Research Article

On the Eccentric Connectivity Polynomial of $\mathcal{F}$ -Sum of Connected Graphs

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Received 17 February 2020; Revised 21 April 2020; Accepted 27 April 2020; Published 31 May 2020

Academic Editor: Honglei Xu

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The eccentric connectivity polynomial (ECP) of a connected graph $G = (V(G), E(G))$ is described as $\xi^c(G, y) = \sum_{a \in V(G)} \deg_G(a) y^{ec_G(a)}$, where $ec_G(a)$ and $\deg_G(a)$ represent the eccentricity and the degree of the vertex a , respectively. The eccentric connectivity index (ECI) can also be acquired from $\xi^c(G, y)$ by taking its first derivatives at $y = 1$. The ECI has been widely used for analyzing both the boiling point and melting point for chemical compounds and medicinal drugs in QSPR/QSAR studies. As the extension of ECI, the ECP also performs a pivotal role in pharmaceutical science and chemical engineering. Graph products conveniently play an important role in many combinatorial applications, graph decompositions, pure mathematics, and applied mathematics. In this article, we work out the ECP of $\mathcal{F}$ -sum of graphs. Moreover, we derive the explicit expressions of ECP for well-known graph products such as generalized hierarchical, cluster, and corona products of graphs. We also apply these outcomes to deduce the ECP of some classes of chemical graphs.

1. Introduction

Let G be an n -vertex simple and connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For a given graph G , the order and size are symbolized by $|V(G)|$ and $|E(G)|$, respectively. The degree of $a \in V(G)$ is the number of adjacent vertices to a in G , and it is represented by $\deg_G(a)$. For $a_1, a_2 \in V(G)$, the distance between a_1 and a_2 , denoted with $d_G(a_1, a_2)$, is defined as the length of the shortest path among a_1 and a_2 in G , and the eccentricity $ec_G(a_1)$ is the largest distance among a_1 and any other vertex a_2 of G . We use notions $\mathcal{P}_n$ and $\mathcal{C}_n$ for the n -vertex path and cycle, respectively. The line graph denoted by $L(G)$ of G is the graph whose vertices are the edges of the original graph; two vertices e_1 and e_2 are connected if and only if they share a common end vertex in G . The joint $G + G'$ of graphs G and

G' is the graph union $G \cup G'$ including all the edges joining $V(G)$ and $V(G')$.

A molecular descriptor is a numeric measure of a graph which characterizes its topology. In organic chemistry, topological invariants have established many applications in pharmaceutical drug design, QSAR/QSPR studies, chemical documentation, and isomer discrimination. Some effective topological classes such as degree based, degree distance, eccentric connectivity indices, and so on are established as molecular invariants. In recent years, the study of eccentric invariants for chemical molecular structure has become one of the flourishing lines of research in theoretical chemistry.

The ECI of G is a newly discovered distance-based topological invariant which was put forward by Sharma et al. [1] and is defined as follows:

$$\xi^c(G) = \sum_{a \in V(G)} \deg_G(a) ec_G(a). \quad (1)$$

In recent times, a modification of $\xi^c(G)$ is used and famous as the total eccentricity index $\tau(G)$. It can be defined as

$$\tau(G) = \sum_{a \in V(G)} ec_G(a). \quad (2)$$

The study of polynomials has been a valuable tool to describe the complex and classical behavior of dynamical systems. Furthermore, the polynomial approaches have also been utilized to figure out the central problems such as robustness, stability, and controllability. Polynomial theory can also be implemented in nonlinear, uncertain, hybrid, and time-delay systems and model predictive control. In recent years, there have been numerous works on graph polynomials related with various topological indices. For the detailed discussion about the different types of polynomials such as characteristic, chromatic, edge cover, domination, matching, and clique polynomials and their related studies, we refer the readers to [2–7].

Ashraf and Jalali gave the concept of ECP in [8], which can be specified as

$$\xi^c(G, y) = \sum_{a \in V(G)} \deg_G(a) y^{ec_G(a)}. \quad (3)$$

The total eccentricity polynomial of G can be expressed as follows:

$$\tau(G, y) = \sum_{a \in V(G)} y^{ec_G(a)}. \quad (4)$$

It is an easy exercise to see that the eccentric connectivity and total eccentricity indices can be acquired from their related polynomials by taking their first derivatives at $y = 1$.

Graph products play a significant role in many combinatorial applications, graph decompositions into isomorphic subgraphs, and not only in pure mathematics but also in applied mathematics. In [9, 10], Akhter and Imran investigated the degree-based topological invariants for $\mathcal{F}$ -sum of graphs. De et al. [11] presented the results related to the total eccentricity index of some products of graphs. The ECI of $\mathcal{F}$ -sum graphs in the form of different invariants has been computed in [12]. Došlić and Saheli [13] established the ECI of composite graphs. The ECP of several classes of composite graphs, including Cartesian product, symmetric difference, disjunction, join of graphs, and composition of graphs have been computed in [14]. For more information on different aspects of ECP, one can see [15–21].

