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*Research Article*

# Nonlinear Adaptive Boundary Control of the Modified Generalized Korteweg–de Vries–Burgers Equation

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In this paper, we study the nonlinear adaptive boundary control problem of the modified generalized Korteweg–de Vries–Burgers equation (MGKdVB) when the spatial domain is $[0,1]$. Four different nonlinear adaptive control laws are designed for the MGKdVB equation without assuming the nullity of the physical parameters $\nu$, $\mu$, $\gamma_1$, and $\gamma_2$ and depending whether these parameters are known or unknown. Then, using Lyapunov theory, the $L^2$-global exponential stability of the solution is proven in each case. Finally, numerical simulations are presented to illustrate the developed control schemes.

## 1. Introduction

In various problems of fluid dynamics and physics, the evolution of small amplitude long waves is described by the so-called dispersive equations which are governed by nonlinear partial differential equations (PDEs) [1–5]. This topic has attracted the efforts of many researchers from different disciplines to study the dynamics and the control problem of these dispersive equations. The analysis of such equations usually combines rigorous techniques of modern analysis and physics promoting numerous interdisciplinary research articles in the field.

In this paper, we will explore the nonlinear adaptive stabilization of the following form of the dispersive equation called the modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation:

$$\frac{\partial u}{\partial t} + \gamma_1 u^{\alpha}\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial x^3} + \gamma_2\frac{\partial^4 u}{\partial x^4} = 0, \quad x \in (0,1),\ t \geq 0, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(1,t) = f_1(t), \tag{4}$$

$$\frac{\partial^2 u}{\partial x^2}(1,t) = f_2(t), \tag{5}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{6}$$

Here, the variable $u = u(x,t)$ is a real-valued function that may represent the displacement of the medium or the velocity and depends on two real variables $x$ and $t$, where $x \in [0,1]$ is the distance in the direction of propagation of the medium and $t \geq 0$ is the elapsed time. Furthermore, $\alpha$ is assumed throughout this article as a positive integer and the physical parameters $\gamma_1$, $\nu$, $\gamma_2$, and $\mu$ are positive real numbers which represent nonlinearity, dissipation (diffusion), and dispersion coefficients, respectively. Last, $f_1(t)$ and $f_2(t)$ are nonlinear boundary control functions to be designed so that the solutions of the whole system of equations are stable.

Recently, the derivation of the MGKdVB equation (1) when $\alpha = 3$ has been presented by means of the long-wave approximation and perturbation method [6]. Furthermore, the existence and uniqueness of solutions of the MGKdVB equation (1) have been investigated in [7] where linear boundary conditions are considered. Since our main concern in the current work is the adaptive stabilization of the MGKdVB equation (1), we shall assume the existence of a unique regular solution $u(x,t)$ of the MGKdVB equation (1).

As several variants of equation (1) have been actively and continuously used, we are going to highlight some results obtained in the literature. For instance, setting $\mu = 0$, $\alpha = 1$, $\gamma_1 = 1$, and $\gamma_2 = 0$ reduces equation (1) to the well-known Burgers equation. During the last decades, many researchers devoted their time to study this equation including the adaptive and nonadaptive boundary control problem [1, 8–16]. In 1999, the boundary control of the Burgers equation was tackled by designing a nonlinear adaptive control law when the viscosity is unknown [9]. In 2001, an adaptation law for Burgers equation, where the viscosity is unknown was proposed, and the $L^2$- and $H^1$-global stability of the solution were proven [12]. In [14, 15], the generalized Burgers equation ($\alpha \geq 1$, $\mu = 0$, $\gamma_1 = 1$, and $\gamma_2 = 0$) is considered under the action of an adaptive control when all the parameters of the equation are unknown. Then, the $L^2$-regulation of the solution to the steady state solution was achieved. Furthermore, the global control of the generalized Burgers equation was also studied both analytically and numerically. The distributed and boundary stabilization problems of the Burgers equation have been considered in [16].

The so-called Kuramoto–Sivashinsky (KS) is obtained by setting $\mu = 0$, $\gamma_1 = \gamma_2 = 1$, $\alpha = 1$, and $\nu < 0$ in equation (1). Such an equation was first derived in 1978 by Kuramoto [3] to model a chaotic state in a chemical reaction system. It was also derived in 1980 by Sivashinsky [4] to understand the dynamics of flame front propagation. Since then many researchers further explored the KS equation [17–32]. For instance, assuming a space variable $x \in \mathbb{R}$, the Cauchy problem of the KS equation has been proven to admit a unique smooth and exponentially decaying solution that is continuously dependent on its initial data [17]. In turn, a distributed output feedback control has been put forward to obtain the global stabilization of the KS equation [22]. Such a stabilization result has been improved by proposing appropriate boundary controls [23]. In [24], an adaptive control law when the parameters of the KS equation are unknown has been designed and the global asymptotic stability and the convergence of solutions to zero have been established.

In addition, setting $\mu > 0$, $\gamma_1 = \gamma_2 = 1$, $\alpha = 1$, and $\nu < 0$ in equation (1), the MGKdVB equation reduces to the generalized Kuramoto–Sivashinsky (GKS) equation which was used to model turbulence. This equation has received considerable attention by many researchers [19, 20, 26, 27, 30]. For instance, Kudryashov [19] investigated several classes of analytical solutions of the GKS equation and Guo and Xiang [20] proved the existence and uniqueness of the global and smooth solution for the periodic initial value problem of the GKS equation. In 2002, Iosevich and Miller [26] analyzed the case where the dispersive effects cannot be neglected and obtained solutions to the KdV equation as the limit of the GKS equation when the dispersive and dissipative terms are large. Later on, Larkin [27] considered the GKS equation in a bounded domain as a model of long waves in a viscous fluid flow down on an inclined plane and proved that the solutions to a mixed problem for the KdV equation may be obtained as singular limits of solutions to a corresponding mixed problem for the KS equation.

In turn, when $\gamma_1 = 1$ and $\gamma_2 = 0$ in equation (1), the MGKdVB equation leads to the generalized Korteweg–de Vries–Burgers (GKdVB) equation. This equation is useful in modeling many physical phenomena such as the unidirectional propagation of planar waves [33], strain waves and longitudinal deformation in a nonlinear elastic rod [34], and pressure waves in a liquid-gas bubble mixture [5]. Note that there has been a growing interest in the analysis and control of the GKdVB equation [2, 33, 35–41]. Specifically, long-time behavior of periodic solutions of the GKdVB equation has been investigated [2]. Then, the global solutions of the periodic GKdVB equation when $\alpha < 4$ are obtained and the periodic initial value problem is shown to be well-posed for sufficiently large initial data in the case when $\alpha \geq 4$ [33]. In the context of control systems, nonadaptive boundary control laws have been proposed in [38, 40, 41, 42]. Smaoui and Al-Jamal [38] designed three nonadaptive boundary control laws to show that the dynamics of the GKdVB equation is globally exponentially stable in $L^2(0,1)$ and globally asymptotically stable and semiglobally exponentially stable in $H^1(0,1)$ when $\alpha$ is a positive integer. Then, the controllers developed in [38] have been improved when $\alpha$ is even and when $\alpha$ is odd [41]. Later on, Smaoui et al. [40] designed three different nonadaptive boundary control laws, when $\alpha$ is a positive real number, and showed numerically for certain values of $\alpha$ that the proposed controllers outperform those designed in [38]. On the other hand, the adaptive boundary control of the GKdVB equation was discussed in [39, 42] when either the kinematic viscosity $\nu$ or the dynamic viscosity $\mu$ is unknown or when both viscosities $\nu$ and $\mu$ are unknown.

Up to our knowledge, the adaptive boundary stabilization problem of the MGKdVB equation has not been examined in the literature. In fact, as previously discussed, the adaptive control stabilization of the MGKdVB equation (1) has been dealt with in the following specific cases: $\mu = \gamma_2 = 0$ in [9, 12, 14, 15], $\gamma_2 = 0$ in [39, 42], and $\mu = 0$ in [24].

In contrast to [9, 12, 14, 15, 24, 39, 42], the novelty of this paper is to consider the adaptive boundary control of the MGKdVB equation (1) by designing four different nonlinear boundary control laws without assuming the nullity of the physical parameters and depending whether these parameters are known or unknown (i.e., when $\nu$ is unknown or $\gamma_1$ is unknown or $\nu$ and $\gamma_1$ are unknown or when $\nu$, $\gamma_1$, $\mu$, and $\gamma_2$ are unknown). Furthermore, an exhaustive list of numerical simulations will be presented for each proposed control to support and validate the theoretical outcome.

This paper is arranged as follows. In Section 2, the first adaptive boundary nonlinear control law is proposed for the MGKdVB equation when the parameter $\nu$ is unknown. In Section 3, the second adaptive nonlinear boundary control law is designed for the MGKdVB equation when the parameter $\gamma_1$ is unknown. In Section 4, the third adaptive nonlinear boundary control law is designed for the MGKdVB equation when the parameters $\nu$ and $\gamma_1$ are unknown. In Section 5, the fourth adaptive nonlinear boundary control law is designed for the MGKdVB equation when the parameters $\nu$, $\mu$, $\gamma_1$, and $\gamma_2$ are unknown. Also, the global exponential stability of the solutions in $L^2(0,1)$ is presented analytically and numerically for each of the designed controller proposed in Sections 2–5. Section 6 compares the rates of convergence of the solutions of the four adaptive presented controllers provided in Sections 2–5. Finally, some concluding remarks are presented in Section 7.

## 2. Design of the First Nonlinear Adaptive Controller

In this section, we present the first nonlinear adaptive boundary control law for the MGKdVB equation. A control law for the MGKdVB equation is proposed when the parameter $\nu$ is unknown. The next theorem illustrates the first result of our nonlinear adaptive boundary control law.

**Theorem 1.** *The MGKdVB equation given by equation (1), subject to the boundary conditions stated in equations (2)–(5) and the initial condition $u_0 \in L^2(0,1)$, is globally exponentially stable in $L^2(0,1)$, provided that the following nonlinear control law is proposed:*

$$f_1(t) = -\eta_1 u(1,t), \tag{7}$$

$$f_2(t) = \frac{-\gamma_1}{(\alpha+2)(\mu+\gamma_2\eta_1)}u^{\alpha+1}(1,t) - \frac{\gamma_2}{(\mu+\gamma_2\eta_1)}\frac{\partial^3 u}{\partial x^3}(1,t) + \frac{((\mu\eta_1^2/2)+\mu\eta_2)}{(\mu+\gamma_2\eta_1)}u(1,t), \tag{8}$$

*where*

$$\frac{d\eta_1}{dt} = r_1 u^2(1,t), \quad r_1 > 0, \tag{9}$$

$$\frac{d\eta_2}{dt} = r_2 u^2(1,t), \quad r_2 > 0. \tag{10}$$

*Proof 1.* Let

$$V(t) = \frac{1}{2}\int_0^1 u^2(x,t)dx, \tag{11}$$

be a Lyapunov function candidate.

