Hindawi
Complexity
Volume 2020, Article ID 4374508, 9 pages
https://doi.org/10.1155/2020/4374508

Research Article

The d-Shadowing Property and Average Shadowing Property for Iterated Function Systems

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Received 25 August 2019; Revised 4 December 2019; Accepted 25 January 2020; Published 29 April 2020

Academic Editor: Mojtaba Ahmadieh Khanesar

In this paper, we introduce the definitions of $\overline{d}$-shadowing property, $\underline{d}$-shadowing property, topological ergodicity, and strong ergodicity of iterated function systems $\mathrm{IFS}(f_0, f_1)$. Then, we show the following: (1) if $\mathrm{IFS}(f_0, f_1)$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), then $\mathscr{F}^k$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property) for any $k \in \mathbf{Z}^+$; (2) if $\mathscr{F}^k$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property) for some $k \in \mathbf{Z}^+$, then $\mathrm{IFS}(f_0, f_1)$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property); (3) if $\mathrm{IFS}(f_0, f_1)$ has the $\overline{d}$-shadowing property or $\underline{d}$-shadowing property, and f_0 or f_1 is surjective, then $\mathrm{IFS}(f_0, f_1)$ is chain mixing; (4) let f_0, f_1 be open maps. For $\mathrm{IFS}(f_0, f_1)$ with the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), if $A \subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s \in A$, then $\mathrm{IFS}(f_0, f_1)$ is strongly ergodic, and hence, $\mathscr{F}^k$ is strongly ergodic; and (5) for $\mathrm{IFS}(f_0, f_1)$ with the average shadowing property, if $S \subset X$ is dense in X, and s is a quasi-weakly almost periodic point of f_0 or f_1 for any $s \in S$, then $\mathrm{IFS}(f_0, f_1)$ is ergodic.

1. Introduction

In this paper, let $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$. Suppose that X is a compact metric space and $f: X \longrightarrow X$ a continuous map. The set $J \subset \mathbf{N}$ is a syndetic set if there is $N_0 \in \mathbf{Z}^+$ such that $[n, n + N_0] \cap J \neq \varnothing$ for each $n \in \mathbf{N}$. For any $x \in X$, $\varepsilon > 0$, let $B(x, \varepsilon)$ denote the ε-neighborhood of x. $x \in X$ is a *minimal point* of f if for any neighborhood U of x, the set $N(x, U) = \{n \in \mathbf{N}\colon f^n(x) \in U\}$ is syndetic, and the set of all minimal points of f is denoted by $AP(f)$. $x \in X$ is called a *quasi-weakly almost periodic point* of f if for any neighborhood U of x, the set $N(x, U) = \{n \in \mathbf{N}\colon f^n(x) \in U\}$ has positive upper density.

The shadowing property is a very important notion in dynamical systems. Many researchers have found some relationship among various shadowing properties, chain transitivity, transitivity, and ergodicity. Gu [1] proved that if (X, f) has the asymptotic average shadowing property, and f is surjective, then (X, f) is chain transitive. For more recent results about various shadowing properties, one can refer to [2–9] and references therein.

A δ-*ergodic-pseudo-orbit* of f is a sequence $\{x_i\}_{i\geq 0}$ such that for any $i \in \mathbf{N}$,

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\left|\{i \in \mathbf{N} : 0 \leq i < n, d\left(f(x_i), x_{i+1}\right) < \delta\}\right| = 1, \qquad (1)$$

where $|\cdot|$ represents the cardinality.

f is said to have the $\overline{d}$-*shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-ergodic-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ε-shadowed by a true orbit $\{f^i(z)\}_{i\geq 0}$ in a way such that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n}\left|\left\{i \in \mathbf{N}\colon 0 \leq i < n, d\left(f^i(z), x_i\right) < \varepsilon\right\}\right| > \frac{1}{2}, \qquad (2)$$

where f is said to have the $\underline{d}$-*shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-ergodic-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ε-shadowed by a true orbit $\{f^i(z)\}_{i\geq 0}$ in a way such that

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\{i\in\mathbf{N}\colon 0\le i<n, d(f_i(z),x_i)<\varepsilon\}\right|>0. \quad (3)$$

Let X be a metric space and f_0, f_1 be continuous maps on X. The iterated function system IFS(f_0, f_1) is the action of the semigroup generated by $\{f_0, f_1\}$ on X. In this paper, we introduce the definitions of the $\overline{d}$-shadowing property and $\underline{d}$-shadowing property for IFS(f_0, f_1).

An *orbit* of IFS(f_0, f_1) is a sequence $\{f^i_\omega(x)\}_{i\ge0}$, where $\omega=\omega_0\omega_1\omega_2\ldots\in\Sigma^2=\{\alpha=\alpha_0\alpha_1\alpha_2\ldots\colon \alpha_i\in\{0,1\}\}$, and for any $i\in\mathbf{N}$,

$$f^i_\omega(x)=f_{\omega_{i-1}}\circ\cdots\circ f_{\omega_0}(x),$$
$$f^0_\omega(x)=x. \quad (4)$$

A sequence $\{\xi_i\}_{i\ge0}$ is called a *δ-ergodic-pseudo-orbit* for IFS(f_0, f_1) if there is $\omega\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta\right\}\right|=1. \quad (5)$$

IFS(f_0, f_1) is said to have the *$\overline{d}$-shadowing property* if for any $\varepsilon>0$, there is $\delta>0$ such that every δ-ergodic-pseudo-orbit $\{x_i\}_{i\ge0}$ is ε-shadowed by a true orbit $\{f^i_\omega(z)\}_{i\ge0}$ in a way such that

$$\limsup_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f^i_\omega(z),\xi_i\right)<\varepsilon\right\}\right|>\frac{1}{2}. \quad (6)$$

IFS(f_0, f_1) is said to have the *$\underline{d}$-shadowing property* if for any $\varepsilon>0$, there is $\delta>0$ such that every δ-ergodic-pseudo-orbit $\{\xi_i\}_{i\ge0}$ is ε-shadowed by a true orbit $\{f^i_\omega(z)\}_{i\ge0}$ in a way such that

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f^i_\omega(z),\xi_i\right)<\varepsilon\right\}\right|>0. \quad (7)$$

Denote $\mathscr{F}^k=\{X\colon f_{\omega_{k-1}}\circ\cdots\circ f_{\omega_1}\circ f_{\omega_0}\colon \omega_0,\omega_1,\ldots,\omega_{k-1}\in\{0,1\}\}$, where $k\in\mathbf{Z}^+$.

In this paper, the definitions of the shadowing property and average shadowing property for IFS(f_0, f_1) are introduced by Bahabadi [10], and the definition of the asymptotic average shadowing property for IFS(f_0, f_1) is introduced by Nia [11]. Let f_0, f_1 be open maps. For IFS(f_0, f_1) with the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), if $A\subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s\in A$, then IFS(f_0, f_1) is strongly ergodic. Under similar conditions, Niu [2] researched that (X, f) has the average shadowing property and then the conclusion is true. Wang and Niu showed that, for (X, f) with the average shadowing property, if $S\subset X$ is dense in X, and s is a quasi-weakly almost periodic point of f for any $s\in S$, then f is transitivity (see [12]). However, under similar conditions, we will prove that IFS(f_0, f_1) is ergodic. Then, we will come up with a situation that IFS(f_0, f_1) does not have the asymptotic average shadowing property.

According to Bahabadi [10], a finite sequence $\xi_0=x,\ldots,\xi_k=y$ is called a δ-chain of IFS(f_0, f_1) if for any $i=0,\ldots,k-1$, there is $\omega_i\in\{0,1\}$ such that $d(f_{\omega_i}(\xi_i),\xi_{i+1})<\delta$.

