2 Q^* u + \beta_2 U_2^* z - \beta_1 S^* U_1^* y - k_2 Q^* y - \beta_2 U_2^* y) \\
&\quad + B\left(1 - \frac{1}{z}\right)(kU_1^* y - kzU_1^*) + C\left(1 - \frac{1}{u}\right)(k_1 U_2^* z - k_1 U_2^* u) \\
&= -(\mu + \alpha_1)S^* \frac{(1 - x)^2}{x} + \beta_1 S^* U_1^*\left(1 - xy - \frac{1}{x} + y\right) \\
&\quad + A\beta_1 S^* U_1^*(xy - y - x + 1) + Ak_2 Q^*\left(u - y - \frac{u}{y} + 1\right) \\
&\quad + A\beta_2 U_2^*\left(z - y - \frac{z}{y} + 1\right) + BkU_1^*\left(y - z - \frac{y}{z} + 1\right) + Ck_1 U_2^*\left(z - u - \frac{z}{u} + 1\right) \\
&= -(\mu + \alpha_1)S^* \frac{(1 - x)^2}{x} + (\beta_1 S^* U_1^* + A\beta_1 S^* U_1^* + Ak_2 Q^* + A\beta_2 U_2^* + BkU_1^* + Ck_1 U_2^*) \\
&\quad + xy(-\beta_1 S^* U_1^* + A\beta_1 S^* U_1^*) + y(\beta_1 S^* U_1^* - A\beta_1 S^* U_1^* - Ak_2 Q^* - A\beta_2 U_2^* + BkU_1^*) \\
&\quad + u(Ak_2 Q^* - Ck_1 U_2^*) + z(A\beta_2 U_2^* - BkU_1^* + Ck_1 U_2^*) - xA\beta_1 S^* U_1^* \\
&\quad - \frac{u}{y} Ak_2 Q^* - \frac{z}{y} A\beta_2 U_2^* - \frac{y}{z} BkU_1^* - \frac{z}{u} Ck_1 U_2^* - \frac{1}{x}\beta_1 S^* U_1^*.
\end{aligned} \tag{20}$$

The variables with nonnegative coefficients in (20) are xy, y, u, and z. If all the coefficients are positive, then $\dot{V}$ is positive. If all the coefficients of xy, y, u, and z are equal to zero, then we get

$$\begin{aligned} A\beta_1 S^* U_1^* - \beta_1 S^* U_1^* &= 0, \\ \beta_1 S^* U_1^* - A\beta_1 S^* U_1^* - Ak_2 Q^* - A\beta_2 U_2^* + BkU_1^* &= 0, \\ Ak_2 Q^* - Ck_1 U_2^* &= 0, \\ A\beta_2 U_2^* - BkU_1^* + Ck_1 U_2^* &= 0. \end{aligned} \tag{21}$$

By (21), we obtain

$$\begin{aligned} A &= 1, \\ B &= \frac{\beta_2 U_2^* + k_2 Q^*}{kU_1^*}, \\ C &= \frac{k_2 Q^*}{k_1 U_2^*}. \end{aligned} \tag{22}$$

Hence, we have

$$\begin{aligned} \dot{V} = &-(\mu + \alpha_1) S^* \frac{(1-x)^2}{x} + \beta_1 S^* U_1^* \left(2 - x - \frac{1}{x}\right) \\ &+ k_2 Q^* \left(3 - \frac{u}{y} - \frac{y}{z} - \frac{z}{u}\right) + \beta_2 U_2^* \left(2 - \frac{z}{y} - \frac{y}{z}\right). \end{aligned} \tag{23}$$

It is clear that $-(\mu + \alpha_1) S^* ((1-x)^2/x) \le 0$ if $x > 0$ and $-(\mu + \alpha_1) S^* ((1-x)^2/x) = 0$ if only if $x = 1$. By the relationships between the arithmetic mean and the geometric mean, we get $2 - x - (1/x) \le 0$ if $x > 0$ and $2 - x - (1/x) = 0$ if and only if $x = 1$; $3 - (u/y) - (y/z) - (z/u) \le 0$ for $x > 0, y > 0, u > 0$ and $3 - (u/y) - (y/z) - (z/u) = 0$ if and only if $y = z = u$; $2 - (z/y) - (y/z) \le 0$ for $z > 0, y > 0$ and $2 - (z/y) - (y/z) = 0$ if and only if $z = y$. Therefore, $\dot{V} \le 0$ if $x, y, z, u > 0$ and $\dot{V} = 0$ if and only if $x = 1$ and $y = z = u$. Substituting $S = S^*$ and $(U_1/U_1^*) = (U_2/U_2^*) = (Q/Q^*)$ into the first equation of system (3), we get $0 = \Lambda - \beta_1 S^* U_1 - (\mu + \alpha_1) S^*$. It follows from the first equation of (16) that $U_1 = U_1^*$. Therefore, the maximum invariant set of system (2) on set $\{(x, y, z, u): \dot{V} = 0\}$ is the singleton $(1, 1, 1, 1)$. This implies that the largest invariant set where $\dot{V} = 0$ is the singleton $\{(S^*, U_1^*, U_2^*, Q^*, R^*)\}$. Thus, by LaSalle's invariance principle in [42], the drug-addiction equilibrium E^* of model (3) is globally asymptotically stable when $R_0 > 1$. This completes the proof. □

4. Global Dynamics of the R-D Model (2)

4.1. Positivity and Boundedness of the Solutions

Theorem 4. *Let $(S(l,t), U_1(l,t), U_2(l,t), Q(l,t), R(l,t))$ be a solution of system (2) and $S(l,t), U_1(l,t), U_2(l,t), Q(l,t), R(l,t) \in \mathbf{C}(\overline{\Omega} \times [0,T)) \cap \mathbf{C}^{2,1}(\Omega \times [0,T))$, where T is the maximal existing time. If $S(l,0) > 0, U_1(l,0) > 0, U_2(l,0) > 0$, $Q(l,0) > 0$, and $R(l,0) > 0$, then $S(l,t) > 0$, $U_1(l,t) > 0, U_2(l,t) > 0, Q(l,t) > 0$, and $R(l,t) > 0$, for all $(l,t) \in \Omega \times [0,T)$.*

Proof. By the first equation of model (2), we get

$$\frac{\partial S}{\partial t} \ge d_1 \Delta S - \beta_1 S U_1 - (\mu + \alpha_1) S, \tag{24}$$

that is,

$$\frac{\partial S}{\partial t} - d_1 \Delta S + (\beta_1 U_1 + \mu + \alpha_1) S \ge 0. \tag{25}$$

Since $S(l,0) > 0 \equiv 0$, by Lemma 2.4.1 in [33], we get $S(l,t) > 0$.

