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Research Article

Percolation Theories for Multipartite Networked Systems under Random Failures

Qing Cai,[1] **Sameer Alam**,[1] **Mahardhika Pratama**,[2] **and Zhen Wang**[3]

[1]*School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore*
[2]*School of Computer Science and Engineering, Nanyang Technological University, Singapore*
[3]*School of Mechanical Engineering and Center for Optical Imagery Analysis and Learning (OPTIMAL), Northwestern Polytechnical University, Xi'an, China*

Correspondence should be addressed to Sameer Alam; sameeralam@ntu.edu.sg, Mahardhika Pratama; pratama@ieee.org, and Zhen Wang; w-zhen@nwpu.edu.cn

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Real-world complex systems inevitably suffer from perturbations. When some system components break down and trigger cascading failures on a system, the system will be out of control. In order to assess the tolerance of complex systems to perturbations, an effective way is to model a system as a network composed of nodes and edges and then carry out network robustness analysis. Percolation theories have proven as one of the most effective ways for assessing the robustness of complex systems. However, existing percolation theories are mainly for multilayer or interdependent networked systems, while little attention is paid to complex systems that are modeled as multipartite networks. This paper fills this void by establishing the percolation theories for multipartite networked systems under random failures. To achieve this goal, this paper first establishes two network models to describe how cascading failures propagate on multipartite networks subject to random node failures. Afterward, this paper adopts the largest connected component concept to quantify the networks' robustness. Finally, this paper develops the corresponding percolation theories based on the developed network models. Simulations on computer-generated multipartite networks demonstrate that the proposed percolation theories coincide quite well with the simulations.

1. Introduction

It is universally acknowledged that complex systems are ubiquitous in our lives [1]. Complex systems like city transportation systems [2] and power supplier systems [3] are indispensable infrastructures to human life. In order to better understand complex systems so as to facilitate better service providing, an effective way is to model a complex system as a complex network composed of nodes and edges with the nodes denoting the system components and the edges representing the interactions between system components [4]. For example, a power grid system can be represented by a network in which a node denotes a power station and an edge denotes the transmission line between two stations. Complex network modeling and analysis have proven as a potent instrument for system control [5–7] and have received great popularity in the last two decades [8, 9].

Note that complex systems in reality will inevitably suffer from external and/or internal unpredictable perturbations which can trigger cascading failures wreaking havoc on system structures and functionalities [5, 10]. A dramatic event in history was the Italian blackout that happened in 2003 [11]. It had been reported that the blackout was triggered by the breakdown of several power lines caused by a storm. It was until the seminal work done in [11] that the science underlying the event had been disclosed from the perspective of network robustness analysis. Network robustness analysis now has proven to be an effective approach to evaluating the robustness of complex systems so as to help prevent unseen system disasters [12–14]. Due to its significant economical values, many efforts have been made towards networked system robustness analysis. Existing studies can be roughly categorised into two classes, i.e., simulation-based studies [15–17] and theoretical studies [11, 18, 19]. Simulation-based studies investigate the robustness of networked systems by carrying out computer simulations. Their main drawback is that they cannot uncover the governing principles for system robustness. To overcome this drawback, theoretical studies came out and amongst which are the percolation theories [1, 20].

Percolation on networks can not only measure the systems' robustness but also provide the mathematical explanations for the systems' robustness behaviours. Note that real-world systems are not independent but are organized in a layer-layer interdependent way, which are widely modeled as multilayer networks [18, 21, 22]. The component failures in one layer of a multilayer networked system can induce the failures in other layers and cascading failures could eventually occur. The work in [11] indicates that multilayer networked systems are vulnerable to perturbations and have sparked the research enthusiasm on the percolation theories for multilayer networked systems. Albeit the maturity of percolation theories for multilayer networked systems, little attention is paid to multipartite networked systems. Many complex systems like ecosystems [23, 24], certain control systems [1], and metabolic systems [25] can be modeled as multipartite networks. To explore the robustness of multipartite networked systems is also of great significance.

The structure difference between multilayer and multipartite networks renders the applications of existing percolation theories to multipartite networked systems infeasible. With regard to this, for a given multipartite networked system, this paper first establishes two network cascading models to prescribe the way how cascading failures propagate on multipartite networks when initial node failures occur. Afterward, this paper develops the percolation theories for assessing the robustness of multipartite networked systems under random node failures based on the largest connected component concept.

The remainder of this paper is structured as follows. Section 2 provides related preliminaries including basic network notations, network robustness evaluation metrics, and percolation theory for single-layer networked systems. Section 3 presents the research problem and motivation. Section 4 delineates in detail the proposed percolation theories for analyzing the robustness of multipartite networked systems subject to random node failures. Section 5 validates the correctness of the proposed theories through simulations on random multipartite networks with Poisson degree distributions. Section 6 concludes the paper.

2. Preliminaries

2.1. Network Notations. Generally, a network is mathematically denoted by $G = \{V, E\}$, where V and E represent the sets of nodes and edges, respectively. The edges between nodes can be depicted by the adjacency matrix $\mathbf{A}_{N\times N}$ of G with N being the number of nodes in G. The matrix $\mathbf{A}$ is usually symmetric and binary. Let e_{ij} be the entry of $\mathbf{A}$. If there is an edge between nodes i and j, then $e_{ij} = 1$; otherwise, it is equal to 0.

Regarding complex network analytics, one of the most concerned properties is the node degree. For a network G, the degree k_i of node i is defined as the number of edges attached to it. Generally, k_i can be calculated as $k_i = \sum_{j=0}^{N} e_{ij}$. Another property is the degree distribution $P(k)$ of G which specifies the probability for a node to have degree k. With k_i and $P(k)$, we have the mean degree $\langle k\rangle$ of G, which can be calculated as $\langle k\rangle = (1/N)\sum_{i=1}^{N} k_i = \sum_{k=0}^{N-1} kP(k)$.

2.2. Multipartite and Multilayer Networked Systems

Definition 1. Let us consider a complex system that can be modeled as a network G whose node set V consists of L subsets, i.e., $V = \{S_1, S_2, \ldots, S_L\}$. If $\forall a, b \in [1, L]$, G satisfies the following conditions:

$$\begin{cases} S_a \cap S_b = \varnothing, & \text{if } a \neq b, \\ e_{ij} \in \{0,1\}, & \text{if } i \in S_a \wedge j \in S_b \wedge b \in \{a-1, a+1\}, \\ e_{ij} \equiv 0, & \text{if } i \in S_a \wedge j \in S_{a\pm k}, \forall k \in [2, L-1], \\ e_{ij} \equiv 0, & \text{if } i, j \in S_a. \end{cases} \quad (1)$$

Then, we say this system is a multipartite networked system and the network G is called a multipartite network.

Remark 1. The node set S_a of a multipartite networked system can also be called a partite set. Equation (1) indicates that an edge of a multipartite network only happens between two nodes with one coming from partite set S_a and the other one from partite set S_{a+1} or S_{a-1}.

Definition 2. Let us consider another system which can be modeled as a network G, and G is composed of L subnetworks, i.e., $G = \{G_1, G_2, \ldots, G_L\}$. The node set $V \subset G$ can also be divided into L subsets with S_i being the node set for network G_i. If G satisfies the following conditions:

$$\begin{cases} S_a \cap S_b = \varnothing, & \text{if } a \neq b, \\ e_{ij} \in \{0,1\}, & \text{if } i \in S_a \wedge j \in S_b \wedge b \in \{a, a-1, a+1\}, \end{cases} \quad (2)$$

then we call this system a multilayer networked system and the network G is called a multilayer network [26, 27].