Inspired by Yang et al. [22], we continue the research on finding the indices and polynomials of $\mathcal{F}$ -sum of graphs. This article is arranged as follows. First, we find the ECP of the generalized hierarchical product of graphs, and as applications, we give explicit outcomes for chemical graphs such as a truncated cube and linear phenylene. Furthermore, we compute the ECP of $\mathcal{F}$ -sum of graphs and give its

applications in Section 3. Finally, we determine the ECP of the cluster and the corona products of graphs in the last section.

Proposition 1 (see [23]). *Let $\mathcal{C}_l$ and $\mathcal{P}_l$ denote the l -vertex cycle and path, respectively. Then,*

$$\begin{aligned} (1) \quad \xi^c(\mathcal{C}_l, y) &= \begin{cases} 2ly^{((l-1)/2)} & \text{if } l \text{ is odd,} \\ 2ly^{(l/2)} & \text{if } l \text{ is even.} \end{cases} \\ (2) \quad \tau(\mathcal{C}_l, y) &= \begin{cases} ly^{((l-1)/2)} & \text{if } l \text{ is odd,} \\ ly^{(l/2)} & \text{if } l \text{ is even.} \end{cases} \\ (3) \quad \xi^c(\mathcal{P}_l, y) &= \begin{cases} 2y^{l-1} + 4 \sum_{i=2}^{(l-1)/2} y^{l-i} + 2y^{(l-1)/2} & \text{if } l \text{ is odd,} \\ 2y^{l-1} + 4 \sum_{i=2}^{l/2} y^{l-i} & \text{if } l \text{ is even.} \end{cases} \\ (4) \quad \tau(\mathcal{P}_l, y) &= \begin{cases} 2 \sum_{i=1}^{(l-1)/2} y^{l-i} + y^{(l-1)/2} & \text{if } l \text{ is odd,} \\ 2 \sum_{i=1}^{l/2} y^{l-i} & \text{if } l \text{ is even.} \end{cases} \end{aligned}$$

2. Generalized Hierarchical Product

Barrière et al. [24] brought in the concept of the generalized hierarchical product, that is, a generalization of the (standard) hierarchical product and Cartesian product of graphs [25]. The generalized hierarchical product of G and G' with $\emptyset \neq \mathcal{U} \subseteq V(G)$ is represented by $G(\mathcal{U}) \Pi G'$. It has vertex set $V(G) \times V(G')$ and $(a_1, b_1)(a_2, b_2) \in E(G(\mathcal{U}) \Pi G')$ if $a_1 = a_2 \in \mathcal{U}$ and $b_1, b_2 \in E(G')$ or $b_1 = b_2$ and $a_1 a_2 \in E(G)$. For $a_1, a_2 \in V(G)$, a path between a_1 and a_2 through $\mathcal{U}$ is a $a_1 a_2$ -path in G including some vertex $c \in \mathcal{U}$ (vertex c could be the vertex a_1 or a_2). For $a_1, a_2 \in V(G)$, the distance between these vertices through $\mathcal{U}$, symbolized as $d_{G(\mathcal{U})}(a_1, a_2)$, is the length of a smallest $a_1 a_2$ -path through $\mathcal{U}$. It is easily seen that, if one of the vertices a_1 and a_2 belongs to $\mathcal{U}$, then $d_{G(\mathcal{U})}(a_1, a_2) = d_G(a_1, a_2)$ (see [26]). Now, in Lemma 1, we describe the distinct properties of this product.

Lemma 1 (see [24]). *Let G and G' be graphs with $\mathcal{U} \subseteq V(G)$. Then,*

$$\begin{aligned} (a) \quad |V(G(\mathcal{U}) \Pi G')| &= |V(G)||V(G')| \quad \text{and} \\ |E(G(\mathcal{U}) \Pi G')| &= |E(G)||V(G')| + |E(G)||\mathcal{U}|. \\ (b) \quad \deg_{G(\mathcal{U}) \Pi G'}(a_1, b_1) &= \{ \deg_G(a_1) + \deg_{G'}(b_1) \text{ if } a_1 \in \mathcal{U}, \deg_G(a_1) \text{ if } a_1 \in V(G) \setminus \mathcal{U}. \\ (c) \quad d_{G(\mathcal{U}) \Pi G'}((a_1, b_1), (a_2, b_2)) &= \{ d_{G(\mathcal{U})}(a_1, a_2) \\ &\quad + d_{G'}(b_1, b_2) \text{ if } b_1 \neq b_2, d_G(a_1, a_2) \text{ if } b_1 = b_2. \\ (d) \quad ec_{G(\mathcal{U}) \Pi G'}(a_1, b_1) &= ec_{G(\mathcal{U})}(a_1) + ec_{G'}(b_1). \end{aligned}$$

In the next theorem, we give the expression of the ECP of $G(\mathcal{U}) \Pi G'$ in terms of eccentric connectivity and total eccentricity polynomials of $G(\mathcal{U})$ and G' .