Let

$$\begin{aligned} g_{U_1} &= \beta_1 S U_1 + k_2 Q + \beta_2 U_2 - (\mu + \delta_1 + k) U_1, \\ g_{U_2} &= kU_1 - (\mu + \delta_2 + k_1 + \beta_2) U_2, \\ g_Q &= k_1 U_2 - (\mu + k_2 + \alpha_2) Q. \end{aligned} \tag{26}$$

Then, the second to the fourth equations of system (2) can be rewritten as

$$\begin{aligned} &\frac{\partial U_1}{\partial t} - d_2 \Delta U_1 - \left[g_{U_1}(l,t,U_1,U_2,Q) - g_{U_1}(l,t,0,U_2,Q) + g_{U_1}(l,t,0,U_2,Q) - g_{U_1}(l,t,0,0,Q) + g_{U_1}(l,t,0,0,Q) - g_{U_1}(l,t,0,0,0)\right] \\ &= \frac{\partial U_1}{\partial t} - d_2 \Delta U_1 - \left[(\beta_1 S - \mu - \delta_1 - k) U_1 + \beta_2 U_2 + k_2 Q\right] = 0, \\ &\frac{\partial U_2}{\partial t} - d_3 \Delta U_2 - \left[g_{U_2}(l,t,U_1,U_2,Q) - g_{U_2}(l,t,0,U_2,Q) + g_{U_2}(l,t,0,U_2,Q) - g_{U_2}(l,t,0,0,Q) + g_{U_2}(l,t,0,0,Q) - g_{U_2}(l,t,0,0,0)\right] \\ &= \frac{\partial U_2}{\partial t} - d_3 \Delta U_2 - \left[(kU_1 - (\mu + \delta_2 + k_1 + \beta_2) U_2\right] = 0, \\ &\frac{\partial Q}{\partial t} - d_4 \Delta Q - \left[g_Q(l,t,U_1,U_2,Q) - g_Q(l,t,0,U_2,Q) + g_Q(l,t,0,U_2,Q) - g_Q(l,t,0,0,Q) + g_Q(l,t,0,0,Q) - g_Q(l,t,0,0,0)\right] \\ &= \frac{\partial Q}{\partial t} - d_4 \Delta Q - \left[(k_1 U_2 - (\mu + k_2 + \alpha_2) Q\right] \\ &= 0. \end{aligned} \tag{27}$$

Define operator $\mathscr{L}_i$ $(i = 1, 2, 3)$ on $\Omega \times [0, T)$ as follows:

$$\begin{aligned}\mathscr{L}_1 U_1 &= \frac{\partial U_1}{\partial t} - d_2 \Delta U_1,\\ \mathscr{L}_2 U_2 &= \frac{\partial U_2}{\partial t} - d_3 \Delta U_2,\\ \mathscr{L}_3 Q &= \frac{\partial Q}{\partial t} - d_4 \Delta Q.\end{aligned} \tag{28}$$

We now consider the following parabolic system:

$$\begin{cases} \mathscr{L}_1 U_1 + h_{11} U_1 + h_{12} U_2 + h_{13} Q = 0,\\ \mathscr{L}_2 U_2 + h_{21} U_1 + h_{22} U_2 + h_{23} Q = 0,\\ \mathscr{L}_3 Q + h_{31} U_1 + h_{32} U_2 + h_{33} Q = 0,\\ U_1(l, 0) > 0,\\ U_2(l, 0) > 0,\\ Q(l, 0) > 0,\\ \dfrac{\partial U_1}{\partial t} = \dfrac{\partial U_2}{\partial t} = \dfrac{\partial Q}{\partial t} = 0, \end{cases} \tag{29}$$

where

$$\begin{aligned} h_{11} &= -(\beta_1 S - \mu - \delta_1 - k),\\ h_{12} &= -\beta_2,\\ h_{13} &= -k_2, h_{21} = -k,\\ h_{22} &= -(\mu + \delta_2 + k_1 + \beta_2),\\ h_{23} &= 0,\\ h_{31} &= 0,\\ h_{32} &= -k_1,\\ h_{33} &= \mu + k_2 + \alpha_2. \end{aligned} \tag{30}$$

Applying Theorem 4.2.4 in [33] to model (29), we get $U_1(l, t) \geq 0, U_2(l, t) \geq 0$, and $Q(l, t) \geq 0$. So, by the second equation of model (2), we get

$$\frac{\partial U_1}{\partial t} \geq d_2 \Delta U_1 - (\mu + \delta_1 + k) U_1. \tag{31}$$

That is,

$$\frac{\partial U_1}{\partial t} - d_2 \Delta U_1 + (\mu + \delta_1 + k) U_1 \geq 0. \tag{32}$$

Since $U_1(l, 0) > 0 \equiv 0$, according to Lemma 2.4.1 in [33], $U_1(l, t) > 0$. Analogously,

$$\begin{aligned} \frac{\partial U_2}{\partial t} &\geq d_3 \Delta U_2 - (\mu + \delta_2 + k_1 + \beta_2) U_2,\\ \frac{\partial Q}{\partial t} &\geq d_4 \Delta Q - (\mu + k_2 + \alpha_2) Q. \end{aligned} \tag{33}$$

That is,

$$\begin{aligned} \frac{\partial U_2}{\partial t} - d_3 \Delta U_2 + (\mu + \delta_2 + k_1 + \beta_2) U_2 &\geq 0,\\ \frac{\partial Q}{\partial t} - d_4 \Delta Q + (\mu + k_2 + \alpha_2) Q &\geq 0. \end{aligned} \tag{34}$$

Similarly, we obtain $U_2(l, t) > 0$ and $Q(l, t) > 0$. If $R(l, t) > 0$ does not hold, then there exist $l_1 \in \Omega, t_1 \in [0, T)$ such that $R(l_1, t_1) = 0, \dot{R}(l_1, t_1) \leq 0$. By the fifth equation of model (2), we get

$$\dot{R}(l_1, t_1) = \alpha_1 S(l_1, t_1) + \alpha_2 Q(l_1, t_1). \tag{35}$$

Since $\forall l \in \Omega, t \in [0, T), S(l, t) > 0, Q(l, t) > 0$, $\dot{R}(l_1, t_1) = \alpha_1 S(l_1, t_1) + \alpha_2 Q(l_1, t_1) > 0$. This is a contradiction. Therefore, $R(l, t) > 0$ for all $l \in \Omega, t \in [0, T)$. This completes the proof. □

Theorem 5. *Let* $(S(l, t), U_1(l, t), U_2(l, t), Q(l, t), R(l, t))$ *be a solution of model (2) and* $S(l, t), U_1(l, t), U_2(l, t), Q(l, t), R(l, t) \in \mathbf{C}(\overline{\Omega} \times [0, T)) \cap \mathbf{C}^{2,1}(\Omega \times [0, T))$, *where* T *is the maximum time of existence. If* $S(l, 0) > 0, U_1(l, 0) > 0, U_2(l, 0) > 0$, $Q(l, 0) > 0$, *and* $R(l, 0) > 0$, *then the solution* $(S(l, t), U_1(l, t), U_2(l, t), Q(l, t), R(l, t))$ *is bounded on* $\overline{\Omega}$.

