Research Article

Symmetry Groups, Similarity Reductions, and Conservation Laws of the Time-Fractional Fujimoto–Watanabe Equation Using Lie Symmetry Analysis Method

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In this paper, the time-fractional Fujimoto–Watanabe equation is investigated using the Riemann–Liouville fractional derivative. Symmetry groups and similarity reductions are obtained by virtue of the Lie symmetry analysis approach. Meanwhile, the time-fractional Fujimoto–Watanabe equation is transformed into three kinds of reduced equations and the third of which is based on Erdélyi–Kober fractional integro-differential operators. Furthermore, the conservation laws are also acquired by Ibragimov's theory.

1. Introduction

Nowadays, nonlinear partial differential equations (NPDEs) have become more and more significant in fluid mechanics, mathematical physics, oceanography, and so on [1–4]. As we know, NPDEs are usually of integer order and researchers have proposed abundant methods to obtain solutions of NPDEs, including inverse scattering transformation [5], Riemann–Hilbert method [6–10], Hirota direct method [11, 12], Darboux transformation [13, 14], Bäcklund transformation [15], Frobenius integrable decompositions [16, 17], and so on [18–24]. As a generalization, the notions of fractional derivatives are put forward and the classical are Riemann–Liouville and Caputo fractional derivatives. Nonlinear fractional differential equations (NFDEs) are also introduced and have a large number of applications in mathematical physics and automation. The exploration of solutions for NFDEs is a crucial aspect. A variety of methods are presented, for instance, the first integral method [25], functional variable method [26], auxiliary equation method [27], and exponential function method [28].

The Lie symmetry method also provides a way to seek solutions for NPDEs and NFDEs [29–32]. This is a way of

using known (old) solutions to find new ones. If we obtain a solution for NPDEs or NFDEs, then by using group transformations, the new solutions can be derived. It means that if a NPDE or NFDE has a solution, then it will actually have infinitely many solutions. This method is so effective. Researchers obtain analytical solutions and conservation laws to equations with the help of this method, for example, seventh-order time-fractional Sawada–Kotera–Ito equation, time-fractional fifth-order modified Sawada–Kotera equation, Burridge–Knopoff equation, and so on [33–37].

The time-fractional Fujimoto–Watanabe equation [38] is

$$D_t^\gamma u = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 3u^2 u_x, \quad (1)$$

where $D_t^\gamma u$ is the Riemann–Liouville fractional derivative of $u = u(x, t)$ with respect to time variable t .

The Fujimoto–Watanabe equation is one important equation and applied in some fields [38]. Its analytical solutions are obtained, and these solutions can reveal many different natural phenomena [39, 40]. For instance, its traveling wave solutions describe the propagation status of water waves in mathematical physics and oceanography. In geography, specialists can predict natural disasters with the

help of its solutions. In fluid mechanics, researchers acquire its period solutions and study its dynamical behaviors [41].

This paper is organized as follows. In Section 2, we introduce basic concepts and properties about the Riemann–Liouville fractional derivative. In Section 3, symmetry groups are obtained with the help of the Lie symmetry analysis approach. In Section 4, similarity reductions are derived and the time-fractional Fujimoto–Watanabe equation is transformed into three kinds of reduced equations. In Section 5, based on Ibragimov’s theory, the conservation laws of the time-fractional Fujimoto–Watanabe equation are constructed. In Section 6, some conclusions are given.

2. Basic Concept and Properties of the Riemann–Liouville Fractional Derivative

Definition 1 (Riemann–Liouville fractional derivative) (see [42]). Assuming $f = f(x, t)$ is a real-valued function, where x is the space variable and t is the time variable, then the Riemann–Liouville fractional derivative of f of order $\gamma (\gamma > 0)$ is defined as follows:

$$D_t^\gamma f(x, t) = \frac{\partial^\gamma f}{\partial t^\gamma} = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\gamma-1} f(x, s) ds, & n-1 < \gamma < n, \\ \frac{\partial^n f}{\partial t^n}, & \gamma = n, \end{cases} \quad (2)$$

where $\Gamma(x)$ is the gamma function.