Theorem 1. *Let G and G' be graphs with $\mathcal{U} \subseteq V(G)$. Then,*

$$\begin{aligned} \xi^c(G(\mathcal{U}) \Pi G', y) &= \xi^c(G(\mathcal{U}), y) \tau(G', y) \\ &\quad + \varepsilon(G(\mathcal{U}), y) \xi^c(G', y), \end{aligned} \quad (5)$$

where $\varepsilon(G(\mathcal{U}), y) = \sum_{a \in \mathcal{U}} y^{ec_{G(\mathcal{U})}(a)}$.

Proof. From Lemma 1 and formula (3), we get

$$\begin{aligned}
\xi^c(G(\mathcal{U}) \sqcap G', y) &= \sum_{(a,b) \in V(G(\mathcal{U}) \sqcap G')} \deg_{G(\mathcal{U}) \sqcap G'}(a, b) y^{ec_{G(\mathcal{U}) \sqcap G'}(a,b)} \\
&= \sum_{a \in \mathcal{U}} \sum_{b \in V(G')} (\deg_G(a) + \deg_{G'}(b)) y^{(ec_{G(\mathcal{U})}(a) + ec_{G'}(b))} \\
&\quad + \sum_{a \in V(G) \setminus \mathcal{U}} \sum_{b \in V(G')} \deg_G(a) y^{(ec_{G(\mathcal{U})}(a) + ec_{G'}(b))} \\
&= \sum_{a \in \mathcal{U}} \deg_G(a) y^{ec_{G(\mathcal{U})}(a)} \sum_{b \in V(G')} y^{ec_{G'}(b)} + \sum_{a \in \mathcal{U}} y^{ec_{G(\mathcal{U})}(a)} \sum_{b \in V(G')} \deg_{G'}(b) y^{ec_{G'}(b)} \\
&\quad + \sum_{a \in V(G) \setminus \mathcal{U}} \deg_G(a) y^{ec_{G(\mathcal{U})}(a)} \sum_{b \in V(G')} y^{ec_{G'}(b)} \\
&= \xi^c(G(\mathcal{U}), y) \tau(G', y) + \varepsilon(G(\mathcal{U}), y) \xi^c(G', y).
\end{aligned} \tag{6}$$

This finishes the proof.

The Cartesian product $G \sqcap G'$ of graphs G and G' has the vertex set $V(G) \times V(G')$ and $(a_1, b_1)(a_2, b_2) \in E(G \sqcap G')$ if $a_1 = a_2$ and $b_1 b_2 \in E(G')$ or $b_1 = b_2$ and $a_1 a_2 \in E(G)$. By using Theorem 1, we can deduce the following theorem by putting $\mathcal{U} = V(G)$. $\square$

Theorem 2 (see [14]). *Let G and G' be two connected graphs. Then,*

$$\xi^c(G \sqcap G', y) = \xi^c(G, y) \tau(G', y) + \tau(G, y) \xi^c(G', y). \tag{7}$$

Example 1. The molecular graph of truncated cube can be represented as a generalized hierarchical product of $G(\mathcal{U})$ and $\mathcal{P}_2$ (depicted in Figure 1), with $\mathcal{U} = \{a_1, a_4, a_9, a_{12}\}$. It is easy task to see that $\tau(\mathcal{P}_2, y) = \xi^c(\mathcal{P}_2, y) = 2y$ and

$$\begin{aligned}
\varepsilon(G(\mathcal{U}), y) &= \sum_{a \in \mathcal{U}} y^{ec_{G(\mathcal{U})}(a)} = 4y^5, \\
\xi^c(G(\mathcal{U}), y) &= \sum_{a \in V(G)} \deg_G(a) y^{ec_{G(\mathcal{U})}(a)} = 32y^5.
\end{aligned} \tag{8}$$

Then by Theorem 1, we have $\xi^c(G(\mathcal{U}) \sqcap \mathcal{P}_2, y) = (32y^5)(2y) + (4y^5)(2y) = 72y^6$.

Example 2. The linear phenylene F_n with $n > 1$ benzene ring is the graph that is also recognized by $\mathcal{P}_{3n}(\mathcal{U}) \sqcap \mathcal{P}_2$ (depicted in Figure 2), with $\mathcal{U} = \{a_{3k} \mid 1 \leq k \leq n\} \cup \{a_{3k+1} \mid 0 \leq k \leq n-1\}$. By tedious computation, we get $\tau(\mathcal{P}_2, y) = \xi^c(\mathcal{P}_2, y) = 2y$ and

$$\begin{aligned}
\varepsilon(\mathcal{P}_{3n}(\mathcal{U}), y) &= \sum_{a \in \mathcal{U}} y^{ec_{\mathcal{P}_{3n}(\mathcal{U})}(a)} = \begin{cases} 2 \sum_{i=1}^{n/2} y^{3(n-i)} (y^2 + 1), & \text{if } n \text{ is even,} \\ 2 \sum_{i=1}^{(n-1)/2} y^{3(n-i)} (y^2 + 1), & \text{if } n \text{ is odd,} \end{cases} \\
\xi^c(\mathcal{P}_{3n}(\mathcal{U}), y) &= \sum_{a \in V(G)} \deg_G(a) y^{ec_{G(\mathcal{U})}(a)} = \begin{cases} 2y^{3n-1} + 4 \sum_{i=2}^{3n/2} y^{3n-i}, & \text{if } n \text{ is even,} \\ 2y^{3n-1} + 4 \sum_{i=2}^{(3n+1)/2} y^{3n-i} + 2y^{(3n+1)/2}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned} \tag{9}$$

FIGURE 1: The molecular graph of truncated cube.