Differentiating $V(t)$ with respect to time, we obtain

$$\frac{dV}{dt} = \int_0^1 \frac{\partial u}{\partial t}(x,t)u(x,t)dx. \tag{12}$$

Referring to equation (1), one can write equation (12) as follows:

$$\begin{aligned}\frac{dV}{dt} &= \int_0^1 u\left(\nu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^3 u}{\partial x^3} - \gamma_1 u^\alpha\frac{\partial u}{\partial x} - \gamma_2\frac{\partial^4 u}{\partial x^4}\right)dx \\ &= \nu\int_0^1 u\frac{\partial^2 u}{\partial x^2}dx - \mu\int_0^1 u\frac{\partial^3 u}{\partial x^3}dx - \gamma_1\int_0^1 u^{\alpha+1}\frac{\partial u}{\partial x}dx \\ &\quad - \gamma_2\int_0^1 u\frac{\partial^4 u}{\partial x^4}dx.\end{aligned} \tag{13}$$

Integrating by parts and using the boundary conditions (2)–(5), we obtain

$$\begin{aligned}\frac{dV}{dt} \le{}& -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx + \nu u(1,t)f_1(t) - \mu u(1,t)f_2(t) \\ &+ \frac{\mu}{2}f_1^2(t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) \\ &+ \gamma_2 f_1(t)f_2(t).\end{aligned} \tag{14}$$

Using control laws (7) and (8), we obtain

$$\frac{dV}{dt} \le -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx - \nu\eta_1 u^2(1,t) - \mu\eta_2 u^2(1,t). \tag{15}$$

Applying Poincaré's inequality gives

$$\frac{dV}{dt} \le -\nu\int_0^1 u^2 dx - \nu\eta_1 u^2(1,t) - \mu\eta_2 u^2(1,t). \tag{16}$$

Now let $E(t)$ be the following energy function:

$$E(t) = V(t) + \frac{1}{2\nu r_1}(\nu\eta_1 - a)^2 + \frac{1}{2\mu r_2}(\mu\eta_2 - b)^2, \quad a \ge 0, b \ge 0. \tag{17}$$

It is clear from this equation that $E(t) \ge 0$, $\forall t \ge 0$. Differentiating $E(t)$ with respect to time yields:

$$\frac{dE}{dt} = \frac{dV}{dt} + \frac{1}{r_1}\frac{d\eta_1}{dt}(\nu\eta_1 - a) + \frac{1}{r_2}\frac{d\eta_2}{dt}(\mu\eta_2 - b). \tag{18}$$

By virtue of inequality (16) and equations (9) and (10), we deduce from equation (18) that

$$\frac{dE}{dt} \le -\nu\int_0^1 u^2 dx - au^2(1,t) - bu^2(1,t). \tag{19}$$

Since $-au^2(1,t) - bu^2(1,t) \le 0$, for all $t \ge 0$ then equation (19) reduces to

$$\frac{dE}{dt} \le -\nu\int_0^1 u^2 dx \le 0, \quad \forall t \ge 0. \tag{20}$$

Thereby, $E(t) \le E(0)$, $\forall t \ge 0$ and hence both of $\eta_1$ and $\eta_2$ are bounded for all $t > 0$. This implies that $u(1,t) \in L^2(0,\infty)$. Invoking inequality (16), we obtain

$$\frac{dV}{dt} \le -2\nu V(t) - (\nu\eta_1 + \mu\eta_2)u^2(1,t). \tag{21}$$

Applying Gronwall–Bellman's inequality, we obtain

$$V(t) \le V(0)e^{-2\nu t} + C\int_0^t e^{-2\nu(t-\tau)}u^2(1,\tau)d\tau, \tag{22}$$

where $C = \sup_{t>0}|\nu\eta_1(t) + \mu\eta_2(t)|$. Since $u(1,t) \in L^2(0,\infty)$, we conclude that $V(t)$ approaches zero as $t \longrightarrow \infty$. Therefore, $u(x,t)$ exponentially tends to zero as $t \longrightarrow \infty$. □

In the next subsection, we give a numerical presentation of the dynamical behavior of the MGKdVB equation by applying the first nonlinear adaptive boundary control law presented in equations (7) and (8).

*2.1. Numerical Solutions.* Numerical solutions for the MGKdVB equation (i.e., equations (1)–(6)) under the presence of the controls $f_1(t)$ and $f_2(t)$ as presented by equations (7) and (8) were simulated using COMSOL Multiphysics software. The solutions are computed for $\alpha = 1, 2, 3$, and 4.

From equations (7) and (8), one can observe that the first adaptive control law proposed in Theorem 1 does not require the preknowledge of the kinematic viscosity $\nu$ which is assumed to be unknown. Nevertheless, for simulation purposes, the value of $\nu$ is set to be 0.01 in the MGKdVB equation (i.e., equation (1)). The dynamic viscosity $\mu$ is chosen to be 0.001. The parameters $\gamma_1$ and $\gamma_2$ are set to be 1 and 0.0005, respectively. Moreover, $r_1$ and $r_2$ are chosen to equal 0.2. The initial values of $\eta_1$ and $\eta_2$ are set to be such as $\eta_1(0) = 0.2$ and $\eta_2(0) = 0.1$. Moreover, let the initial datum $u(x, 0) = \sin(\pi x)$. Figures 1(a)–1(d) show a 3D view of the solution of the MGKdVB equation utilizing the control law presented in Theorem 1 for several values of $\alpha$. Figure 2 shows the $L^2$-norm of the solutions. It can be noticed from these figures that as $\alpha$ increases, the decay rate of the solution to the steady state solution decreases. This is due to the effect of the nonlinear term which causes the instability in the behavior of the MGKdVB equation.

Figures 3 and 4 depict the behavior of the functions $\eta_1$ and $\eta_2$ that are utilized in the first control law. Figure 3 shows that, for $0 < t \le 5$, $\dot{\eta}_1$ decreases as the value of $\alpha$ increases from 2 to 4 and increases after $t \simeq 5$. Figure 4 indicates that $\dot{\eta}_2$ behaves similar to $\dot{\eta}_1$ as $\alpha$ increases. The choice of the feedback gains $r_1$ and $r_2$ will definitely affect the values of $\eta_1$ and $\eta_2$ and therefore the stability of the solution. Fixing the initial conditions, $\eta_1(0)$ and $\eta_2(0)$, and increasing the values of $r_1$ and $r_2$ will increase the values of $\eta_1$ and $\eta_2$ which in turn speeds up the convergence of the solution to zero as can be deduced from inequality (21).

It should be noted that the parameters of the controllers $f_1(t)$ and $f_2(t)$ are $\eta_1$, $\eta_2$, $\gamma_1$, $\gamma_2$, $\mu$, and $\alpha$. For given values of $\gamma_1$, $\gamma_2$, $\mu$, and $\alpha$, the parameters $\eta_1$ and $\eta_2$ can be computed from the uncoupled system of ODE equations (9) and (10) once the value at the right boundary $u(1, t)$ is known, the control gains $r_1$ and $r_2$ are chosen, and the initial conditions $\eta_1(0)$ and $\eta_2(0)$ are fixed. When these parameters are evaluated, the controllers $f_1(t)$ and $f_2(t)$ can be easily computed using equations (7) and (8).

## 3. Design of the Second Nonlinear Adaptive Controller

In this section, we present the second nonlinear adaptive boundary control law for the MGKdVB equation. A control scheme for the MGKdVB equation is proposed when the parameter $\gamma_1$ is unknown. Furthermore, numerical results will be also discussed to reveal the effectiveness of this control law.

The following theorem illustrates the result of our second nonlinear adaptive boundary control.

**Theorem 2.** *The MGKdVB equation given by equation (1) with the boundary conditions given by equations (2)–(5) and the initial condition $u_0 \in L^2(0, 1)$ is globally exponentially stable in the $L^2(0, 1)$, by considering the following nonlinear control law:*

$$f_1(t) = -\frac{\eta_1}{\nu} u(1, t), \tag{23}$$

$$\begin{aligned} f_2(t) = {} & \frac{\mu \eta_1^2}{2\nu(\gamma_2 \eta_1 + \nu\mu)} u(1, t) - \frac{\nu \gamma_2}{\gamma_2 \eta_1 + \nu\mu} \frac{\partial^3 u}{\partial x^3}(1, t) \\ & + \frac{\nu \eta_2}{\gamma_2 \eta_1 + \nu\mu} u^{2\alpha+1}(1, t), \end{aligned} \tag{24}$$

*where*

$$\frac{d\eta_1}{dt} = r_1 u^2(1, t), \quad r_1 > 0, \tag{25}$$

$$\frac{d\eta_2}{dt} = r_2 u^{2\alpha+2}(1, t), \quad r_2 > 0. \tag{26}$$

*Proof 2.* Consider

$$V(t) = \frac{1}{2} \int_0^1 u^2(x, t) dx, \tag{27}$$

and differentiate $V(t)$ with respect to time and arguing as for estimate (14), we obtain

$$\begin{aligned} \frac{dV}{dt} \le {} & -\nu \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx + \nu u(1, t) f_1(t) - \mu u(1, t) f_2(t) \\ & + \frac{\mu}{2} f_1^2(t) - \frac{\gamma_1}{\alpha + 2} u^{\alpha+2}(1, t) - \gamma_2 u(1, t) \frac{\partial^3 u}{\partial x^3}(1, t) \\ & + \gamma_2 f_1(t) f_2(t). \end{aligned} \tag{28}$$

Using the control law given by equations (23) and (24), inequality (28) becomes

$$\begin{aligned} \frac{dV}{dt} \le {} & -\nu \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx - \eta_1 u^2(1, t) - \eta_2 u^{2\alpha+2}(1, t) \\ & - \frac{\gamma_1}{\alpha + 2} u^{\alpha+2}(1, t). \end{aligned} \tag{29}$$

By using Poincaré's inequality, we obtain

$$\frac{dV}{dt} \le -\nu \int_0^1 u^2 dx - \eta_1 u^2(1, t) - \eta_2 u^{2\alpha+2}(1, t) - \frac{\gamma_1}{\alpha + 2} u^{\alpha+2}(1, t). \tag{30}$$

The last term of inequality (30) can be bounded as follows:

(1) If $u^{\alpha+2}(1, t) \le 0$, then $u(1, t) \le 0$, and in this case $\alpha$ is odd. Therefore,

$$-\frac{\gamma_1}{\alpha + 2} u^{\alpha+2}(1, t) = \left(\frac{\sqrt{\gamma_1}}{\sqrt{\alpha + 2}} u^{\alpha+1}(1, t)\right)\left(\frac{-\sqrt{\gamma_1}}{\sqrt{\alpha + 2}} u(1, t)\right) \ge 0. \tag{31}$$

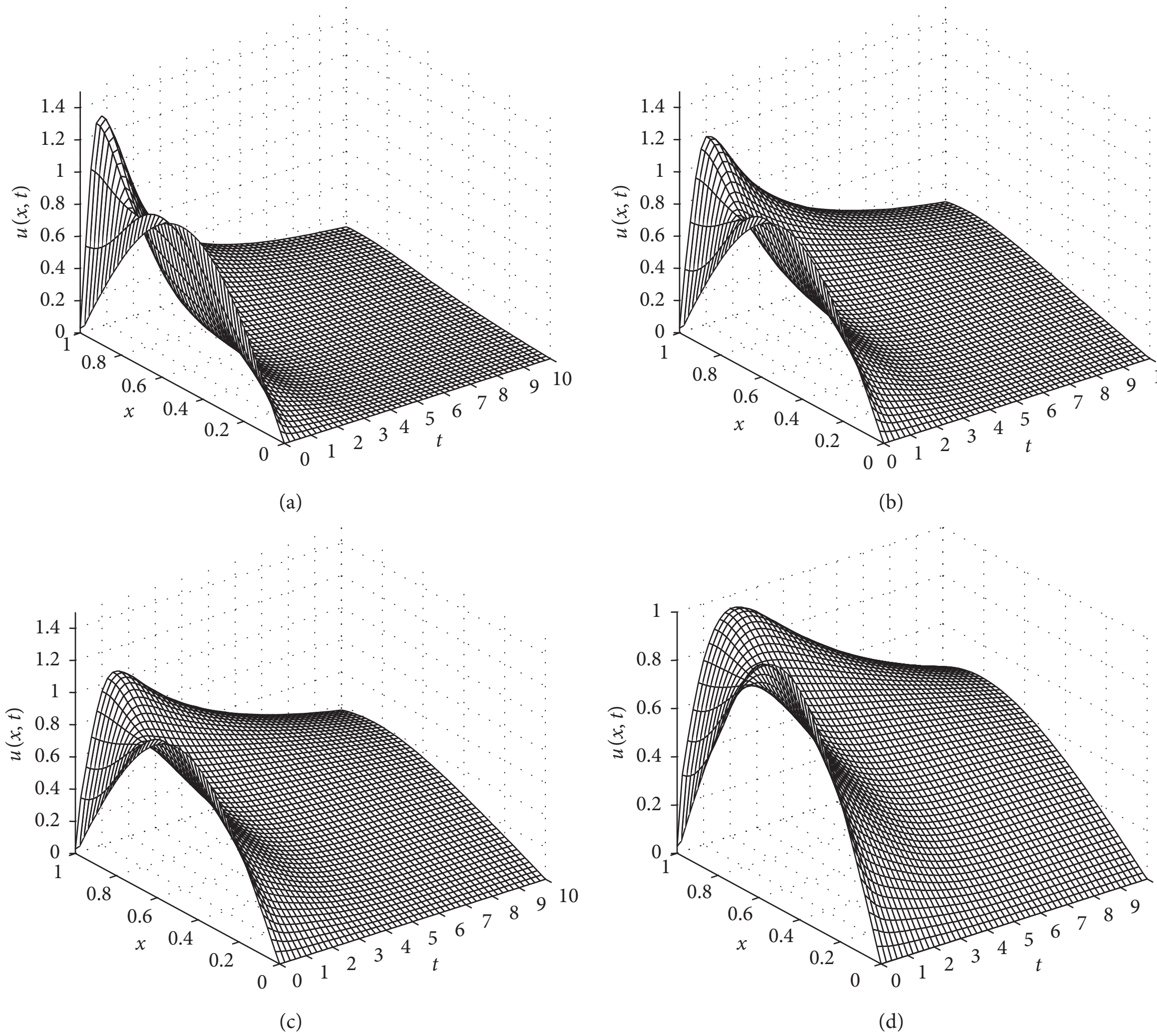


Figure 1: A 3D view of the dynamics of the MGKdVB equation utilizing the first nonlinear adaptive controller $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$. (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$.

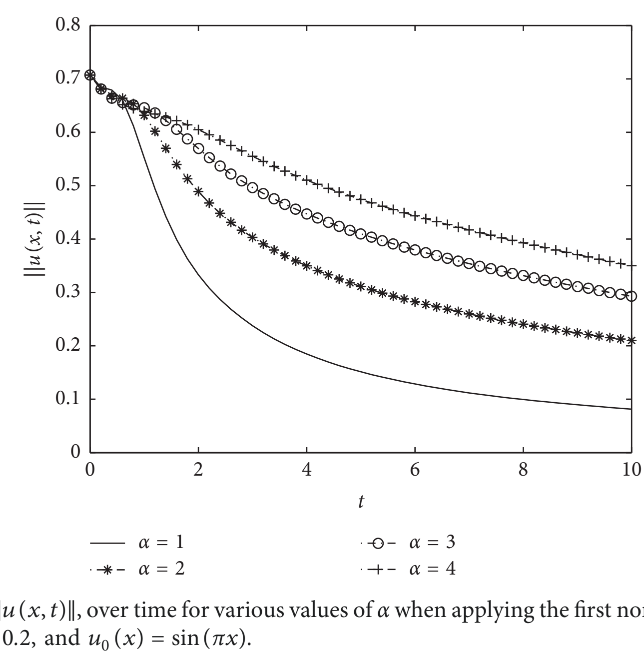


Figure 2: The $L^2$-norm of $u(x,t)$, $\|u(x,t)\|$, over time for various values of $\alpha$ when applying the first nonlinear adaptive controller $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$.

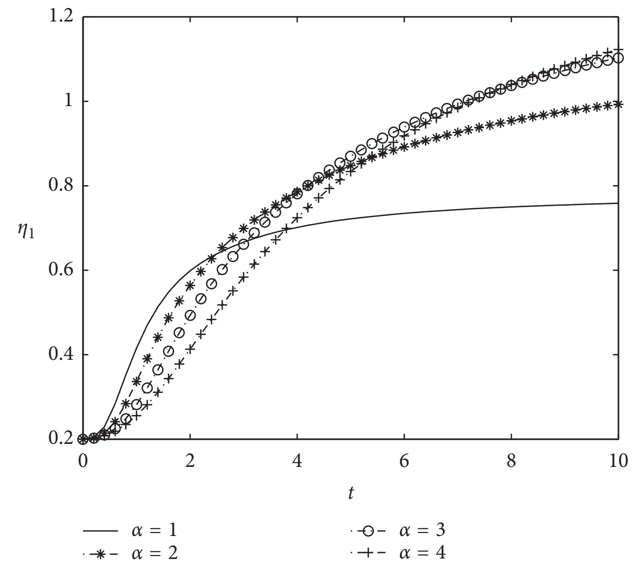


Figure 3: $\eta_1$ over time for various values of $\alpha$ when applying the first nonlinear adaptive controller.

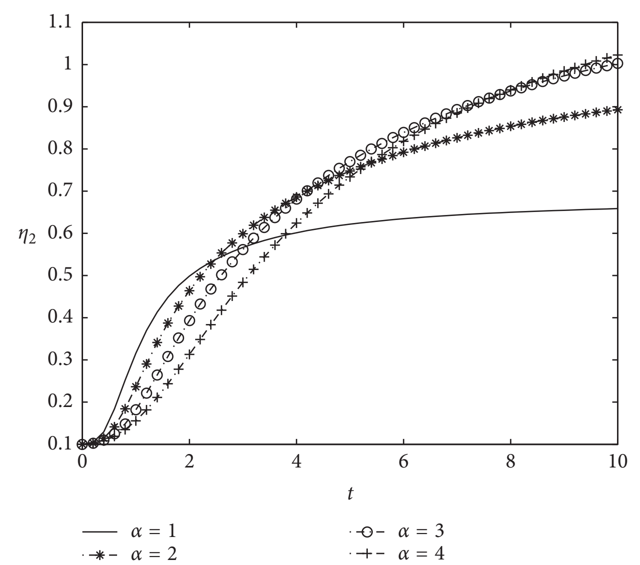


Figure 4: $\eta_2$ over time for various values of $\alpha$ when applying the first nonlinear adaptive controller.

By using Young's inequality, we obtain

$$-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) = \left(\frac{\sqrt{\gamma_1}}{\sqrt{\alpha+2}}u^{\alpha+1}(1,t)\right)\left(\frac{-\sqrt{\gamma_1}}{\sqrt{\alpha+2}}u(1,t)\right) \leq \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t). \tag{32}$$

(2) If $u^{\alpha+2}(1,t) \geq 0$, then

$$-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) \leq \left(\frac{\sqrt{\gamma_1}}{\sqrt{\alpha+2}}u^{\alpha+1}(1,t)\right)\left(\frac{\sqrt{\gamma_1}}{\sqrt{\alpha+2}}u(1,t)\right) \leq \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t). \tag{33}$$

Thus, in both cases, we have

$$-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) \leq \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t). \tag{34}$$

Combining (30) and (34) gives

$$\frac{dV}{dt} \leq -\nu\int_0^1 u^2 dx - \eta_1 u^2(1,t) - \eta_2 u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t). \tag{35}$$

Now, let us define a nonnegative energy function $E(t)$ as follows:

$$E(t) = V(t) + \frac{1}{2r_1}\left(\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right)^2 + \frac{1}{2r_2}\left(\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right)^2, \tag{36}$$

where $a > 0$.

Differentiating $E(t)$, we obtain

$$\frac{dE}{dt} = \frac{dV}{dt} + \frac{1}{r_1}\frac{d\eta_1}{dt}\left(\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right) + \frac{1}{r_2}\frac{d\eta_2}{dt}\left(\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right). \tag{37}$$

Using inequality (35) and equations (25) and (26), we have

$$\begin{aligned}\frac{dE}{dt} \leq{}& -\nu\int_0^1 u^2 dx - \eta_1 u^2(1,t) - \eta_2 u^{2\alpha+2}(1,t) \\ &+ \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t) \\ &+ u^2(1,t)\left(\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right) \\ &+ u^{2\alpha+2}(1,t)\left(\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right).\end{aligned} \tag{38}$$

The latter leads to

$$\frac{dE}{dt} \leq -\nu\int_0^1 u^2 dx - a u^{2\alpha+2}(1,t). \tag{39}$$

Noting that $-au^{2\alpha+2}(1,t) \leq 0$ for all $t \geq 0$, the estimate (39) gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -\nu \int_0^1 u^2 \mathrm{d}x, \tag{40}$$

that is,

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -2\nu V(t). \tag{41}$$

Hence, $E(t) \le E(0)$, $\forall t \ge 0$. One can conclude that $\eta_1$ and $\eta_2$ are bounded functions for all $t > 0$ and thus $u(1,t) \in L^2(0,\infty) \cap L^{(2\alpha+2)}(0,\infty)$ when $\alpha$ is a positive integer.