The Riemann–Liouville fractional derivative has many properties, for instance,

$$(a) D_t^\gamma t^\eta = \frac{\Gamma(\eta+1)t^{\eta-\gamma}}{\Gamma(\eta-\gamma+1)}, \quad \eta > 0,$$

$$(b) D_t^\gamma f[g(t)] = f_g[g(t)] \cdot D_t^\gamma g(t) = D_t^\gamma f[g(t)] \cdot (g_t)^\gamma,$$

$$(c) D_t^\gamma [f(t) \cdot g(t)] = D_t^\gamma f(t) \cdot g(t) + f(t) \cdot D_t^\gamma g(t), \quad (3)$$

where f and g are real-valued functions, $f_g = (df/dg)$, and $g_t = (dg/dt)$.

From Definition 1, we find that the Riemann–Liouville fractional derivative is a generalized form of the ordinary integer-order derivative. Property (b) is the composite rule of the Riemann–Liouville fractional derivative. Property (c) is the Leibniz rule of the Riemann–Liouville fractional derivative.

3. Symmetry Group of the Time-Fractional Fujimoto–Watanabe Equation

In this section, we seek symmetry groups for the time fractional Fujimoto–Watanabe equation with the help of the Lie symmetry analysis method.

The general time-fractional differential equation is as follows:

$$D_t^\gamma u = L(x, t, u, u_x, u_{xx}, u_{xxx}, \dots), \quad (4)$$

where $u = u(x, t)$, the subscripts represent partial derivatives, i.e., $u_x = (\partial u / \partial x)$, $u_{xx} = (\partial^2 u / \partial x^2)$, $u_{xxx} = (\partial^3 u / \partial x^3)$, $\dots$, and the corresponding one-parameter transformations are

$$\bar{x} = x + \varepsilon \lambda(x, t, u) + O(\varepsilon^2),$$

$$\bar{t} = t + \varepsilon \mu(x, t, u) + O(\varepsilon^2),$$

$$\bar{u} = u + \varepsilon \varphi(x, t, u) + O(\varepsilon^2),$$

$$D_t^\gamma \bar{u} = D_t^\gamma u + \varepsilon \varphi^\gamma(x, t, u) + O(\varepsilon^2),$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial x} + \varepsilon \varphi^x(x, t, u) + O(\varepsilon^2), \quad (5)$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon \varphi^{xx}(x, t, u) + O(\varepsilon^2),$$

$$\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} = \frac{\partial^3 u}{\partial x^3} + \varepsilon \varphi^{xxx}(x, t, u) + O(\varepsilon^2),$$

$\dots$

where $\varepsilon \ll 1$ is an infinitesimal parameter and $\lambda = \lambda(x, t, u)$, $\mu = \mu(x, t, u)$, and $\varphi = \varphi(x, t, u)$ are real-valued infinitesimal functions with respect to variables x, t , and u . $\varphi^\gamma = \varphi^\gamma(x, t, u)$, $\varphi^x = \varphi^x(x, t, u)$, $\varphi^{xx} = \varphi^{xx}(x, t, u)$, and $\varphi^{xxx} = \varphi^{xxx}(x, t, u)$ are extended infinitesimal functions. These extended infinitesimal functions can be determined using the following expressions:

$$\begin{aligned} \varphi^\gamma &= D_t^\gamma(\varphi) + \lambda D_t^\gamma(u_x) - D_t^\gamma(\lambda u_x) + D_t^\gamma(D_t(\mu)u) \\ &\quad - D_t^{\gamma+1}(\mu u) + \mu D_t^{\gamma+1}(u), \end{aligned} \quad (6)$$

$$\begin{aligned} \varphi^x &= D_x(\varphi) - u_x D_x(\lambda) - u_t D_x(\mu), \\ \varphi^{xx} &= D_x(\varphi^x) - u_{xx} D_x(\lambda) - u_{xt} D_x(\mu), \\ \varphi^{xxx} &= D_x(\varphi^{xx}) - u_{xxx} D_x(\lambda) - u_{xxt} D_x(\mu), \\ &\dots \end{aligned} \quad (7)$$

where D_t^γ represents the total Riemann–Liouville fractional derivative with respect to t , D_x represents the total derivative with respect to x , and D_t represents the total derivative with respect to t , i.e.,

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \dots, \quad (8)$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{xtt} \frac{\partial}{\partial u_{xx}} + \dots.$$

In order to obtain the symmetry groups, let the infinitesimal generator be as follows:

$$V = \lambda(x, t, u) \frac{\partial}{\partial x} + \mu(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u}, \quad (9)$$

where V is sometimes called vector field.