Remark 2. In the literature, a multilayer network can also be called an interdependent network or a network of networks [21, 28]. We can see from the above definitions that the only structural difference between a multilayer network and a multipartite network is that parallel edges (edges between nodes from the same node set) are not allowed in a multipartite network.

2.3. Networked System Robustness Evaluation. Network robustness analysis has proven as an effective tool for assessing the robustness of networked systems under disturbances [29, 30]. Regarding system robustness analysis, the foremost issue is how one defines or quantifies the robustness of

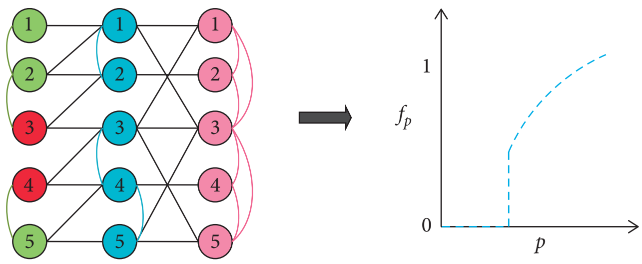

Figure 1: An illustration to the common idea shared by most existing metrics for quantifying the robustness of a networked system under node perturbations.

a networked system. Hitherto, a handful of robustness metrics have been proposed by researchers, and most of them share the common idea as delineated in Figure 1.

In the left panel of Figure 1 is a multilayer network G. Assume that G is under node perturbations and $1-p$ fraction of nodes is removed (marked in red in the figure). The node removals will trigger the cascading breakdown of other nodes and edges. Based on a prescribed cascading model which defines the way how cascading failures propagate on G, G finally reaches a stable stage (the final remaining part of G) in which no node/edge removal is possible. Then, one counts f_p, the fraction of effective nodes, in the stable stage. One then gets the robustness curve drawn in the right panel of Figure 1 in which f_p is shown with respect to p under different values.

In the literature, f_p is widely defined as the fraction of non-zero-degree nodes in the stable stage of G [15, 16, 24] after removing $1-p$ fraction of nodes. Based on this kind of definition for f_p, the robustness of a network G then can be quantified by the node robustness index R_n devised in [16] or by the area index R_A used in [15, 24]. The R_n index is calculated as $R_n = \sum_{p=0}^{1} f_p$, while the R_A index is measured as the area of the region covered by the robustness curve and the X-Y axis (see the dark region in the right panel of Figure 1). The larger the value of R_n or R_A is, the higher robustness the focal network has.

The above definition for f_p is effective for network robustness analysis. However, scientists argue that this kind of definition may not reflect the real robustness of networks. In reality, the breakup of some components of a complex system will fragment the system into pieces and it is argued that only the largest piece will keep functioning. As per this assumption, scientists suggest the definition of f_p as the fraction of nodes in the largest connected component (LCC, the subnetwork that contains the most nodes) in the stable stage G. The LCC-based definition for f_p has been widely recognized [11, 21, 31], and this paper also adopts this kind of definition for network robustness analysis.

Note that how to select the $1-p$ fraction of nodes to be removed from a network depends on how one models the perturbations on the network [32–34]. Figure 1 only illustrates network robustness under node failures. In reality, failures can occur to the edges of a network. Meanwhile, many networks have been reported to possess community structures [35, 36]; therefore, failures can also occur to network communities. Related works can be found in [37, 38].

2.4. Percolation on Single-Layer Networked Systems. Percolation theories have gained large popularity for analyzing the robustness of networked systems [32, 39]. In the following we will illustrate the percolation theory for analyzing the robustness of single-layer networked systems.

Given that a fraction $1-p$ of nodes from a given single-layer network G is randomly removed, the node removal breaks G into small parts and there exists the largest one, i.e., the LCC. Percolation theory aims to mathematically figure out the fraction of nodes in the LCC, hereafter denoted by P^∞, with respect to p and the degree distribution $P(k)$ of G [18, 40, 41], i.e., it aims to derive the relation $P^\infty = F(p, P(k))$ with $F(\cdot)$ being a map or a function. Before presenting the mathematical derivations, we first present in the following some related definitions.

Definition 3 (generating function). A generating function for a degree distribution $P(k)$ is defined as

$$G_0(x) = \sum_{k=0}^{\infty} x^k P(k), \tag{3}$$

where k is the degree of a node and x is an arbitrary placeholder.

Definition 4 (excess degree distribution). Following a randomly chosen edge we reach a node s. Define the excess degree distribution $P^E(k)$ as the probability for node s to have k extra neighbours.

Remark 3. The excess degree distribution practically denotes the probability for a randomly chosen node to have degree $k+1$. In the literature [1, 42], $P^E(k)$ is widely calculated as

$$P^E(k) = \frac{(k+1)P(k+1)}{\langle k \rangle}. \tag{4}$$

Remark 4. Analogous to equation (3), the generating function for $P^E(k)$ is formulated as

$$G_1(x) = \sum_{k=0}^{\infty} x^k P^E(k) = \frac{G_0'(x)}{G_0'(1)}, \tag{5}$$

with $G_0'(x)$ being the first-order derivative of $G_0(x)$.

With the above definitions, the percolation theory for calculating P^∞ for a single-layer network G can be mathematically written as

$$P^\infty = p \sum_{k=0}^{\infty} P(k)\left(1-u^k\right) = p\left[1-G_0(u)\right], \tag{6}$$

with u being the probability for node s (reached by following a randomly chosen edge) not to be connected to the LCC via its neighbouring node d.

The probability variable u is calculated by the following transcendental equation:

$$u = \sum_{k=0}^{\infty}\left(1-p+pu^k\right)P^E(k) = 1-p+pG_1(u). \tag{7}$$

There exists a critical value of p, denoted by p_c. Once the fraction of node removal surpasses $1 - p_c$, then the focal network G will break down, i.e., $P^{\infty} = 0$. The critical value appears when the right and left panels of equation (7) meet with each other tangentially at $u = 1$. By calculating the derivative of equation (7), we have

$$\frac{\mathrm{d}u}{\mathrm{d}u} = \frac{\mathrm{d}\left(p_c G_1(u)\right)}{\mathrm{d}u}\Big|_{u=1}, \tag{8}$$

which further leads to

$$p_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}, \tag{9}$$

in which $\langle k^2 \rangle = \sum k^2 P(k)$ is the second moment of $P(k)$. Generally, the smaller the value of p_c is, the more robust the focal network is.

3. Problem Definition and Research Motivation

3.1. Problem Definition. This paper is dedicated to mathematically investigating the robustness of multipartite networks. Below we present the mathematical definition for our studied problem.

Definition 5 (research problem). Consider a system that is modeled as an L-partite network G. Denote G_{ij}, $G_{ij} \subset G$, as the bipartite network composed of node sets S_i and S_j with $j \in \{i-1, i+1\}$. Assume that G is under random attack and $1 - p_i$ fraction of nodes is randomly removed from $S_i \subset G$ for all $i \in [1, L]$ and $p_i \in [0, 1]$. For a given network cascading model which specifies how cascading failures propagate on G, G then reaches a stable stage. Consider the LCC in the stable stage of G. Then, the research problem is to derive the relation

$$P_i^{\infty} = F\left(p_i, P_{12}(k), \ldots, P_{ij}(k), \ldots, P_{L,L-1}(k)\right), \tag{10}$$

with P_i^{∞} being the fraction of nodes remaining in $S_i \subset G$ which also belongs to the LCC and $P_{ij}(k)$ being the degree distribution for the nodes in $S_i \subset G_{ij}$.

3.2. Research Motivation. From the definitions given in Section 2.1, one may argue that a multipartite network can be regarded as a simplified multilayer network and in turn a multilayer network can be regarded as a relaxation of a multipartite network. Therefore, one may think that models and theories developed for multilayer networks will work for multipartite networks. In what follows, we elaborate in detail our research motivations for proposing the percolation theories for multipartite networks.