Proof. By model (2), we get

$$\begin{aligned} &\frac{\partial S}{\partial t} + \frac{\partial U_1}{\partial t} + \frac{\partial U_2}{\partial t} + \frac{\partial Q}{\partial t} - d_1 \Delta S - d_2 \Delta U_1 - d_3 \Delta U_2 - d_4 \Delta Q\\ &= \Lambda - \mu(S + U_1 + U_2 + Q) - (\alpha_1 S + \delta_1 U_1 + \delta_2 U_2 + \alpha_2 Q). \end{aligned} \tag{36}$$

It then follows that

$$\begin{aligned} &\int_\Omega \left(\frac{\partial S}{\partial t} + \frac{\partial U_1}{\partial t} + \frac{\partial U_2}{\partial t} + \frac{\partial Q}{\partial t} - d_1 \Delta S - d_2 \Delta U_1 - d_3 \Delta U_2 - d_4 \Delta Q \right) \mathrm{d}l\\ &= \int_\Omega [\Lambda - \mu(S + U_1 + U_2 + Q) - (\alpha_1 S + \delta_1 U_1 + \delta_2 U_2 + \alpha_2 Q)] \mathrm{d}l. \end{aligned} \tag{37}$$

By Green's formulas and Newman boundary $(\partial S/\partial t) = (\partial U_1/\partial t) = (\partial U_2/\partial t) = (\partial Q/\partial t) = 0, l \in \partial\Omega, t > 0$, we have

$$\begin{aligned} d_1\int_\Omega \Delta S\mathrm{d}l &= d_1\int_{\partial\Omega}\frac{\partial S}{\partial n}\mathrm{d}s = 0,\\ d_2\int_\Omega \Delta U_1\mathrm{d}l &= d_2\int_{\partial\Omega}\frac{\partial U_1}{\partial n}\mathrm{d}s = 0,\\ d_3\int_\Omega \Delta U_2\mathrm{d}l &= d_3\int_{\partial\Omega}\frac{\partial U_2}{\partial n}\mathrm{d}s = 0,\\ d_4\int_\Omega \Delta Q\mathrm{d}l &= d_4\int_{\partial\Omega}\frac{\partial Q}{\partial n}\mathrm{d}s = 0. \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \int_\Omega\left(\frac{\partial S}{\partial t}+\frac{\partial U_1}{\partial t}+\frac{\partial U_1}{\partial t}+\frac{\partial Q}{\partial t}\right)\mathrm{d}l &= \int_\Omega\left[\Lambda-\mu\left(S+U_1+U_2+Q\right)-\left(\alpha_1 S+\delta_1 U_1+\delta_2 U_2+\alpha_2 Q\right)\right]\mathrm{d}l\\ &\le \int_\Omega\left[\Lambda-\mu\left(S+U_1+U_2+Q\right)\right]\mathrm{d}l\\ &= \Lambda|\Omega|-\mu\int_\Omega\left(S+U_1+U_2+Q\right)\mathrm{d}l. \end{aligned} \tag{39}$$

Let $\int_\Omega (S+U_1+U_2+Q)\mathrm{d}l = F(t)$. Then, (39) becomes

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t}\le \Lambda|\Omega|-\mu F(t). \tag{40}$$

Which indicates that $0\le F(t)\le(\Lambda/\mu)|\Omega|+F(0)e^{-\mu t}$, here

$$\begin{aligned} F(0) &= \int_\Omega\left(S(l,0)+U_1(l,0)+U_2(l,0)+Q(l,0)\right)\mathrm{d}l\\ &\le \int_\Omega\left\|S(l,0)+U_1(l,0)+U_2(l,0)+Q(l,0)\right\|_\infty\mathrm{d}l\\ &= \left\|S(l,0)+U_1(l,0)+U_2(l,0)+Q(l,0)\right\|_\infty|\Omega|. \end{aligned} \tag{41}$$

This shows that $F(t)=\int_\Omega(S+U_1+U_2+Q)\mathrm{d}l$ is bounded. Let $Z=(\Lambda/\mu)|\Omega|+F(0)$, then

$$F(t)=\int_\Omega\left(S+U_1+U_2+Q\right)\mathrm{d}l\le Z. \tag{42}$$

By Theorem 3.1 in [43], there is a positive constant Z^* depending on Z so that

$$\left\|S(l,t)+U_1(l,t)+U_2(l,t)+Q(l,t)\right\|_{L^\infty(\Omega)}\le Z^*. \tag{43}$$

Hence, we obtain that $S(l,t), U_1(l,t), U_2(l,t)$, and $Q(l,t)$ are uniformly bounded on $\overline{\Omega}$.

For the last equation of model (2), let $S(l,t)$ and $Q(l,t)$ be bounded by $\overline{S}$ and $\overline{Q}$. Then,

$$\frac{\partial R}{\partial t}\le \alpha_1\overline{S}+\alpha_2\overline{Q}-\mu R. \tag{44}$$

Hence, $R(t)\le((\alpha_1\overline{S}+\alpha_2\overline{Q})/\mu)+\|R(l,0)\|_\infty$ on $\overline{\Omega}$. This completes the proof. □

Let $\mathbf{L}:=\mathbf{C}(\overline{\Omega};\mathbb{R})$ be a Banach space with a supremum norm:

$$\|w\|_\infty := \sup|w(l)|,\quad \forall w\in\mathbf{C}(\overline{\Omega};\mathbb{R}). \tag{45}$$

Define B: $\mathbf{L}^5\longrightarrow\mathbf{L}^5$, where $\mathbf{L}^5=\mathbf{L}\times\mathbf{L}\times\mathbf{L}\times\mathbf{L}\times\mathbf{L}$, and let $B_i\,(i=1,2,3,4)$ be a linear operator on $\mathbf{L}$ defined by

$$\begin{aligned} B_i w(l) &:= d_i\Delta w(l),\\ D(B_i) &:= \left\{w\in\mathbf{L}\colon \Delta w\in\mathbf{L},\ \frac{\partial w}{\partial n}=0 \text{ on } \partial\Omega\right\}. \end{aligned} \tag{46}$$