The third-order prolongation of V is

$$\begin{aligned} pr^{(3)}V = & \lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u} + \varphi^y \frac{\partial}{\partial u_t^y} + \varphi^x \frac{\partial}{\partial u_x} \\ & + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}}, \end{aligned} \quad (10)$$

where $u_t^y = D_t^y u$.

Assume

$$\begin{aligned} \Delta(x, t, u, u^{(3)}) \\ = D_t^y u - L(x, t, u, u_x, u_{xx}, u_{xxx}, \dots) \\ = D_t^y u - u^3 u_{xxx} - 3u^2 u_x u_{xx} - 3u^2 u_x. \end{aligned} \quad (11)$$

In order to satisfy the invariance condition of Lie symmetry, $pr^{(3)}V$ needs to meet the following identity:

$$pr^{(3)}V(\Delta)|_{\Delta=0} = 0. \quad (12)$$

By direct calculation, we obtain

$$\begin{aligned} [\phi^y - u^3 \phi^{xxx} - 3u^2 u_x \phi^{xx} - (3u^2 u_{xx} + 3u^2) \phi^x \\ - (3u^2 u_{xxx} + 6uu_x u_{xx} + 6uu_x) \phi]|_{\Delta=0} = 0. \end{aligned} \quad (13)$$

From equation (7), we have

$$\begin{aligned} \phi^x &= \phi_x + \phi_u u_x - \lambda_x u_x - \lambda_u u_x^2 - \mu_x u_t - \mu_u u_x u_t, \\ \phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \lambda_{xx})u_x + (\phi_{uu} - 2\lambda_{xu})u_x^2 - \lambda_{uu}u_x^3 \\ &\quad + (\phi_u - 2\lambda_x)u_{xx} - 3\lambda_u u_x u_{xx} - \mu_{xx}u_t - 2\mu_x u_{xt} \\ &\quad - 2\mu_{xu}u_x u_t - 2\mu_u u_x u_{xt} - \mu_{uu}u_{xx}u_t - \mu_{uu}u_x^2 u_t, \\ \phi^{xxx} &= \phi_{xxx} + (3\phi_{xuu} - \lambda_{xxx})u_x + 3(\phi_{xuu} - \lambda_{xuu})u_x^2 \\ &\quad + (\phi_{uuu} - 3\lambda_{uuu})u_x^3 - \lambda_{uuu}u_x^4 + 3(\phi_{xu} - \lambda_{xx})u_{xx} \\ &\quad + 3(\phi_{uu} - 3\lambda_{xu})u_x u_{xx} - 6\lambda_{uu}u_x^2 u_{xx} + (\phi_u - 3\lambda_x)u_{xxx} \\ &\quad - 3\lambda_u u_{xx}^2 - \mu_{xxx}u_t - 3\mu_{xxu}u_x u_t - 3\mu_{xx}u_{xt} \\ &\quad - 6\mu_{xu}u_x u_{xt} - 3\mu_x u_{xxt} - 3\mu_{xuu}u_x^2 u_t - 3\mu_{xu}u_{xx}u_t \\ &\quad - 3\mu_{uu}u_x^2 u_{xt} - 3\mu_u u_{xx}u_{xt} - 3\mu_u u_x u_{xxt} \\ &\quad - 3\mu_{uu}u_x u_{xx}u_t - \mu_u u_{xxx}u_t - \mu_{uuu}u_x^3 u_t - 4\lambda_u u_x u_{xxx}. \end{aligned} \quad (14)$$

Definition 2 (Generalized Leibniz rule of the Riemann–Liouville fractional derivative). Assuming $u = u(t)$ and $v = v(t)$ are real-valued functions, then the generalized Leibniz rule is

$$D_t^\gamma [u(t) \cdot v(t)] = \sum_{n=0}^{\infty} \binom{\gamma}{n} D_t^{\gamma-n} [u(t)] \cdot D_t^n [v(t)], \quad \gamma > 0, \quad (15)$$

where

$$\binom{\gamma}{n} = \frac{(-1)^{n-1} \gamma \Gamma(n-\gamma)}{\Gamma(1-\gamma) \Gamma(n+1)}. \quad (16)$$

