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Research Article

On a Thermoelastic Laminated Timoshenko Beam: Well Posedness and Stability

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In this paper, we are concerned with a linear thermoelastic laminated Timoshenko beam, where the heat conduction is given by Cattaneo's law. We firstly prove the global well posedness of the system. For stability results, we establish exponential and polynomial stabilities by introducing a stability number χ.

1. Introduction

In this paper, we address the following thermoelastic laminated Timoshenko beam in $(0, 1) \times (0, \infty)$:

$$
\begin{cases}
\rho \omega_{tt} + G(\psi - \omega_x)_x + \delta \theta_x = 0, \\[2mm]
I_\rho (3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) = 0, \\[2mm]
I_\rho s_{tt} - D s_{xx} + G(\psi - \omega_x) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}\beta s_t = 0, \\[2mm]
\rho_3 \theta_t + q_x + \delta \omega_{xt} = 0, \\[2mm]
\tau q_t + \alpha q + \theta_x = 0,
\end{cases}
\tag{1}
$$

which subject to the following boundary conditions:

$$
\begin{cases}
\omega_x(0, t) = \psi(0, t) = s(0, t) = \theta(0, t) = 0, & t \in (0, \infty), \\[2mm]
\omega_x(1, t) = \psi(1, t) = s(1, t) = \theta(1, t) = 0, & t \in (0, \infty),
\end{cases}
\tag{2}
$$

and initial conditions

$$
\begin{cases}
\omega(x, 0) = \omega_0(x), \psi(x, 0) = \psi_0(x), s(x, 0) = s_0(x), \theta(x, 0) = \theta_0(x), & x \in (0, 1), \\[2mm]
q(x, 0) = q_0(x), \omega_t(x, 0) = \omega_1(x), \psi_t(x, 0) = \psi_1(x), s_t(x, 0) = s_1(x), & x \in (0, 1),
\end{cases}
\tag{3}
$$

where $\rho, G, I_\rho, D, \gamma, \beta, \rho_3, \delta, \tau$, and α are positive constants. $\theta(x, t)$ represents the difference temperature and $q(x, t)$ is the heat flux.

Laminated beam, which is a relevant research subject due to the high applicability of such materials in the industry, was firstly introduced by Hansen and Spies, see, for instance

[1, 2]. They introduced a mathematical model for two-layered beams with structural damping due to the interfacial slip which is given by

$$\begin{cases} \rho \omega_{tt} + G(\psi - \omega_x)_x = 0, \\ \\ I_\rho (3s_{tt} - \psi_{tt}) - G(\psi - \omega_x) - D(3s_{xx} - \psi_{xx}) = 0, \\ \\ I_\rho s_{tt} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t - Ds_{xx} = 0, \end{cases} \quad (4)$$

where the coefficients $\rho, G, I_\rho, D, \gamma,$ and β are positive constants and represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. The function $\omega(x, t)$ denotes the transversal displacement, $\psi(x, t)$ represents the rotational displacement, and $s(x, t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. The third equation describes the dynamics of the slip.

Up till now, there are some results concerning laminated beam equations, which are mainly concerned with global existence and stability of the related system. By adding suitable damping effects, such as internal damping, (boundary) frictional damping, and viscoelastic damping, it was shown that if the linear damping terms are added in two of the three equations, system (4) is exponentially stable under the "equal wave speeds" assumption $(\rho/I_\rho) = (G/D)$. But if the damping terms are added in the three equations, then the system decays exponentially without the equal wave speeds assumption, see, for example, [3–17]. For thermoelastic laminated Timoshenko beam, there are few published works, we can mention the results due to Liu and Zhao [18] and Apalara [19]. In [18], the authors considered the following laminated beams with past history

$$\begin{cases} \rho \varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0, \\ \\ I_\rho (3\omega - \psi)_{tt} - D(3\omega - \psi)_{xx} + \int_0^\infty g(s)(3\omega - \psi)_{xx}(t - s)ds - G(\psi - \varphi_x) - \theta = 0, \\ \\ I_\rho \omega_{tt} - D\omega_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t = 0, \\ \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3\omega - \psi)_t = 0, \end{cases} \quad (5)$$

together with the following boundary conditions:

$$\begin{cases} \varphi_x(0, t) = \psi(0, t) = \omega(0, t) = \theta(0, t) = 0, & t \in (0, \infty), \\ \varphi(1, t) = \psi_x(1, t) = \omega_x(1, t) = \theta_x(1, t) = 0, & t \in (0, \infty). \end{cases} \quad (6)$$