Definition 1 (see [10]). IFS(f_0, f_1) is as follows:

(1) Chain transitive if for any $x, y\in X$ and any $\delta>0$, there is a δ-chain of IFS(f_0, f_1) from x to y

(2) Transitive if for any nonempty open sets $U, V\subset X$, there are $\omega\in\Sigma^2$ and $n\in\mathbf{N}$ such that $f^n_\omega(U)\cap V\ne\varnothing$

(3) Mixing if for any nonempty open sets $U, V\subset X$, there are $\omega\in\Sigma^2$ and $N\in\mathbf{N}$ such that $f^n_\omega(U)\cap V\ne\varnothing$ for any $n\ge N$

(4) Chain mixing if for any $x, y\in X$ and any $\delta>0$, there is $N\in\mathbf{Z}^+$ such that for any $n\ge N$, there is a δ-chain of IFS(f_0, f_1) from x to y consisting of exactly n elements

For any $A\subset\mathbf{N}$, define the positive upper density of A by

$$\overline{d}(A)=\limsup_{n\longrightarrow\infty}\frac{1}{n}|A\cap\{0,1,\ldots,n-1\}|. \quad (8)$$

Define the lower density of A by

$$\underline{d}(A)=\liminf_{n\longrightarrow\infty}\frac{1}{n}|A\cap\{0,1,\ldots,n-1\}|. \quad (9)$$

If $A\subset\mathbf{N}$, A^c is the complementary set of A.

ω in the pseudo-orbit of the shadowing property,$\overline{d}$-shadowing property, and $\underline{d}$-shadowing property for IFS(f_0, f_1) is the same as the one chosen in the shadowing orbit, while the two ωs in the definitions of the average shadowing property and asymptotic average shadowing property may be different.

2. The $\overline{d}$-Shadowing Property and $\underline{d}$-Shadowing Property for IFS(f_0, f_1)

Bahabadi [10] introduced the definition of the shadowing property for IFS(f_0, f_1). In this paper, we will introduce the definitions of the $\overline{d}$-shadowing property and $\underline{d}$-shadowing property for IFS(f_0, f_1) and give some results.

Definition 2. A sequence $\{\xi_i\}_{i\ge0}$ is called a δ-pseudo-orbit for IFS(f_0, f_1) if there is $\omega\in\Sigma^2$ such that for any $i\in\mathbf{N}$,

$$d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta. \quad (10)$$

Definition 3. IFS(f_0, f_1) has the shadowing property if for any $\varepsilon>0$, there is $\delta>0$ such that every δ-pseudo-orbit $\{\xi_i\}_{i\ge0}$ is ε-shadowed by a point $z\in X$, i.e., there is $z\in X$ such that for any $i\in\mathbf{N}$,

$$d\left(f^i_\omega(z),\xi_i\right)<\varepsilon. \quad (11)$$

Example 1. Suppose that $f_0(x)=0$, $f_1(x)=1$, $x\in[0,1]$. For any $\varepsilon>0$, there is $\delta=\varepsilon>0$. Let $\{\xi_i\}_{i\ge0}$ be a δ-ergodic-pseudo-orbit for IFS(f_0, f_1). Therefore, there is $\omega=\omega_0\omega_1\omega_2\cdots\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta\right\}\right|=1. \quad (12)$$

Then, there is $z=\xi_0$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_\omega^i(z),\xi_i\right)<\varepsilon\right\}\right|=1. \tag{13}$$

Obviously, IFS(f_0, f_1) has the $\overline{d}$-shadowing property and $\underline{d}$-shadowing property.

Lemma 1. *For any $k\in\mathbf{Z}^+$, and any $\varepsilon>0$, there is $0<\delta<(\varepsilon/k)$ such that there is any δ-chain of length $k+1$ for IFS$(f_0, f_1)\{\xi_0,\xi_1,\ldots,\xi_k\}$, i.e., there is $\omega\in\Sigma^2$ such that $d(f_{\omega_i}(\xi_i),\xi_{i+1})<\delta$ satisfies*

$$d\left(f_\omega^i(\xi_0),\xi_i\right)<\varepsilon,\quad \forall\, 0\le i\le k. \tag{14}$$

Proof. Fix $k\in\mathbf{Z}^+$ and let $\varepsilon>0$. Since f_0 and f_1 are uniformly continuous, for any $\omega^1=\omega_0^1\omega_1^1\cdots\in\Sigma^2$, $f_{\omega^1}^i$ is also uniformly continuous, where $i=0,1,\ldots,k$. Then for $(\varepsilon/k)>0$, there is $0<\delta<(\varepsilon/k)$ such that $d(f_{\omega^1}^i(x), f_{\omega^1}^i(y))<(\varepsilon/k)$ for any $x,y\in X$ satisfies $d(x,y)<\delta$. Since $\{\xi_0,\xi_1,\ldots,\xi_k\}$ is a δ-chain of IFS(f_0, f_1), there is $\omega\in\Sigma^2$ such that

$$d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta,\quad \le i\le k-1. \tag{15}$$

Put $\omega^1=\omega$. Then,

$$\begin{aligned}
&d\left(f_\omega^j(\xi_0),\xi_j\right)\\
&\le d\left(f_{\omega_{j-1}}\cdots f_{\omega_1}f_{\omega_0}(\xi_0), f_{\omega_{j-1}}\cdots f_{\omega_1}(\xi_1)\right)+\cdots+\\
&\quad d\left(f_{\omega_{j-1}}\left(\xi_{j-1}\right),\xi_j\right)\\
&\quad<\frac{\varepsilon}{k}+\frac{\varepsilon}{k}+\cdots+\frac{\varepsilon}{k}+\delta<k\cdot\frac{\varepsilon}{k}=\varepsilon,
\end{aligned} \tag{16}$$

where $0\le j\le k$.

Theorem 1

(1) If IFS(f_0, f_1) has the $\underline{d}$-shadowing property, then $\mathscr{F}^k$ has the $\underline{d}$-shadowing property for any $k\in\mathbf{Z}^+$

(2) If IFS(f_0, f_1) has the $\underline{d}$-shadowing property, then $\mathscr{F}^k$ has the $\overline{d}$-shadowing property for any $k\in\mathbf{Z}^+$

Proof. Fix $k\in\mathbf{Z}^+$ and let $\varepsilon>0$. We can find a number $\delta>0$ satisfying Lemma 1. We will show that (1) is true. We can find a number $\delta_1>0$ such that each δ_1-ergodic-pseudo-orbit of IFS(f_0, f_1) is δ-shadowed by a true orbit along a set with positive lower density, and $\delta>\delta_1$. Let $\{x_i\}_{i\ge 0}$ be a δ_1-ergodic-pseudo-orbit of $\mathscr{F}^k$. That is to say, there is $\omega\in\Sigma^2$ such that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\frac{1}{n}\Big|\Big\{i\in\mathbf{N}\colon 0\le i<n, d\Big(&f_{\omega_{(i+1)k-1}}\circ\cdots\circ\\
&f_{\omega_{ik}}(x_i),x_{i+1}\Big)<\delta_1\Big\}\Big|=1.
\end{aligned} \tag{17}$$

Let

$$\begin{aligned}
\{\xi_i\}_{i\ge 0}=\Big\{&x_0, f_{\omega_0}(x_0),\ldots, f_{\omega_{k-2}}\circ\cdots\circ f_{\omega_0}(x_0), x_1,\\
&f_{\omega_k}(x_1),\ldots, f_{\omega_{2k-2}}\circ\cdots\circ f_{\omega_k}(x_1),\ldots, x_n, f_{\omega_{nk}}(x_n),\\
&\ldots, f_{\omega_{(n+1)k-2}}\circ\cdots\circ f_{\omega_{nk}}(x_n),\ldots\Big\}.
\end{aligned} \tag{18}$$

Obviously, $\{\xi_i\}_{i\ge 0}$ is a δ_1-ergodic-pseudo-orbit of IFS(f_0, f_1). Then, there is $z\in X$ such that

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_\omega^i(z),\xi_i\right)<\delta\right\}\right|>0. \tag{19}$$

Put $A=\{i\in\mathbf{N}\colon d(f_\omega^{ki}(z),x_i)<\varepsilon\}$, $A_1=\{i\in\mathbf{N}\colon d(f_\omega^i(z),\xi_i)<\delta\}$, $B_1=$ %$\{i\in\mathbf{N}\colon d(f_{\omega_{(i+1)k-1}}\circ\cdots\circ f_{\omega_{ik}}(x_i),x_{i+1})<$ δ_1%$\}$, *an* $dM=\{i=kl+j\colon l\in B_1, j=1,\cdots,k\}$.

Obviously, $\underline{d}(A_1)>0, d(M)=1, d(B_1)=1$, and then $\underline{d}(A_1\cap M)=\underline{d}(A_1)>0$. Now, to prove the theorem, we need to show the following claim.

Claim 1. Put $(|A\cap\{0,1,\ldots,n-1\}|/n)=(s_n/n)$ and $(|A_1\cap M\cap\{0,1,\ldots,nk-1\}|-k/nk)=(t_n/nk)$, then $(s_n/n)\ge(t_n/nk)$.