Using inequality (35), we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} \le& -2\nu V(t) - \left(\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t) \\ &- \left(\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{(2\alpha+2)}(1,t).\end{aligned} \tag{42}$$

Exploiting the Gronwall–Bellman's inequality, we obtain

$$V(t) \le V(0)e^{-2\nu t} + C\int_0^t e^{-2\nu(t-\tau)}\left[u^2(1,\tau) + u^{2\alpha+2}(1,\tau)\right]\mathrm{d}\tau, \tag{43}$$

where $C=\max\{\sup_{t>0}|\eta_1(t)-(\gamma_1/(2(\alpha+2)))|, \sup_{t>0}|\eta_2(t)-(\gamma_1/(2(\alpha+2)))|\}$.

Using the fact that $u(1,t) \in L^2(0,\infty) \cap L^{(2\alpha+2)}(0,\infty)$, we conclude that $V(t) \longrightarrow 0$ exponentially as $t \longrightarrow \infty$. Hence, $u(x,t)$ exponentially tends to zero as $t \longrightarrow \infty$. □

In the next subsection, we give a numerical presentation of the dynamical behavior of the MGKdVB equation when using the second nonlinear adaptive boundary control law presented in equations (23) and (24).

*3.1. Numerical Solutions.* Numerical solutions for the MGKdVB equation (i.e., equations (1)–(8)) with the controls $f_1(t)$ and $f_2(t)$ as presented by equations (23) and (24) were simulated using COMSOL Multiphysics software. The solutions are computed for $\alpha = 1, 2, 3$, and 4.

It follows from equations (23) and (24) that the pre-knowledge of the parameter $\gamma_1$ is not required in the second adaptive control law proposed in Theorem 2. However, the value of $\gamma_1$ is set to be 1 in the MGKdVB equation (i.e., equation (1)) for simulation purposes. The kinematic viscosity $\nu$ is chosen to be 0.01 and the dynamic viscosity $\mu$ is set to be 0.001. The parameter $\gamma_2$ is chosen to be 0.0005. Moreover, $r_1$ and $r_2$ are chosen to equal 0.2. The initial values of $\eta_1$ and $\eta_2$ are set to be such as $\eta_1(0) = \eta_2(0) = 1$ and $u(x,0) = \sin(\pi x)$. Figures 5(a)–5(d) show a 3D view of the solution of the MGKdVB equation under the action of the control law presented in Theorem 2 for different values of $\alpha$. Moreover, the $L^2$-norm of these solutions is presented in Figure 6. It can be concluded from the figures that as $\alpha$ increases, the convergence rate of the solution to the steady state solution becomes smaller. This is due to the presence of the nonlinear term $(u^\alpha(\partial u/\partial x))$ which dominates over the other terms in the MGKdVB equation.

Figures 7 and 8 present the behavior of the functions $\eta_1$ and $\eta_2$ that act in the second control law. Indeed, Figure 7 shows that the function $\eta_1$ grows as the value of $\alpha$ increases from 1 to 4. Unlike Figure 7, Figure 8 shows that the function $\eta_2$ decreases as the value of $\alpha$ increases from 1 to 4. Also, it should be noted that the choice of the feedback gains $r_1$ and $r_2$ has an impact on the values of $\eta_1$ and $\eta_2$ and consequently on the stability of the solution. One can observe that fixing the initial conditions, $\eta_1(0)$ and $\eta_2(0)$, and increasing the values of $r_1$ and $r_2$ will increase the values of $\eta_1$ and $\eta_2$ which in turn speeds up the convergence of the solution to zero, as shown in estimate (42).

## 4. Design of the Third Nonlinear Adaptive Controller

In this section, we present the third nonlinear adaptive boundary control law for the MGKdVB equation. A control scheme for the MGKdVB equation is proposed when the parameters $\nu$ and $\gamma_1$ are unknown. The following theorem illustrates the result of our third nonlinear adaptive boundary control law.

**Theorem 3.** *The MGKdVB equation given by equation (1) with the boundary conditions given by equations (2)–(5) and the initial condition $u_0 \in L^2(0,1)$ is globally exponentially stable in the $L^2(0,1)$, by considering the following nonlinear control law:*

$$f_1(t) = 0, \tag{44}$$

$$f_2(t) = -\frac{\gamma_2}{\mu}\frac{\partial^3 u}{\partial x^3}(1,t) + \eta_1 u(1,t) + \eta_2 u^{2\alpha+1}(1,t), \tag{45}$$

*where*

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}t} = r_1 u^2(1,t), \quad r_1 > 0, \tag{46}$$

$$\frac{\mathrm{d}\eta_2}{\mathrm{d}t} = r_2 u^{2\alpha+2}(1,t), \quad r_2 > 0. \tag{47}$$

*Proof 3.* Let

$$V(t) = \frac{1}{2}\int_0^1 u^2(x,t)\mathrm{d}x, \tag{48}$$

be a Lyapunov function candidate. Differentiating $V(t)$ and utilizing a similar reasoning as in Section 2, we have the following inequality:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} \le& -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x + \nu u(1,t)f_1(t) - \mu u(1,t)f_2(t) \\ &+ \frac{\mu}{2}f_1^2(t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) \\ &+ \gamma_2 f_1(t) f_2(t).\end{aligned} \tag{49}$$

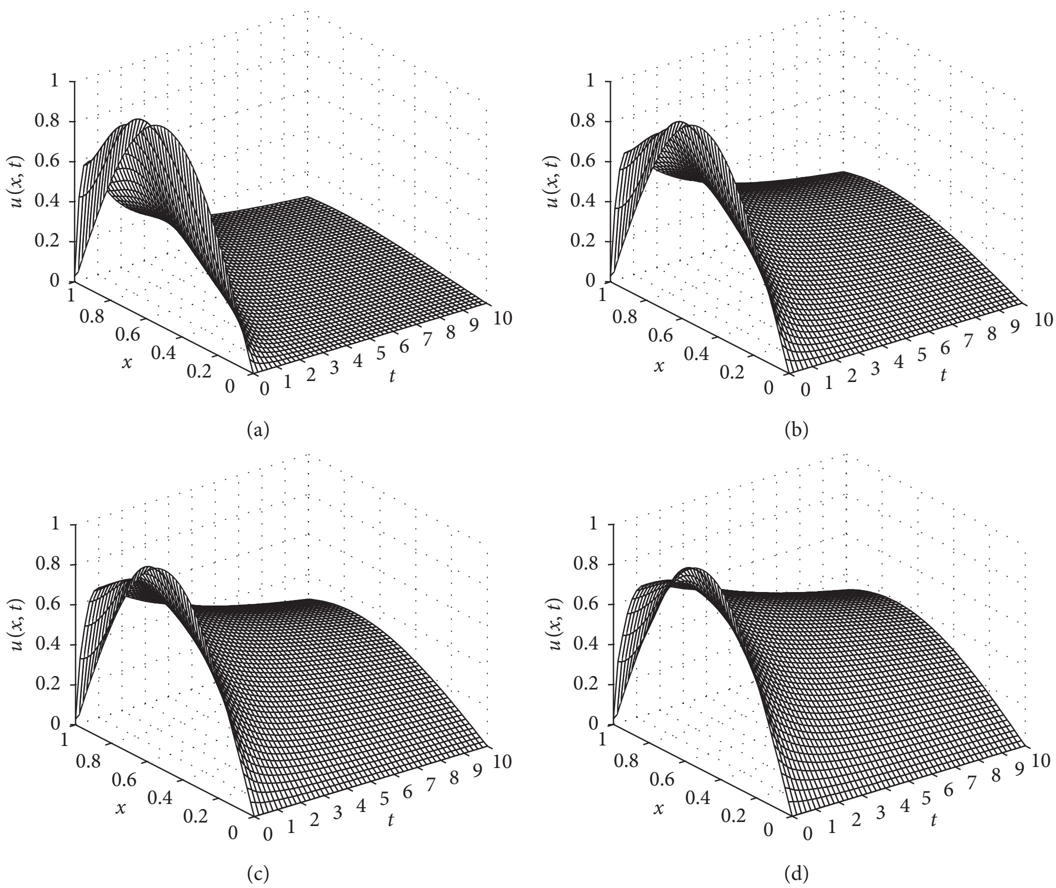


Figure 5: A 3D view of the dynamics of the MGKdVB equation utilizing the second nonlinear adaptive controller $\nu = 0.01$, $\mu = 0.001$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$. (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$.

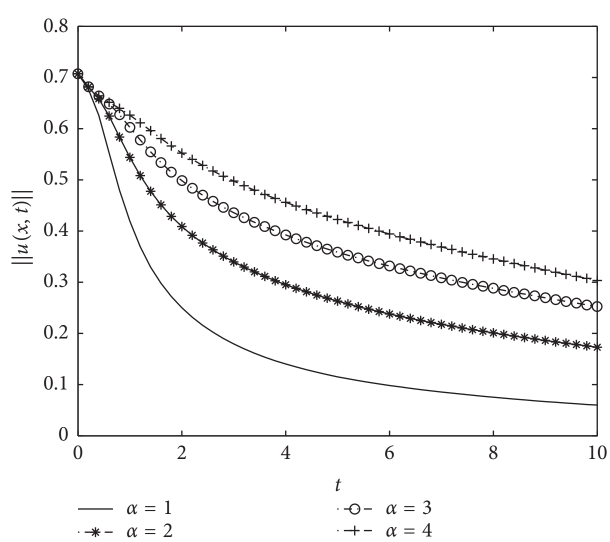


Figure 6: The $L^2$-norm of $u(x,t)$, $\|u(x,t)\|$, over time for various values of $\alpha$ when applying the second nonlinear adaptive controller $\mu = 0.001$, $\nu = 0.01$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$.