Then, by [23], we obtain that B_i are the infinitesimal generators of a strongly continuous semigroup $\{e^{tB_i}\}_{t\ge0}$ in $\mathbf{L}$. For $\forall\varphi:=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)\in\mathbf{L}^5$, there are

$$\begin{aligned} B\varphi(l) &:= \left(B_1\varphi_1(l), B_2\varphi_2(l), B_3\varphi_3(l), B_4\varphi_4(l), 0\right)^T,\\ D(B) &:= D(B_1)\times D(B_2)\times D(B_3)\times D(B_4)\times\mathbf{L}. \end{aligned} \tag{47}$$

They are also the infinitesimal generator of a strongly continuous semigroup $\{e^{tB}\}_{t\ge0}$ in $\mathbf{Y}:=\mathbf{L}^5$, where

$$\begin{aligned} e^{tB} &= \left(e^{tB_1}, e^{tB_2}, e^{tB_3}, e^{tB_4}, \mathbf{1}\right)^T,\\ e^{tB}\varphi &= \left(e^{tB_1}\varphi_1, e^{tB_2}\varphi_2, e^{tB_3}\varphi_3, e^{tB_4}\varphi_4, \varphi_5\right)^T, \end{aligned} \tag{48}$$

since $\mathbf{Y}$ is a Banach space with norm

$$\left\|(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)^T\right\|_{\mathbf{Y}} := \|\varphi_1\|_{\mathbf{L}}+\|\varphi_2\|_{\mathbf{L}}+\|\varphi_3\|_{\mathbf{L}}+\|\varphi_4\|_{\mathbf{L}}+\|\varphi_5\|_{\mathbf{L}}. \tag{49}$$

Let G be a nonlinear operator on $\mathbf{Y}$ defined by

$$G(\varphi) := \left(g_1(\varphi), g_2(\varphi), g_3(\varphi), g_4(\varphi), g_5(\varphi)\right), \tag{50}$$

where

$$
\begin{aligned}
g_1(\varphi) &= \Lambda - \beta_1\varphi_1\varphi_2 - (\mu + \alpha_1)\varphi_1,\\
g_2(\varphi) &= \beta_1\varphi_1\varphi_2 + k_2\varphi_4 + \beta_2\varphi_3 - (\mu + \delta_1 + k)\varphi_2,\\
g_3(\varphi) &= k\varphi_2 - (\mu + \delta_2 + k_1 + \beta_2)\varphi_3,\\
g_4(\varphi) &= k_1\varphi_3 - (\mu + k_2 + \alpha_2)\varphi_4,\\
g_5(\varphi) &= \alpha_1\varphi_1 + \alpha_2\varphi_4 - \mu\varphi_5.
\end{aligned} \tag{51}
$$

Then, model (2) can be rewritten into a more abstract form in $\mathbf{Y}$ as follows:

$$
\begin{aligned}
&\frac{d\varphi(t)}{dt} = B\varphi(t) + G(\varphi(t)),\\
&\varphi(t) \coloneqq \left(S(\cdot,t), U_1(\cdot,t), U_2(\cdot,t), Q(\cdot,t), R(\cdot,t)\right)^T,\\
&\varphi(0) \coloneqq \left(S_0(\cdot), U_{10}(\cdot), U_{20}(\cdot), Q_0(\cdot,t), R_0(\cdot,t)\right)^T.
\end{aligned} \tag{52}
$$

By Proposition 4.16 in [44], it can be obtained that the unique continuously differentiable solution φ: $(0, T_{\max}] \longrightarrow \mathbf{Y}$ of the above equations has a maximum interval of existence $(0, T_{\max}]$ so that

$$
\varphi(t) = e^{tB}\varphi_0 + \int_0^t e^{(t-s)B} G(\varphi(s))\mathrm{d}s, \tag{53}
$$

and either $T_{\max} \longrightarrow \infty$ or $\limsup_{t \longrightarrow T_{\max}-0} \|\varphi(t)\|_{\mathbf{Y}} \longrightarrow +\infty$. By Theorem 5, we obtain $T_{\max} = +\infty$ holds. Therefore, $\varphi(t)$ is a global solution.

4.2. Global Stability of the R-D Model. Since the R equation of model (2) is uncoupled with the other equations, model (2) can be reduced by ignoring R. It is clear that the R-D heroin epidemic model (2) has a drug-free equilibrium $E_r = ((\Lambda/\mu + \alpha_1), 0, 0, 0)$ and a unique positive drug-addiction equilibrium $E_r^* = (S^*, U_1^*, U_2^*, Q^*)$ if $R_0 > 1$.

Theorem 6. *If $R_0 \le 1$, the drug-free equilibrium E_r is globally asymptotically stable.*

Proof. We give a Lyapunov function as follows:

$$
W_1(t) = \int_\Omega V(\varphi(l,t))\mathrm{d}l, \tag{54}
$$

where V is given by (14) and $\varphi(l,t) = (S(l,t), U_1(l,t), U_2(l,t), Q(l,t))$. Direct computations show that

$$
\begin{aligned}
\frac{dW_1}{\mathrm{d}t} &= \int_\Omega g\mathrm{rad}_\varphi V \cdot \frac{\partial\varphi}{\partial t}\mathrm{d}l\\
&= \int_\Omega \left(1 - \frac{S_0}{S}, 1, 1, 1\right) \cdot \left(\dot{S} + d_1\Delta S, \dot{U}_1 + d_2\Delta U_1, \dot{U}_2 + d_3\Delta U_2, \dot{Q} + d_4\Delta Q\right)\mathrm{d}l\\
&= \int_\Omega \left[\left(1 - \frac{S_0}{S}\right)\dot{S} + \dot{U}_1 + \dot{U}_2 + \dot{Q}\right]\mathrm{d}l + \int_\Omega \left(1 - \frac{S_0}{S}\right) d_1\Delta S\mathrm{d}l\\
&\quad + \int_\Omega d_2\Delta U_1\mathrm{d}l + \int_\Omega d_3\Delta U_2\mathrm{d}l + \int_\Omega d_4\Delta Q\mathrm{d}l\\
&= \int_\Omega \frac{dV}{dt}\mathrm{d}l + \int_\Omega \left(1 - \frac{S_0}{S}\right) d_1\Delta S\mathrm{d}l.
\end{aligned} \tag{55}
$$