FIGURE 2: The linear phenylene $F_n = \mathcal{P}_{3n}(\mathcal{U}) \square \mathcal{P}_2$.

Then by Theorem 1, we have

$$\xi^c(\mathcal{P}_{3n}(\mathcal{U}) \square \mathcal{P}_2, y) = \begin{cases} 4y^{3n} + 8 \sum_{i=2}^{3n/2} y^{3n-i+1} + 4 \sum_{i=1}^{n/2} y^{3(n-i)+1} (y^2 + 1), & \text{if } n \text{ is even,} \\ 4y^{3n-1} + 8 \sum_{i=2}^{(3n+1)/2} y^{3n-i+1} + 4y^{3(n+1)/2} & \text{if } n \text{ is odd.} \\ + 4 \sum_{i=1}^{(n-1)/2} y^{3(n-i)+1} (y^2 + 1), & \end{cases} \quad (10)$$

3. $\mathcal{F}$ -Sum of Graphs

Eliasi and Taeri initiated the notion of $\mathcal{F}$ -sum of a graph in [27], and they studied the Wiener index of it. Some explicit expressions of various PI indices have been presented for $\mathcal{F}$ -sum graphs in [28]. Metsidik et al. [29] studied the Wiener indices of $\mathcal{F}$ -sum graphs.

The subdivision graph of G , symbolized by $\mathcal{S}(G)$, can be sketched from G by interchanging each edge of G with a path $\mathcal{P}_2$. The line superposition graph $\mathcal{Q}(G)$ of G can be attained from G by placing a new vertex into each edge of G and then linking with edges of each pair of new vertices on adjacent edges of G . The triangle parallel graph of G is symbolized by $\mathcal{R}(G)$, and it can be constructed from G by interchanging each edge of G with a triangle.

The total graph of G is represented by $\mathcal{T}(G)$, which has edges and vertices of G as its own vertices, and adjacency in $\mathcal{T}(G)$ is described as the adjacency or incidence of the related elements (vertices and edges) of G .

Let $\mathcal{F}$ be one of the aforementioned operations $\mathcal{S}$, $\mathcal{T}$, $\mathcal{Q}$, or $\mathcal{R}$. The $\mathcal{F}$ -sum of graphs G and G' is represented by

$G +_{\mathcal{F}} G'$. It has vertex set $(V(G) \cup E(G)) \times V(G')$, and $(a_1, b_1)(a_2, b_2) \in E(G +_{\mathcal{F}} G')$ if and only if $a_1 = a_2 \in V(G)$ and $b_1 b_2 \in E(G')$ or $b_1 = b_2 \in V(G')$ and $a_1 a_2 \in E(\mathcal{F}(G))$.

Lemma 2 (see [12]). *Let G be a connected graph. If $\mathcal{U} = V(G)$, then*

- (a) $|V(\mathcal{S}(G))| = |V(G)| + |E(G)|$ and $|E(\mathcal{S}(G))| = 2|E(G)|$
- (b) $\deg_{\mathcal{S}(G)(\mathcal{U})}(a) = \begin{cases} \deg_G(a) & \text{if } a \in \mathcal{U}, \\ 2 & \text{if } a \in V(\mathcal{S}(G)) \setminus \mathcal{U}. \end{cases}$
- (c) $ec_{\mathcal{S}(G)(\mathcal{U})}(a) = \begin{cases} 2ec_G(a) & \text{if } a \in \mathcal{U}, \\ 2ec_{L(G)}(a) + 1 & \text{if } a \in V(\mathcal{S}(G)) \setminus \mathcal{U}. \end{cases}$

Theorem 3. *Let $G(n \geq 2)$ and G' be graphs. Then,*

$$\begin{aligned} \xi^c(G +_{\mathcal{S}} G', y) &= \xi^c(G, y^2) \tau(G', y) \\ &\quad + 2y \tau(L(G), y^2) \tau(G', y) + \tau(G, y^2) \xi^c(G', y). \end{aligned} \quad (11)$$