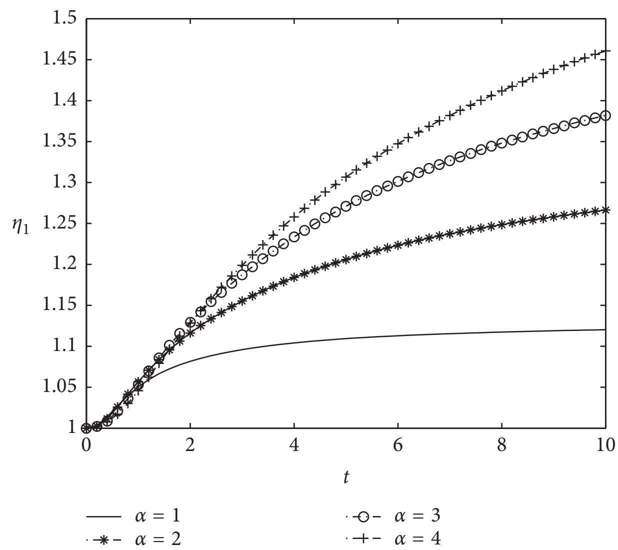


Figure 7: $\eta_1$ over time for various values of $\alpha$ when applying the second nonlinear adaptive controller.

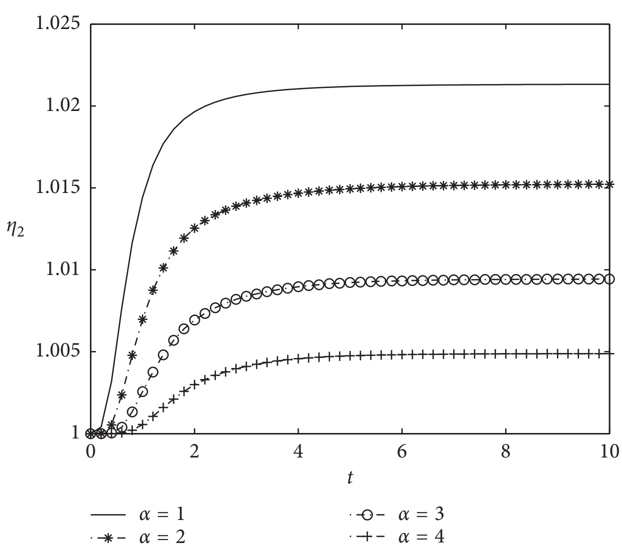


Figure 8: $\eta_2$ over time for various values of $\alpha$ when applying the second nonlinear adaptive controller.

Using the control law given by equations (44) and (45), the above inequality yields:

$$\frac{dV}{dt} \leq -\nu \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx - \mu\eta_1 u^2(1,t) - \mu\eta_2 u^{2\alpha+2}(1,t) - \frac{\gamma_1}{\alpha+2} u^{\alpha+2}(1,t). \tag{50}$$

Thanks to Poincaré's inequality, we obtain

$$\frac{dV}{dt} \leq -\nu \int_0^1 u^2 dx - \mu\eta_1 u^2(1,t) - \mu\eta_2 u^{2\alpha+2}(1,t) - \frac{\gamma_1}{\alpha+2} u^{\alpha+2}(1,t). \tag{51}$$

Arguing as for inequality (34), the last term in (51) can be bounded as follows:

$$-\frac{\gamma_1}{\alpha+2} u^{\alpha+2}(1,t) \leq \frac{\gamma_1}{2(\alpha+2)} u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)} u^2(1,t). \tag{52}$$

Thus, inequality (51) gives

$$\frac{dV}{dt} \leq -\nu \int_0^1 u^2 dx - \mu\eta_1 u^2(1,t) - \mu\eta_2 u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)} u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)} u^2(1,t), \tag{53}$$

that is,

$$\frac{dV}{dt} \leq -\nu \int_0^1 u^2 dx - \left(\mu\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right) u^2(1,t) - \left(\mu\eta_2 + \frac{\gamma_1}{2(\alpha+2)}\right) u^{2\alpha+2}(1,t). \tag{54}$$

Now, let us define a nonnegative energy function $E(t)$ as follows:

$$E(t) = V(t) + \frac{1}{2\mu r_1}\left(\mu\eta_1 - a - \frac{\gamma_1}{2(\alpha+2)}\right)^2 + \frac{1}{2\mu r_2}\left(\mu\eta_2 - b - \frac{\gamma_1}{2(\alpha+2)}\right)^2, \tag{55}$$

where $a, b > 0$.

Differentiating $E(t)$ with respect to time, we obtain

$$\frac{dE}{dt} = \frac{dV}{dt} + \frac{1}{r_1}\frac{d\eta_1}{dt}\left(\mu\eta_1 - a - \frac{\gamma_1}{2(\alpha+2)}\right) + \frac{1}{r_2}\frac{d\eta_2}{dt}\left(\mu\eta_2 - b - \frac{\gamma_1}{2(\alpha+2)}\right). \tag{56}$$

Using inequality (54) and equations (46) and (47), we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} \leq& -\nu\int_0^1 u^2\mathrm{d}x + \left(\mu\eta_1 - a - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t)\\ &-\left(\mu\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t)\\ &+\left(\mu\eta_2 - b - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t)\\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t).\end{aligned} \tag{57}$$

Inequality (57) leads to

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\nu\int_0^1 u^2\mathrm{d}x - au^2(1,t) - bu^{2\alpha+2}(1,t). \tag{58}$$

Noting that $-au^2(1,t) - bu^{2\alpha+2}(1,t) \leq 0$, inequality (58) reduces to

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -2\nu V(t). \tag{59}$$

Hence, $E(t) \leq E(0)$, $\forall t \geq 0$. One can conclude that $\eta_1$ and $\eta_2$ are bounded functions for all $t > 0$ and thus $u(1,t) \in L^2(0,\infty) \cap L^{(2\alpha+2)}(0,\infty)$ when $\alpha$ is a positive integer.

From inequality (54), we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} \leq& -2\nu V(t) - \left(\mu\eta_1 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t)\\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t).\end{aligned} \tag{60}$$

Exploiting the Gronwall–Bellman's inequality, we have

$$V(t) \leq V(0)e^{-2\nu t} + C\int_0^t e^{-2\nu(t-\tau)}\left[u^2(1,\tau) + u^{2\alpha+2}(1,\tau)\right]\mathrm{d}\tau, \tag{61}$$

where $C = \max\{\sup_{t>0}|\mu\eta_1(t) - (\gamma_1/(2(\alpha+2)))|, \sup_{t>0}|\mu\eta_2(t) - (\gamma_1/(2(\alpha+2)))|\}$.

Using the fact that $u(1,t) \in L^2(0,\infty) \cap L^{(2\alpha+2)}(0,\infty)$, we can show that $V(t)$ tends to zero as $t \longrightarrow \infty$. Hence, $u(x,t)$ exponentially tends to zero as $t \longrightarrow \infty$. □

*4.1. Numerical Solutions.* Numerical solutions for the MGKdVB equation (i.e., equations (1)–(6)) in the presence of the control $f_1(t)$ and $f_2(t)$ as presented by equations (59) and (60) were simulated using COMSOL Multiphysics software. The solutions are computed for $\alpha = 1, 2, 3$, and 4.

We deduce from equations (44) and (45) that the third adaptive control law presented in Theorem 3 is independent of the values of the kinematic viscosity $\nu$ and the parameter $\gamma_1$ which are assumed to be unknown. For simulation purposes, the values of $\nu$ and $\gamma_1$ are chosen to be 0.01 and 1, respectively. The dynamic viscosity $\mu$ is set to be 0.001 and the parameter $\gamma_2$ is chosen to equal 0.0005. Moreover, $r_1$ and $r_2$ are chosen to be 0.2. The initial values of $\eta_1$ and $\eta_2$ are set to be such as $\eta_1(0) = \eta_2(0) = 1$, and $u(x,0) = \sin(\pi x)$. Figures 9(a)–9(d) show a 3D view of the solution of the MGKdVB equation utilizing the control law presented in Theorem 3 for different values of $\alpha$. Figure 10 shows the $L^2$-norm of these solutions. It demonstrates how the nonlinear term $(u^\alpha(\partial u/\partial x))$ affects the dynamics of the MGKdVB equation. When the value of $\alpha$ increases, the solution takes a longer time to approach the steady state solution.

Figures 11 and 12 depict the behavior of the functions $\eta_1$ and $\eta_2$ that are utilized in the third control law. Figure 11 indicates that the function $\eta_1$ gets bigger as $\alpha$ increases from 1 to 4. Figure 12 indicates that $\eta_2$ decreases as $\alpha$ increases unlike the behavior of $\eta_1$. Again similar to the previous two controllers, the choice of the feedback gains $r_1$ and $r_2$ will definitely affect the values of $\eta_1$ and $\eta_2$ and therefore the stability of the solution. Fixing the initial conditions, $\eta_1(0)$ and $\eta_2(0)$, and increasing the values of $r_1$ and $r_2$ will increase the values of $\eta_1$ and $\eta_2$ which in turn speeds up the convergence of the solution to zero as can be deduced from inequality (54).

## 5. Design of the Fourth Nonlinear Adaptive Controller

In this section, we present the fourth nonlinear adaptive boundary control law for the MGKdVB equation. A control scheme for the MGKdVB equation is proposed when all the parameters are unknown. The following theorem presents the result of our fourth nonlinear adaptive boundary control law.

**Theorem 4.** *The MGKdVB equation given by equation (1) with the boundary conditions given by equations (2)–(5) and the initial condition $u_0 \in L^2(0,1)$ is globally exponentially stable in the $L^2(0,1)$-sense, under the presence of the following nonlinear control law:*

$$f_1(t) = 0, \tag{62}$$

$$f_2(t) = (1+\eta_1)\frac{\partial^3 u}{\partial x^3}(1,t) + \eta_2 u(1,t) + \eta_3 u^{2\alpha+1}(1,t), \tag{63}$$

*where*

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}t} = r_1 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t), \quad r_1 > 0, \tag{64}$$

$$\frac{\mathrm{d}\eta_2}{\mathrm{d}t} = r_2 u^2(1,t), \quad r_2 > 0, \tag{65}$$

$$\frac{\mathrm{d}\eta_3}{\mathrm{d}t} = r_3 u^{2\alpha+2}(1,t), \quad r_3 > 0. \tag{66}$$