(M1) Limitation of Percolation Theory for Multilayer Networks with One-to-One Correlations

The percolation theory introduced in Section2.4 is for single-layer networks and therefore does not work for multilayer networks. In view of this, the authors in [11] first investigated the robustness of multilayer networks with one-to-one (O2O) correlations, i.e., each node in one subnetwork depends on one and only one node in its coupled subnetwork. Similar network models can be found in [18, 21].

Figure 2 exhibits the dynamic cascading model established in [11] for analyzing the robustness of multilayer networks with O2O correlations. This model assumes that for each subnetwork contained in a multilayer network, only the nodes that belong to the LCC will survive perturbations and cascading failures will not stop until a mutual LCC is reached.

Based on the model illustrated in Figure 2, the authors further devised the corresponding percolation theory. Obviously, the model shown in Figure 2 together with its corresponding theory is not applicable to multipartite networks. The reason is obvious as a multipartite network does not obey the O2O correlation hypothesis.

(M2) Limitation of Percolation Theory for Multilayer Networks with One-to-Many Correlations

Real-world multilayer networks are often one-to-many (O2M) correlated, i.e., each node in one subnetwork depends on more than one node in its coupled subnetwork. With regard to this, studies on the robustness of multilayer networks with O2M correlations came out [31, 43, 44].

Figure 3 shows the dynamic cascading model proposed in [18] for analyzing the robustness of multilayer networks with O2M correlations. Given a two-layer network consisting of subnetworks A and B, let $\widetilde{P}_A(k)$ (and $\widetilde{P}_B(k)$) be the degree distribution of the nodes in A (and B) that have correlations with nodes in B (and A). Assume that a fraction $1 - p_A$ and a fraction $1 - p_B$ of nodes are randomly removed from A and B, respectively. With the model shown in Figure 3 the authors then have devised the corresponding percolation theory which is mathematically written as

$$\begin{cases} P_A^{\infty} = u_A\left[1 - G_{A0}\left(1 - u_A\left(1 - f_A\right)\right)\right], \\ P_B^{\infty} = u_B\left[1 - G_{B0}\left(1 - u_B\left(1 - f_B\right)\right)\right], \end{cases} \tag{11}$$

where $\widetilde{G}_{A0}$ and $\widetilde{G}_{B0}$ are, respectively, the generating functions of $\widetilde{P}_A(k)$ and $\widetilde{P}_B(k)$. The corresponding variables f_A, f_B, u_A, and u_B are calculated as

$$\begin{cases} f_A = G_{A1}\left[1 - u_A\left(1 - f_A\right)\right], \\ f_B = G_{B1}\left[1 - u_B\left(1 - f_B\right)\right], \\ u_A = p_A\left[1 - \widetilde{G}_{A0}\left(1 - P_B^{\infty}\right)\right], \\ u_B = p_B\left[1 - \widetilde{G}_{B0}\left(1 - P_A^{\infty}\right)\right]. \end{cases} \tag{12}$$

Note that a multipartite network can be regarded as a simplified multilayer network with O2M correlations. However, the model shown in Figure 3 still does not work for multipartite networks. The reasons are twofold. On the one hand, the model shown in Figure 3 considers the LCC of the network in each layer, while the nodes in each "layer" of a multipartite network are disconnected from each other, and thus the LCC does not exist. On the other hand, the percolation theory based on the model shown in Figure 3 involves the degree distributions $P_A(k)$ and $P_B(k)$. Bear in mind that $P_A(k)$ denotes the degree distribution of the nodes in network A that have connections with each other. Thus, we have $P_A(k) = P_B(k) = \cdots \equiv 0$ for a multipartite network, and the substitution of this condition into equation (11) will lead to the following result:

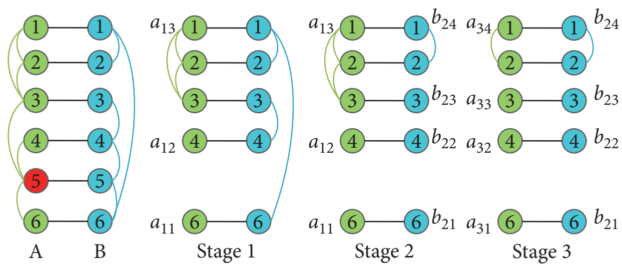

Figure 2: Dynamic network model for analyzing the robustness of multilayer networked systems with one-to-one correlations. Initially, node 5 from network A is removed. Due to the O2O correlations, node 5 in network B is also eliminated in Stage 1 and network A breaks into three clusters, denoted by a_{11}, a_{12}, and a_{13}. In Stage 2, nodes 4 and 6 both in networks A and B together with the links attached to them are removed, and network B breaks into four clusters, namely, b_{21}, b_{22}, b_{23}, and b_{24}. In Stage 3, node 3 in network A is removed and network A breaks into four clusters. Because in Stage 3 no further link elimination and network breaking occur, the cluster consisting of a_{34} and b_{24}, also the LCC, survives.

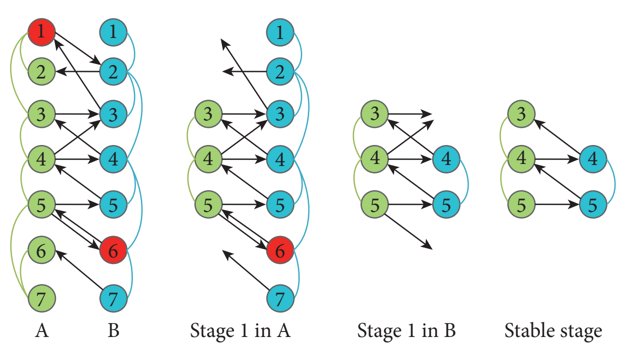

Figure 3: Dynamic network model for analyzing the robustness of multilayer networked systems with one-to-many correlations. Arrows represent the support links connecting a support node in one network to the dependent node in the other network. Initially, the attacks are on node 1 in A and node 6 in B. For network A in Stage 1, node 7 fails because it has no support links, and nodes 2 and 6 are eliminated because they do not belong to the LCC of A. For network B in Stage 1, nodes 1, 2, and 7 fail because of no support, and node 3 is removed since it is not in the LCC of B. The above processes continue until a stable state is reached.

$$\begin{cases} P_A^{\infty} = p_A\left[1 - G_{A0}\left(1 - P_B^{\infty}\right)\right], \\ P_B^{\infty} = p_B\left[1 - G_{B0}\left(1 - P_A^{\infty}\right)\right]. \end{cases} \quad (13)$$

For one thing, the above equation only works for bipartite networks. For another thing, the extension of the above equation definitely does not fit for multipartite networks because equation (11) is based on the dynamic model shown in Figure 3 which is not applicable to multipartite networks. One more thing is that the derivations of equation (11) are very complicated. Therefore, new models and theories for analyzing the robustness of multipartite networks are desirable, and this is the very motivation of this work.

4. Proposed Models and Theories

4.1. Global Model for Robustness Analysis. Although the models exhibited in Figures 2 and 3 are not feasible for multipartite networked systems, their ideas could provide us inspirations. Equipped with the concept of LCC, we first establish a simple cascading model, which we call it the global model, for multipartite networks.

Definition 6 (global model). Consider an L-partite network G with S_i being its i-th node set. Assume that G is under random attack and $1 - p_i$ fraction of nodes is randomly removed from $S_i \subset G$ for $\forall i \in [1, L]$ and $p_i \in [0, 1]$. The edges attached to the removed nodes are also removed. The removal of nodes and edges fragments G into small parts, and the largest one is regarded as the LCC of G and only the LCC will survive in the final stable stage.

