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Research Article

Stabilization of a Class of Complex Chaotic Systems by the Dynamic Feedback Control

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The stabilization problem of the complex chaotic system is investigated in this paper. First, a systematic method is proposed, by which a given complex chaotic system can be transformed into its equivalent real chaotic system. Then, both simple and physical controller is designed for the corresponding real chaotic system by the dynamic feedback control method, thereby the controller for the original complex chaotic system is obtained. Especially, for some complex system, the controller is obtained by the linear feedback control method. Finally, two illustrative examples with numerical simulations are used to verify the validity and effectiveness of the theoretical results.

1. Introduction

It is well known that the first chaotic system was proposed by Lorenz in 1963. From then on, many works have been done about both theoretical results and applications, see Refs. [1–15] and the references therein. Complex chaotic system whose state variables belong to complex space is another important type of chaotic dynamical system, which has been widely investigated in both theorem and applications and has become a hot topic in recent years, for details see Refs. [16–24]. Especially, the encryption effect is better due to the fact that the complex chaotic system is composed of real and imaginary numbers. Since the dynamic behavior of the complex system is more complicated than that of the real chaotic system, the control problems of such system is very difficult. Many researchers usually adopted this strategy; they firstly transfer the complex chaotic system into its corresponding real chaotic system by separating the real parts and imaginary parts of the complex state variables and then they investigate the control method of the real chaotic system. Ultimately, the control problems of such complex system were realized. However, on one hand, there is lack of a systematic method in the first step, i.e., for a specific complex chaotic system, a specific method is applied to transform it into its equivalent real chaotic system. How to find a systematic method by which the complex chaotic system can be transformed into its equivalent real chaotic system is not only important in theory but also significant in applications; thus, it stimulates our work in this paper.

On the other hand, most of the controllers designed in the aforementioned existing results are complicated; thereby, they are hard to be performed in real applications. As a matter of fact, how to design a both simple and physical controller to realize the control problems of the complex chaotic systems is also important both in theory and applications. Among the existing methods, the dynamic feedback control method and the linear feedback control method are widely applied, and thus these two methods are adopted by this study.

Motivated by the above conclusions, the stabilization problem of the complex chaotic system is studied by the dynamic feedback control method. The main contributions of this paper are given as follows:

(1) A systematic method is proposed, which can be used to transform a given complex chaotic system into its equivalent real chaotic system

(2) Both simple and physical controller is designed for the original complex chaotic system is obtained by the dynamic feedback control method and the linear feedback control method, respectively, and numerical simulations are performed to verify the above theoretical results

Before ending this section, we present some notations used in this paper. $\mathbb{R}^n$ is the n dimensional Euclidean space, $\mathbb{C}^n$ is the n dimensional complex space, I_n denotes the $n \times n$ identity matrix, $\mathscr{M}_{m\times n}$ denotes the set of $m \times n$ real matrices, $\otimes$ is the Kronecker product, and $\propto$ is the semitensor product (STP), i.e., let $M_{m\times n} \in \mathscr{M}_{m\times n}$, $N_{p\times q} \in \mathscr{N}_{p\times q}$, and $t = \operatorname{lcm}\{n, p\}$ be the least common multiple of n and p. The STP of M and N is defined as

$$M \propto N = \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathscr{M}_{mt/n\times qt/p}, \tag{1}$$

where

$$M \otimes N = \begin{pmatrix} M_{11}N & M_{12}N & \cdots & M_{1n}N \\ \vdots & \vdots & \vdots & \vdots \\ M_{m1}N & M_{m2}N & \cdots & M_{mn}N \end{pmatrix} \in \mathscr{M}_{mp\times np}. \tag{2}$$

$\operatorname{Re}(\cdot)$ and $\operatorname{Im}(\cdot)$ represent the real part and the imaginary part of $(\cdot)$, respectively, and i is the imaginary unit, i.e., $i^2 = -1$.

2. Preliminary

Consider the following controlled chaotic system:

$$\dot{w} = G(w) + bv, \tag{3}$$

where $w \in \mathbb{R}^n$ is the state, $G(w) \in \mathbb{R}^n$ is continuous function with $G(0) = 0$, $b \in \mathbb{R}^{n\times r}$ is a constant matrix, and $v \in \mathbb{R}^r$ is the controller to be designed.