Proof. Without loss of generality, suppose that $d(z,x_0)<\varepsilon$, then the claim is true when $n=1$. Assume $(s_n/n)\ge(t_n/nk)$, then we will show that $(s_{n+1}/n+1)\ge(t_{n+1}/(n+1)k)$. For any $m=kp+q\in A_1\cap M\cap\{0,\ldots,nk-1\}$, where $p\in\mathbf{N}$, $0<q\le k$, we have $d(f_\omega^m(z),\xi_m)<\delta$, and the sequence $\left\{f_\omega^{m-1}(z),\xi_m,\cdots,\xi_{k(p+1)-1},x_{p+1}\right\}$ is a δ-chain. According to Lemma 1, $d(f_\omega^{k(p+1)}(z),x_{p+1})<\varepsilon$. So, $p+1\in s_{n+1}$. Therefore, $s_{n+1}\ge(|A_1\cap M\cap\{0,1,\ldots,nk\}|/k)$ and $(s_{n+1}/n+1)\ge(|A_1\cap M\cap\{0,1,\cdots,(n+1)k-1\}|-k/(n+1)k)=(t_{n+1}/(n+1)k)$

Obviously,

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty}\frac{|A_1\cap M\cap\{0,1,\ldots,nk-1\}|-k}{nk}\\
&\quad=\liminf_{n\longrightarrow\infty}\frac{|A_1\cap M\cap\{0,1,\ldots,nk-1\}|}{nk}.
\end{aligned} \tag{20}$$

Then, $\underline{d}(A)\ge\underline{d}(A_1\cap M)>0$. So (1) is true. Now suppose that IFS(f_0, f_1) has the $\overline{d}$-shadowing property. In this case, the proof is the same as for (1) except that $\overline{d}(A_1\cap M)=\overline{d}(A_1)>(1/2)$. Then, we can see that $\overline{d}(A)>(1/2)$.

Theorem 2

(1) If $\mathscr{F}^k$ has the $\underline{d}$-shadowing property for some $k\in\mathbf{Z}^+$, then IFS(f_0, f_1) has the $\underline{d}$-shadowing property

(2) If $\mathscr{F}^k$ has the $\overline{d}$-shadowing property for some $k\in\mathbf{Z}^+$, then IFS(f_0, f_1) has the $\overline{d}$-shadowing property

Proof. Suppose that $\mathscr{F}^k$ has the $\underline{d}$-shadowing property for some $k\in\mathbf{Z}^+$. Let $\varepsilon>0$, and we can find a number $\varepsilon_1>0$ satisfying Lemma 1. Since $\mathscr{F}^k$ has the $\underline{d}$-shadowing property, there is $\delta_1>0$ such that any δ_1-ergodic-pseudo-orbit of $\mathscr{F}^k$ is

ε_1-shadowed by a true orbit along a set with positive lower density, and $\varepsilon_1 > \delta_1$. For δ_1, we can get $\delta > 0$ by Lemma 1. Let $\{\xi_i\}_{i\geq 0}$ be a δ-ergodic-pseudo-orbit of IFS(f_0, f_1), and $\{x_i\}_{i\geq 0} = \{\xi_{ki}\}_{i\geq 0}$. Therefore, there is $\omega \in \Sigma^2$ such that

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\left|\left\{i \in \mathbf{N}: 0 \leq i < n, d\left(f_{\omega_i}(\xi_i), \xi_{i+1}\right) < \delta\right\}\right| = 1. \quad (21)$$

Let $M = \{i \in \mathbf{N}: 0 \leq i < n, d(f_{\omega_{ik+j}}(\xi_{ik+j}), \xi_{ik+j+1}) < \delta, j = 0, \ldots, k-1\}$, then

$$\lim_{n\longrightarrow\infty} \frac{1}{n}|M| = 1. \quad (22)$$

Therefore, if $i \in M$, then the sequence $\{\xi_{ik}, \xi_{ik+1}, \ldots, \xi_{(i+1)k}\}$ is a δ-chain. According to Lemma 1, $d(f_{\omega_{(i+1)k-1}} \circ \cdots \circ f_{\omega_{ik}}(x_i), \ x_{i+1}) = d(f_{\omega_{(i+1)k-1}} \circ \cdots \circ f_{\omega_{ik}}(\xi_{ik}), \xi_{(i+1)k}) < \delta_1$, so $\{x_i\}_{i\geq 0}$ is a δ_1-ergodic-pseudo-orbit of $\mathscr{F}^k$. There is $z \in X$ such that

$$\liminf_{n\longrightarrow\infty} \frac{1}{n}\left|\left\{i \in \mathbf{N}: 0 \leq i < n, d\left(f_\omega^{ki}(z), x_i\right) < \varepsilon_1\right\}\right| > 0. \quad (23)$$

Put $A = \{i \in \mathbf{N}: d(f_\omega^{ki}(z), x_i) < \varepsilon_1\}$ and $A_1 = \{i \in \mathbf{N}: d(f_\omega^{i}(z), \xi_i) < \varepsilon\}$, then $\underline{d}(A\cap M) = \underline{d}(A) > 0$. There is $m \in A\cap M$ such that $d(f_\omega^{km}(z), \xi_{km}) = d(f_\omega^{km}(z), x_m) < \varepsilon_1$, and then the sequence $\{f_\omega^{km-1}(z), x_m, \xi_{km+1}, \cdots, \xi_{km+k-1}\}$ is a ε_1-chain. According to Lemma 1, we have $d(f_\omega^{km+j}(z), \xi_{km+j}) < \varepsilon$, where $j = 0, \ldots, k-1$. Now, to prove the theorem, we need to show the following claim.

Claim 2. Put $(|A\cap M\cap\{0, 1, \ldots, n-1\}|/n) = (s_n/n)$ and $(|A_1\cap\{0, 1, \ldots, nk-1\}|/nk) = (t_n/nk)$, then $(t_n/nk) \geq (s_n/n)$.

Proof. Without loss of generality, suppose that $1 \in M$ and $d(z, x_0) < \varepsilon_1$, then the claim is true when $n = 1$. Assume that $(t_n/nk) \geq (s_n/n)$, then we will show that $(t_{n+1}/(n+1)k) \geq (s_{n+1}/n+1)$. Obviously, $s_{n+1} = s_n$ or $s_{n+1} = s_n + 1$. If $s_{n+1} = s_n + 1$, then $d(f_\omega^{kn}(z), x_n) < \varepsilon_1$, and we can see that $d(f_\omega^{kn+j}(z), \xi_{kn+j}) < \varepsilon$, where $j = 0, \ldots, k-1$. So $t_{n+1} = t_n + k$ and $(t_{n+1}/(n+1)k) \geq (s_{n+1}/n+1)acs$. If $s_{n+1} = s_n$, then $t_{n+1} \geq t_n$ and $(t_{n+1}/(n+1)k) \geq (s_{n+1}/n+1)$.

Then, $\underline{d}(A_1) \geq \underline{d}(A\cap M) > 0$. So (1) is true. Now suppose that $\mathscr{F}^k$ has the $\overline{d}$-shadowing property for some $k \in \mathbf{Z}^+$. In this case, the proof is the same as for (1) except that $\overline{d}(A\cap M) = \overline{d}(A) > (1/2)$. Then, we can see that $\overline{d}(A_1) > (1/2)$.

Corollary 1. *Let IFS(f_0, f_1) be a iterated function system, then the following statements are equivalent:*

(1) IFS(f_0, f_1) has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property)

(2) $\mathscr{F}^k$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property) for some $k \in \mathbf{Z}^+$

(3) $\mathscr{F}^k$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property) for any $k \in \mathbf{Z}^+$

Lemma 2. *Let $A, B \subset \mathbf{N}$; (1) if $\underline{d}(A) = \alpha$ and $\overline{d}(B) > 1-\alpha$, then $\overline{d}(A\cap B) > 0$; (2) if $\overline{d}(A) = \alpha$ and $\underline{d}(B) > 1-\alpha$, then $\overline{d}(A\cap B) > 0$. Here, $0 \leq \alpha \leq 1$.*

Proof. Firstly, we will show (1). Suppose on the contrary that $\overline{d}(A\cap B) = 0$. Then, $\overline{d}(B) \leq \overline{d}(A^c)$. Therefore, we can see $\underline{d}(A) + \overline{d}(A^c) \geq \underline{d}(A) + \overline{d}(B) > \alpha + 1 - \alpha = 1$. But $\underline{d}(A) + \overline{d}(A^c) = 1$, which contradicts the hypothesis. Hence, $\overline{d}(A\cap B) > 0$. The proof for the second case is the same as for (1).