Definition 3 (Generalized composite (chain) rule). Assuming $u = u(t)$ and $v = v(t)$ are real-valued functions, then the generalized composite rule is

$$\frac{d^n u[v(t)]}{dt^n} = \sum_{i=0}^n \sum_{j=0}^i \binom{i}{j} \frac{1}{i!} [-v(t)]^j \frac{d^n}{dt^n} [(v(t))^{i-j}] \frac{d^i u(v)}{dv^i}. \quad (17)$$

Because of equation (11), equation (6) can be rewritten as

$$\begin{aligned} \phi^y &= D_t^y(\phi) - \gamma D_t(\mu) \cdot \frac{\partial^y u}{\partial t^y} - \sum_{n=1}^{\infty} \binom{\gamma}{n} D_t^n(\lambda) \cdot D_t^{\gamma-n}(u_x) \\ &\quad - \sum_{n=1}^{\infty} \binom{\gamma}{n+1} D_t^{n+1}(\mu) \cdot D_t^{\gamma-n}(u). \end{aligned} \quad (18)$$

According to equations (11) and (13), we obtain

$$D_t^y(\phi) = \frac{\partial^y \phi}{\partial t^y} + \phi_u \frac{\partial^y u}{\partial t^y} - u \frac{\partial^y \phi_u}{\partial t^y} + \sum_{n=1}^{\infty} \binom{\gamma}{n} \frac{\partial^y \phi_u}{\partial t^y} \cdot D_t^{\gamma-n}(u) + \phi, \quad (19)$$

where

$$\begin{aligned} \phi &= \sum_{n=2}^{\infty} \sum_{i=2}^n \sum_{j=2}^i \sum_{k}^{j-1} \binom{\gamma}{n} \binom{n}{i} \binom{j}{k} \frac{1}{j!} \frac{t^{n-\gamma}}{\Gamma(n+1-\gamma)} \\ &\quad (-u)^k \frac{\partial^j \phi}{\partial t^i} (u^{j-k}) \frac{\partial^{n-i+j} \phi}{\partial t^{n-i} \partial u^j}. \end{aligned} \quad (20)$$

According to equations (3)b and (3)c, equation (14) can be rewritten as

$$\begin{aligned} \phi^y &= \frac{\partial^y \phi}{\partial t^y} + (\phi_u - \gamma D_t(\mu)) \frac{\partial^y u}{\partial t^y} - u \frac{\partial^y \phi_u}{\partial t^y} + \phi \\ &\quad + \sum_{n=1}^{\infty} \left[\binom{\gamma}{n} \frac{\partial^n \phi_u}{\partial t^n} - \binom{\gamma}{n+1} D_t^{n+1}(\mu) \right] D_t^{\gamma-n}(u) \\ &\quad - \sum_{n=1}^{\infty} \binom{\gamma}{n} D_t^n(\lambda) \cdot D_t^{\gamma-n}(u_x). \end{aligned} \quad (21)$$

Substituting equations (14) and (21) into equation (13) and equating the coefficients of all powers of partial

derivatives of u to 0, we obtain a set of determining equations as follows:

$$\begin{aligned}
\lambda_t &= \lambda_u = 0, \\
\mu_x &= \mu_u = 0, \\
\varphi_{uu} &= 0, \\
\varphi_u - 2\lambda_x &= 0, \\
3\lambda_x - \gamma\mu_t - 3\varphi_u &= 0, \\
\frac{\partial^\gamma \varphi}{\partial t^\gamma} - u \frac{\partial^\gamma \varphi_u}{\partial t^\gamma} - u^3 \varphi_{xxx} - 3u^2 \varphi_x &= 0, \\
\left(\frac{\gamma}{n} \right) \frac{\partial^n \varphi_u}{\partial t^n} - \left(\frac{\gamma}{n+1} \right) D_t^{n+1}(\mu) &= 0.
\end{aligned} \tag{22}$$

Solving equation (22), we acquire the solution as follows:

$$\begin{aligned}
\lambda(x, t, u) &= \gamma C_1 x + C_2, \\
\mu(x, t, u) &= -3C_1 t, \\
\varphi(x, t, u) &= 2\gamma C_1 u,
\end{aligned} \tag{23}$$

where C_1 and C_2 are arbitrary constants.