Example 1. Figure 4 gives a graphical example of the global model for defining the way how failures propagate on multipartite networks. We can observe from Figure 4 that the global model practically takes a multipartite network as a whole. When a multipartite network is under attack, the global model directly calculates the LCC in the network.

4.2. Local Model for Robustness Analysis. The global model is simple and straightforward. However, it may not reflect the dynamics of all types of multipartite networks. Inspired by the models proposed in [11, 18], we further develop another cascading model which we call it the local model.

Definition 7 (local model). Consider an L-partite network G with S_i being its i-th node set. Assume that G is under random attack and $1 - p_i$ fraction of nodes is randomly removed from $S_i \subset G$ for $\forall i \in [1, L]$ and $p_i \in [0, 1]$. The edges attached to the removed nodes are also removed. The removal of nodes and edges fragments $G_{12} \subset G$ into small parts, and the nodes outside the LCC of G_{12}, together with their edges, are removed. The removal of nodes and edges further fragments $G_{23} \subset G$ into small parts, and nodes outside the LCC of G_{23} are removed. The node removal and network fragmentation processes repeat recursively on $G_{ij} \subset G$ for all $i \in [1, L - 1]$ until G reaches a stable stage in which no node/edge removal and network fragmentations are possible. Then, the remaining subnetwork in the stable stage is the LCC of G.

Example 2. Figure 5 presents a graphical example of the proposed local model for depicting the dynamic process on a multipartite network under node failures. It can be noticed from Figure 5 that the local model considers the LCC in each bipartite network contained in a multipartite network. In the final stable stage, the local model focuses on the LCC that contains nodes from every partite set of the multipartite network in question.

4.3. Variables and Notations. Before starting depicting our proposed theories for analyzing the robustness of

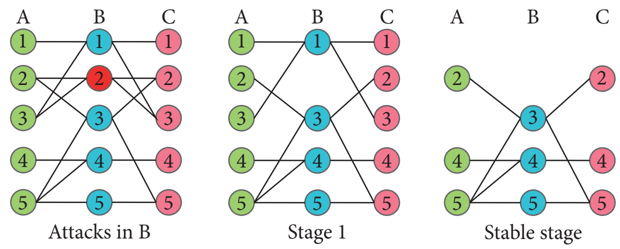

Figure 4: An example of the global model for describing the cascading failures on multipartite networked systems subject to node failures. Initially, node 2 from partite set B of a tripartite network is removed. In Stage 1, the removal of node 2 breaks the focal network into two clusters, and all the nodes that are not in the LCC are removed. In the final stage, only the nodes in the LCC are remaining.

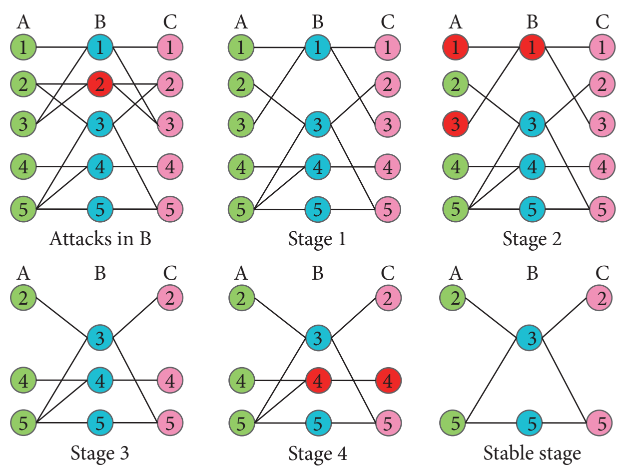

Figure 5: An example of the local model for describing the cascading failures on multipartite networked systems subject to node failures. Initially, node 2 from partite set B is removed. In Stage 1, the local model considers the LCC of the bipartite network G_{AB} containing the nodes in A and B. In Stage 2, the nodes that are not in the LCC of G_{AB} are removed. In Stage 3, the local model considers the LCC of the bipartite network G_{BC} containing the nodes in B and C. In Stage 4, the nodes that are not in the LCC of G_{BC} are removed. The above process continues until no further node removal is possible. The remaining subnetwork in the stable stage is considered as the LCC of the focal network.

multipartite networks, we first list out related variables and notations that will be heavily used in our later derivations.

Given an L-partite network G with S_i being its i-th partite set and $n_i = |S_i|$ being the number of nodes in S_i, let $P_i(k)$ be the degree distribution of the nodes in S_i. Denote $G_{ij} \subset G$ as the bipartite network consisting of partite sets S_i and S_j with $j = \{i-1, i+1\}$. Let $P_{ij}(k)$ be the degree distribution of the nodes in $S_i \subset G_{ij}$ that have connections with nodes in $S_i \subset G_{ij}$.

Based on the definition of excess degree distribution, we, respectively, define the excess degree distributions of $P_i(k)$ and $P_{ij}(k)$ as $P_i^E(k)$ and $P_{ij}^E(k)$. Then, based on the definition of generating function, we define $G_i^0(x) = \sum_{k=0}^{\infty} x^k P_i(k)$ and $G_{ij}^0(x) = \sum_{k=0}^{\infty} x^k P_{ij}(k)$, respectively, as the generating functions for $P_i(k)$ and $P_{ij}(k)$. Analogously, we define $G_i^1(x) = \sum_{k=0}^{\infty} x^k P_i^E(k)$ and $G_{ij}^1(x) = \sum_{k=0}^{\infty} x^k P_{ij}^E(k)$, respectively, as the generating functions for $P_i^E(k)$ and $P_{ij}^E(k)$.

4.4. Percolation Theory Based on the Global Model.

With all the above defined variables, for an L-partite networked system, we propose the following percolation theory for calculating P_i^{∞} with respect to the global model presented in Definition 6.

Definition 8 (probability vector $\mathbf{u}$). Consider the situation that $1-p_i$ fraction of nodes is randomly removed from S_i of an L-partite network G for $\forall i \in [1, L]$ with $p_i \in [0,1]$. For $j \in \{i-1, i+1\}$, define a probability vector $\mathbf{u} = \{u_{12}, u_{21}, \ldots, u_{ij}, \ldots, u_{L,L-1}\}$, with u_{ij} being the probability for a node in S_i not to be connected to the LCC of G via a node in S_j.

Theorem 1 *(percolation theory based on the global model). Consider an L-partite network G with $L \geq 3$; we randomly remove $1-p_i$ fraction of nodes from S_i for $\forall i \in [1, L]$ with $p_i \in [0,1]$. Based on the global model given in Definition 4, G reaches a stable stage. Define the probability vector $\mathbf{u}$. In the limit of $n_i \longrightarrow \infty$, the fraction P_i^{∞} of nodes in S_i that also belongs to the LCC in the stable stage of G is calculated as*

$$\begin{cases} P_1^{\infty} = p_1\left[1 - G_{12}^0(u_{12})\right], \\ P_i^{\infty} = p_i\left[1 - \prod_{\{j=i-1,i+1\}} G_{ij}^0\left(u_{ij}\right)\right], \quad \forall i \in [2, L-1], \\ P_L^{\infty} = p_L\left[1 - G_{L,L-1}^0\left(u_{L,L-1}\right)\right], \end{cases} \tag{14}$$

with the variable $u_{ij} \in \mathbf{u}$ being calculated as

$$\begin{cases} u_{21} = 1 - p_1 + p_1 G_{12}^1(u_{12}), \\ u_{ij} = 1 - p_j + p_j G_{ji}^1\left(u_{ji}\right) G_{j,j+\delta}^1\left(u_{j,j+\delta}\right), \\ u_{L-1,L} = 1 - p_L + p_L G_{L,L-1}^1\left(u_{L,L-1}\right), \end{cases} \tag{15}$$

where $\delta = -1$, if $j = i-1$, and $\delta = 1$, if $j = i+1$.