Theorem 3. *f_0 or f_1 is surjective,*

(1) If IFS(f_0, f_1) has the $\underline{d}$-shadowing property, then IFS(f_0, f_1) is chain transitive

(2) If IFS(f_0, f_1) has the $\overline{d}$-shadowing property, then IFS(f_0, f_1) is chain transitive

Proof. Without loss of generality, suppose that f_1 is surjective. Then, there is a sequence $\{y_i\}_{i\geq 0} = \{y_0 = y, y_1, y_2, \cdots\}$ such that for any $i > 0$, $f_1(y_i) = y_{i-1}$. Let $\varepsilon > 0$ be arbitrary, and $x, y \in X$. Firstly, we will prove that (1) is true. We can find a number $\delta > 0$ as in the definition of the $\underline{d}$-shadowing property for IFS(f_0, f_1). Put $a_1 = 2$, $a_i = 2^{a_1+\cdots+a_{i-1}}$. Construct a sequence as follows: $\{\xi_i\}_{i\geq 0} = \{x, f_0(x), \quad f_0^{a_1}(x), y_{a_2}, y_{a_2-1}, \cdots, y, x, f_0(x), \cdots, f_0^{a_3}(x), y_{a_4}, \cdots, y, \cdots\}$. Then, $\{\xi_i\}_{i\geq 0}$ is a δ-ergodic-pseudo-orbit for IFS(f_0, f_1), i.e., there is $\omega = \omega_0\omega_1\omega_2\cdots \in \Sigma^2$ such that

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\left|\left\{i \in \mathbf{N}: 0 \leq i < n, d\left(f_{\omega_i}(\xi_i), \xi_{i+1}\right) < \delta\right\}\right| = 1, \quad (24)$$

where

$$\omega_i = \begin{cases} 0, & d(f_0(\xi_i), \xi_{i+1}) < \delta, \\ 1, & d(f_1(\xi_i), \xi_{i+1}) \geq \delta. \end{cases} \quad (25)$$

Put $A_1 = \left\{i \in \mathbf{N}: \xi_i \in \left\{f_0^j(x)\right\}_{j\geq 0}\right\}$ and $A_2 = \{i \in \mathbf{N}: \xi_i \in \left\{f_1^j(y)\right\}_{j\geq 0}\}$. Then, $\overline{d}(A_1) = 1, \overline{d}(A_2) = 1$. Since IFS$(f_0, f_1)$ has the $\underline{d}$-shadowing property, there is $z \in X$ such that

$$\liminf_{n\longrightarrow\infty} \frac{1}{n}\left|\left\{i \in \mathbf{N}: 0 \leq i < n, d\left(f_\omega^i(z), \xi_i\right) < \varepsilon\right\}\right| > 0. \quad (26)$$

According to Lemma 2, there are $i_0, j_0, l, s \in \mathbf{N}, l < s$ such that

$$d\left(f_\omega^l(z), \xi_l\right) < \varepsilon, d\left(f_\omega^s(z), \xi_s\right) < \varepsilon, \xi_l = f_0^{i_0}(x), \xi_s = y_{j_0}. \quad (27)$$

So $\{x, f_0(x), f_0^2\ (x), \cdots, f_0^{i_0-1}(x), f_\omega^l(z), \cdots, f_\omega^{s-1}(z), y_{j_0}, y_{j_0-1}, \cdots, y_1, y\}$ is an ε-chain from x to y. Hence, IFS(f_0, f_1) is chain transitive.

Then, we will prove that (2) is true. Construct a sequence as follows:

$$\{\xi_i\}_{i\geq 0} = \left\{x, y, x, f_0(x), y_1, y, x, f_0(x), f_0^2(x), y_2, y_1, y, \cdots\right\}. \quad (28)$$

Then, $\{\xi_i\}_{i\geq 0}$ is a δ-ergodic-pseudo-orbit for IFS(f_0, f_1), i.e., there is $\omega = \omega_0\omega_1\omega_2\cdots \in \Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta\right\}\right|=1, \quad (29)$$

where

$$\omega_i=\begin{cases}0, & d\left(f_0(\xi_i),\xi_{i+1}\right)<\delta,\\ 1, & d\left(f_1(\xi_i),\xi_{i+1}\right)\ge\delta.\end{cases} \quad (30)$$

Then, $\underline{d}(A_1)=(1/2), \underline{d}(A_2)=(1/2)$. Since $\mathrm{IFS}(f_0,f_1)$ has the $\overline{d}$-shadowing property, there is $z\in X$ such that

$$\limsup_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_\omega^i(z),\xi_i\right)<\varepsilon\right\}\right|>\frac{1}{2}. \quad (31)$$

Then, the proof of this case is the same as for (1).

Theorem 4. *If $\mathrm{IFS}(f_0,f_1)$ has the $\underline{d}$-shadowing property or $\overline{d}$-shadowing property, and f_0 or f_1 is surjective, then $\mathrm{IFS}(f_0,f_1)$ is chain mixing.*

Proof. The result is obtained by Corollary 1, Theorem 3, and Theorem 2.3 in [13].

Lemma 3 (see [10]). *$\mathrm{IFS}(f_0,f_1)$ is chain transitive and has the shadowing property, and then $\mathrm{IFS}(f_0,f_1)$ is transitive.*

Corollary 2. *$\mathrm{IFS}(f_0,f_1)$ has the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property) and the shadowing property, one of f_0,f_1 is surjective, and then $\mathrm{IFS}(f_0,f_1)$ is mixing.*

Proof. The result is obtained by Theorem 4 and Lemma 3.

$\mathrm{IFS}(f_0,f_1)$ is ergodic if for any pair of nonempty open subsets $U,V\subset X$, there is $\omega\in\Sigma^2$ such that $N(U,V)=\{n\in\mathbf{N}\colon f_\omega^n(U)\cap V\ne\varnothing\}$ has the positive upper density. $\mathrm{IFS}(f_0,f_1)$ is strongly ergodic if for any pair of nonempty open subsets $U,V\subset X$, there is $\omega\in\Sigma^2$ such that $N(U,V)=\{n\in\mathbf{N}\colon f_\omega^n(U)\cap V\ne\varnothing\}$ is a syndetic set.

A map $f\colon X\longrightarrow X$ is an open map if for any open set $U\subset X$, $f(U)$ is also an open set.

Theorem 5. *Let f_0,f_1 be open maps. For $\mathrm{IFS}(f_0,f_1)$ with the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), if $A\subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s\in A$, then $\mathrm{IFS}(f_0,f_1)$ is strongly ergodic.*

Proof. Suppose that U and V are two nonempty open subsets of X, then there are $x\in U\cap A, y\in V\cap A$, *an* $d\,\varepsilon>0$ such that $B(x,\varepsilon)\subset U, B(y,\varepsilon)\subset V$ and J_x^p, J_y^q are syndetic sets, where $J_x^p=\{i\in N\colon f_p^i(x)\in B(x,(\varepsilon/2))\}, J_y^q=\{i\in N\colon f_q^i(y)\in B(y,(\varepsilon/2))\}$, $p,q=0$ or 1. Since J_x^p, J_y^q are syndetic sets, we can find $N_x,N_y\in\mathbf{Z}^+$ such that $J_x^p\cap[n,n+N_x]\ne\varnothing, J_y^q\cap[n,n+N_y]\ne\varnothing$ for any $n\in\mathbf{N}$. Firstly, we will prove if $\mathrm{IFS}(f_0,f_1)$ has the $\underline{d}$-shadowing property, then the conclusion is true. Put $N_0=\max\left\{N_x,N_y\right\}, a_0=0, a_1=N_0$, $a_2=2^{a_0+a_1},\ldots,a_n=2^{a_0+a_1+a_2+\cdots+a_{n-1}}$, *an* $d\,E_n=a_0+a_1+\cdots+a_n$. Put

$$E=\bigcup_{n=0}^{+\infty}\{E_{2n},E_{2n}+1,\ldots,E_{2n+1}-1\},$$
$$F=\bigcup_{n=0}^{+\infty}\{E_{2n+1},E_{2n+1}+1,\ldots,E_{2n+2}-1\}. \quad (32)$$