It follows by Green's identity that

$$
\begin{aligned}
\int_\Omega \left(1 - \frac{S_0}{S}\right) d_1\Delta S\mathrm{d}l &= \int_{\partial\Omega} \left(1 - \frac{S_0}{S}\right) d_1\frac{\partial S}{\partial n}\mathrm{d}s - \int_\Omega d_1\nabla\left(1 - \frac{S_0}{S}\right)\nabla S\mathrm{d}l\\
&= -\int_\Omega d_1 S_0 \frac{|\nabla S|^2}{S^2}\mathrm{d}l \le 0.
\end{aligned} \tag{56}
$$

Taking (56) into (55) and by Theorem 2, we obtain that if $R_0 \le 1$, $(\mathrm{d}V/\mathrm{d}t) \le 0$, $\int_\Omega (\mathrm{d}V/\mathrm{d}t)\mathrm{d}l \le 0$. Furthermore, if $R_0 \le 1$, $(\mathrm{d}W_1/\mathrm{d}t) < 0$. By LaSalle's invariance principle in [42], the drug-free equilibrium E_r is globally asymptotically stable. This completes the proof. □

Theorem 7. *If $R_0 > 1$, then the drug-addiction equilibrium E_r^* is globally asymptotically stable.*

Proof. We give the following Lyapunov function:

$$
W_2(t) = \int_\Omega V_2(\varphi(l,t))\mathrm{d}l, \tag{57}
$$

where $\varphi(l,t) = (S(l,t), U_1(l,t), U_2(l,t), Q(l,t))$ and

$$V_2 = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \left(U_1 - U_1^* - U_1^* \ln \frac{U_1}{U_1^*}\right) + \frac{\beta_2 U_2^* + k_2 Q^*}{k U_1^*}\left(U_2 - U_2^* - U_2^* \ln \frac{U_2}{U_2^*}\right) + \frac{k_2 Q^*}{k_1 U_2^*}\left(Q - Q^* - Q^* \ln \frac{Q}{Q^*}\right). \tag{58}$$

Direct calculation yields

$$\begin{aligned}
\frac{dW_2}{dt} &= \int_\Omega g\mathrm{rad}_\varphi V_2 \cdot \frac{\partial \varphi}{\partial t} dl \\
&= \int_\Omega \left(1 - \frac{S^*}{S}, 1 - \frac{U_1^*}{U_1}, \frac{\beta_2 U_2^* + k_2 Q^*}{k U_1^*}\left(1 - \frac{U_2^*}{U_2}\right), \frac{k_2 Q^*}{k_1 U_2^*}\left(1 - \frac{Q^*}{Q}\right)\right) \\
&\quad \cdot \left(\dot{S} + d_1 \Delta S, \dot{U}_1 + d_2 \Delta U_1, \dot{U}_2 + d_3 \Delta U_2, \dot{Q} + d_4 \Delta Q\right) dl \\
&= \int_\Omega \left[\left(1 - \frac{S^*}{S}\right)\dot{S} + \left(1 - \frac{U_1^*}{U_1}\right)\dot{U}_1 + \frac{\beta_2 U_2^* + k_2 Q^*}{k U_1^*}\left(1 - \frac{U_2^*}{U_2}\right)\dot{U}_2 + \frac{k_2 Q^*}{k_1 U_2^*}\left(1 - \frac{Q^*}{Q}\right)\dot{Q}\right] dl \\
&\quad + \int_\Omega \left(1 - \frac{S^*}{S}\right) d_1 \Delta S dl + \int_\Omega \left(1 - \frac{U_1^*}{U_1}\right) d_2 \Delta U_1 dl \\
&\quad + \int_\Omega \frac{\beta_2 U_2^* + k_2 Q^*}{k U_1^*}\left(1 - \frac{U_2^*}{U_2}\right) d_3 \Delta U_2 dl + \int_\Omega \frac{k_2 Q^*}{k_1 U_2^*}\left(1 - \frac{Q^*}{Q}\right) d_4 \Delta Q dl \\
&= \int_\Omega \frac{dV_2}{dt} dl - d_1 S^* \int_\Omega \frac{|\nabla S|^2}{S^2} dl - d_2 U_1^* \int_\Omega \frac{|\nabla U_1|^2}{U_1^2} dl \\
&\quad - d_3 U_2^* \frac{\beta_2 U_2^* + k_2 Q^*}{k U_1^*} \int_\Omega \frac{|\nabla U_2|^2}{U_2^2} dl - d_4 Q^* \frac{k_2 Q^*}{k_1 U_2^*} \int_\Omega \frac{|\nabla Q|^2}{Q^2} dl.
\end{aligned} \tag{59}$$

By Theorem 3, we know that if $R_0 > 1$, then $(dV_1/dt) \le 0$, $\int_\Omega (dV_2/dt) dl \le 0$. Therefore, if $R_0 > 1$, then $(dW_2/dt) < 0$. By LaSalle's invariance principle in [42], the drug-addiction equilibrium E_r^* is globally asymptotically stable. This completes the proof. □

5. Numerical Simulations

In this section, we shall carry some numerical simulations to illustrate our analytic results by using the parameter values in Table 1. We fix $\Lambda = 1, \mu = 0.02, \beta_2 = 0.0011, k = 0.0095$, $\alpha_2 = 0.0001, \delta_1 = 0.01$, and $\delta_2 = 0.005$.

If we choose $\beta_1 = 0.0002, k_2 = 0.00008, k_1 = 0.008$, and $\alpha_1 = 0.02$, then $R_0 \approx 0.13 < 1$ and $E_0 = ((\Lambda/\mu + \alpha_1), 0, 0, 0, (\alpha_1 \Lambda/\mu(\mu + \alpha_1)))$. Give different initial values $I_1 = (5, 5, 5, 5, 5)$ and $I_2 = (5, 10, 3, 6, 1)$, respectively, and we can see that the drug-free equilibrium E_0 is globally asymptotically stable (Figure 2). When $\alpha_1 = \mu$, we get $(\Lambda/\mu + \alpha_1) = (\alpha_1 \Lambda/\mu(\mu + \alpha_1))$, this is verified by the figure. We can see all solutions of the system converge to the drug-free equilibrium $(25, 0, 0, 0, 25)$. If we keep β_1, k_2, and k_1 unchanged and let $\alpha_1 = 0.08 > \mu = 0.02$, then $R_0 = 0.05 < 1$. According to the above discussion, E_0 is asymptotically stable, and when $\alpha_1 > \mu$, $(\Lambda/\mu + \alpha_1) < (\alpha_1 \Lambda/\mu(\mu + \alpha_1))$. Figure 3 not only illustrates the stability of E_0, but also shows the number of people who are permanently immunized against drugs, $(R(t))$ is greater than the number of susceptible people $(S(t))$ in the equilibrium E_0 when $\alpha_1 > \mu$. By Figures 2 and 3, we can not only clearly see that $U_1(t)$ declined sharply and got to zero finally, but also see that all solutions of the system infinitely close to the drug-free equilibrium E_0. It verifies the existence of E_0.