Based on the above results, the infinitesimal generator can be rewritten as

$$V = (\gamma C_1 x + C_2) \frac{\partial}{\partial x} - 3C_1 t \frac{\partial}{\partial t} + 2\gamma C_1 u \frac{\partial}{\partial u}. \tag{24}$$

If we let

$$\begin{aligned}
V_1 &= \gamma x \frac{\partial}{\partial x} - 3t \frac{\partial}{\partial t} + 2\gamma u \frac{\partial}{\partial u}, \\
V_2 &= \frac{\partial}{\partial x},
\end{aligned} \tag{25}$$

then V can also be rewritten as

$$V = C_1 V_1 + C_2 V_2. \tag{26}$$

Introducing Lie bracket operation, i.e., for arbitrary vector fields A and B , $[A, B] = AB - BA$.

From Table 1, we can find V_1 and V_2 are closed obviously. Consequently, the symmetry groups of the time-fractional Fujimoto–Watanabe equation can be spanned by $\{V_1, V_2\}$.

4. Similarity Reductions for the Time-Fractional Fujimoto–Watanabe Equation

In this section, we investigate the similarity reductions for the time-fractional Fujimoto–Watanabe equation. Thus, we can obtain reduced equations.

Because the symmetry groups are spanned by V_1 and V_2 , we need to discuss in two cases:

Case 1: for $V = (\partial/\partial x)$, we need to solve the following system of equations:

TABLE 1: Lie bracket operation of V_1 and V_2 .

$[V_i, V_j]$	V_1	V_2
V_1	0	$-\gamma V_2$
V_2	γV_2	0

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \tag{27}$$

From equation (27), we arrive at t and u are similarity variables. We can assume the solution of equation (1) has the form $u = f(t)$.

Substituting $u = f(t)$ into equation (1), we have the following reduced equation:

$$D_t^\gamma u = 0. \tag{28}$$

Solving equation (28), we obtain the group invariant solutions:

$$u = u(x, t) = f(t) = lt^{\gamma-1}, \tag{29}$$

where l is an arbitrary constant.

Case 2 (method 1): for $V = \gamma x (\partial/\partial x) - 3t (\partial/\partial t) + 2\gamma u (\partial/\partial u)$, similar to Case 1, we also need to solve the following system of equations:

$$\frac{dx}{\gamma x} = \frac{dt}{-3t} = \frac{du}{2\gamma u}. \tag{30}$$

From equation (30), we arrive at similarity variables as follows:

$$\begin{aligned}
\eta &= tx^{3/\gamma}, \\
f &= ux^{-2}.
\end{aligned} \tag{31}$$

We can assume the solution of equation (1) has the form

$$u = x^2 f(\eta) = x^2 f(tx^{3/\gamma}). \tag{32}$$

Substituting equation (32) into equation (1), we have the following reduced equation:

$$\begin{aligned}
D_\eta^\gamma f(\eta) &= 12f^4 + 6f^3 + \left(\frac{27}{\gamma^3} + \frac{81}{\gamma^2} + \frac{78}{\gamma} \right) \eta f^3 f' \\
&+ \left(\frac{81}{\gamma^3} + \frac{81}{\gamma^2} \right) \eta^2 f^3 f'' + \frac{27}{\gamma^3} \eta^3 f^3 f''' \\
&+ \left(\frac{81}{\gamma^3} + \frac{81}{\gamma^2} \right) \eta^2 f^2 f'^2 + \frac{81}{\gamma^3} \eta^3 f^2 f' f'' + \frac{9}{\gamma} \eta f^2 f',
\end{aligned} \tag{33}$$

where $\eta = tx^{3/\gamma}$.

For Case 2, we have another method to obtain the reduced equation. We need to use Erdélyi-Kober fractional integro-differential operators.