Proof. We start by considering P_1^{∞}. As the probability for a node $a \in S_1$ not to be connected to the LCC via a node $b \in S_2$ is u_{12}, the probability ϕ_{12} for nodes in S_1 not to be connected to the LCC via nodes in S_2 is calculated as

$$\phi_{12} = \sum_{k=0}^{n_2} P_{12}(k) u_{12}^k. \tag{16}$$

In the limit of $n_i \longrightarrow \infty$, we have

$$\lim_{n_2 \longrightarrow \infty} \phi_{12} = \sum_{k=0}^{\infty} P_{12}(k) u_{12}^k = G_{12}^0(u_{12}). \tag{17}$$

Note that $1-p_i$ fraction of nodes is removed from S_i for $\forall i \in [1, L]$; thus, p_i fraction of nodes is remaining in S_i.

Therefore, in the LCC of G, the remaining fraction of nodes in S_1 is calculated as

$$P_1^{\infty} = p_1(1-\phi_{12}) = p_1\left[1 - G_{12}^0(u_{12})\right]. \quad (18)$$

Analogously, we can prove the correctness of the expression of P_L^{∞} as given in Theorem 1. Now, let us consider P_i^{∞}.

Note that a node $a \in S_i$ can simultaneously have neighbours in S_{i-1} and S_{i+1}. If nodes in S_i do not belong to the LCC, then the nodes in S_i should not be connected to the LCC via their neighbours. Based on the above analysis, we know that the probability $\phi_{i,i-1}$ for nodes in S_i not to be connected to the LCC via nodes in S_{i-1} is

$$\phi_{i,i-1} = \sum_{k=0}^{n_{i-1}} P_{i,i-1}(k) u_{i,i-1}^k. \quad (19)$$

Analogously, the probability $\phi_{i,i+1}$ for nodes in S_i not to be connected to the LCC via nodes in S_{i+1} is

$$\phi_{i,i+1} = \sum_{k=0}^{n_{i+1}} P_{i,i+1}(k) u_{i,i+1}^k. \quad (20)$$

Then, the probability ϕ_i for nodes in S_i not to be connected to the LCC via their neighbours is $\phi_i = \phi_{i,i-1} \cdot \phi_{i,i+1}$. Therefore, in the limit of $n_i \longrightarrow \infty$, we further have

$$\lim_{n_i \longrightarrow \infty} \phi_i = G_{i,i-1}^0(u_{i,i-1}) G_{i,i+1}^0(u_{i,i+1}), \quad (21)$$

based on which we can figure out P_i^{∞} as

$$P_i^{\infty} = p_i(1-\phi_i) = p_i\left[1 - G_{i,i-1}^0(u_{i,i-1}) G_{i,i+1}^0(u_{i,i+1})\right]. \quad (22)$$

To this end, the first part of Theorem 1, i.e., the part regarding P_i^{∞}, is proved. Next, we give the proof for the second part regarding the probability vector $\mathbf{u}$.

We start by analyzing u_{21}, which denotes the probability for the event that a node $b \in S_2$ is not connected to the LCC of G via a node $a \in S_1$ to happen. Note that if node a is removed, which happens with a probability $1 - p_1$, then the above event obviously happens. Now, consider the situation that node a is not removed. Then, the aforementioned event happens if a is not connected to the LCC via its neighbours. As a consequence, in the limit of $n_i \longrightarrow \infty$, we have

$$u_{21} = 1 - p_1 + p_1 \sum_{k=0}^{n_2} P_{12}^E(k) u_{12}^k = 1 - p_1 + p_1 \cdot G_{12}^1(u_{12}). \quad (23)$$

Analogously, we can prove the correctness of the expression of $u_{L-1,L}$. Next, we consider u_{ij} with $j = i+1$.

Note that nodes in S_i have connections with nodes in S_j. The event that a node $a \in S_i$ is not connected to the LCC via a node $b \in S_j$, which happens with the probability u_{ij}, can happen under two situations: (S1) node b has been removed; (S2) node b is not removed and b itself is not connected to the LCC via its own neighbours. Situation S1 happens with the probability $\psi_{S1} = 1 - p_j$. Thus, the key step for working out u_{ij} then lies in the calculation of the probability ψ_{S2} for situation S2 to happen.

Because a node $b \in S_j$ can simultaneously have neighbours in S_i and S_{j+1}, if b is not connected to the LCC via its neighbours, then b should neither be connected to the LCC via nodes in S_i nor via nodes in S_{j+1}. Assume that node b has k neighbours in S_i; then, the probability φ_{ji} for node b not to be connected to the LCC via nodes in S_i is

$$\varphi_{ji} = u_{ji}^k. \quad (24)$$

Analogously, when node b has k neighbours in S_{j+1}, the probability $\varphi_{j,j+1}$ for node b not to be connected to the LCC via nodes in S_{j+1} is

$$\varphi_{j,j+1} = u_{j,j+1}^k. \quad (25)$$

Recall the definition of excess degree distribution; the probabilities for node b to, respectively, have k neighbours in S_i and k neighbours in S_{j+1} are $P_{ji}^E(k)$ and $P_{j,j+1}^E(k)$. As a consequence, the probability ψ_{S2} can be calculated as

$$\psi_{S2} = p_j \sum_{k=0}^{n_i} P_{ji}^E(k) \varphi_{ji} \sum_{k=0}^{n_{j+1}} P_{j,j+1}^E(k) \varphi_{j,j+1}. \quad (26)$$

Note that situations S1 and S2 are two independent events; therefore, in the limit of $n_i \longrightarrow \infty$, the probability u_{ij} is calculated as

$$\begin{aligned} u_{ij} &= \psi_{S1} + \psi_{S2} \\ &= 1 - p_j + p_j G_{ji}^1(u_{ji}) G_{j,j+1}^1(u_{j,j+1}). \end{aligned} \quad (27)$$

The proof of u_{ij} for $j = i-1$ is omitted as it is analogous to that of u_{ij} for $j = i+1$ as presented above. To this end, Theorem 1 is proved. □

Remark 5. Note that when analyzing ψ_{S2}, it is easy to derive the wrong expression of ψ_{S2} in the following way:

$$\begin{aligned} \psi_{S2} &= p_j \sum_{k=0}^{\infty} P_j^E(k) \sum_{m=0}^{k} \binom{k}{m} u_{ji}^m u_{j,j+1}^{k-m} \\ &= p_j G_j^1(u_{ji} + u_{j,j+1}). \end{aligned} \quad (28)$$

The idea of the above derivations is that m out of k neighbours of a node $b \in S_j$ are not connected to the LCC via nodes in S_i, and the remaining $k - m$ neighbours of b are not connected to the LCC via nodes in S_{j+1}. However, the event that node b has k neighbours in S_i and the event that node b has k neighbours in S_{j+1} are independent. Therefore, variable m does not need to run over k for $k \in [0, \infty)$.

Remark 6. From Theorem 1, one can easily derive the percolation theory for L-partite networked systems with $L = 2$, i.e., bipartite networked systems. Specifically, one can have

$$\begin{cases} P_1^{\infty} = p_1\left[1 - G_{12}^0(u_{12})\right], \\ P_2^{\infty} = p_2\left[1 - G_{21}^0(u_{21})\right], \end{cases} \quad (29)$$

with u_{12} and u_{21}, respectively, being calculated as

$$\begin{cases} u_{12} = 1 - p_2 + p_2 G_{21}^1(u_{21}), \\ u_{21} = 1 - p_1 + p_1 G_{12}^1(u_{12}). \end{cases} \quad (30)$$

4.5. Percolation Theory Based on the Local Model. When calculating the LCC, the global model takes a multipartite network as a whole while the local model by contrast analyzes its subnetworks. The local model requires that the final LCC should encompass nodes from every partite set. Therefore, the above proposed percolation theory does not work for multipartite networked systems with respect to the local model. In what follows we elucidate our proposed percolation theory based on the local model.