We can see that $\overline{d}(E)=1$ and $\overline{d}(F)=1$. For $(\varepsilon/2)>0$, we can find a number $\varepsilon_1>0$ satisfying Lemma 1, where $k=N_0+1$. There is $\delta>0$ such that every δ-ergodic-pseudo-orbit of $\mathrm{IFS}(f_0,f_1)$ is ε_1-shadowed by a true orbit along a set with positive lower density, and $\varepsilon_1>\delta$. Put $\{\xi_i\}_{i\ge0}$ as follows:

$$\xi_i=\begin{cases}f_p^{i-E_{2n}}(x), & E_{2n}\le i<E_{2n+1},\\ f_q^{i-E_{2n+1}}(y), & E_{2n+1}\le i<E_{2n+2}.\end{cases} \quad (33)$$

That is to say, $\{\xi_i\}_{i\ge0}=\left\{x,f_p(x),\cdots,f_p^{a_1-1}(x),y,f_q(y),\cdots,f_q^{a_2-1}(y),x,\cdots,f_p^{a_3-1}(x),\cdots\right\}$. Then, $\{\xi_i\}_{i\ge0}$ is a δ-ergodic-pseudo-orbit for $\mathrm{IFS}(f_0,f_1)$, i.e., there is $\omega=\omega_0\omega_1\omega_2\cdots\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)<\delta\right\}\right|=1, \quad (34)$$

where

$$\omega_i=\begin{cases}p, & \xi_i=f_p^{i-E_{2n}}(x),\\ q, & \xi_i=f_q^{i-E_{2n+1}}(y).\end{cases} \quad (35)$$

Since $\mathrm{IFS}(f_0,f_1)$ has the $\underline{d}$-shadowing property, there is $z\in X$ such that

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_\omega^i(z),\xi_i\right)<\varepsilon_1\right\}\right|>0. \quad (36)$$

Let $B=\{0\le i<n\colon d(f_\omega^i(z),\xi_i)<\varepsilon_1\}$, then $\underline{d}(B)>0$. According to Lemma 2, both $B\cap E$ and $B\cap F$ are infinite. Therefore, we can find $n_0\in B\cap E$ and $m_0\in B\cap F$ such that $E_{2t_0+1}-n_0\ge 2N_0$ and $E_{2s_0+2}-m_0\ge 2N_0$, where $t_0,s_0\in\mathbf{N}$, $t_0<s_0$. Suppose on the contrary that for any $n_0\in B\cap E$, $E_{2t+1}-n_0<2N_0$, where $t\in\mathbf{N}$. We can see

$$\begin{aligned}\underline{d}(B)&\\ &=\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}\colon 0\le i<n, d\left(f_\omega^i(z),\xi_i\right)<\varepsilon_1\right\}\right|\\ &\le\lim_{t\longrightarrow\infty}\frac{1}{E_{2t+1}}\left|B\cap\{0,1,\ldots,E_{2t+1}-1\}\right|\\ &\le\lim_{t\longrightarrow\infty}\frac{a_0+2N_0+a_2+2N_0+a_4+\cdots+a_{2t}+2N_0}{E_{2t+1}}\\ &\le\lim_{t\longrightarrow\infty}\frac{a_0+a_1+\cdots+a_{2t}+2N_0}{E_{2t+1}}\\ &=\lim_{t\longrightarrow\infty}\frac{a_0+a_1+\cdots+a_{2t}+2N_0}{a_0+a_1+\cdots+a_{2t}+a_{2t+1}}\\ &=\lim_{t\longrightarrow\infty}\frac{a_0+a_1+\cdots+a_{2t}+2N_0}{a_0+a_1+\cdots+a_{2t}+2^{a_0+a_1+\cdots+a_{2t}}}\\ &=0,\end{aligned} \quad (37)$$

which is in contrast with $\underline{d}(B)>0$. Another proof can be obtained similarly. Since $n_0\in B\cap E$ and $m_0\in B\cap F$, $d(f_\omega^{n_0}(z),\xi_{n_0})<\varepsilon_1$ and $d(f_\omega^{m_0}(z),\xi_{m_0})<\varepsilon_1$, where $\xi_{n_0}=f_p^{n_0-E_{2t_0}}(x)$ and $\xi_{m_0}=f_q^{m_0-E_{2s_0+1}}(y)$. Owing to $x,y\in A$, there are $0\le n^*$ and $m^*\le N_0$ such that $d(f_p^{n_0-E_{2t_0}+n^*}(x),x)<(\varepsilon/2)$ and $d(f_q^{m_0-E_{2s_0+1}+m^*}(y),y)<(\varepsilon/2)$. Obviously, $\{f_\omega^{n_0-1}(z),\xi_{n_0},\xi_{n_0+1},\cdots,\xi_{n_0+n^*},\cdots,\xi_{n_0+N_0}\}$ and $\{f_\omega^{m_0-1}(z),\xi_{m_0},\xi_{m_0+1},\cdots,\xi_{m_0+m^*},\cdots,\xi_{m_0+N_0}\}$ are ε_1-chains. According to Lemma 1, $d(f_\omega^{n_0+n^*}(z), f_p^{n_0-E_{2t_0}+n^*}(x))<\varepsilon/2andd(f_\omega^{m_0+m^*}(z), f_q^{m_0-E_{2s_0+1}+m^*}(y))<(\varepsilon/2)$. Therefore, $d(f_\omega^{n_0+n^*}(z),x)<\varepsilon$and $d(f_\omega^{m_0+m^*}(z),y)<\varepsilon$. Then, $f_\omega^{n_0+n^*}(z)\in B(x,\varepsilon)\subset U$ and $f_\omega^{m_0+m^*}(z)\in B(y,\varepsilon)\subset V$. Obviously, $m_0+m^*>n_0+n^*$. Put $N^*=m_0+m^*-(n_0+n^*)$ and $\omega^1=\omega_0^1\omega_1^1\cdots\omega_{N^*-1}^1\cdots=\omega_{n_0+n^*}\cdots\omega_{m_0+m^*-1}\cdots$, then $f_{\omega^1}^{N^*}(f_\omega^{n_0+n^*}(z))=f_\omega^{m_0+m^*}(z)\in B(y,\varepsilon)\subset V$. Hence, $f_{\omega^1}^{N^*}(U)\cap V\neq\varnothing$.

We write $W=f_{\omega^1}^{N^*}(U)\cap V\neq\varnothing$. As f_0,f_1 are open sets, $f_{\omega^1}^{N^*}(U)$ is an open set. Therefore, W is an open set. Then, there are $z^*\in W\cap A$ and $\omega^2\in\Sigma^2$ such that $J=\{n\in\mathbf{N}: f_{\omega^2}^n(z^*)\in W\}$ is a syndetic set. For any $m\in J$, $f_{\omega^2}^m(W)\cap W\neq\varnothing$. There is $\omega^3=\omega_0^3\omega_1^3\cdots=\omega_0^1\omega_1^1\cdots\omega_{N^*-1}^1\omega^2\in\Sigma^2$ such that $\varnothing\neq f_{\omega^2}^m(W)\cap W=f_{\omega^2}^m(f_{\omega^1}^{N^*}(U)\cap V)\cap f_{\omega^1}^{N^*}(U)\cap V\subset f_{\omega^3}^{N^*+m}(U)\cap V$. Since $\{N^*+m: m\in J\}\subset N(U,V)=\{n\in\mathbf{N}: f_{\omega^3}^{N^*+m}(U)\cap V\neq\varnothing\}$, $N(U,V)$ is a syndetic set. As U and V are arbitrary, IFS(f_0,f_1) is strongly ergodic.

Then, we will prove if IFS(f_0,f_1) has the $\overline{d}$-shadowing property, then the conclusion is true. Put $a_0=0, a_1=N_0$, $a_2=2\cdot N_0,\ldots,a_n=n\cdot N_0$, and $E_n=a_0+a_1+\cdots+a_n$.

$$E=\bigcup_{n=0}^{+\infty}\{E_{2n},E_{2n}+1,\cdots,E_{2n+1}-1\},$$
$$F=\bigcup_{n=0}^{+\infty}\{E_{2n+1},E_{2n+1}+1,\cdots,E_{2n+2}-1\}. \tag{38}$$

We can see that $\underline{d}(E)=(1/2)$ and $\underline{d}(F)=(1/2)$. Then, the proof of this case is the same as for case one.