Case 2 (method 2): for $V = \gamma x(\partial/\partial x) - 3t(\partial/\partial t) + 2\gamma u(\partial/\partial u)$, we also have the following similarity variables:

$$\begin{aligned}\eta &= xt^{\gamma/3}, \\ f &= ut^{(2/3)\gamma}.\end{aligned}\quad (34)$$

Then, the solution of equation (1) has the form

$$u = t^{-(2/3)\gamma} f(\eta) = t^{-(2/3)\gamma} f(xt^{\gamma/3}). \quad (35)$$

Assuming $n-1 < \gamma < n$ and substituting equation (35) into Definition 1, we have

$$D_t^\gamma u(x, t) = \frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\xi)^{n-\gamma-1} \xi^{-(2/3)\gamma} f(x\xi^{\gamma/3}) d\xi \right] \quad (n-1 < \gamma < n). \quad (36)$$

Introducing variable transformation,

$$\zeta = \frac{t}{\xi}. \quad (37)$$

$$d\xi = -\frac{t}{\zeta^2} d\zeta. \quad (38)$$

Substituting equations (37) and (38) into equation (36), we derive

Thus,

$$\begin{aligned}D_t^\gamma u(x, t) &= \frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^n}{\partial t^n} \left[t^{n-(5/3)\gamma} \frac{1}{\Gamma(n-\gamma)} \int_1^\infty (\zeta-1)^{n-\gamma-1} \zeta^{-(n-(5/3)\gamma+1)} f(\zeta^{-(\gamma/3)} \eta) d\zeta \right] \\ &= \frac{\partial^n}{\partial t^n} \left[t^{n-(5/3)\gamma} \left(K_{-(3/\gamma)}^{1-(2/3)\gamma, n-\gamma} f \right) (\eta) \right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[\frac{\partial}{\partial t} \left(t^{n-(5/3)\gamma} \left(K_{-(3/\gamma)}^{1-(2/3)\gamma, n-\gamma} f \right) (\eta) \right) \right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-(5/3)\gamma-1} \left(n - \frac{5}{3}\gamma + \frac{\gamma}{3} \eta \frac{d}{d\eta} \right) \left(K_{-(3/\gamma)}^{1-(2/3)\gamma, n-\gamma} f \right) (\eta) \right],\end{aligned}\quad (39)$$

where the definition of the Erdélyi-Kober fractional integral operator is as follows:

$$\left(K_k^{i,j} f \right) (\eta) = \begin{cases} \frac{1}{\Gamma(j)} \int_1^\infty (\xi-1)^{j-1} \xi^{-(i+j)} f(\xi^{1/k} \eta) d\xi, & j > 0, \\ f(\eta), & j = 0. \end{cases} \quad (40)$$

Repeating the same procedure $n-1$ times, we derive

$$\begin{aligned}D_t^\gamma u(x, t) &= \frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-(5/3)\gamma-1} \left(n - \frac{5}{3}\gamma + \frac{\gamma}{3} \eta \frac{d}{d\eta} \right) \left(K_{-(3/\gamma)}^{1-(2/3)\gamma, n-\gamma} f \right) (\eta) \right] \\ &= \dots \\ &= t^{-(5/3)\gamma} \left[\prod_{l=0}^{n-1} \left(1 - \frac{5}{3}\gamma + l + \frac{\gamma}{3} \eta \frac{d}{d\eta} \right) \left(K_{-(3/\gamma)}^{1-(2/3)\gamma, n-\gamma} f \right) (\eta) \right] \\ &= t^{-(5/3)\gamma} \left(P_{-(3/\gamma)}^{1-(5/3)\gamma, \gamma} f \right) (\eta),\end{aligned}\quad (41)$$

where the definition of the Erdelyi–Kober fractional differential operator is as follows:

$$\begin{aligned} (P_k^{i,j} f)(\eta) &= \prod_{l=0}^{n-1} \left(i + l - \frac{1}{k} \eta \frac{d}{d\eta} \right) (K_k^{i+j, n-j} f)(\eta), \\ n &= \begin{cases} [j] + 1, & j \notin \mathbb{N}, \\ j, & j \in \mathbb{N}. \end{cases} \end{aligned} \quad (42)$$

Consequently, the time-fractional Fujimoto–Watanabe equation is transformed into the following fractional ordinary differential equation:

$$\left(P_{-(5/3)\gamma}^{1-(5/3)\gamma, \gamma} f \right)(\eta) = f^3 f''' + 3f^2 f' f'' + 3f^2 f', \quad (43)$$

where $\eta = xt^{(\gamma/3)}$.

5. Conservation Laws of the Time-Fractional Fujimoto–Watanabe Equation

Conservation laws have always been one significant aspect of the investigation on NPDEs and NFDEs. In this section, we construct conservation laws for the time-fractional Fujimoto–Watanabe equation. In this process, Ibragimov's theory plays a key role [43].