If we choose parameters $\beta_1 = 0.01, k_2 = 0.008, k_1 = 0.01$, $\alpha_1 = 0.02$, and initial values $I_3 = (25, 25, 25, 25, 25)$ and $I_4 = (25, 5, 16, 12, 3)$, then $R_0 \approx 6.5 > 1$. Hence, the endemic equilibrium E^* is global asymptotically stable (Figure 4). This also verifies the existence of E^*.

6. Discussion

In this paper, we formulated a novel R-D heroin epidemic model, which incorporates the relapse compartment and permanent immunization compartment. With the help of

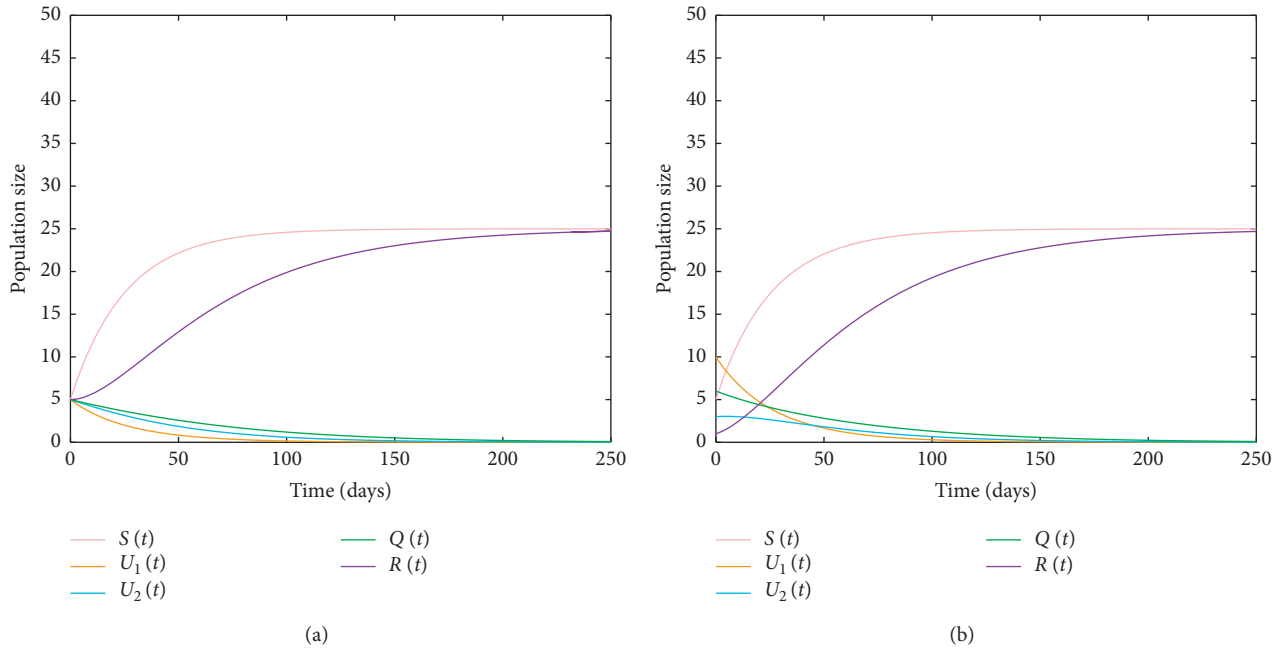


FIGURE 2: The stability of E_0 with different initial values when $\alpha_1 = \mu$. (a) The stability of E_0 with an initial value I_1. (b) The stability of E_0 with an initial value I_2.

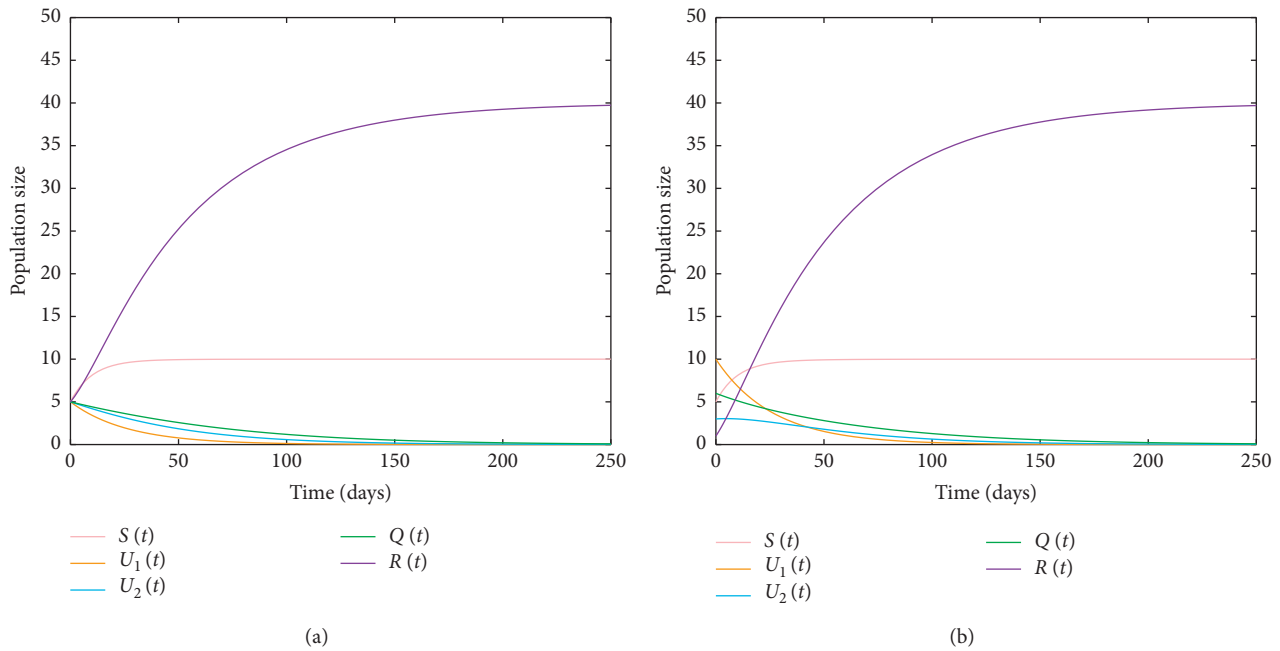


FIGURE 3: The stability of E_0 with different initial values when $\alpha_1 \neq \mu$. (a) The stability of E_0 with an initial value I_1. (b) The stability of E_0 with an initial value I_2.

the next-generation matrix method, we obtained the basic reproduction number R_0. We obtained the global dynamics of the model by constructing some suitable Lyapunov functions. It is shown when $R_0 \leq 1$, the drug-free equilibrium is globally asymptotically stable; that is, the drug abuse will be eradicated; when $R_0 > 1$, the endemic equilibrium is globally asymptotically stable, which means that drug abuse will be permanent.