Lemma 1 (see [18]). *Consider system (3). If $(G(w), b)$ can be stabilized, then the designed controller v is of the following form:*

$$v = K(t)w, \tag{4}$$

where $K = k(t)b^T$, and the feedback gain $k(t)$ is updated by the following equation:

$$\dot{k}(t) = -\|w\|^2. \tag{5}$$

3. Problem Formulation

Consider the following controlled complex chaotic system:

$$\dot{p} = f(p) + bu, \tag{6}$$

where

$$p = \begin{pmatrix} z \\ x \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}, \quad x = \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix}, \tag{7}$$

$z \in \mathbb{C}^m$ and $x \in \mathbb{R}^{n-m}$ are the state, $m \geq 1$,

$$f(p) = f(x, z, \overline{z}) = \begin{pmatrix} M(x)z + H(i)x \\ N(z, \overline{z}, x) \end{pmatrix}, \tag{8}$$

where $\overline{z}$ is the conjugate of z, $M(x) \in \mathbb{R}^{m\times m}$, $H(i) \in \mathbb{C}^{m\times(n-m)}$ is a complex constant matrix, and $N(x, z, \overline{z}) \in \mathbb{R}^{n-m}$, $b \in \mathbb{R}^{n\times r}$, $u \in \mathbb{C}^r$ is the designed controller, i.e.,

$$H(i) = \begin{pmatrix} h_1(i) \\ h_2(i) \\ \vdots \\ h_m(i) \end{pmatrix}, \quad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \tag{9}$$

where $h_j(i) \in \mathbb{C}^{n-m}$, $j = 1, \ldots, m$, $b_1 \in \mathbb{R}^{m\times s}$, $b_2 \in \mathbb{R}^{(n-m)\times(r-s)}$, $1 \leq s \leq r$,

$$u = \begin{pmatrix} u_z \\ u_x \end{pmatrix}, \quad u_z = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}, \quad u_x = \begin{pmatrix} u_{k+1} \\ u_{k+2} \\ \vdots \\ u_r \end{pmatrix}, \tag{10}$$

that is, $u_z \in \mathbb{C}^k$, $u_x \in \mathbb{R}^{r-k}$, and $1 \leq k < r$.

Remark 1. The complex chaotic system of equation (6) is very common, which covers a lot of complex chaotic systems, such as complex Lorenz system and complex hyperchaotic Lorenz system.

The goal of this paper is to investigate the stabilization of system (6), i.e., how to design a controller u to guarantee

$$\lim_{t\longrightarrow\infty} p(t) = 0. \tag{11}$$

4. Main Results

4.1. A Systematic Method which is Used to Transform a Complex System into its Equivalent Real System. In this section, a systematic method proposed, by which a complex system can be transformed into its equivalent real system.

Theorem 1. *Consider the complex chaotic system (6). Its equivalent real system is described as the following form:*

$$\dot{y} = F(y) + BU, \tag{12}$$

where $y \in \mathbb{R}^{m+n}$ is the state, $F(y) \in \mathbb{R}^{m+n}$ is continuous function with $F(0) = 0$, $B \in \mathbb{R}^{(m+n)\times(k+r)}$ is a constant matrix, and $U \in \mathbb{R}^{k+r}$ is the controller to be designed.

Proof. Let $z_j = y_{2j-1} + y_{2j} \times i$, $j = 1, \ldots, m$, and $y_{2m+l} = x_{m+l}$, $1 \le l \le n - m$; then, the equivalent real system (12) is obtained, i.e.,

$$\begin{aligned}
y &= \begin{pmatrix} y_z \\ y_x \end{pmatrix}, \\
y_z &= \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2m} \end{pmatrix}, \\
y_x &= \begin{pmatrix} y_{2m+1} \\ y_{2m+2} \\ \vdots \\ y_{m+n} \end{pmatrix}, \\
F(y) &= \begin{pmatrix} F_1(y) \\ F_2(y) \\ \vdots \\ F_{m+n}(y) \end{pmatrix}, \\
\begin{pmatrix} F_1(y) \\ F_2(y) \\ \vdots \\ F_{2m}(y) \end{pmatrix} &= M^*(x) \times y_z + H^* \times y_x, \\
M^*(x) &= M(x) \otimes I_2, \\
H^* &= \begin{pmatrix} \mathrm{Re}(h_1(i)) \\ \mathrm{Im}(h_1(i)) \\ \vdots \\ \mathrm{Re}(h_m(i)) \\ \mathrm{Im}(h_m(i)) \end{pmatrix}, \\
B &= \begin{pmatrix} B_1 & 0 \\ 0 & b_2 \end{pmatrix}, \\
B_1 &= b_1 \otimes I_2, \\
U &= \begin{pmatrix} U_z \\ U_x \end{pmatrix}, \\
U_z &= \begin{pmatrix} \mathrm{Re}(u_1) \\ \mathrm{Im}(u_1) \\ \vdots \\ \mathrm{Re}(u_k) \\ \mathrm{Im}(u_k) \end{pmatrix}, \\
U_2 &= u_x.
\end{aligned} \tag{13}$$