Assume

$$v(x, t) = D_t^\gamma u - u^3 u_{xxx} - 3u^2 u_x u_{xx} - 3u^2 u_x. \quad (44)$$

Introduce a new dependent variable $\xi = \xi(x, t)$ and a formal Lagrangian:

$$\mathcal{F} = \xi v = \xi [D_t^\gamma u - u^3 u_{xxx} - 3u^2 u_x u_{xx} - 3u^2 u_x]. \quad (45)$$

We can present action integral of equation (45) in the following form:

$$\int_0^T \int_b^a \mathcal{F}(x, t, u, \xi, D_t^\gamma u, u_x, u_{xx}, u_{xxx}) dx dt. \quad (46)$$

The corresponding adjoint equation of equation (44) is obtained as follows:

$$\begin{aligned} v^* &= \frac{\delta \mathcal{F}}{\delta u} \\ &= (D_t^\gamma)^* \xi + 6uu_x^2 \xi_x + 6u^2 u_x \xi_{xx} + 3u^2 \xi_x \\ &\quad + 6u^2 u_{xx} \xi_x + u^3 \xi_{xxx} = 0, \end{aligned} \quad (47)$$

where

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^\gamma)^* \frac{\partial}{\partial u_t^\gamma} + \sum_{n=1}^{\infty} (-1)^n D_{i_1} \dots D_{i_n} \frac{\partial}{\partial u_{i_1 \dots i_n}} \\ &= \frac{\partial}{\partial u} + (D_t^\gamma)^* \frac{\partial}{\partial u_t^\gamma} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}}, \end{aligned} \quad (48)$$

$(D_t^\gamma)^*$ is the adjoint operator of D_t^γ , and it is defined as follows:

$$(D_t^\gamma)^* = L_T^{n-\gamma} D_t^n, \quad (49)$$

where

$$L_T^{n-\gamma} u(x, t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^T \frac{u(x, \zeta)}{(\xi - t)^{1+\gamma-n}} d\zeta, \quad n-1 < \gamma < n. \quad (50)$$

In order to construct conservation laws, we need to find conservation vector $V^\# = (V^x, V^t)$ which satisfies the following conservation law equation:

$$[D_x(V^x) + D_t(V^t)]|_{(1)} = 0, \quad (51)$$

where D_x and D_t are the total derivatives with respect to x and t .

According to Ibragimov's theory, conservation vectors can be acquired as follows:

$$\begin{aligned} V^t &= \mu \mathcal{F} + \sum_{l=0}^{n-1} (-1)^l D_t^{\gamma-1-l} (\mathcal{R}) D_t^l \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) - (-1)^n J \left[\mathcal{R}, D_t^n \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) \right], \\ V^x &= \lambda \mathcal{F} + \mathcal{R} \left[\frac{\partial \mathcal{F}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xx}} \right) + D_x D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x (\mathcal{R}) \left[\frac{\partial \mathcal{F}}{\partial u_{xx}} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x D_x (\mathcal{R}) \left[\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right], \end{aligned} \quad (52)$$

where $n-1 < \gamma < n$ and $\mathcal{R} = \varphi - \lambda u_x - \mu u_t$ s.t. infinitesimal generator $V = \lambda(\partial/\partial x) + \mu(\partial/\partial t) + \varphi(\partial/\partial u)$, and J operator is defined by

$$J(u, v) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \int_t^T \frac{u(x, \zeta) v(x, \lambda)}{(\lambda - \zeta)^{1+\gamma-n}} d\lambda d\zeta. \quad (53)$$

Case 1: infinitesimal generator $V_1 = \gamma x(\partial/\partial x) - 3t(\partial/\partial t) + 2\gamma u(\partial/\partial u)$, $\lambda = \gamma x$, $\mu = -3t$, $\varphi = 2\gamma u$, and $\mathcal{R}_1 = 2\gamma u - \gamma x u_x + 3t u_t$. Then, we have