Theorem 6. *Let f_0,f_1 be open maps. For IFS(f_0,f_1) with the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), if $A\subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s\in A$, then $\mathcal{F}^k$ is strongly ergodic.*

Proof. According to the condition, for any $s\in S$, s is a minimal point of f_0 or f_1. For any $k\in\mathbf{Z}^+$, it is well known that $AP(f)=AP(f^k)$ and then s is a minimal point of f_0^k or f_1^k. We can combine Corollary 1 and Theorem 5. Then, $\mathcal{F}^k$ is strongly ergodic.

A point $x\in X$ is called a stable point of IFS(f_0,f_1) if for any $\varepsilon>0$, there is $\delta>0$ satisfying $d(f_\omega^n(x),f_\omega^n(y))<\varepsilon$ for any $y\in X$ with $d(x,y)<\delta$ and any $\omega\in\Sigma^2, n\in\mathbf{N}$. IFS$(f_0,f_1)$ is called Lyapunov stable if any point of X is a stable point of IFS(f_0,f_1).

Theorem 7. *For IFS(f_0,f_1) with the $\underline{d}$-shadowing property (respectively, $\overline{d}$-shadowing property), if IFS(f_0,f_1) is Lyapunov stable, and f_0 or f_1 is surjective, then IFS(f_0,f_1) is transitive. Hence, $\mathcal{F}^k$ is transitive for any $k\in\mathbf{Z}^+$.*

Proof. Suppose that U and V are two nonempty open subsets of X, then there are $x\in U, y\in V$, and $\varepsilon>0$ such that $B(x,\varepsilon)\subset U$ and $B(y,\varepsilon)\subset V$, where $x,y\in X$ are stable points of IFS(f_0,f_1). There is $\varepsilon_1>0$ such that for any $u,v\in X$,

$$d(u,v)<\varepsilon_1, \text{then } d(f_\omega^n(u),f_\omega^n(v))<\frac{\varepsilon}{2}, \quad \text{for all } n=0,1,2\ldots. \tag{39}$$

Without loss of generality, suppose that f_1 is surjective. Then, there is a sequence $\{y_{-j}\}_{j\ge0}$ such that $y_{-j+1}=f(y_{-j})$ for all $j\in\mathbf{Z}^+$ and $y_0=y$. Firstly, we will prove if IFS(f_0,f_1) has the $\underline{d}$-shadowing property, then the conclusion is true. Put $a_0=0, a_1=1, a_2=2^{a_0+a_1}, \ldots, a_n=2^{a_0+a_1+a_2+\cdots+a_{n-1}}$, and $e_n=a_0+a_1+\cdots+a_n$ Put

$$E=\bigcup_{n=0}^{+\infty}\{e_{2n},e_{2n}+1,\cdots,e_{2n+1}-1\},$$
$$F=\bigcup_{n=0}^{+\infty}\{e_{2n+1},e_{2n+1}+1,\cdots,e_{2n+2}-1\}, \tag{40}$$

then we can see that $\overline{d}(E)=1$ and $\overline{d}(F)=1$. For $\varepsilon_1>0$, we can find a number $\delta>0$ as in the definition of the $\underline{d}$-shadowing property for IFS(f_0,f_1). Construct a sequence $(\xi_i)_{i\ge0}$ as follows:

$$\xi_i=\begin{cases} f_0^{i-e_{2n}}(x), & e_{2n}\le i<e_{2n+1},\\ y_{i-e_{2n+2}}, & e_{2n+1}\le i<e_{2n+2}.\end{cases} \tag{41}$$

That is to say, $\{\xi_i\}_{i\ge0}=\{x,y_{-1},x,\cdots,f_0^{a_3-1}(x),y_{-a_4},\cdots y_{-1},\cdots\}$. Then, $\{\xi_i\}_{i\ge0}$ is a δ-ergodic-pseudo-orbit for IFS(f_0,f_1), i.e., there is $\omega^1=\omega_0^1\omega_1^1\omega_2^1\cdots\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}: 0\le i<n, d\left(f_{\omega_i^1}(\xi_i),\xi_{i+1}\right)<\delta\right\}\right|=1, \tag{42}$$

where

$$\omega_i^1=\begin{cases} 0, & \xi_i=f_0^{i-e_{2n}}(x),\\ 1, & \xi_i=y_{i-e_{2n+2}}.\end{cases} \tag{43}$$

Since IFS(f_0,f_1) has the $\underline{d}$-shadowing property, there is $z_0\in X$ such that

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\left|\left\{i\in\mathbf{N}: 0\le i<n, d\left(f_{\omega^1}^i(z_0),\xi_i\right)<\varepsilon_1\right\}\right|>0. \tag{44}$$

Let $B=\{0\le i<n: d(f_{\omega^1}^i(z_0),\xi_i)<\varepsilon_1\}$, then $\underline{d}(B)>0$. According to Lemma 2, both $B\cap E$ and $B\cap F$ are infinite. Therefore, there are $n_0\in B\cap E$ and $m_0\in B\cap F$ such that $e_{2t_0}\le n_0<e_{2t_0+1}$ and $e_{2s_0+1}\le m_0<e_{2s_0+2}$, where $t_0,s_0\in\mathbf{N}, t_0<s_0$; then, $d(f_{\omega^1}^{n_0}(z_0),\xi_{n_0})<\varepsilon_1$ and $d(f_{\omega^1}^{m_0}(z_0),\xi_{m_0})<\varepsilon_1$, where $\xi_{n_0}=f_0^{n_0-e_{2t_0}}(x)$ and $\xi_{m_0}=y_{m_0-e_{2s_0+2}}$. According to (2.1), $d(f_{\omega^1}^{e_{2s_0+2}}(z_0),y)=d(f_{\omega^1_{e_{2s_0+2}-1}}\circ\cdots\circ f_{\omega^1_{m_0}}(f_{\omega^1}^{m_0}(z_0)),\quad f_{\omega^1_{e_{2s_0+2}-1}}\circ$

$\cdots \circ f_{\omega^1_{m_0}}(y_{m_0-e_{2s_0+2}})) < (\varepsilon/2)$ and $d(f_{\omega^1_{e_{2s_0+2}-1}} \circ \cdots \circ f_{\omega^1_{n_0}}(f^{n_0}_{\omega^1}(z_0)), f_{\omega^1_{e_{2s_0+2}-1}} \circ \cdots \circ f_{\omega^1_{n_0}}(f_0^{n_0-e_{2t_0}}(x))) = d(f^{e_{2s_0+2}}_{\omega^1}(z_0), f_{\omega^1_{e_{2s_0+2}-1}} \circ \cdots \circ f_{\omega^1_{n_0}}(f_0^{n_0-e_{2t_0}}(x))) = d(f^{e_{2s_0+2}}_{\omega^1}(z_0), f_{\omega^1_{e_{2s_0+2}-1}} \circ \cdots \circ f_{\omega^1_{n_0}}(f_{\omega^1_{n_0-1}} \circ \cdots \circ f_{\omega^1_{e_{2t_0}}}(x))) < (\varepsilon/2)$.

Then, $d(y, f_{\omega^1_{e_{2s_0+2}-1}} \circ \cdots \circ f_{\omega^1_{n_0}}(f_{\omega^1_{n_0-1}} \circ \cdots \circ f_{\omega^1_{e_{2t_0}}}(x))) < \varepsilon$. Put $\omega^2 = \omega^2_0 \omega^2_1 \cdots = \omega^1_{e_{2t_0}} \omega^1_{e_{2t_0+1}} \cdots \in \Sigma^2$ and $N_0 = e_{2s_0+2} - e_{2t_0}$, then $f^{N_0}_{\omega^2}(U) \cap V \neq \varnothing$. Hence, IFS$(f_0, f_1)$ is transitive. According to Corollary 1, $\mathcal{F}^k$ is transitive for any $k \in \mathbf{Z}^+$.

Then, we will prove if IFS(f_0, f_1) has the $\overline{d}$-shadowing property, then the conclusion is true. Put $a_0 = 0, a_1 = 1, a_2 = 2, \ldots, a_n = n$, and $E_n = a_0 + a_1 + \cdots + a_n$.

$$E = \bigcup_{n=0}^{+\infty} \{E_{2n}, E_{2n} + 1, \ldots, E_{2n+1} - 1\},$$
$$F = \bigcup_{n=0}^{+\infty} \{E_{2n+1}, E_{2n+1} + 1, \ldots, E_{2n+2} - 1\}. \quad (45)$$

We can see that $\underline{d}(E) = (1/2)$ and $\underline{d}(F) = (1/2)$. Then, the proof of this case is the same as for case one.