Theorem 2 *(percolation theory based on the local model). Consider an L-partite network G with $L \geq 3$; we randomly remove $1 - p_i$ fraction of nodes from S_i for $\forall i \in [1, L]$ with $p_i \in [0, 1]$. Based on the local model given in Definition 5, G reaches a stable stage. Define the probability vector* $\mathbf{u}$. *In the limit of $n_i \longrightarrow \infty$, the fraction P_i^∞ of nodes in S_i which also belongs to the LCC in the stable stage of G is calculated as*

$$\begin{cases} P_1^\infty = p_1\left[1 - G_{12}^0(u_{12})\right], \\ P_i^\infty = p_i \prod_{j=\{i-1,i+1\}} \left[1 - G_{ij}^0\left(u_{ij}\right)\right], \quad i \in [2, L-1], \\ P_L^\infty = p_L\left[1 - G_{L,L-1}^0\left(u_{L,L-1}\right)\right], \end{cases} \quad (31)$$

with the variable u_{ij} being calculated as

$$\begin{cases} u_{21} = 1 - p_1 + p_1 G_{12}^1(u_{12}), \\ u_{ij} = 1 - p_j\left[1 - G_{ji}^1\left(u_{ji}\right)\right]\left[1 - G_{j,j+\delta}^1\left(u_{j,j+\delta}\right)\right], \\ u_{L-1,L} = 1 - p_L + p_L G_{L,L-1}^1\left(u_{L,L-1}\right), \end{cases} \quad (32)$$

where $\delta = -1$, if $j = i - 1$, and $\delta = 1$, if $j = i + 1$.

Proof. We only present the proof for P_i^∞, since the proofs for P_1^∞ and P_L^∞ are exactly the same as that presented in Theorem 1. As mentioned earlier, a node $a \in S_i$ can simultaneously have neighbours in S_{i-1} and S_{i+1}. Based on the local model given in Definition 5, we see that as long as there exists one neighbour of node a that connects a to the LCC, then node a definitely belongs to the LCC. According to the proof of Theorem 1, we know that in the limit of $n_i \longrightarrow \infty$, the probability $\phi_{i,i-1}$ for nodes in S_i not to be connected to the LCC via nodes in S_{i-1} is

$$\phi_{i,i-1} = G_{i,i-1}^0\left(u_{i,i-1}\right). \quad (33)$$

Analogously, we have $\phi_{i,i+1} = G_{i,i+1}^0(u_{i,i+1})$. Therefore, the probability for node a to be connected to the LCC via at least one neighbour in S_{i-1} is $1 - \phi_{i,i-1}$. Analogously, the probability for node a to be connected to the LCC via at least one neighbour in S_{i+1} is $1 - \phi_{i,i+1}$. Consequently, the probability $\overline{\phi_i}$ for nodes in S_i to be connected to the LCC via their neighbours is calculated as

$$\overline{\phi_i} = 1 - \phi_i = \left(1 - \phi_{i,i-1}\right)\left(1 - \phi_{i,i+1}\right). \quad (34)$$

Therefore, in the limit of $n_i \longrightarrow \infty$, P_i^∞ is calculated as

$$\begin{aligned} P_i^\infty &= p_i \overline{\phi_i} = p_i\left(1 - \phi_{i,i-1}\right)\left(1 - \phi_{i,i+1}\right) \\ &= p_i \prod_{j=\{i-1,i+1\}} \left[1 - G_{ij}^0\left(u_{ij}\right)\right]. \end{aligned} \quad (35)$$

To this end, the correctness of P_i^∞ as given in Theorem 2 is proved. Next, we give the proof for the probability variable u_{ij}. The proofs for variables u_{21} and $u_{L-1,L}$ are omitted as they are the same as that given in the proof for Theorem 1. We first analyze u_{ij} with $j = i + 1$.

Recall that the variable u_{ij} denotes the probability that a node $a \in S_i$ is not connected to the LCC via a node $b \in S_j$. Analogous to what are analyzed in the proof for Theorem 1, the event that $a \in S_i$ is not connected to the LCC via $b \in S_j$ also happens under two situations: (S1) node b is removed, which happens with the probability $\psi_{S1} = 1 - p_j$; (S2) b remains and b is not connected to the LCC via its neighbours. Note that the probability ψ_{S2} for situation S2 to happen cannot be calculated in the way demonstrated in the proof of Theorem 1. The reason is that the local model requires that the LCC contains nodes from S_i for $\forall i \in [1, L]$, while the global model does not require this condition.

Because node $b \in S_j$ has neighbours in S_i and S_{j+1}, the probabilities φ_{ji} and $\varphi_{j,j+1}$ are, respectively, calculated as $\varphi_{ji} = u_{ji}^k$ and $\varphi_{j,j+1} = u_{j,j+1}^k$. Therefore, the probability φ_j for $b \in S_j$ neither to be connected to the LCC via neighbours in S_i nor via nodes in S_{j+1} is

$$\begin{aligned} \varphi_i = &\sum_{k=0}^{n_i} P_{ji}^E(k)\varphi_{ji} + \sum_{k=0}^{n_{j+1}} P_{j,j+1}^E(k)\varphi_{j,j+1} \\ &- \sum_{k=0}^{n_i} P_{ji}^E(k)\varphi_{ji} \sum_{k=0}^{n_{j+1}} P_{j,j+1}^E(k)\varphi_{j,j+1}. \end{aligned} \quad (36)$$

In the limit of $n_i \longrightarrow \infty$, φ_i can be further calculated as

$$\begin{aligned} \lim_{n_i \longrightarrow \infty} \varphi_i = &G_{ji}^1\left(u_{ji}\right) + G_{j,j+1}^1\left(u_{j,j+1}\right) \\ &- G_{ji}^1\left(u_{ji}\right)G_{j,j+1}^1\left(u_{j,j+1}\right). \end{aligned} \quad (37)$$

As $\psi_{S2} = p_j\varphi_i$, in the limit of $n_i \longrightarrow \infty$, the probability u_{ij} is calculated as

$$\begin{aligned} u_{ij} &= \psi_{S1} + \psi_{S2} = 1 - p_j + p_j\varphi_i \\ &= 1 - p_j\left[1 - G_{ji}^1\left(u_{ji}\right)\right]\left[1 - G_{j,j+1}^1\left(u_{j,j+1}\right)\right]. \end{aligned} \quad (38)$$

Based on the same token shown above, we can work out u_{ij} for $j = i - 1$ as

$$u_{ij} = 1 - p_j\left[1 - G_{ji}^1\left(u_{ji}\right)\right]\left[1 - G_{j,j-1}^1\left(u_{j,j-1}\right)\right], \quad (39)$$

and therefore Theorem 2 is proved. □

5. Numerical Simulations

5.1. Random Multipartite Networks. The proposed theories exhausted in Section 4 theoretically investigate the robustness of multipartite networks with arbitrary degree distributions in face of random node failures. Here we generate multipartite networks with Poisson degree distributions to validate the correctness of the proposed theories. The reasons for doing so are twofold. First, generating multipartite networks with Poisson degree distributions is easy to implement. Second, a Poisson distribution has very good mathematical properties. To be specific, for a Poisson distribution $P(k) = e^{-\langle k\rangle}(\langle k\rangle^k/k!)$ with $\langle k\rangle$ being its expectation, it is easy to prove that

$$G_0(x) = \sum_{k=0}^{\infty} x^k P(k) = \sum_{k=0}^{\infty} x^k e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!} = e^{\langle k\rangle(x-1)}, \tag{40}$$

$$G_1(x) = \frac{G_0'(x)}{G_0'(1)} = e^{\langle k\rangle(x-1)} = G_0(x). \tag{41}$$