□

Remark 2. Since the complex chaotic system (6) is equivalent to the corresponding real system (12). Thus, the stabilization problem of system (12) is investigated and the controller U is designed. Moreover, the controller u of system (6) is obtained by

$$u = \begin{pmatrix} 1 & i \end{pmatrix} \propto U = \begin{pmatrix} U_1 + U_2 \times i \\ U_3 + U_4 \times i \\ \vdots \\ U_{2r-1} + U_{2r} \times i \end{pmatrix}. \tag{14}$$

Remark 3. For a given general controlled complex chaotic system of the following form:

$$\dot{z} = f(z) + bu, \tag{15}$$

where $z \in \mathbb{C}^n$ is the state, $f(z) \in \mathbb{C}^n$ is continuous function with $f(0) = 0$, and $b \in \mathbb{R}^{n\times r}$ and $u \in \mathbb{C}^r$ are the designed controller, $r \ge 1$.

By Theorem 1, its equivalent real system is obtained as follows:

$$\dot{y} = F(y) + BU, \tag{16}$$

where $y \in \mathbb{R}^{2n}$ is the state, $F(y) \in \mathbb{R}^{2n}$ is continuous function with $F(0) = 0$, $B \in \mathbb{R}^{2n\times 2r}$ is a constant matrix, and $U \in \mathbb{R}^{2r}$ is the controller to be designed, i.e.,

$$y = \begin{pmatrix} \mathrm{Re}(z_1) \\ \mathrm{Im}(z_1) \\ \vdots \\ \mathrm{Re}(z_n) \\ \mathrm{Im}(z_n) \end{pmatrix},$$

$$F(y) = \begin{pmatrix} \mathrm{Re}(f_1(z)) \\ \mathrm{Im}(f_1(z)) \\ \vdots \\ \mathrm{Re}(f_n(z)) \\ \mathrm{Im}(f_n(z)) \end{pmatrix}, \tag{17}$$

$$B = b \otimes I_2,$$

$$U = \begin{pmatrix} \mathrm{Re}(u_1) \\ \mathrm{Im}(u_1) \\ \vdots \\ \mathrm{Re}(u_r) \\ \mathrm{Im}(u_r) \end{pmatrix}.$$

If the complex chaotic system in which there exist model uncertainty and external disturbance, we present the following result.

Corollary 1. *Consider the following complex chaotic system with both model uncertainty and external disturbance:*

$$\dot{p} = f(p) + u_d + bu, \tag{18}$$

where p, $f(p)$, b, and u are given in equations (7)–(10), respectively,

$$u_d = \Delta f(p) + d(t) = \begin{pmatrix} u_{d1} \\ u_{d2} \\ \vdots \\ u_{dn} \end{pmatrix}, \tag{19}$$

where $\Delta f(p) \in \mathbb{C}^n$ stands for the model uncertainty and $d(t) \in \mathbb{R}^n$ is the external disturbance. Its equivalent real system is described as the following form:

$$\dot{y} = F(y) + BU + U_D, \tag{20}$$

where y, $F(y)$, B, and U are the same as those in Theorem 1, and

$$U_D = \begin{pmatrix} \mathrm{Re}(u_{d1}) \\ \mathrm{Im}(u_{d1}) \\ \vdots \\ \mathrm{Re}(u_{dn}) \\ \mathrm{Im}(u_{dn}) \end{pmatrix}. \tag{21}$$

4.2. Stabilization of the Complex Chaotic System. In this section, the stabilization of the complex chaotic system is investigated by the dynamic feedback control method and the linear feedback control method, respectively, and the conclusions are presented.