Proof. Followed by formula (3) and Lemma 2, we have

$$\begin{aligned}
\xi^c(\mathcal{S}(G)(\mathcal{U}), y) &= \sum_{a \in V(\mathcal{S}(G))} \deg_{\mathcal{S}(G)}(a) y^{ec_{\mathcal{S}(G)}(a)} \\
&= \sum_{a \in \mathcal{U}} \deg_{\mathcal{S}(G)}(a) y^{ec_{\mathcal{S}(G)}(a)} + \sum_{a \in V(\mathcal{S}(G)) \setminus \mathcal{U}} \deg_{\mathcal{S}(G)}(a) y^{ec_{\mathcal{S}(G)}(a)} \\
&= \sum_{a \in \mathcal{U}} \deg_G(a) y^{2ec_G(a)} + 2 \sum_{a \in V(\mathcal{S}(G)) \setminus \mathcal{U}} y^{ec_{\mathcal{S}(G)}(a)} \\
&= \sum_{a \in \mathcal{U}} \deg_G(a) y^{2ec_G(a)} + 2 \sum_{a \in V(L(G))} y^{2ec_{L(G)}(a)+1} \\
&= \xi^c(G, y^2) + 2y\tau(L(G), y^2).
\end{aligned} \tag{12}$$

$$\varepsilon(\mathcal{S}(G)(\mathcal{U}), y) = \sum_{a \in \mathcal{U}} y^{ec_{\mathcal{S}(G)}(a)} = \sum_{a \in V(G)} y^{2ec_G(a)} = \tau(G, y^2). \tag{13}$$

Combining these results with (5), we get the desired result. This finishes the proof. $\square$

Lemma 3 (see [12]). *Let G be a connected graph. If $\mathcal{U} = V(G)$, then*

$$\begin{aligned}
(a) \quad & |V(\mathcal{R}(G))| = |V(G)| + |E(G)| \quad \text{and} \\
& |E(\mathcal{R}(G))| = 3|E(G)|. \\
(b) \quad & \deg_{\mathcal{R}(G)(\mathcal{U})}(a) = \begin{cases} 2\deg_G(a) & \text{if } a \in \mathcal{U}, \\ 2 & \text{if } a \in V(\mathcal{R}(G)) \setminus \mathcal{U}. \end{cases} \\
(c) \quad & ec_{\mathcal{R}(G)(\mathcal{U})}(a) = \begin{cases} ec_G(a) & \text{if } a \in \mathcal{U}, \\ ec_{L(G)}(a) + 1 & \text{if } a \in V(\mathcal{R}(G)) \setminus \mathcal{U}. \end{cases}
\end{aligned}$$

Theorem 4. *Let $G(n \geq 2)$ and G' be graphs. Then,*

$$\begin{aligned}
\xi^c(G +_{\mathcal{R}} G', y) &= 2\xi^c(G, y^2)\tau(G', y) \\
&\quad + 2y\tau(L(G), y)\tau(G', y) + \tau(G, y)\xi^c(G', y).
\end{aligned} \tag{14}$$

Proof. With the help of definition of the ECP and Lemma 3, we have

$$\begin{aligned}
\xi(\mathcal{R}(G)(\mathcal{U}), y) &= \sum_{a \in V(\mathcal{R}(G))} \deg_{\mathcal{R}(G)}(a) y^{ec_{\mathcal{R}(G)}(a)} \\
&= \sum_{a \in \mathcal{U}} \deg_{\mathcal{R}(G)}(a) y^{ec_{\mathcal{R}(G)}(a)} + \sum_{a \in V(\mathcal{R}(G)) \setminus \mathcal{U}} \deg_{\mathcal{R}(G)}(a) y^{ec_{\mathcal{R}(G)}(a)} \\
&= \sum_{a \in \mathcal{U}} \deg_G(a) y^{2ec_G(a)} + 2 \sum_{a \in V(\mathcal{R}(G)) \setminus \mathcal{U}} y^{ec_{\mathcal{R}(G)}(a)} \\
&= 2 \sum_{a \in \mathcal{U}} \deg_G(a) y^{2ec_G(a)} + 2 \sum_{a \in V(L(G))} y^{ec_{L(G)}(a)+1} \\
&= 2(\xi^c(G, y^2) + y\tau(L(G), y)).
\end{aligned} \tag{15}$$

$$\varepsilon(\mathcal{R}(G)(\mathcal{U}), y) = \sum_{a \in \mathcal{U}} y^{ec_{\mathcal{R}(G)}(a)} = \sum_{a \in V(G)} y^{ec_G(a)} = \tau(G, y). \tag{16}$$

Combining these results with (5), we get following:

$$\begin{aligned}
\xi^c(G +_{\mathcal{R}} G', y) &= 2(\xi^c(G, y^2) + y\tau(L(G), y))\tau(G', y) \\
&\quad + \tau(G, y)\xi^c(G', y) \\
&= 2\xi^c(G, y^2)\tau(G', y) + 2y\tau(L(G), y)\tau(G', y) \\
&\quad + \tau(G, y)\xi^c(G', y).
\end{aligned} \tag{17}$$