*Proof 4.* Letting

$$V(t) = \frac{1}{2}\int_0^1 u^2(x,t)\mathrm{d}x, \tag{67}$$

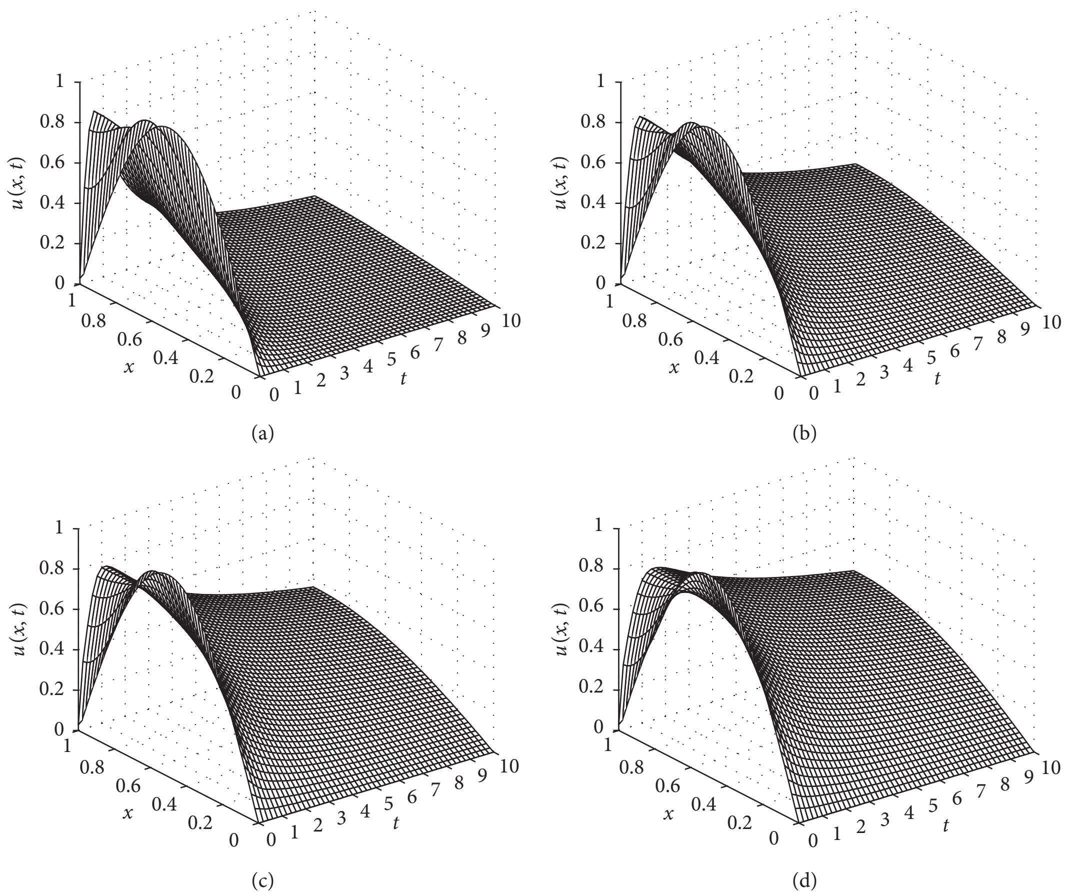


Figure 9: A 3D view of the dynamics of the MGKdVB equation utilizing the third nonlinear adaptive controller $\mu = 0.001$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$. (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$.

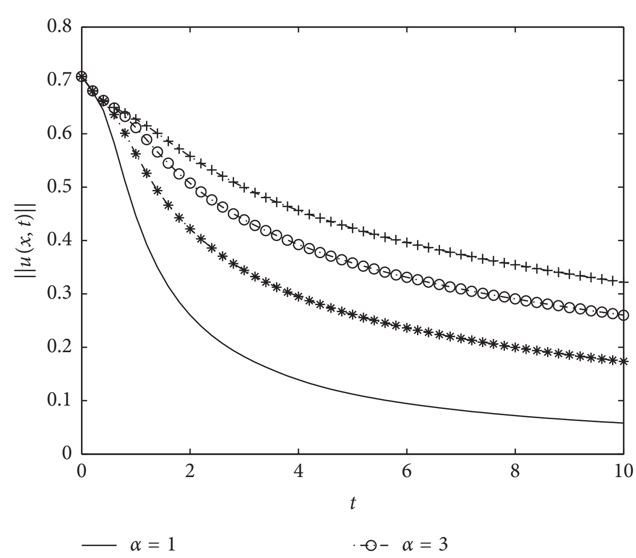


Figure 10: The $L^2$-norm of $u(x,t)$, $\|u(x,t)\|$, over time for various values of $\alpha$ when applying the third nonlinear adaptive controller $\mu = 0.001$, $\gamma_2 = 0.0005$, $r_1 = 0.2$, $r_2 = 0.2$, and $u_0(x) = \sin(\pi x)$.

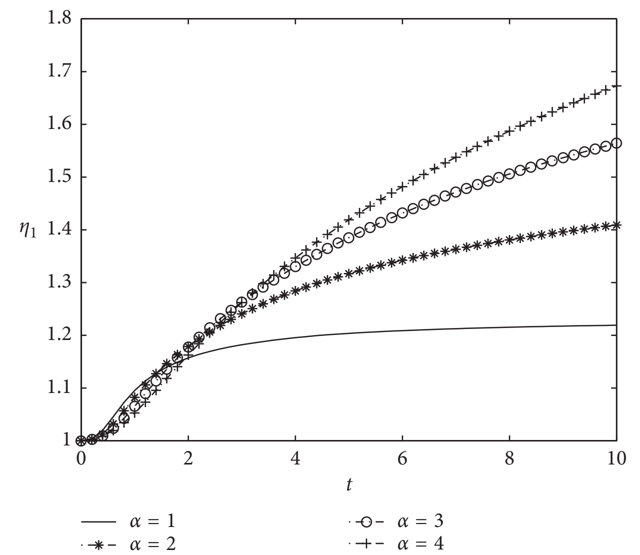


FIGURE 11: $\eta_1$ over time for various values of $\alpha$ when applying the third nonlinear adaptive controller.

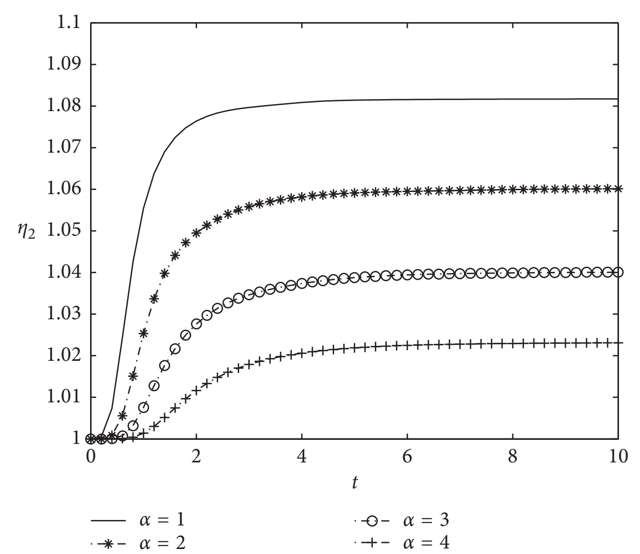


FIGURE 12: $\eta_2$ over time for various values of $\alpha$ when applying the third nonlinear adaptive controller.

be a Lyapunov function candidate and differentiating $V(t)$, we obtain as in the previous sections the following inequality:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x + \nu u(1,t)f_1(t) - \mu u(1,t)f_2(t)\\ &+\frac{\mu}{2}f_1^2(t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\\ &+\gamma_2 f_1(t)f_2(t).\end{aligned} \tag{68}$$

Using the control law given by equations (44) and (45), the above inequality yields:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x - \mu u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\\ &-\mu\eta_1 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) - \mu\eta_2 u^2(1,t)\\ &-\mu\eta_3 u^{2\alpha+2}(1,t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t)\\ &-\gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t),\\ \text{or } \frac{\mathrm{d}V}{\mathrm{d}t} &\le -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x + u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\\ &\cdot\left(-\mu-\mu\eta_1-\gamma_2\right) - \mu\eta_2 u^2(1,t) - \mu\eta_3 u^{2\alpha+2}(1,t)\\ &-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t).\end{aligned} \tag{69}$$

In light of (34), the last term of inequality (69) can be expanded as follows:

$$-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) \le \frac{\gamma_1}{2(\alpha+2)}u^{2\alpha+2}(1,t) + \frac{\gamma_1}{2(\alpha+2)}u^2(1,t). \tag{70}$$

Thus, inequality (69) can be written as follows:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -\nu\int_0^1\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x - u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\left(\mu+\mu\eta_1+\gamma_2\right)\\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t) - \left(\mu\eta_3 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t).\end{aligned} \tag{71}$$

Applying Poincaré's inequality in (71), we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -\nu\int_0^1 u^2\mathrm{d}x - u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\left(\mu+\mu\eta_1+\gamma_2\right)\\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t) - \left(\mu\eta_3 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t).\end{aligned} \tag{72}$$

Now, we define an energy function $E(t)$ as follows:

$$\begin{aligned}E(t) &= V(t) + \frac{1}{2\mu r_1}\left(\mu+\mu\eta_1+\gamma_2\right)^2\\ &+\frac{1}{2\mu r_2}\left(\mu\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right)^2\\ &+\frac{1}{2\mu r_3}\left(\mu\eta_3 - b - \frac{\gamma_1}{2(\alpha+2)}\right)^2,\end{aligned} \tag{73}$$

where $a, b > 0$.

Taking the derivative of $E(t)$, we obtain

$$\frac{dE}{dt} = \frac{dV}{dt}(t) + \frac{1}{r_1}\frac{d\eta_1}{dt}(\mu + \mu\eta_1 + \gamma_2) + \frac{1}{r_2}\frac{d\eta_2}{dt}\left(\mu\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right) + \frac{1}{r_3}\frac{d\eta_3}{dt}\left(\mu\eta_3 - b - \frac{\gamma_1}{2(\alpha+2)}\right). \quad (74)$$

Thanks to inequality (72) and equations (64)–(66), we have

$$\begin{aligned}\frac{dE}{dt} \le& -v\int_0^1 u^2 dx - u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)(\mu + \mu\eta_1 + \gamma_2) \\ &-\left(\mu\eta_3 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t) \\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(\alpha+2)}\right)u^2(1,t) \\ &+ u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)(\mu + \mu\eta_1 + \gamma_2) \\ &+ u^2(1,t)\left(\mu\eta_2 - a - \frac{\gamma_1}{2(\alpha+2)}\right) \\ &+ u^{2\alpha+2}(1,t)\left(\mu\eta_3 - b - \frac{\gamma_1}{2(\alpha+2)}\right).\end{aligned} \quad (75)$$

The above inequality gives

$$\frac{dE}{dt} \le -v\int_0^1 u^2 dx - au^2(1,t) - bu^{2\alpha+2}(1,t). \quad (76)$$