Theorem 2. *Consider system (12). If $(F(y), B)$ can be stabilized, then the controller U is designed of the following form:*

$$U = K(t)y, \tag{22}$$

where $K = k(t)B^T$, and $k(t)$ is updated by

$$\dot{k}(t) = -\|y\|^2. \tag{23}$$

Remark 4. According to equation (14), the controller u for system (6) is obtained; thus, the stabilization of system (6) is realized.

If system (12) has some special structure, the stabilization problem of such system can be realized by the linear feedback control method, and the result is proposed.

Theorem 3. *Consider system (12) of the following form:*

$$y = \begin{pmatrix} Y \\ X \end{pmatrix},$$

$$F(y) = \begin{pmatrix} A(X)Y \\ G(X,Y) \end{pmatrix}, \tag{24}$$

$$B = \begin{pmatrix} B_Y \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} \dot{Y} &= A(X)Y + B_Y U, \\ \dot{X} &= G(X,Y), \end{aligned} \tag{25}$$

with

$$\dot{X} = G(X,0), \tag{26}$$

is globally asymptotically stable. If $(F(y), B)$ is controllable, then the designed controller U is of the following form:

$$U = K(X)Y, \tag{27}$$

where $K(X)$ meets the matrix $B_Y K(X) + A(X)$ is Hurwitz whatever X is.

Proof. Since $(F(y), B)$ is controllable, thus $(A(X), B_Y)$ is also controllable whatever X is. According to the pole assignment theory, the controller U given in (27) is as requested, which completes the proof. □

5. Illustrative Examples with Numerical Simulations

In this section, we shall take two complex chaotic systems for example to show how to apply the obtained theoretical results, and then numerical simulations are performed to

verify the effectiveness and the validity of the aforementioned theoretical results.

Example 1. The controlled complex Lorenz chaotic system [25], which is presented as follows:

$$\dot{p} = f(p) + bu, \tag{28}$$

where

$$\begin{aligned} p &= \begin{pmatrix} z \\ x \end{pmatrix}, \\ z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\ x &= x_3, \end{aligned} \tag{29}$$

i.e., $m = 2$, $n = 3$, and

$$\begin{aligned} f(p) = f(x, z, \overline{z}) &= \begin{pmatrix} M(x)z + H(i)x \\ N(x, z, \overline{z}) \end{pmatrix}, \\ b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = b_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ M(x) &= \begin{pmatrix} -10 & 10 \\ 110 - x_3 & -1 \end{pmatrix}, \\ H(i) &= 0, \\ N(x, z, \widehat{z}) &= -2x_3 + \frac{1}{2(\overline{z}_1 z_2 + z_1 \overline{z}_2)}. \end{aligned} \tag{30}$$

According to Theorem 1, the equivalent real system is obtained as follows:

$$\dot{y} = F(y) + BU, \tag{31}$$

where

$$\begin{aligned} y &= \begin{pmatrix} y_z \\ y_x \end{pmatrix}, \\ y_z &= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \\ y_x &= y_5, \\ F(y) &= \begin{pmatrix} F_1(y) \\ F_2(y) \\ \vdots \\ F_5(y) \end{pmatrix}, \\ \begin{pmatrix} F_1(y) \\ F_2(y) \\ F_3(y) \\ F_4(y) \end{pmatrix} &= M(x) \otimes I_2 \times y_z \\ &= \begin{pmatrix} -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \\ 110 - y_5 & 0 & -1 & 0 \\ 0 & 110 - y_5 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \\ B = B_1 = b_1 \otimes I_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ U = U_z &= \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \end{aligned} \tag{32}$$

i.e.,

$$\begin{aligned} \dot{y}_1 &= -10y_1 + 10y_3, \\ \dot{y}_2 &= -10y_2 + 10y_4, \\ \dot{y}_3 &= (110 - y_5)y_1 - y_3 + U_1, \\ \dot{y}_4 &= (110 - y_5)y_2 - y_4 + U_2, \\ \dot{y}_5 &= -2y_5 + y_1 y_3 + y_2 y_4. \end{aligned} \tag{33}$$

Note that if $y_3 = y_4 = 0$, then the following subsystem

$$\begin{aligned} \dot{y}_1 &= -10y_1, \\ \dot{y}_2 &= -10y_2, \\ \dot{y}_5 &= -2y_5, \end{aligned} \tag{34}$$

is globally asymptotically stable; thus, $(F(y), B)$ can be stabilized.