This completes the proof. $\square$

Lemma 4 (see [12]). *Let G be a connected graph. If $\mathcal{U} = V(\mathcal{Q})$, then*

$$\begin{aligned}
(a) \quad & |V(\mathcal{Q}(G))| = |V(G)| + |E(G)| \quad \text{and} \\
& |E(\mathcal{Q}(G))| = 2|E(G)| + |E(LG)|. \\
(b) \quad & \deg_{\mathcal{Q}(G)(\mathcal{U})}(a) = \begin{cases} \deg_G(a) & \text{if } a \in \mathcal{U}, \\ \deg_{L(G)}(a) + 2 & \text{if } a \in V(\mathcal{Q}(G)) \setminus \mathcal{U}. \end{cases} \\
(c) \quad & ec_{\mathcal{Q}(G)(\mathcal{U})}(a) = \begin{cases} ec_G(a) + 1 & \text{if } a \in \mathcal{U}, \\ ec_{L(G)}(a) + 1 & \text{if } a \in V(\mathcal{Q}(G)) \setminus \mathcal{U}. \end{cases}
\end{aligned}$$

Theorem 5. Let $G(n \geq 2)$ and G' be graphs. Then,

$$\begin{aligned} \xi^c(G +_Q G', y) &= y(\xi^c(G, y) + \xi^c(L(G), y) \\ &\quad + y\tau(L(G), y))\tau(G', y) + y\tau(G, y)\xi^c(G', y). \end{aligned} \quad (18)$$

Proof. By formula (3) and Lemma 4, we have

$$\begin{aligned} \xi^c(Q(G)(\mathcal{U}), y) &= \sum_{a \in V(Q(G))} \deg_{Q(G)}(a) y^{ec_{Q(G)}(a)} \\ &= \sum_{a \in \mathcal{U}} \deg_{Q(G)}(a) y^{ec_{Q(G)}(a)} + \sum_{a \in V(Q(G)) \setminus \mathcal{U}} \deg_{Q(G)}(a) y^{ec_{Q(G)}(a)} \\ &= \sum_{a \in \mathcal{U}} \deg_G(a) y^{ec_G(a)+1} + \sum_{a \in V(L(G))} (\deg_{L(G)}(a) + 1) y^{ec_{L(G)}(a)+1} \\ &= y \sum_{a \in \mathcal{U}} \deg_G(a) y^{ec_G(a)} + \sum_{a \in V(L(G))} \deg_{L(G)}(a) y^{ec_{L(G)}(a)+1} + \sum_{a \in V(L(G))} y^{ec_{L(G)}(a)+1} \\ &= y\xi^c(G, y) + y\xi^c(L(G), y) + y\tau(L(G), y). \end{aligned} \quad (19)$$

$$\varepsilon(Q(G)(\mathcal{U}), y) = \sum_{a \in \mathcal{U}} y^{ec_{Q(G)}(a)} = \sum_{a \in V(G)} y^{ec_G(a)+1} = y\tau(G, y). \quad (20)$$

Combining these results with (5), we get the required result. This accomplishes the proof. $\square$

Lemma 5 (see [12]). Let G be a connected graph. If $\mathcal{U} = V(G)$, then

$$\begin{aligned} (a) \quad &|V(\mathcal{T}(G))| = |V(G)| + |E(G)| \quad \text{and} \\ &|E(\mathcal{T}(G))| = 3|E(G)| + |E(L(G))|. \\ (b) \quad &\deg_{\mathcal{T}(G)(\mathcal{U})}(a) = \begin{cases} 2\deg_G(a) & \text{if } a \in \mathcal{U}, \\ \deg_{L(G)}(a) + 1 & \text{if } a \in V(\mathcal{T}(G)) \setminus \mathcal{U}. \end{cases} \\ (c) \quad &ec_{\mathcal{T}(G)(\mathcal{U})}(a) = \begin{cases} ec_G(a) & \text{if } a \in \mathcal{U}, \\ ec_{L(G)}(a) + 1 & \text{if } a \in V(\mathcal{T}(G)) \setminus \mathcal{U}. \end{cases} \end{aligned}$$

Theorem 6. Let $G(n \geq 2)$ and G' be graphs. Then,

$$\begin{aligned} \xi^c(G +_{\mathcal{T}} G', y) &= y(2\xi^c(G, y) + \xi^c(L(G), y) \\ &\quad + \tau(L(G), y))\tau(G', y) + \tau(G, y)\xi^c(G', y). \end{aligned} \quad (21)$$

Proof. With the help of (3) and Lemma 5, we have

$$\begin{aligned} \xi^c(\mathcal{T}(G)(\mathcal{U}), y) &= \sum_{a \in V(\mathcal{T}(G))} \deg_{\mathcal{T}(G)}(a) y^{ec_{\mathcal{T}(G)}(a)} \\ &= \sum_{a \in \mathcal{U}} \deg_{\mathcal{T}(G)}(a) y^{ec_{\mathcal{T}(G)}(a)} + \sum_{a \in V(\mathcal{T}(G)) \setminus \mathcal{U}} \deg_{\mathcal{T}(G)}(a) y^{ec_{\mathcal{T}(G)}(a)} \\ &= 2 \sum_{a \in \mathcal{U}} \deg_G(a) y^{ec_G(a)+1} + \sum_{a \in V(L(G))} (\deg_{L(G)}(a) + 1) y^{ec_{L(G)}(a)+1} \\ &= 2y \sum_{a \in \mathcal{U}} \deg_G(a) y^{ec_G(a)} + \sum_{a \in V(L(G))} \deg_{L(G)}(a) y^{ec_{L(G)}(a)+1} + \sum_{a \in V(L(G))} y^{ec_{L(G)}(a)+1} \\ &= 2y\xi^c(G, y) + y\xi^c(L(G), y) + y\tau(L(G), y). \end{aligned} \quad (22)$$

$$\varepsilon(\mathcal{T}(G)(\mathcal{U}), y) = \sum_{a \in \mathcal{U}} y^{ec_{\mathcal{T}(G)}(a)} = \sum_{a \in V(G)} y^{ec_G(a)} = \tau(G, y). \quad (23)$$