Noting that $-au^2(1,t) - bu^{2\alpha+2}(1,t) \le 0$, inequality (76) yields:

$$\frac{dE}{dt} \le -v\int_0^1 u^2 dx. \quad (77)$$

One can conclude that $E(t) \le E(0)$, $\forall t \ge 0$. Hence, $\eta_1$, $\eta_2$, and $\eta_3$ are bounded functions for all $t > 0$. Thus, from (64)–(66), we obtain

$$u(1,t) \in L^2(0,\infty) \cap L^{2(\alpha+1)}(0,\infty). \quad (78)$$

From inequality (72), we have

$$\begin{aligned}\frac{dV}{dt} \le& -2vV(t) - u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)(\mu + \mu\eta_1 + \gamma_2) \\ &-\left(\mu\eta_2 - \frac{\gamma_1}{2(2\alpha+2)}\right)u^2(1,t) \\ &-\left(\mu\eta_3 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,t).\end{aligned} \quad (79)$$

Exploiting Gronwall–Bellman's inequality, we obtain

$$\begin{aligned}\frac{dV}{dt} \le& V(0)e^{-2vt} + \int_0^t e^{-2v(t-\tau)}\bigg[u(1,\tau)\frac{\partial^3 u}{\partial x^3}(1,\tau) \\ &(\mu + \mu\eta_1 + \gamma_2) - \left(\mu\eta_2 - \frac{\gamma_1}{2(2\alpha+2)}\right)u^2(1,\tau) \\ &-\left(\mu\eta_3 - \frac{\gamma_1}{2(\alpha+2)}\right)u^{2\alpha+2}(1,\tau)\bigg]d\tau.\end{aligned} \quad (80)$$

Thus,

$$\begin{aligned}\frac{dV}{dt} \le& V(0)e^{-2vt} + C\int_0^t e^{-2v(t-\tau)}\bigg[\left|u(1,\tau)\frac{\partial^3 u}{\partial x^3}(1,\tau)\right| \\ &+ \left|u^2(1,\tau)\right| + \left|u^{2\alpha+2}(1,\tau)\right|\bigg]d\tau,\end{aligned} \quad (81)$$

where

$$\begin{aligned}C = \max\bigg\{&\sup_{t>0}\left|\mu\eta_2(t) - \frac{\gamma_1}{2(\alpha+2)}\right|, \sup_{t>0}\left|\mu\eta_3(t) - \frac{\gamma_1}{2(\alpha+2)}\right|, \\ &\sup_{t>0}\left|\mu + \mu\eta_1(t) + \gamma_2\right|\bigg\}.\end{aligned} \quad (82)$$

Hence, using the fact that $u(1,t) \in L^2(0,\infty) \cap L^{2(\alpha+1)}(0,\infty)$, one can show that $V(t)$ tends to zero as $t \longrightarrow \infty$. Therefore, $u(x,t)$ exponentially tends to zero as $t \longrightarrow \infty$. □

*5.1. Numerical Solutions.* Numerical solutions for the MGKdVB equation (i.e., equations (1)–(6)) in the presence of the control $f_1(t)$ and $f_2(t)$ as presented by equations (62) and (63) were simulated using COMSOL Multiphysics software. The solutions are computed for $\alpha = 1, 2, 3$, and 4.

The fourth adaptive control law designed in equations (62) and (63) is independent of the values of the kinematic viscosity $v$, the dynamic viscosity $\mu$, $\gamma_1$, and $\gamma_2$. Although these parameters are assumed to be unknown, we set in our numerical simulations the values of $v$, $\mu$, $\gamma_1$, and $\gamma_2$ as 0.01, 0.001, 1, and 0.0005, respectively. Moreover, $r_1$, $r_2$, and $r_3$ are chosen to equal 0.2, 0.2, and 0.5, respectively, while the initial values of $\eta_1$, $\eta_2$, and $\eta_3$ are set to be such as $\eta_1(0) = \eta_2(0) = \eta_3(0) = 1$ and $u(x,0) = \sin(\pi x)$. A 3D view of the solution of the MGKdVB equation coupled with the control law presented in Theorem 4 is shown in Figures 13(a)–13(d) for several values of $\alpha$. Figure 14 shows the $L^2$-norm of these solutions.

Figures 15–17 show the behavior of the functions $\eta_1$, $\eta_2$, and $\eta_3$ that are given in the fourth control law. For instance, it can be concluded from Figure 15 that the function $\eta_1$ increases as the value of $\alpha$ increases from 1 to 4. Furthermore, Figure 16 indicates that $\eta_2$ also increases as $\alpha$ increases from 2 to 4. Also, one can notice from Figure 17 that $\eta_3$ decreases as $\alpha$ increases. Again similar remarks to the previous controllers can be made here; the choice of the feedback gains $r_1$, $r_2$, and $r_3$ will definitely affect the values of $\eta_1$, $\eta_2$, and $\eta_3$ and

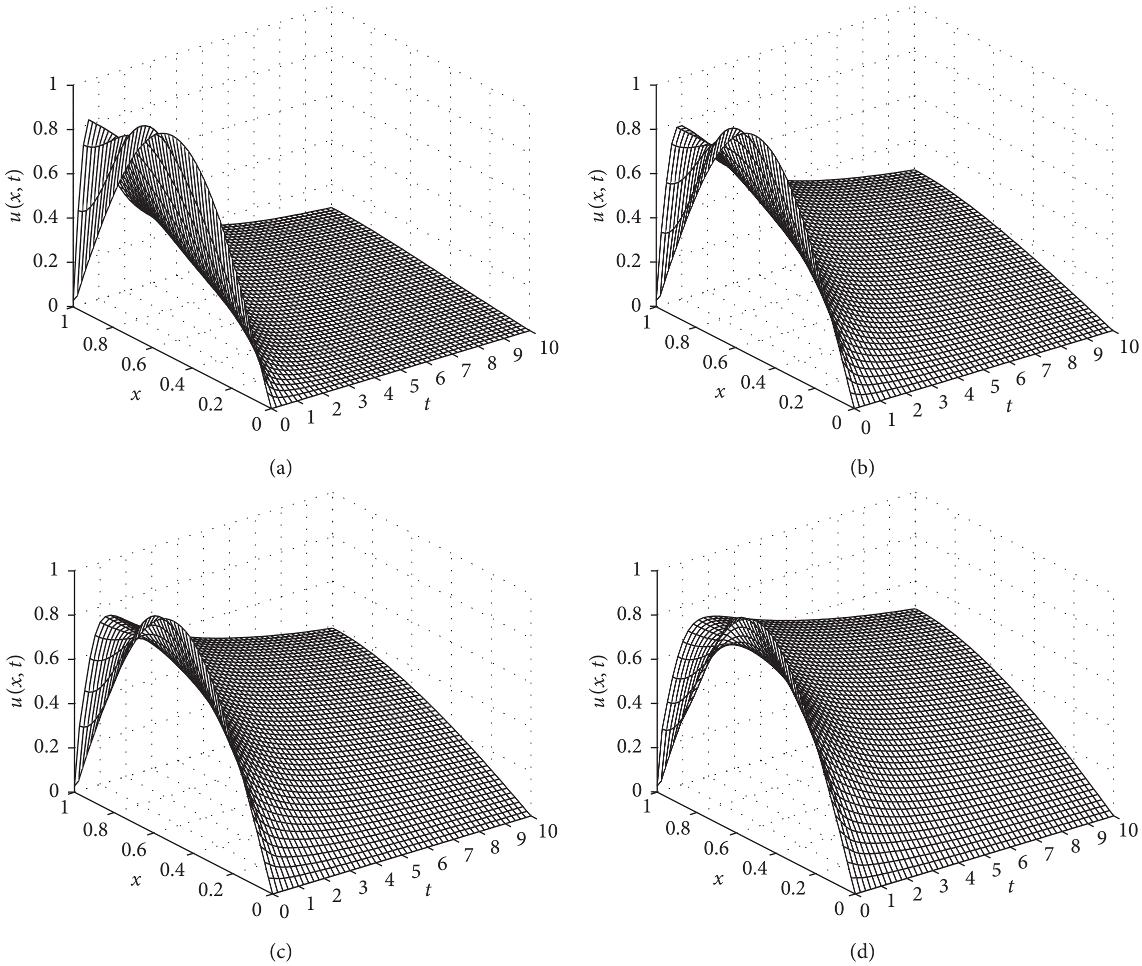


FIGURE 13: A 3D view of the dynamics of the MGKdVB equation utilizing the fourth nonlinear adaptive controller $r_1 = 0.2$, $r_2 = 0.2$, $r_3 = 0.4$, and $u_0(x) = \sin(\pi x)$. (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$.

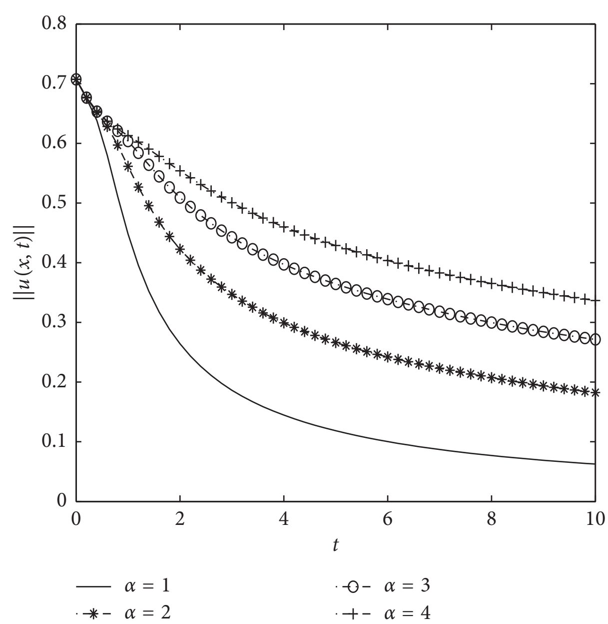


FIGURE 14: The $L^2$-norm of $u(x,t)$, $\|u(x,t)\|$, over time for various values of $\alpha$ when applying the fourth nonlinear adaptive controller $r_1 = 0.2$, $r_2 = 0.2$, $r_3 = 0.4$, and $u_0(x) = \sin(\pi x)$.

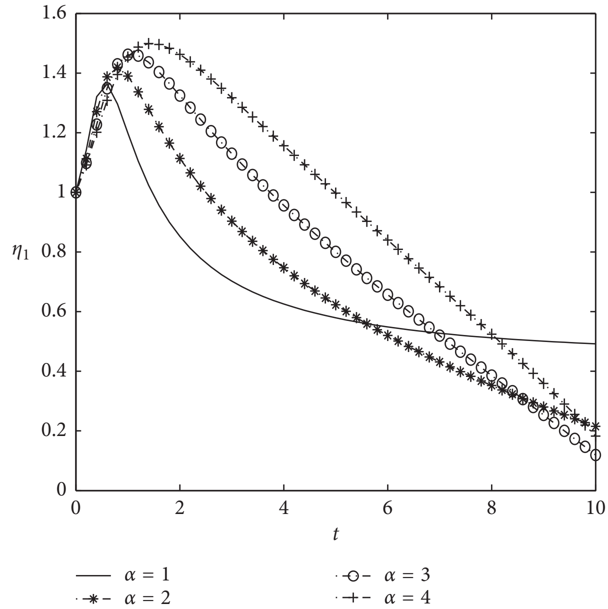


Figure 15: $\eta_1$ over time for various values of $\alpha$ when applying the fourth nonlinear adaptive controller.