According to Theorem 2, the controller U is designed as follows:

$$U = K(t)y = k(t)\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} y = k(t)\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} k(t)y_3 \\ k(t)y_4 \end{pmatrix}, \tag{35}$$

where $K = k(t)B^T$ and $\dot{k}(t) = -\|y\|^2$. Thus,

$$u = \begin{pmatrix} 1 & i \end{pmatrix} \propto U = \begin{pmatrix} 1 & i \end{pmatrix} \propto \begin{pmatrix} k(t)y_3 \\ k(t)y_4 \end{pmatrix} = k(t)z_2. \tag{36}$$

Numerical simulation is carried out with the initial conditions: $y(0) = [-5, 3, -2, -6, 7], k(0) = -1$. Figure 1 shows y_1, y_2, and y_3 are asymptotically stable, and Figure 2 shows y_4 and y_5 are asymptotically stable, which implies the $z(t)$ and $x(t)$ are stabilized. Figure 3 shows the feedback gain $k(t)$ approaches to constant.

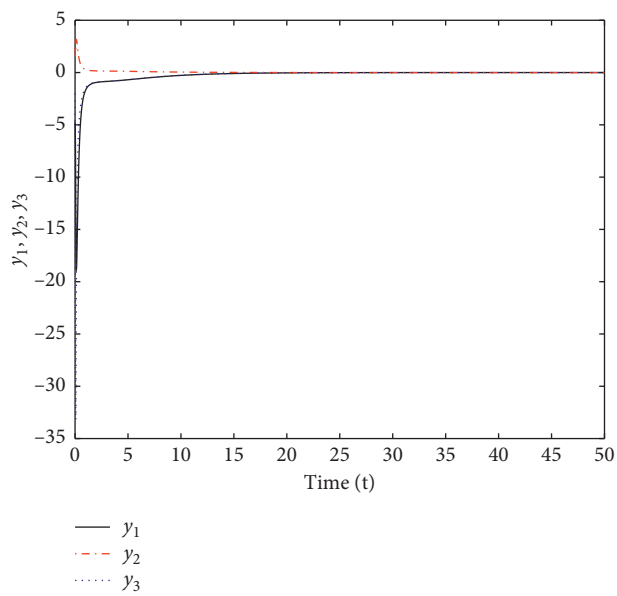

FIGURE 1: y_1, y_2, and y_3 are asymptotically stable.

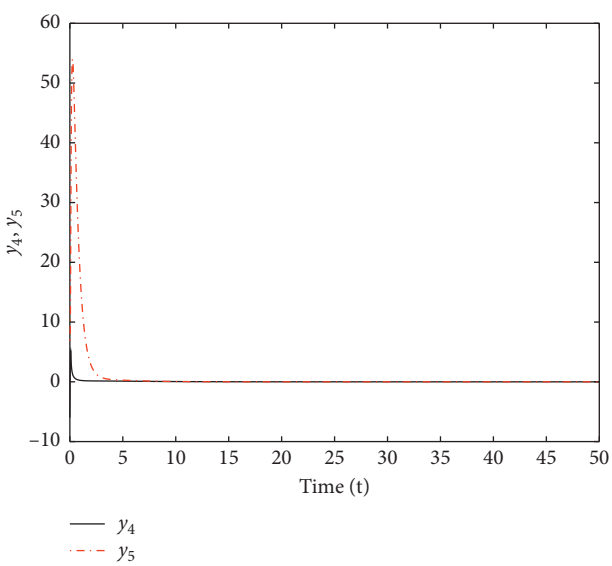

FIGURE 2: y_4 and y_5 are asymptotically stable.