$$\begin{aligned}
 V^t &= -3t\mathcal{F} + \sum_{l=0}^{n-1} (-1)^l D_t^{\gamma-1-l}(\mathcal{R}_1) D_t^l \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) - (-1)^n J \left[\mathcal{R}_1, D_t^n \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) \right] \\
 &= \sum_{l=0}^{n-1} (-1)^l D_t^{\gamma-1-l} (2\gamma u - \gamma x u_x + 3t u_t) D_t^l (\xi) - (-1)^n J [2\gamma u - \gamma x u_x + 3t u_t, D_t^n (\xi)], \\
 V^x &= \gamma x \mathcal{F} + \mathcal{R}_1 \left[\frac{\partial \mathcal{F}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xx}} \right) + D_x D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x (\mathcal{R}_1) \left[\frac{\partial \mathcal{F}}{\partial u_{xx}} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x D_x (\mathcal{R}_1) \left[\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right] \\
 &= (2\gamma u - \gamma x u_x + 3t u_t) [-3u^2 u_{xx} \xi - 3u^2 \xi - D_x (-3u^2 u_x \xi) + D_{xx} (-u^3 \xi)] \\
 &\quad + D_x (2\gamma u - \gamma x u_x + 3t u_t) [-3u^2 u_x \xi - D_x (-u^3 \xi)] + D_x D_x (2\gamma u - \gamma x u_x + 3t u_t) [-u^3 \xi] \\
 &= -(2\gamma u - \gamma x u_x + 3t u_t) (3u^2 \xi + 3u^2 u_x \xi_x + 3u^2 u_{xx} \xi + u^3 \xi_{xx}) + (\gamma u_{xx} - \gamma x u_{xx} + 3t u_{xt}) (u^3 \xi_x) + (\gamma x u_{xxx} - 3t u_{xxt}) (u^3 \xi).
 \end{aligned} \tag{54}$$

Case 2: infinitesimal generator $V_2 = (\partial/\partial x)$, $\lambda = 1$, $\mu = 0$, $\varphi = 0$, and $\mathcal{R}_2 = -u_x$. Then, we have

$$\begin{aligned}
 V^t &= \sum_{l=0}^{n-1} (-1)^l D_t^{\gamma-1-l}(\mathcal{R}_2) D_t^l \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) - (-1)^n J \left[\mathcal{R}_2, D_t^n \left(\frac{\partial \mathcal{F}}{\partial u_t^\gamma} \right) \right] \\
 &= \sum_{l=0}^{n-1} (-1)^{l+1} D_t^{\gamma-1-l} (u_x) D_t^l (\xi) - (-1)^{n+1} J [u_x, D_t^n (\xi)], \\
 V^x &= \mathcal{F} + \mathcal{R}_2 \left[\frac{\partial \mathcal{F}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xx}} \right) + D_x D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x (\mathcal{R}_2) \left[\frac{\partial \mathcal{F}}{\partial u_{xx}} - D_x \left(\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right) \right] + D_x D_x (\mathcal{R}_2) \left[\frac{\partial \mathcal{F}}{\partial u_{xxx}} \right] \\
 &= (-u_x) [-3u^2 u_{xx} \xi - 3u^2 \xi - D_x (-3u^2 u_x \xi) + D_{xx} (-u^3 \xi)] + D_x (-u_x) [-3u^2 u_x \xi - D_x (-u^3 \xi)] + D_x D_x (-u_x) [-u^3 \xi] \\
 &= (u_x) (3u^2 \xi + 3u^2 u_x \xi_x + 3u^2 u_{xx} \xi + u^3 \xi_{xx}) - (u_{xx}) (u^3 \xi_x) + (u_{xxx}) (u^3 \xi) \\
 &= 3u^2 u_x \xi + 3u^2 u_x u_{xx} \xi + 3u^2 u_x^2 \xi_x + u^3 u_x \xi_{xx} - u^3 u_{xx} \xi_x + u^3 u_{xxx} \xi.
 \end{aligned} \tag{55}$$

6. Conclusions

In this investigation, we explore the time-fractional Fujimoto–Watanabe equation in the perspective of the Riemann–Liouville derivative. We acquire symmetry groups and similarity reductions by means of the Lie symmetry approach. Meanwhile, three kinds of reduced equations have been obtained. For the second infinitesimal generator, we use two distinct methods to derive two different reduced equations, one of which is based on fractional integral operators and fractional differential operators. At last, with the help of Ibragimov's

theory, the conservation laws are constructed. These results reveal this approach is very effective to obtain reduced equations for the fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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