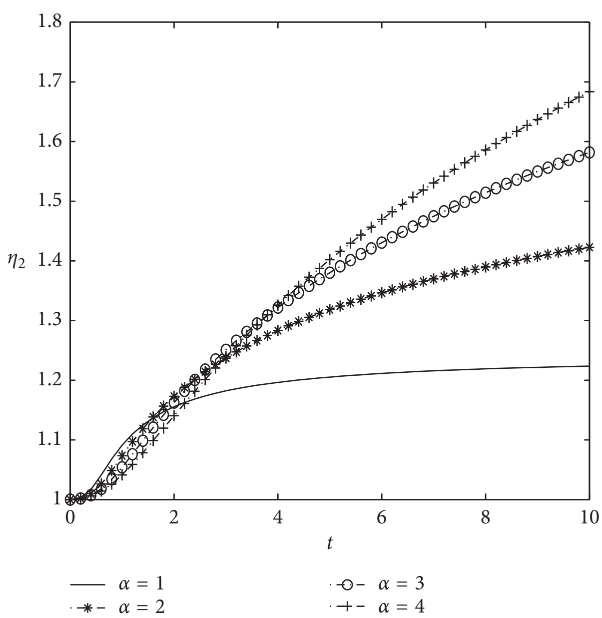


Figure 16: $\eta_2$ over time for various values of $\alpha$ when applying the fourth nonlinear adaptive controller.

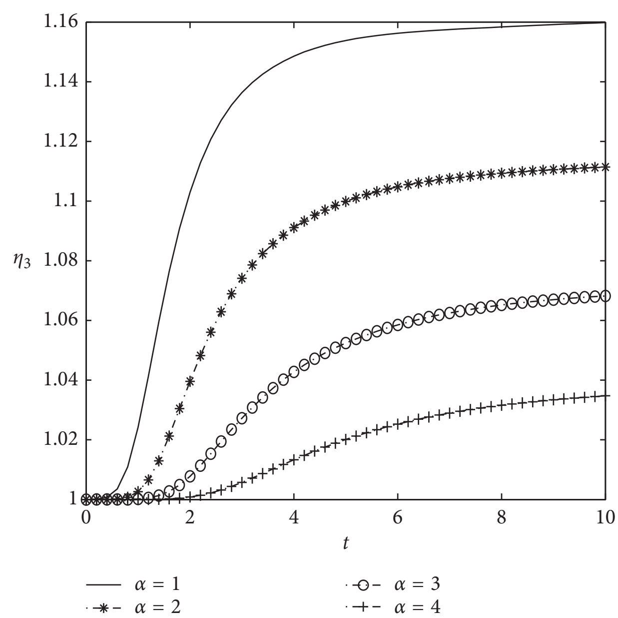


Figure 17: $\eta_3$ over time for various values of $\alpha$ when applying the fourth nonlinear adaptive controller.

therefore the stability of the solution. Fixing the initial conditions, $\eta_1(0)$, $\eta_2(0)$, and $\eta_3(0)$, and increasing the values of $r_1$, $r_2$, and $r_3$, will definitely increase the values of $\eta_1$, $\eta_2$, and $\eta_3$ which in turn increase the convergence rate of the solution to zero, as can be shown from inequality (69).

Obviously, the parameters of the controller $f_2(t)$ are $\eta_1$, $\eta_2$, $\eta_3$, and $\alpha$. Fixing $\alpha$, the parameters $\eta_1$, $\eta_2$, and $\eta_3$ can be computed from the uncoupled system of ODE equations (64)–(66) once the value at the right boundary $u(1,t)$ and $(\partial^3 u/\partial x^3)(1,t)$ are known, the control gains $r_1$, $r_2$, and $r_3$ are chosen, and the initial conditions $\eta_1(0)$, $\eta_2(0)$, and $\eta_3(0)$ are fixed. Then, the controller $f_2(t)$ can be simply computed using equation (63). This clearly shows that the computations in the fourth adaptive boundary controller are also reasonable.

*Remark 1.* The design of the four adaptive boundary controllers proposed in Sections 2–5 involves low-computational complexity since it is computationally bounded by the time it takes to solve the uncoupled system of first order differential equations.

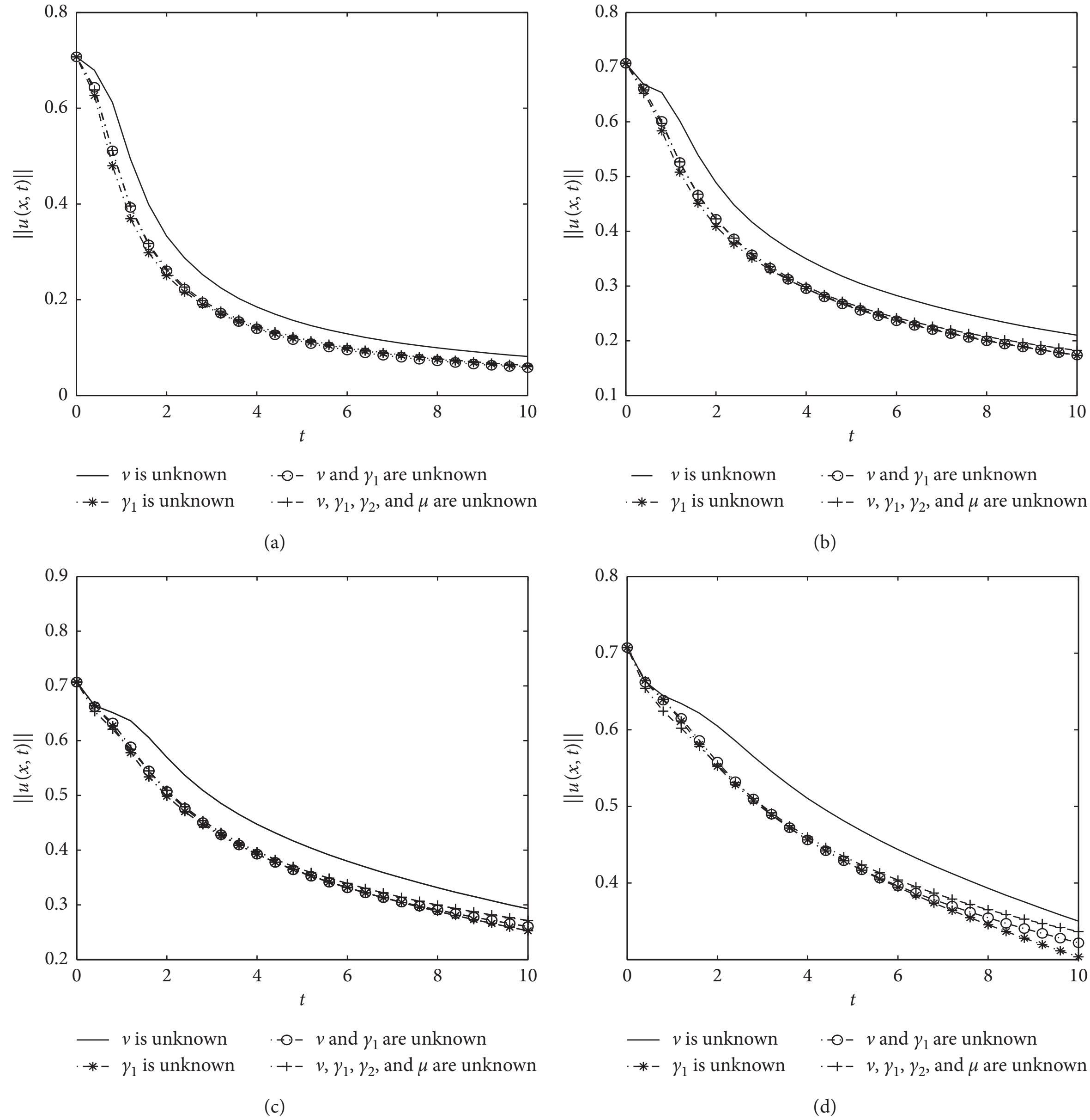


Figure 18: The $L^2$-norm of $u(x,t)$, $\|u(x,t)\|$, over time for various values of $\alpha$; comparison between the behavior of the equation for the fourth nonlinear adaptive controller, $u_0(x) = \sin(\pi x)$. (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$.

## 6. Comparison of the Nonlinear Adaptive Controllers Proposed in Sections 2–5

In this section, we compare the performances of the nonlinear controllers presented in Sections 2–5 for several values of $\alpha$. This comparison illustrates the efficiency of the adaptive control laws when either $\nu$ or $\gamma_1$ is unknown, when both $\nu$ and $\gamma_1$ are unknown, and when all the parameters are unknown.

The $L^2$-norm of the solutions $u(x,t)$ of the MGKdVB equation will be used for comparison. Figures 18(a)–18(d) show the $L^2$-norm of $u(x,t)$ over time for several values of $\alpha$ when $u_0(x) = \sin(\pi x)$. These figures show that, for all values of $\alpha$, the solutions of the MGKdVB equation obtained utilizing the first control law given when $\nu$ is unknown takes a longer time to reach the steady state solution than the other control laws. A careful look at the figures also demonstrates that, for $\alpha = 1, 2$, and 3, solutions of the MGKdVB equation obtained utilizing the second, third, and fourth control laws seem to have a similar decay rate. Figure 18(d) indicates that, when $\alpha = 4$, the second control law which is proposed when $\gamma_1$ is unknown outperforms the other control laws.

## 7. Concluding Remarks

The adaptive boundary stabilization problem of the MGKdVB equation (1) was tackled in this paper. Four different nonlinear adaptive control schemes were introduced for this equation when the physical parameters $\nu$, $\mu$, $\gamma_1$, and $\gamma_2$ are unknown and positive and when $\alpha$ is a positive integer. The global exponential stability of the solution in

$L^2(0,1)$ has been established analytically and numerically. In addition, the rates of convergence of the four presented controllers were compared.

For future work, we should investigate the adaptive and nonadaptive stabilization of the MGKdVB equation (1) under the presence of a time-delay in the boundary control. It would be also interesting to investigate the same problem when equation (1) itself is subject to the effect of the time delay in one of the terms, especially the nonlinear one $u^{\alpha}(\partial u/\partial x)$.

## Data Availability

The numerical data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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