According to Theorem 3, the controller U is designed as follows:

$$U = K(X)Y = \begin{pmatrix} 0 & 0 & (X-110) & 0 \\ 0 & 0 & 0 & (X-110) \end{pmatrix} Y = \begin{pmatrix} (X-110)Y_1 \\ (X-110)Y_2 \end{pmatrix}, \tag{37}$$

where

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \tag{38}$$

$$X = y_5.$$

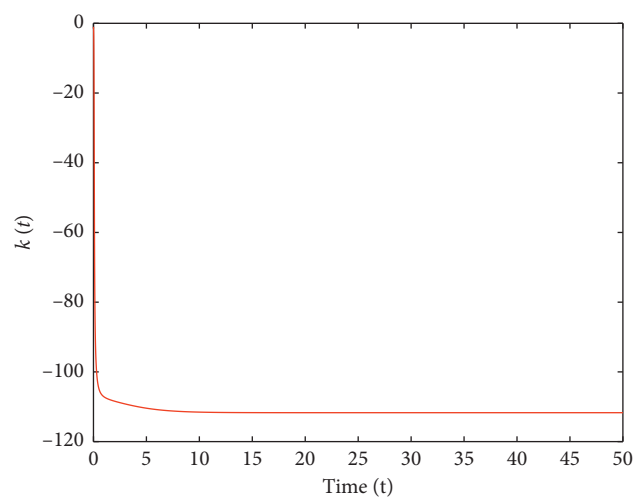

FIGURE 3: $k(t)$ approaches to constant.

Thus,

$$u = \begin{pmatrix} 1 & i \end{pmatrix} \propto U = \begin{pmatrix} 1 & i \end{pmatrix} \propto \begin{pmatrix} (x_3 - 110)y_1 \\ (x_3 - 110)y_2 \end{pmatrix} = (x_3 - 110)z_1. \tag{39}$$

Numerical simulation is carried out with the initial conditions: $y(0) = [-5, 3, -2, -6, 7]$. Figure 4 shows y_1, y_2, and y_3 are asymptotically stable, and Figure 5 shows y_4 and y_5 are asymptotically stable, which implies the $z(t)$ and $x(t)$ are stabilized.

Example 2. The controlled complex hyperchaotic Lorenz system [26], which is presented as follows:

$$\dot{p} = f(p) + bu, \tag{40}$$

where

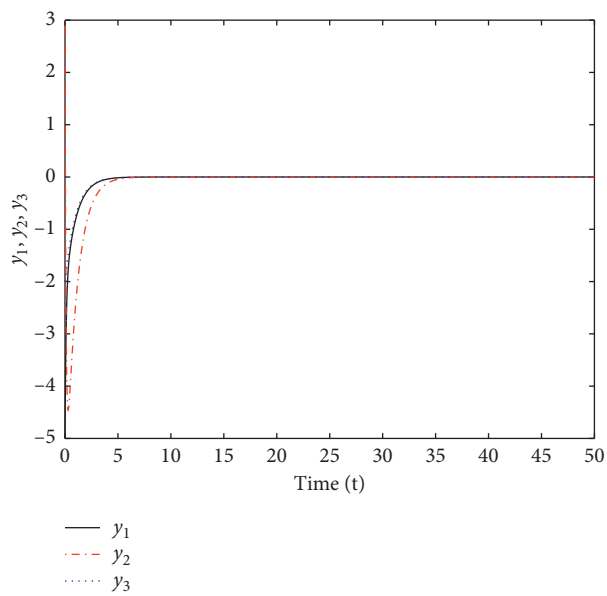

FIGURE 4: y_1, y_2, and y_3 are asymptotically stable.

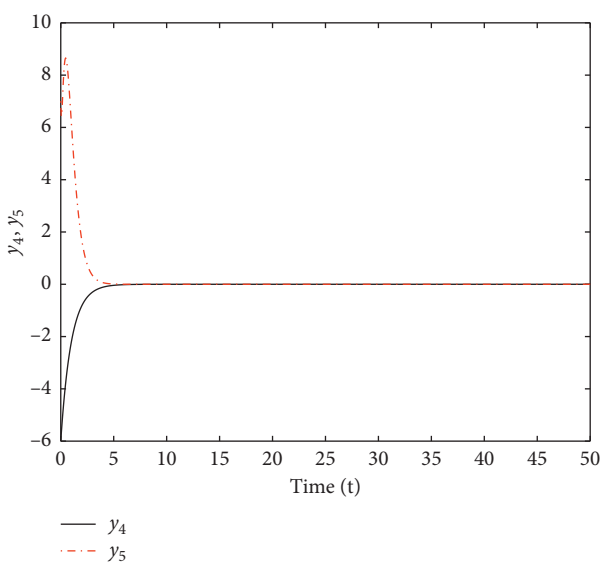

FIGURE 5: y_4 and y_5 are asymptotically stable.