3. The Average Shadowing Property for IFS(f_0, f_1)

Bahabadi [10] introduced the definition of the average shadowing property for IFS(f_0, f_1). In this section, we will research the relationship among the average shadowing property, ergodicity, and strong ergodicity.

Definition 4. A sequence $\{\xi_i\}_{i \geq 0}$ is called a δ-average-pseudo-orbit for IFS(f_0, f_1) if there are $\omega \in \Sigma^2$ and $N \in \mathbf{Z}^+$ such that for every $n \geq N$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d\left(f_{\omega_i}(\xi_i), \xi_{i+1}\right) < \delta. \quad (46)$$

Definition 5. IFS(f_0, f_1) has the average shadowing property (ASP) if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{\xi_i\}_{i \geq 0}$ is ε-shadowed on average by a point $z \in X$, i.e., there is $\omega \in \Sigma^2$ such that

$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i_\omega(z), \xi_i\right) < \varepsilon. \quad (47)$$

Theorem 8. *For IFS(f_0, f_1) with the ASP, if $S \subset X$ is dense in X, and s is a quasi-weakly almost periodic point of f_0 or f_1 for any $s \in S$, then IFS(f_0, f_1) is ergodic.*

Proof. Suppose that $U, V \subset X$ are two nonempty open sets. We can find two points $u \in U$, $v \in V$, and $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$ and $B(v, \varepsilon) \subset V$. There is $\varepsilon_0 > 0$ such that $x \in B(x, \varepsilon_0) \subset B(u, (\varepsilon/2))$, $y \in B(y, \varepsilon_0) \subset B(v, (\varepsilon/2))$, where $x, y \in S$. Therefore, $J^p_x = \left\{i \in \mathbf{N}\colon f^i_p(x) \in B(u, (\varepsilon/2))\right\}$ and $J^q_y = \left\{i \in \mathbf{N}\colon f^i_q(y) \in B(v, (\varepsilon/2))\right\}$ have the positive upper density, where $p, q = 0$ or 1. We can find $k_1, k_2 \in \mathbf{N}$ such that $\overline{d}(J^p_x) = \alpha > (4/k_1) \geq (4/k) > 0$, $\overline{d}(J^q_y) = \beta > 4/k_2 \geq 4/k > 0$, where $k = \max\{k_1, k_2\}$. Since IFS(f_0, f_1) has the ASP, there is $\delta > 0$ such that every δ-average-pseudo-orbit of IFS(f_0, f_1) is $(\varepsilon/2k)$-shadowed on average by a point in X. We can find a number $N_0 \in \mathbf{N}$ such that $(3\ D/N_0) < \delta$, and $N_0 = t \cdot k$, where $D = \mathrm{diam}(X)$, $t \in \mathbf{Z}^+$. Define the sequence $\{\xi_i\}_{i \geq 0}$ as follows: $\{\xi_i\}_{i \geq 0} = \{x, f_p(x), \cdots, f^{N_0-1}_p(x), y, f_q(y), \cdots, f^{N_0-1}_q(y), f^{N_0}_p(x), \cdots, f^{2N_0-1}_p(x), f^{N_0}_q(y), \cdots\}$.

We can choose

$$\omega_i = \begin{cases} p, & d\left(f_p(\xi_i), \xi_{i+1}\right) < \delta, \\ q, & d\left(f_p(\xi_i), \xi_{i+1}\right) \geq \delta. \end{cases} \quad (48)$$

$\omega = \omega_0 \omega_1 \omega_2 \cdots \in \Sigma^2$. Obviously, for $n \geq N_0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d\left(f_{\omega_i}(\xi_i), \xi_{i+1}\right) \leq \frac{[n/N_0] \cdot 3D}{n} \leq \frac{3D}{N_0} < \delta. \quad (49)$$

So $\{\xi_i\}_{i \geq 0}$ is a δ-average-pseudo-orbit for IFS(f_0, f_1). Hence, there are $z \in X$ and $\omega^1 \in \Sigma^2$ such that

$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i_{\omega^1}(z), \xi_i\right) < \frac{\varepsilon}{2k}. \quad (50)$$

Since $\overline{d}(J^p_x) = \alpha > (4/k_1) \geq (4/k) > 0$, $\overline{d}(J^q_y) = \beta > (4/k_2) \geq (4/k) > 0$, $k = \max\{k_1, k_2\}$, we have $\overline{d}(W^p_x) > (2/k)$, $\overline{d}(W^q_y) > (2/k)$, where $W^p_x = \left\{i \in \mathbf{N}\colon \xi_i \in B(u, (\varepsilon/2))\right\}$, $W^q_y = \left\{i \in \mathbf{N}\colon \xi_i \in B(v, (\varepsilon/2))\right\}$. Put $F = \left\{i \in \mathbf{N}\colon d(f^i_{\omega^1}(z), \xi_i) \geq (\varepsilon/2)\right\}$. Then, $\overline{d}(F) < (1/k)$. Suppose on the contrary that $\overline{d}(F) \geq (1/k)$, then

$$\begin{aligned} &\limsup_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d\left(f^i_{\omega^1}(z), \xi_i\right) \\ &\geq \limsup_{n \longrightarrow \infty} \frac{1}{n} \Big(|F \cap \{0, 1, \ldots, n-1\}| \cdot \frac{\varepsilon}{2} \\ &\quad + \left|F^c \cap \{0, 1, \ldots, n-1\}\right| \cdot 0\Big) \\ &\geq \frac{1}{k} \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2k}, \end{aligned} \quad (51)$$

which is in contrast with (50). So $\overline{d}(F) < (1/k)$. Therefore, $\underline{d}(F^c) = 1 - \overline{d}(F) \geq 1 - (1/k) = (k - 1/k)$. According to Lemma 2, $\overline{d}(W^p_x \cap F^c) > 0$ and $\overline{d}(W^q_y \cap F^c) > 0$. There are $i_0, j_0, l, s, s > l$ such that $f^{i_0}_p(x) \in B(u, (\varepsilon/2))$, $f^{j_0}_q(y) \in B(v, (\varepsilon/2))$, and $d(f^l_{\omega^1}(z), \xi_l) < (\varepsilon/2)$, $d(f^s_{\omega^1}(z), \xi_s) < (\varepsilon/2)$, where $\xi_l = f^{i_0}_p(x)$, $\xi_s = f^{j_0}_q(y)$. Then, $d(f^l_{\omega^1}(z), u) < \varepsilon$, $d(f^s_{\omega^1}(z), v) < \varepsilon$. Therefore, $f^l_{\omega^1}(z) \in B(u, \varepsilon) \subset U$ and $f_{\omega^1_{s-1}} \circ \cdots \circ f_{\omega^1_l} f^l_{\omega^1}(z) = f^s_{\omega^1}(z) \in B(v, \varepsilon) \subset V$. Let $n_0 = s - l$, and $\omega^2 = \omega^2_0 \omega^2_1 \cdots = \omega^1_l \omega^1_{l+1} \cdots \omega^1_{s-1} \cdots \in \Sigma^2$, then $N(U, V) = \left\{n_0 \in \mathbf{Z}^+\colon f^{n_0}_{\omega^2}(U) \cap V, \omega^2 \in \Sigma^2\right\} \neq \varnothing$. Obviously,

$$\limsup_{n\longrightarrow\infty}\frac{\left|\left\{n_0\colon n_0=s-l, s>l, l\in W_x^p\cap F^c, s\in W_y^q\cap F^c\right\}\cap\{0,\ldots,n-1\}\right|}{n}>0. \tag{52}$$

Therefore, $\overline{d}(N(U,V))>0$. As U and V are arbitrary, $\mathrm{IFS}(f_0,f_1)$ is ergodic.