Combining these results with (5), we get the needed result. This finishes the proof. $\square$

Example 3. For $k \geq 3$, let G be a zig-zag polyhex nanotube $\text{TUHC}_6[2k, 2]$ as shown in Figure 3. By definition, it is

FIGURE 3: The zig-zag polyhex nanotube $TUHC_6[2k, 2]$.

obvious that $G \cong \mathcal{C}_k +_{\mathcal{S}} \mathcal{P}_2$, where $\mathcal{U} = V(\mathcal{C}_k) \subseteq V(\mathcal{S}(\mathcal{C}_k))$. By Theorem 3, we get

$$\xi^c(\mathcal{C}_k +_{\mathcal{S}} \mathcal{P}_2, y) = \begin{cases} 2ky^{k+1}(2y+3), & \text{if } k \text{ is even,} \\ 2ky^k(2y+3), & \text{if } k \text{ is odd.} \end{cases} \quad (24)$$

Example 4. Let L_k be the hexagonal chain [30] with $k \geq 2$ hexagonals (see Figure 4). From definition, it is obvious that $L_k \cong \mathcal{P}_{k+1} +_{\mathcal{S}} \mathcal{P}_2$, with $\mathcal{U} = V(\mathcal{P}_{k+1}) \subseteq V(\mathcal{S}(\mathcal{P}_{k+1}))$. Then, by applying Theorem 3, we get

$$\xi^c(\mathcal{P}_{k+1} +_{\mathcal{S}} \mathcal{P}_2, y) = \begin{cases} 2y^{k+1}(4y^k + 4y + 3) + 4 \sum_{i=1}^{(k-2)/2} y^{2k-2i+1}(2y+3), & \text{if } k \text{ is even,} \\ 4y^{k+1}(2y^k + 1) + 4 \sum_{i=1}^{(k-2)/2} y^{2k-2i+1}(2y+3), & \text{if } k \text{ is odd.} \end{cases} \quad (25)$$

4. Cluster and Corona Product

The cluster product of G and G' , symbolized by $G'\{G\}$, is achieved by using $|V(G)|$ copies of a rooted graph G and a copy of G' and by specifying the root of the l -th copy of G with the l -th vertex of G' , $l = 1, 2, \dots, |V(G)|$. The order and size of $G'\{G\}$ are $|V(G)||V(G')|$ and $(|E(G)| + |V(G)||E(G')|)$, respectively. If we take a as the root vertex of G' and $\mathcal{U} = \{a\} \subset V(G)$, then $G'\{G\} \cong G(\mathcal{U}) \sqcap G' \cong G(\{a\}) \sqcap G'$. Now with the above notions in hand and by applying Theorem 1, we get the following result.

Theorem 7. Let G and G' be graphs such that $\mathcal{U} \subseteq G$. Then,

$$\xi^c(G'\{G\}, y) = \tau(G', y) \sum_{b \in V(G)} \deg_G(b) y^{ec_G(a) + d_G(b, a)} + \xi^c(G', y) y^{ec_G(a)}. \quad (26)$$

Proof. Since $\mathcal{U} = \{a\}$, $\varepsilon(G(\mathcal{U}), y) = y^{ec_G(a)}$ and $\xi^c(G(\mathcal{U}), y) = \sum_{b \in V(G)} \deg_G(b) y^{ec_G(a) + d_G(b, a)}$. So, using Theorem 1, we get the desired result.

Let S_{l+1} be a star graph having $l+1$ vertices, with the root vertex of degree l . For a given graph G' , the graph $G'\{G\}$ can be constructed by taking cluster product of S_{l+1} with G' . This is famous as l -fold bristled graph $\text{Brst}(G')$. With the help of Theorem 1, the ECP of l -fold bristled graph $\text{Brst}(G')$ can be evaluated. $\square$

Corollary 1. Let G be a graph. Then,

$$\xi^c(G^{t^t}, y) = ty(1+y)\tau(G', y) + y\xi^c(G', y). \quad (27)$$

Proof. Let $G \cong S_{t+1}$, with the central vertex of degree t in S_{t+1} as the root vertex. Then, $ec_G(a) = 1$ and $d_G(a, b) = 1$ for all $b \in V(G) \setminus \{a\}$. The desired expression is obtained with the help of above theorem.