$$p = \begin{pmatrix} z \\ x \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \tag{41}$$

that is, $m = 2$, $n = 4$, and

$$f(p) = f(x, z, \overline{z}) = \begin{pmatrix} M(x)z + H(i)x \\ N(x, z, \overline{z}) \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$M(x) = \begin{pmatrix} -14 & 14 \\ 45 - x_3 & -1 \end{pmatrix}, \tag{42}$$

$$H(i) = \begin{pmatrix} 0 & 1 + i \\ 0 & 0 \end{pmatrix},$$

$$N(x, z, \overline{z}) = \begin{pmatrix} -5x_3 + \dfrac{1}{2(\overline{z}_1 z_2 + z_1 \overline{z}_2)} \\ -5.5x_4 + \dfrac{1}{2(\overline{z}_1 z_2 + z_1 \overline{z}_2)} \end{pmatrix}.$$

According to Theorem 1, the equivalent real system is obtained as follows:

$$\dot{y} = F(y) + BU, \tag{43}$$

where

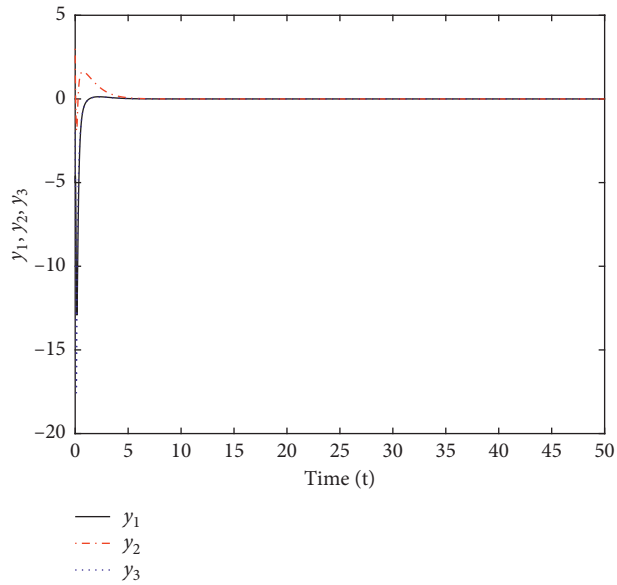

Figure 6: y_1, y_2, and y_3 are asymptotically stable.

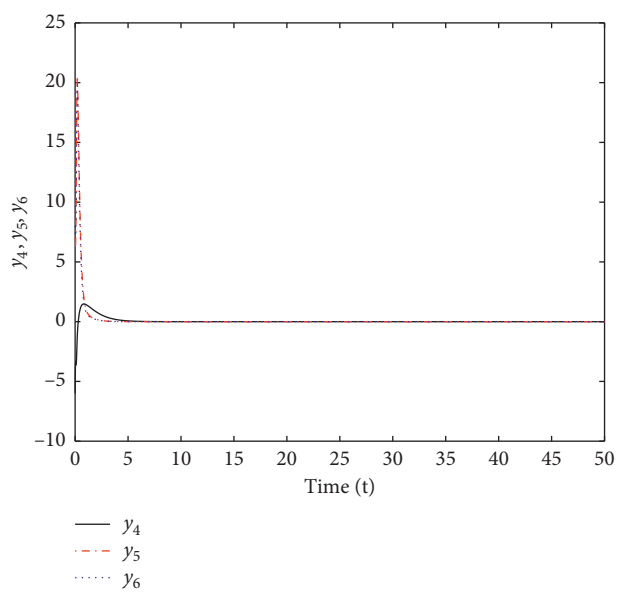

Figure 7: y_4, y_5, and y_6 are asymptotically stable.