Given an empty L-partite network G, for all $i \in [1, L-1]$, we construct G by connecting each pair of nodes with one from S_i and the other one from S_{i+1} with a predefined probability $r_i = (d_i/N)$, where d_i is a constant. Then, it is easy to figure out that the degree distributions $P_{ij}(k)$ and $P_{ji}(k)$ for nodes in $S_i, S_j \subset G_{ij}$ comply with the following Poisson distributions:

$$P_{ij}(k) = \binom{n_j}{k} r_i^k (1-r_i)^{n_j-k} \approx e^{-\langle k_{ij}\rangle}\frac{\langle k_{ij}\rangle^k}{k!}, \tag{42}$$

$$P_{ji}(k) = \binom{n_i}{k} r_i^k (1-r_i)^{n_i-k} \approx e^{-\langle k_{ji}\rangle}\frac{\langle k_{ji}\rangle^k}{k!}, \tag{43}$$

where $\langle k_{ij}\rangle = n_j r_i = (n_j \cdot d_i/N)$ and $\langle k_{ji}\rangle = n_i r_j = (n_i \cdot d_i/N)$.

5.2. Validation for Theorem 1. Without loss of generality, in the simulations, we consider L-partite networks with $L = 3$, i.e., tripartite networks. For a tripartite network G generated in the way illustrated in the previous section, when $1 - p_i$ fraction of nodes is randomly removed from $S_i \subset G$ for all $i \in [1, 3]$ and the cascading failures propagate on G based on the defined global model, then Theorem 1 leads to the following simplified equations:

$$\begin{aligned} P_1^{\infty} &= p_1\left[1 - e^{\langle k_{12}\rangle(u_{12}-1)}\right],\\ P_2^{\infty} &= p_2\left[1 - e^{\langle k_{23}\rangle(u_{23}-1)}e^{\langle k_{21}\rangle(u_{21}-1)}\right],\\ P_3^{\infty} &= p_3\left[1 - e^{\langle k_{32}\rangle(u_{32}-1)}\right], \end{aligned} \tag{44}$$

$$\begin{aligned} u_{21} &= 1 - p_1 + p_1 e^{\langle k_{12}\rangle(u_{12}-1)},\\ u_{23} &= 1 - p_3 + p_3 e^{\langle k_{32}\rangle(u_{32}-1)},\\ u_{12} &= 1 - p_2 + p_2 e^{\langle k_{23}\rangle(u_{23}-1)}e^{\langle k_{21}\rangle(u_{21}-1)},\\ u_{32} &= 1 - p_2 + p_2 e^{\langle k_{23}\rangle(u_{23}-1)}e^{\langle k_{21}\rangle(u_{21}-1)}. \end{aligned} \tag{45}$$

For simplicity, in the simulations, we set the parameters of a tripartite network to be $n_1 = n_2 = n_3 = 5 \times 10^4$, $d_1 = d_2 = d$, and $d = \{3, 6, 9, 12, 15\}$. As a consequence, we have $\langle k_{12}\rangle = \langle k_{21}\rangle = \langle k_{23}\rangle = \langle k_{32}\rangle = \langle k\rangle = d/3$.

Because the parameter p_i affects the robustness of multipartite networks, in order to better demonstrate the simulation results, here we consider two simple cases: (C1) $p_1 = p_2 = p_3 = p$, i.e., we randomly remove the same fraction of nodes from each partite set; (C2) $p_1 = p$, $p_2 = p_3 = 1$, i.e., we only remove nodes from partite set S_1.

Under case 1, the critical value p_c becomes

$$p_c = \sqrt{\frac{1}{\langle k_{23}\rangle\langle k_{32}\rangle + \langle k_{21}\rangle\langle k_{12}\rangle}}. \tag{46}$$

Under case 2, the critical value $p_c = 0$. Interested readers are encouraged to discover this conclusion by themselves.

Figure 6 shows the simulation and theoretical results on the robustness of tripartite networks based on the global model. The theoretical results shown in Figure 6 are obtained by solving equations (44) and (45). During the simulations, p ranges from 0 to 1 at an interval of 0.025. It can be clearly seen from Figure 6 that the theoretical results coincide quite well with the simulations. Under case 1, the critical value p_c given in equation (46) has the simplified form of $p_c = (3/\sqrt{2}\,d)$. Under case 2, $p_c = 0$, which indicates that the focal networks are extremely robust to random node failures. It can be observed from Figure 6 that the values of p_c are in accordance with that of the simulations. The results shown in Figure 6 indicate that multipartite networks are robust to node failures when the global model is of concern.

5.3. Validation for Theorem 2. For a tripartite network with Poisson degree distributions, Theorem 2 leads to the following simplified equations:

$$\begin{aligned} P_1^{\infty} &= p_1\left[1 - e^{\langle k_{12}\rangle(u_{12}-1)}\right],\\ P_2^{\infty} &= p_2\left[1 - e^{\langle k_{23}\rangle(u_{23}-1)}\right]\left[1 - e^{\langle k_{21}\rangle(u_{21}-1)}\right],\\ P_3^{\infty} &= p_3\left[1 - e^{\langle k_{32}\rangle(u_{32}-1)}\right], \end{aligned} \tag{47}$$

$$\begin{aligned} u_{21} &= 1 - p_1 + p_1 e^{\langle k_{12}\rangle(u_{12}-1)},\\ u_{23} &= 1 - p_3 + p_3 e^{\langle k_{32}\rangle(u_{32}-1)},\\ u_{12} &= u_{32} = 1 - p_2 + p_2\Big[e^{\langle k_{23}\rangle(u_{23}-1)} + e^{\langle k_{21}\rangle(u_{21}-1)}\\ &\qquad - e^{\langle k_{23}\rangle(u_{23}-1)}e^{\langle k_{21}\rangle(u_{21}-1)}\Big]. \end{aligned} \tag{48}$$

Figure 7 shows the simulation and theoretical results on the robustness of tripartite networks based on the local model. The theoretical results shown in Figure 7 are obtained by solving equations (47) and (48) with the same parameter settings presented in the previous section. The results recorded in Figure 7 also demonstrate that the proposed theory coincides quite well with the simulations.

By comparing Figures 6 and 7, we can see that the robustness of multipartite networks with respect to the local model shows first-order phase transition, which indicates that multipartite networks with the local model are vulnerable to perturbations. Both Figures 6 and 7 show that larger mean degrees will enhance the networks' robustness.