The corona product $G \square G'$ of graphs G and G' is a graph, which can be drawn by using $|V(G)|$ copies of G' and a copy of G and linking the l -th vertex of G to every vertex in l -th copy of G' , $1 \leq l \leq n$. In Theorem 8, we compute the ECP of $G \square G'$. $\square$

Theorem 8. Let G and G' be graphs. Then,

$$\xi^c(G \square G', y) = (|V(G)|y + y^2(2|E(G)| + |V(G)|))\tau(G', y) + y\xi^c(G', y). \quad (28)$$

Proof. Let $a \in V(\mathcal{P}_1)$ be the root vertex of $(G+a)$, having order $|V(G)|+1$ and size $|E(G)|+|V(G)|$. Also note that $ec_{G+a}(a) = 1$, $\deg_{G+a}(a) = |V(G)|$, $d_{G+a}(a, b) = 1$, $\deg_{G+a}(b) = \deg_G(b) + 1$ for all $b \in V(G+a) \setminus \{a\}$ and $G' \square G \cong G'\{G+a\}$. By using Theorem 7, we have

FIGURE 4: The linear hexagonal chain L_k . FIGURE 5: The graphs B_k , $T_{k,3}$, and $J_{m,7}$.

$$\begin{aligned}\xi^c(G' \square G, y) &= \left(\deg_{G+a}(a)y + \sum_{b \in V(G)} (\deg_G(b) + 1)y^2 \right) \tau(G', y) + y\xi^c(G', y) \\ &= (|V(G)|y + y^2(2|E(G)| + |V(G)|))\tau(G', y) + y\xi^c(G', y).\end{aligned}\quad (29)$$

Example 5. For the bottleneck graph $\mathcal{P}_2 \square G$, $\xi^c(\mathcal{P}_2 \square G, y) = (2|E(G)| + |V(G)|)y^3 + (|V(G)| + 2)y^2$.

Example 6. If we use the vertices a_l , $1 \leq l \leq k$, in the construction of a bridge graph, then it is obtained by connecting

the vertices a_l and a_{l+1} of G_{l+1} by a link for all $l = 1, 2, \dots, k-1$ and is represented by $B(G_1, G_2, \dots, G_n; a_1, a_2, \dots, a_k)$. For $G_1 \cong G_2 \cong \dots \cong G_k$ and $a_1 = a_2 = \dots = a_k = a$, we can define $G_k(G, a) = B(G, G, \dots, G; a, a, \dots, a)$. In particular, let $B_k = G_k(\mathcal{P}_3, a)$ with $\deg_{\mathcal{P}_3}(a) = 2$ and $T_{k,3} = G_k(\mathcal{E}_3, a)$ be a type of bridge graphs. Then, we have $B_k = \mathcal{P}_k \square \mathcal{P}_2$, $T_{k,3} = \mathcal{P}_k \square \mathcal{P}_2$, and $J_{m,k+1} = \mathcal{P}_m \square \mathcal{E}_k$. The graphs B_k , $T_{k,3}$, and $J_{m,7}$ are depicted in Figure 5. By using Theorem 8, we gain the following results:

$$\begin{aligned}\xi^c(B_k, y) &= \begin{cases} 2y^k(y+2) + 2(y+3) \sum_{i=2}^{k/2} y^{k-i+1}, & \text{if } k \text{ is even,} \\ 2y^{(k+1)/2}(y+2)(1+y^{(k-1)/2}) + 2(y+3) \sum_{i=2}^{(k-1)/2} y^{k-i+1}, & \text{if } k \text{ is odd,} \end{cases} \\ \xi^c(T_{k,3}, y) &= \begin{cases} y^k(3y+4) + 3(y+2) \sum_{i=2}^{k/2} y^{k-i+1}, & \text{if } k \text{ is even,} \\ y^{(k+1)/2}(3y+4)(1+y^{(k-1)/2}) + 3(y+2) \sum_{i=2}^{(k-1)/2} y^{k-i+1}, & \text{if } k \text{ is odd,} \end{cases} \\ \xi^c(J_{m,k+1}, y) &= \begin{cases} 3ky^{m+1} + (k+2)y^m + (3ky+k+4) \sum_{i=2}^{m/2} y^{m-i+1}, & \text{if } k \text{ is even,} \\ y^{(m+1)/2}(3ky+k+2)(y^{(m-1)/2}+1) \\ + (3ky+k+4) \sum_{i=2}^{(m-1)/2} y^{m-i+1}, & \text{if } k \text{ is odd.} \end{cases}\end{aligned}\quad (30)$$

5. Conclusions

The numerical description of chemical structures with graph invariants is a valuable graph theory application. This

characterization may be in the form of spectra, polynomials, molecular, or atomic topological indices. It is also feasible to specify graphs by matrices. A well-known example of such matrices is an adjacency matrix. However, the

characterization of graphs by polynomials is a new line of research in modern graph theory. This paper is an effort in this direction, through which the ECP for some product graphs is illustrated by graph structure analysis and a mathematical derivation method.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by UPAR Grant of United Arab Emirates University (UAEU), Al Ain, UAE, via Grant nos. G00002590 and G00003271.

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