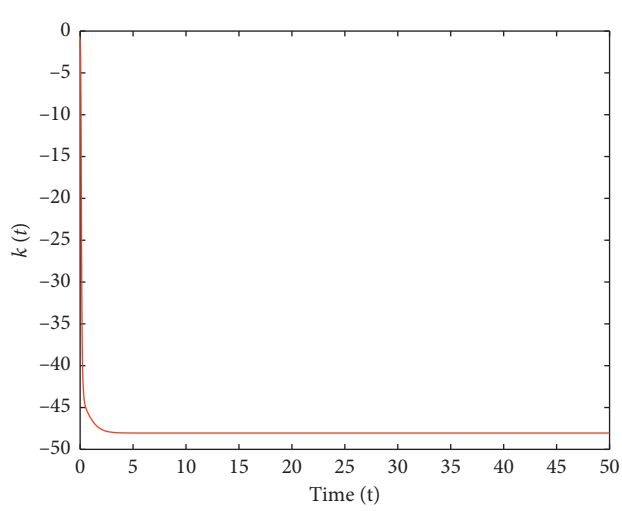

Figure 8: $k(t)$ tends to constant.

$$
\begin{aligned}
y &= \begin{pmatrix} y_z \\ y_x \end{pmatrix}, \\
y_z &= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \\
y_x &= \begin{pmatrix} y_5 \\ y_6 \end{pmatrix}, \\
F(y) &= \begin{pmatrix} F_1(y) \\ F_2(y) \\ \vdots \\ F_6(y) \end{pmatrix}, \\
H^* &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\begin{pmatrix} F_1(y) \\ F_2(y) \\ F_3(y) \\ F_4(y) \end{pmatrix} &= M(x) \otimes I_2 \times y_z + H^* \times y_x \\
&= \begin{pmatrix} -14 & 0 & 14 & 0 \\ 0 & -14 & 0 & 14 \\ 45 - y_5 & 0 & -1 & 0 \\ 0 & 45 - y_5 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_5 \\ y_6 \end{pmatrix} \\
&= \begin{pmatrix} -14(y_1 - y_3) + y_5 \\ -14(y_2 - y_4) + y_6 \\ (45 - y_5)y_1 - y_3 \\ (45 - y_5)y_2 - y_4 \end{pmatrix} \\
B &= B_1 = b_1 \otimes I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
U &= U_z = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},
\end{aligned}
\tag{44}
$$

i.e.,

$$
\begin{aligned}
\dot{y}_1 &= -14(y_1 - y_3) + y_5, \\
\dot{y}_2 &= -14(y_2 - y_4) + y_6, \\
\dot{y}_3 &= (45 - y_5)y_1 - y_3 + U_1, \\
\dot{y}_4 &= (45 - y_5)y_2 - y_4 + U_2, \\
\dot{y}_5 &= -5y_5 + y_1 y_3 + y_2 y_4, \\
\dot{y}_6 &= -5.5y_6 + y_1 y_3 + y_2 y_4.
\end{aligned}
\tag{45}
$$

Notice that if $y_3 = y_4 = 0$, then the following system

$$
\begin{aligned}
\dot{y}_1 &= -14y_1, \\
\dot{y}_2 &= -14y_2, \\
\dot{y}_5 &= -5y_5, \\
\dot{y}_6 &= -5.5y_6,
\end{aligned}
\tag{46}
$$

is globally asymptotically stable; thus, $(F(y), B)$ can be stabilized.

According to Theorem 2, the controller U is designed as follows:

$$
U = K(t)y = k(t)\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} y = k(t)\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} k(t)y_3 \\ k(t)y_4 \end{pmatrix},
\tag{47}
$$

where $K = k(t)B^T$ and $\dot{k}(t) = -\|y\|^2$.

Therefore,

$$
u = \begin{pmatrix} 1 & i \end{pmatrix} \propto U = \begin{pmatrix} 1 & i \end{pmatrix} \propto \begin{pmatrix} k(t)y_3 \\ k(t)y_4 \end{pmatrix} = k(t)z_2. \tag{48}
$$

Numerical simulation is performed with the initial conditions: $y(0) = [-5, 3, -2, -6, 7, -8], k(0) = -1$. Figure 6 shows y_1, y_2, and y_3 are asymptotically stable, and Figure 7 shows y_4, y_5, and y_6 are asymptotically stable, which means that the $z(t)$ and $x(t)$ are stabilized. Figure 8 shows the feedback gain $k(t)$ tends to constant.

6. Conclusions

In conclusion, the stabilization problem of the complex chaotic system has been studied in this paper. First, a systematic method has been proposed, which is applied to transform the complex chaotic system into its equivalent real chaotic system. Then, both simple and physical controllers have been designed for the complex chaotic system by the dynamic feedback control method and the linear feedback control method, respectively. Finally, two illustrative examples with numerical simulations have been performed to verify the validity and effectiveness of the theoretical results.

Data Availability

No data were used in this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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