For the ODE system (3), $k_1, \beta_2, k_2, \alpha_1$, and α_2 represent the detoxification success rate, detoxification failure rate, relapse rate from Q to abusers, permanent withdrawal rates from S to R, and permanent withdrawal rates from Q to R,

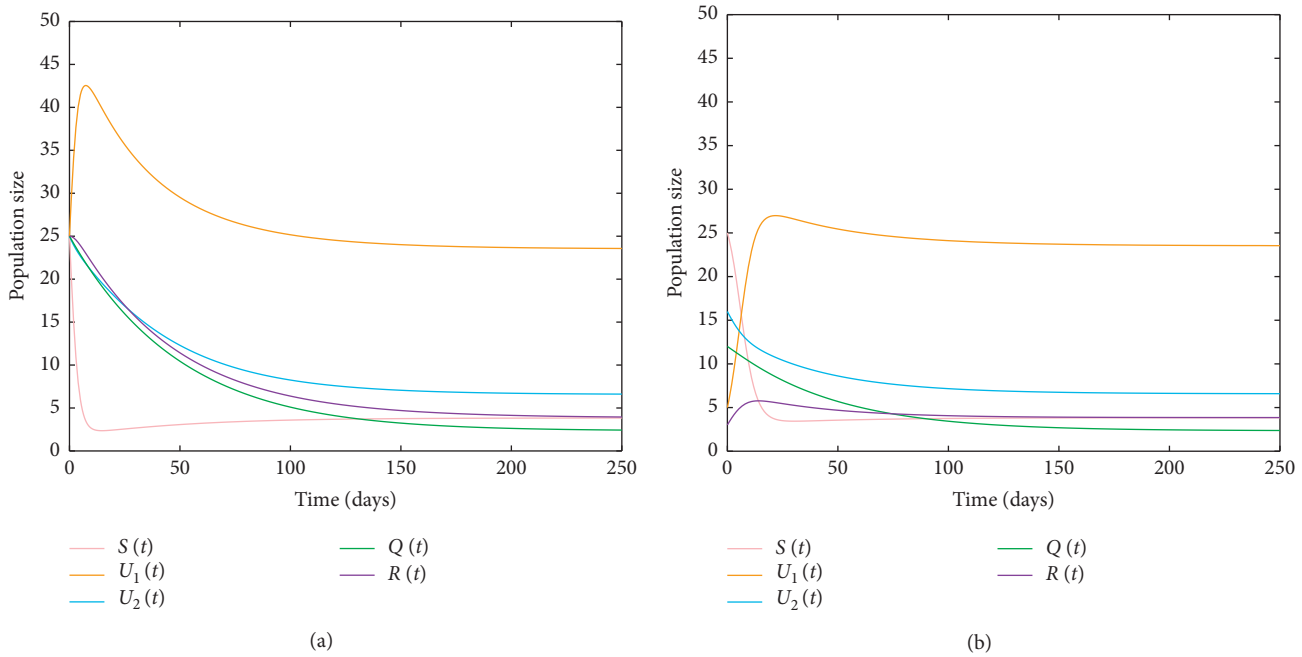
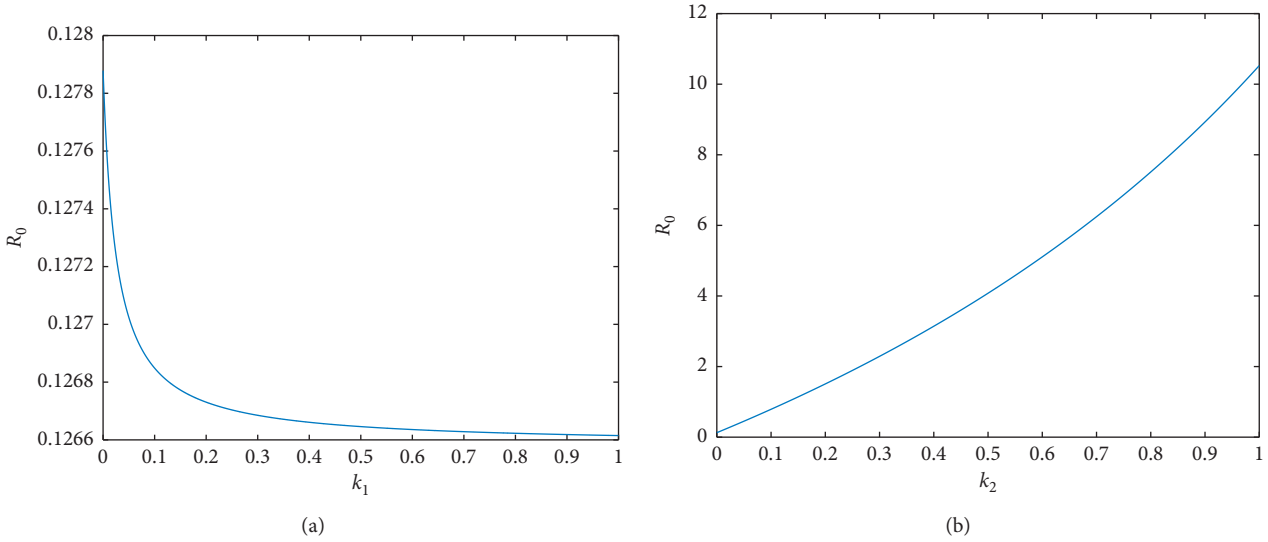


FIGURE 4: The stability of E^* with different initial values. (a) The stability of E^* with an initial value I_3. (b) The stability of E^* with an initial value I_4.