Theorem 9. *Let f_0, f_1 be open maps. For $\mathrm{IFS}(f_0,f_1)$ with the ASP, if $S\subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s\in S$, then $\mathrm{IFS}(f_0,f_1)$ is strongly ergodic.*

Proof. Suppose that U and V are two nonempty open subsets of X. Obviously, the syndetic set has the positive upper density. By Theorem 8, there are $n_0\in\mathbf{N}, \omega^2=\omega_0^2\omega_1^2\cdots\omega_{n_0-1}^2\cdots\in\Sigma^2$ such that $f_{\omega^2}^{n_0}(U)\cap V\neq\varnothing$. We write $W=f_{\omega^2}^{n_0}(U)\cap V\neq\varnothing$. As f_0, f_1 are open sets, $f_{\omega^2}^{n_0}(U)$ is an open set. Therefore, W is an open set. Then, there are $z^*\in W\cap S, \omega^3\in\Sigma^2$ such that $J=\left\{n\in\mathbf{N}\colon f_{\omega^3}^{n}(z^*)\in W\right\}$ is a syndetic set. If $m\in J$, then $f_{\omega^3}^{m}(W)\cap W\neq\varnothing$. There is $\omega^4=\omega_0^4\omega_1^4\cdots=\omega_0^2\omega_1^2\cdots\omega_{n_0-1}^2\omega^3\in\Sigma^2$ such that $\varnothing\neq f_{\omega^3}^{m}(W)\cap W=f_{\omega^3}^{m}\left(f_{\omega^2}^{n_0}(U)\cap V\right)\cap f_{\omega^2}^{n_0}(U)\cap V\subset f_{\omega^4}^{n_0+m}(U)\cap V$. Since $\{n_0+m\colon m\in J\}\subset N(U,V)=\left\{n\in\mathbf{N}\colon f_{\omega^4}^{n_0+m}(U)\cap V\neq\varnothing\right\}$, $N(U,V)$ is a syndetic set. As U and V are arbitrary, $\mathrm{IFS}(f_0,f_1)$ is strongly ergodic.

Lemma 4 (Theorem 3.1 in [13]). *If $\mathrm{IFS}(f_0,f_1)$ has the ASP, then $\mathscr{F}^k$ has the ASP for any $k\in\mathbf{Z}^+$.*

Theorem 10. *Let f_0, f_1 be open maps. For $\mathrm{IFS}(f_0,f_1)$ with the ASP, if $S\subset X$ is dense in X, and s is a minimal point of f_0 or f_1 for any $s\in S$, then $\mathscr{F}^k$ is strongly ergodic.*

Proof. According to the condition, for any $s\in S$, s is a minimal point of f_0 or f_1. For any $k\in\mathbf{Z}^+$, it is well known that $AP(f)=AP(f^k)$ and then s is a minimal point of f_0^k or f_1^k. We can combine Lemma 4 and Theorem 9. Then, $\mathscr{F}^k$ is strongly ergodic.

4. A Remark on the Asymptotic Average Shadowing Property for $\mathrm{IFS}(f_0,f_1)$

The definition of the asymptotic average shadowing property for $\mathrm{IFS}(f_0,f_1)$ is introduced by Nia [11]. Here, we will come up with a situation to show that an example does not have the asymptotic average shadowing property.

Definition 6. A sequence $\{\xi_i\}_{i\geq 0}$ is called an asymptotic average pseudo-orbit for $\mathrm{IFS}(f_0,f_1)$ if there is $\omega\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)=0. \tag{53}$$

Definition 7. $\mathrm{IFS}(f_0,f_1)$ has the asymptotic average shadowing property (AASP) if every asymptotic average pseudo-orbit is shadowed on average by a point $z\in X$, i.e., there is $\omega\in\Sigma^2$ such that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}d\left(f_{\omega}^{i}(z),\xi_i\right)=0. \tag{54}$$

For any $A,B\subset X$,

$$d(A,B)=\min_{a\in A,b\in B}d(a,b). \tag{55}$$

Theorem 11. *If $U\subset X$ is an open set, then U is invariant under $f_t\,(t=0,1)\,(f_t(U)\subset U, t=0,1)$. Assume that there is $x\in X$ such that $\overline{\{f_0^i(x)\}}_{i\geq 0}\subset U$ or $\overline{\{f_1^i(x)\}}_{i\geq 0}\subset U$. Suppose that $y\in X$ is a point whose orbit is metrically separated from U (for any $\omega\in\sum^2$, $d\left(\left\{f_{\omega}^{j}(y)\right\}_{j\geq 0},U\right)>0$). Then, $\mathrm{IFS}(f_0,f_1)$ does not have the AASP.*

Proof. Without loss of generality, suppose $\overline{\{f_0^i(x)\}}_{i\geq 0}\subset U$. Construct $\{\xi_i\}_{i\geq 0}$ as follows: $\{\xi_i\}_{i\geq 0}=\{x,y,x,f_0(x),y,f_1(y),x,f_0(x),f_0^2(x),\ f_0^3(x),y,f_1(y),f_1^2(y),f_1^3(y),x,\ f_0(x),\cdots,f_0^7(x),y,\ldots\}$.

For any $k\in\mathbf{N}$, we can find $l\in\mathbf{N}$ with

$$\sum_{i=1}^{l}2^i\leq k\leq\sum_{i=1}^{l+1}2^i. \tag{56}$$

We can choose

$$\omega_i=\begin{cases}0, & d\left(f_0(\xi_i),\xi_{i+1}\right)=0,\\ 1, & d\left(f_0(\xi_i),\xi_{i+1}\right)>0.\end{cases} \tag{57}$$

$\omega=\omega_0\omega_1\omega_2\cdots\in\Sigma^2$. We can see that

$$\sum_{i=0}^{k-1}d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)\leq 3l\cdot D, D=\operatorname{diam}(X). \tag{58}$$

This implies that

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{i=0}^{n-1}d\left(f_{\omega_i}(\xi_i),\xi_{i+1}\right)\leq\lim_{l\longrightarrow\infty}\frac{3l\cdot D}{2^l}=0. \tag{59}$$

So $\{\xi_i\}_{i\geq 0}$ is an asymptotic average pseudo-orbit of $\mathrm{IFS}(f_0,f_1)$. We will prove that $\mathrm{IFS}(f_0,f_1)$ does not have the AASP. Suppose that $\mathrm{IFS}(f_0,f_1)$ has the AASP. For asymptotic average pseudo-orbit $\{\xi_i\}_{i\geq 0}$, there are $z\in X, \omega^1\in\Sigma^2$ such that $\{\xi_i\}_{i\geq 0}$ is asymptotically shadowed on average by the point z. Meanwhile, the orbit of z has to enter U at some point. Otherwise,

$$\frac{1}{2^n}\sum_{i=0}^{2^n-1}d\left(f_{\omega^1}^{i}(z),\xi_i\right)\geq\frac{1}{2}d\left(\left\{f_{\omega^1}^{j}(z)\right\}_{j\geq 0},U\right)>0,\quad(\forall n\in\mathbf{N}). \tag{60}$$

So IFS(f_0, f_1) does not have the AASP. Therefore, there is $N_0 \in \mathbf{N}$ such that $f_{\omega^1}^{N_0}(z) \in U$. Then, $f_{\omega^1}^{n}(z) \in U$ for any $n \geq N_0$. Therefore, for n that is large enough,

$$\frac{1}{2^n}\sum_{i=0}^{2^n-1} d\left(f_{\omega^1}^{i}(z), \xi_i\right) \geq \frac{1}{4} d\left(\left\{f_1^j(y)\right\}_{j\geq 0}, U\right) > 0, \tag{61}$$

which contradicts with the hypothesis. IFS(f_0, f_1) does not have the AASP.

The following example comes from the study in [13]. It is a vitally important example of IFS(f_0, f_1). It is controversial whether it has the AASP. We can use the above theorem to prove that it does not have the AASP.

Example 2. Let f_0, f_1: $[0,1] \longrightarrow [0,1]$ be two continuous maps such that $f_1(x) > f_0(x) > x$ if and only if $x \in [0, (1/2))$ and $f_0(0) = f_1(0) = (1/4)$. Obviously, we can see that $\overline{\{f_0^i(1/8)\}_{i\geq 0}} \subset U = (0, (1/2))$, $f_t(U) \subset U$, where $t = 0, 1$. And for any $\omega \in \sum^2$, $d(\{f_\omega^j(7/8)\}_{j\geq 0}, U) > 0$. According to Theorem 11, IFS(f_0, f_1) does not have the AASP.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Foundation of Zhuhai College of Jilin University, Autonomous Research Foundation for Colleges and Universities of the Party Central Committee, National Young Science Foundation of China (no. 11701066), and Key Natural Science Foundation of Universities in Guangdong Province (no. 2019KZDXM027).

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