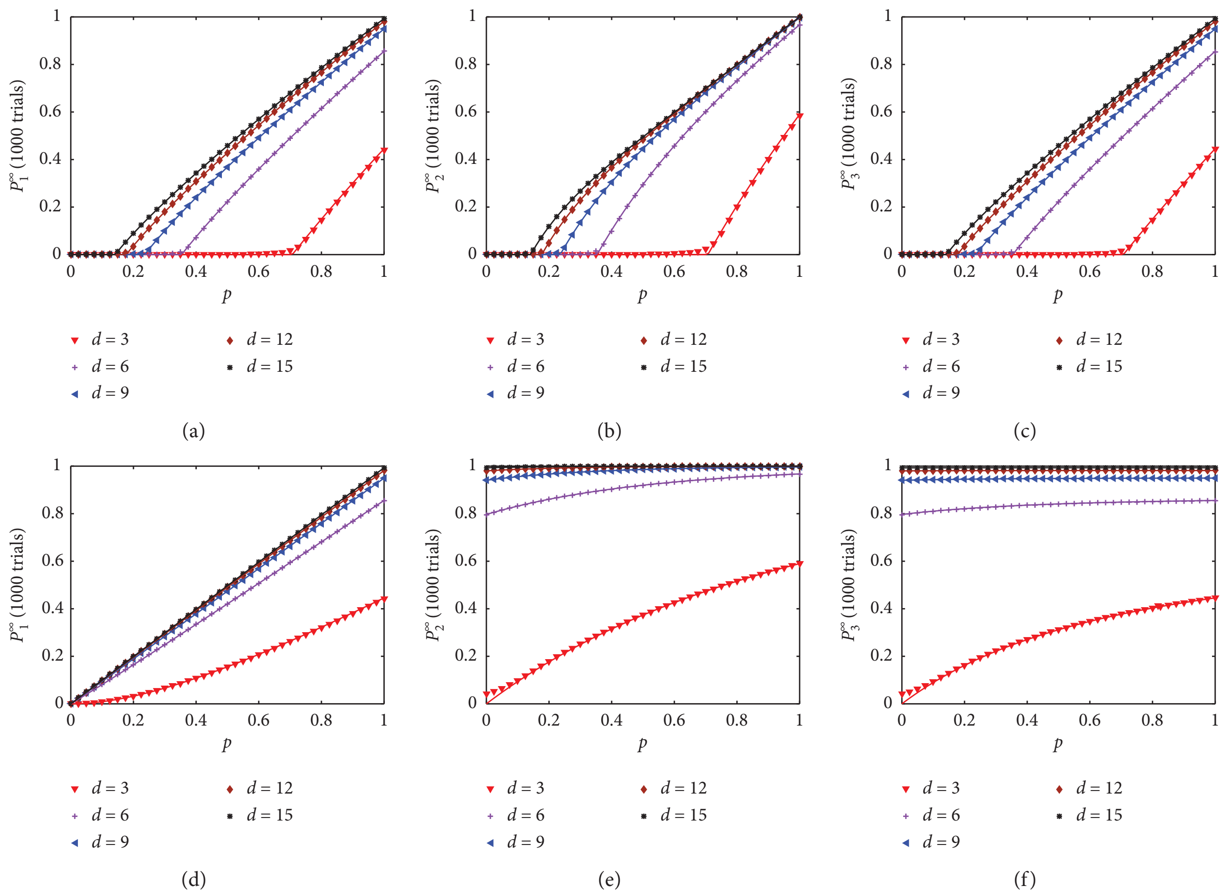

Figure 6: Robustness of tripartite networks based on the global model. Lines denote the theoretical results while symbols represent the simulation results. The first and second rows of the figure, respectively, represent the case that the node removal occurs to S_i for all $i \in [1, 3]$ and for $i = 1$. Simulation results are averaged over 1000 trials.

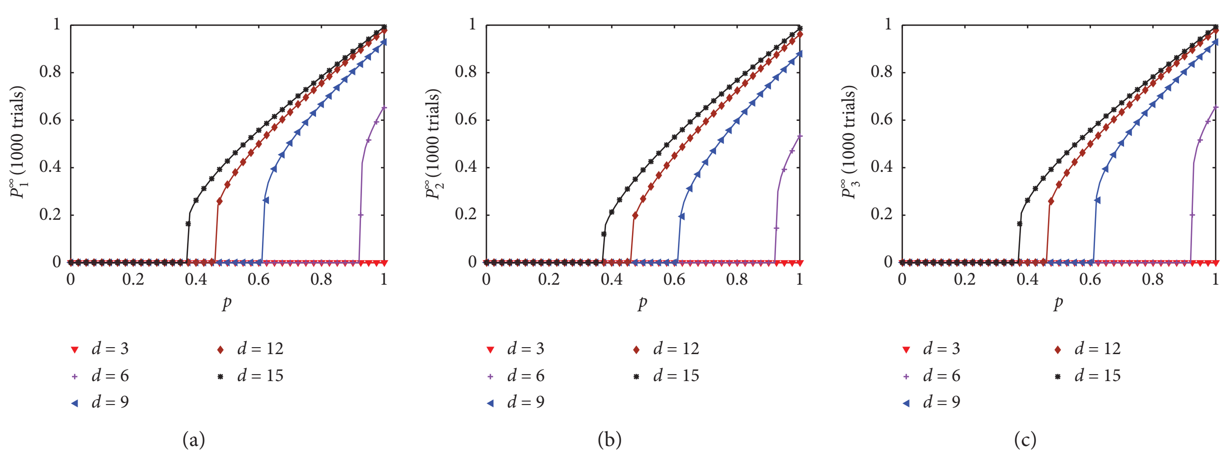

Figure 7: Continued.

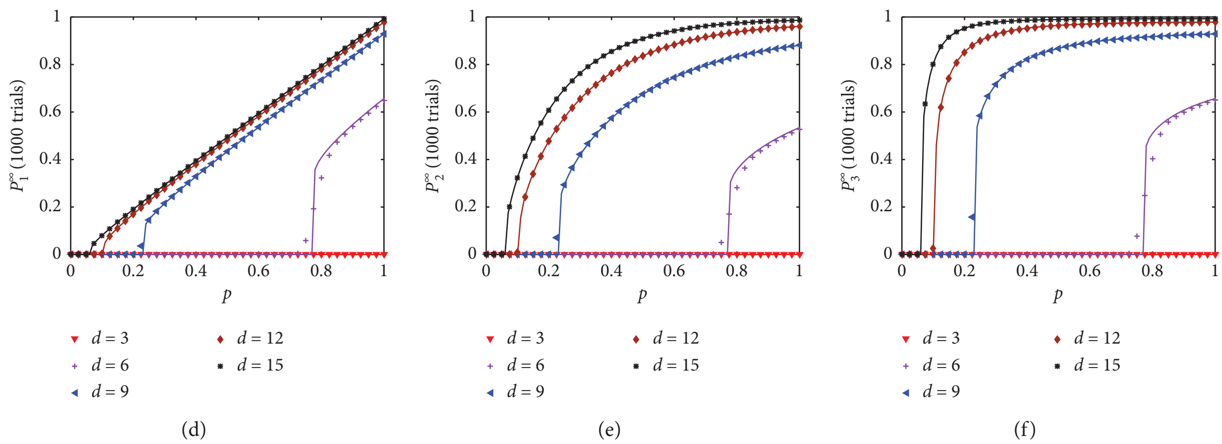

Figure 7: Robustness of tripartite networks based on the local model. Lines denote the theoretical results while symbols represent the simulation results. The first and second rows of the figure, respectively, represent the case that the node removal occurs to S_i for all $i \in [1, 3]$ and for $i = 1$. Simulation results are averaged over 1000 trials.

6. Conclusion

Investigating the robustness of complex systems under perturbations is pivotal. Network robustness analysis provides a potent instrument towards that purpose. While the majority of existing studies on network robustness analysis focus on multilayer networked systems, this paper theoretically studied the robustness of multipartite networked systems. This paper first established two network models for depicting the cascading failures on multipartite networked systems in face of node failures. Equipped with the established network models together with the largest connected component concept, this paper then developed the corresponding percolation theories for analyzing the robustness of multipartite networked systems under random node failures. The proposed theories uncovered the second-order and first-order phase transition phenomena on the robustness of multipartite networked systems. The correctness of the proposed theories had been validated through simulations on multipartite networks with Poisson degree distributions.

Note that complex systems in reality can suffer from target attacks. Although this paper only investigates the robustness of multipartite networked systems under random perturbations, the proposed models and theories provide scientific insights for the target attack scenarios. Meanwhile, the proposed theories could shed new lights on the optimal structure design of robustness network and/or networked systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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