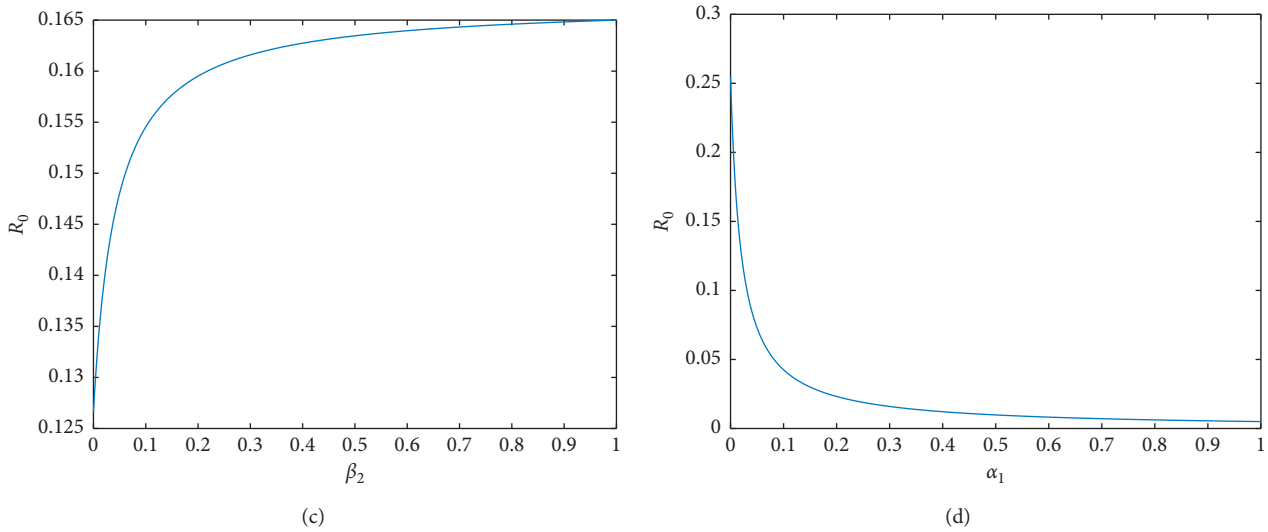


Figure 5: The relationship between R_0 and some parameter values. (a) The relationship between R_0 and k_1. (b) The relationship between R_0 and k_2. (c) The relationship between R_0 and β_2. (d) The relationship between R_0 and α_1.

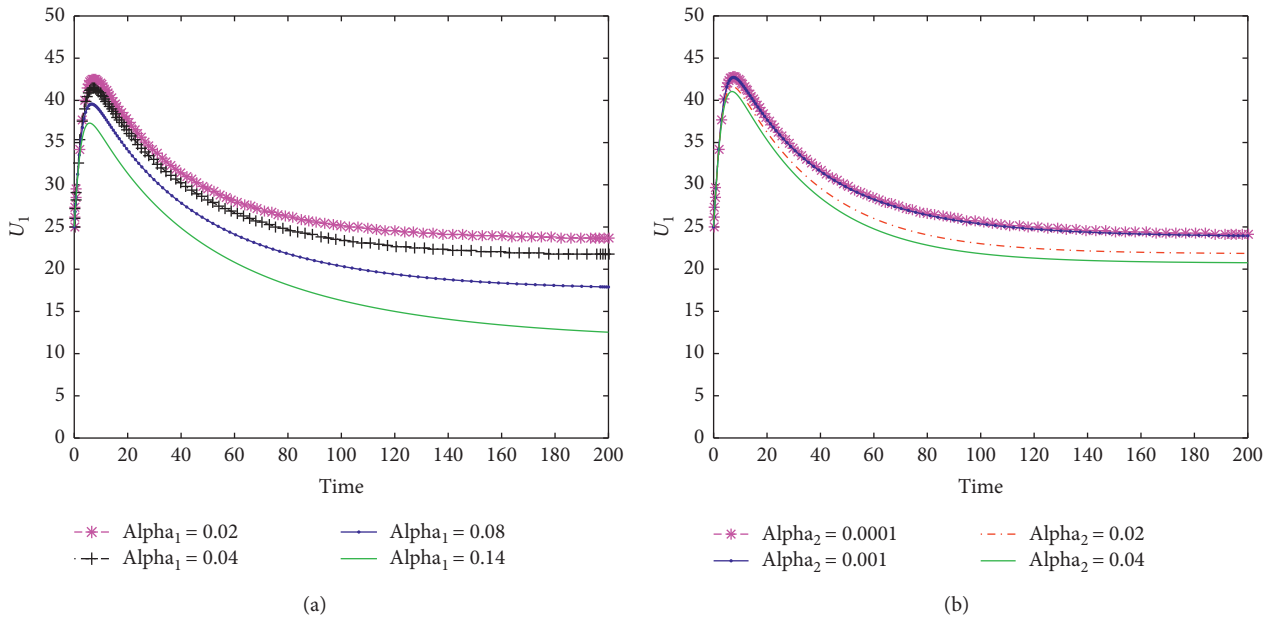


Figure 6: Effect of (a) α_1 and (b) α_2 on U_1.

respectively. Figure 5 shows the relationship between R_0 and k_1, k_2, β_2, and α_1, with other parameter values as given in Table 1. As shown in Figures 5(a)–5(c), R_0 grows with k_2 and β_2 but decreases with k_1, and it means that if we want to control heroin addiction, we should reduce k_2 and β_2 and increase k_1, that is, increase the detoxification success rate and pay attention to people with a history of drug abuse to reduce their mental dependence on heroin. Moreover, as shown in Figure 5(d), R_0 decreases with α_2, and it means that if we want to control the heroin addiction, we should increase publicity to let people understand the harmful effect of heroin and take the initiative to stay away from drugs.

The effect of α_1 and α_2 on U_1 is shown in Figure 6. It indicates that the values of α_1 and α_2 have a significant effect on the number of drug-addiction equilibrium. Let us observe Figure 6(a) first, the larger the α_1 is, the fewer the people who use drugs in equilibrium is, and this tells us that we should not only pay attention to drug abuse but should pay more attention to those who do not use drugs. The government should strengthen publicity to raise people's awareness of drug prevention. Comparing Figure 6(b) with Figure 6(a), we can get that although the effect of α_2 on U_1 is less than the effect of α_1 on U_1, it can still obtain that the larger the α_2 is, the fewer the people who use drugs in equilibrium is;

therefore, we also should pay attention to the people who have already quit drug abuse, so that they can stay away from drugs instead of reusing drugs.

The diffusion phenomena make it more difficult for governments to control drugs. Fortunately, the R-D heroin epidemic model (2) still exists as a drug-free equilibrium, and it convinces us that the spread of drugs can be stopped. Additionally, the global dynamic behaviors of the R-D model (2) show that if $R_0 > 1$ the endemic equilibrium E_r^* is global stability, and if $R_0 \leq 1$, the drug-free equilibrium E_r is global stability, which indicates that the R-D model (2) may contain traveling waves connecting the steady states E_r and E_r^*. We shall conduct further research on this issue in the future.

Data Availability

All the data have been included in the paper; therefore, there are no other data available.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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