They firstly proved the global well posedness of solutions to the system. The main results are the stability of the system. If $\beta \neq 0$, they proved the exponential and polynomial stabilities depending on the behavior of the kernel function g only. If $\beta = 0$, they established exponential stability in case of equal wave speeds assumption and lack of exponential stability in case of nonequal wave speeds assumption. Apalara [19] considered a laminated beam with second sound of the form

$$\begin{cases} \rho \omega_{tt} + G(\psi - \omega_x)_x = 0, \\ \\ I_\rho (3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) + \delta\theta_x = 0, \\ \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \\ \rho_3 \theta_t + q_x + \delta(3s - \psi)_{xt} = 0, \\ \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \quad (7)$$

together with the following boundary conditions:

$$\begin{cases} \omega_x(0, t) = \psi(0, t) = s(0, t) = q(0, t) = 0, & t \in (0, \infty), \\ \omega(1, t) = \psi_x(1, t) = s_x(1, t) = \theta(1, t) = 0, & t \in (0, \infty), \end{cases} \quad (8)$$

and proved the global well posedness and established exponential and polynomial stabilities depending on the parameter

$$\chi_\tau = \left(1 - \frac{\tau\rho_3 G}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau G\delta^2}{\rho I_\rho}. \quad (9)$$

One can also refer to two recent results of laminated beams with thermal damping in [20, 21], and a result of a coupled hyperbolic equations with a heat equation of second sound in [22].

When $s = 0$, system (4) reduces to the well-known Timoshenko system, which have been widely studied. There are so many papers on the Timoshenko system in the literature, most of those results recover the global well posedness, stability, and long-time dynamics by adding some kinds of damping. Here, we recall some works on the thermoelastic Timoshenko system. Muñoz Rivera and Racke [23] considered a Timoshenko system with thermoelastic dissipation and established exponential stability in case of equal wave speed assumption and polynomial stability if wave speeds are nonequal. Almeida Júnior et al. [24] studied

a thermoelastic Timoshenko beam acting on shear force. They obtained the same stability results as in [23]. In addition, they proved that the polynomial decay is optimal. Fernández Sare and Racke [25] considered a Timoshenko system with second sound. They proved that the system is not exponentially stable even if the propagation speeds are equal. The results were generalized by Guesmia et al. [26]. Recently, Santos et al. [27] introduced a stability number χ_r for the system in [25] and established the exponential decay result for $\chi_r = 0$ and polynomial decay for $\chi_r \neq 0$ by using the semigroup method. One can also find a stability result for the Timoshenko system with second sound in Apalara et al. [28]. Feng [29] considered a Timoshenko-Coleman-Gurtin system and studied the long-time dynamics of the system. We at last mention the contribution of Hamadouche and Messaoudi [30] and Aouadi and Boulehmi [31], where the authors considered two classes of nonuniform thermoelastic Timoshenko systems and proved global well posedness and established some stability results.

Our goals in the present work are to study the global well posedness and stability of systems (1)–(3). The main points are summarized as follows:

(i) We prove the global well posedness of systems (1)–(3) by using Lumer–Philips theorem. The main result is presented in Theorem 1.

(ii) We introduce a new stability number denoted by

$$\chi = \tau \delta^2 D - \left(D\rho - GI_\rho \right)\left(\frac{\tau \rho_3 D}{I_\rho} - 1 \right), \tag{10}$$

and we show that the system is exponential stable when $\chi = 0$ and polynomial stable when $\chi \neq 0$. The main results are presented in Theorems 1 and 2.

(iii) The proof of stability results is based on the multiplier method. Since the boundary conditions here we considered are different from those in Apalara [19], so the multipliers we will define are greatly different from the multipliers in Apalara [19].

It follows, from (1), that

$$\frac{d^2}{dt^2} \int_0^1 \omega(x,t)dx = 0, \tag{11}$$

$$\tau \frac{d}{dt} \int_0^1 q(x,t)dx + \alpha \int_0^1 q(x,t)dx = 0.$$

If we denote

$$\overline{\omega}(x,t) = \omega(x,t) - \int_0^1 \omega_0(x) - t \int_0^1 \omega_1(x)dx, \tag{12}$$

$$\overline{q}(x,t) = q(x,t) - e^{-(\alpha/\tau)t} \int_0^1 q_0(x)dx,$$

we easily verify that $(\overline{\omega}, \psi, s, \theta, \overline{q})$ satisfies (1) and in addition,

$$\int_0^1 \overline{\omega}(x,t)dx = 0,$$

$$\int_0^1 \overline{q}(x,t)dx = 0, \tag{13}$$

$$\forall t \geq 0.$$

Hence, Poincaré's inequality holds for $\overline{\omega}$. In the following, we work with $\overline{\omega}$ and $\overline{q}$ but write ω and q for convenience.

The remaining paper is planned as follows. In Section 2, we study the well posedness of the system. In Section 3, we establish the stability results. Throughout this paper, $c > 0$ is a generic constant that changes from one inequality to another.

2. Well Posedness

We start by denoting the vector-valued function by U:

$$U = (\omega, \Phi, 3s - \psi, 3\Lambda - \Psi, s, \Lambda, \theta, q)^T,$$

$$\text{with } \Phi = \omega_t, \tag{14}$$

$\Psi = \psi_t$, and $\Lambda = s_t$.

Then, systems (1)–(3) can be written as

$$\begin{cases} \dfrac{d}{dt} U(t) = \mathscr{A}U, \quad t > 0, \\ \\ U(0) = U_0 = \left(\omega_0, \omega_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, \theta_0, q_0 \right)^T, \end{cases} \tag{15}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}U = \begin{pmatrix} \Phi \\ -\dfrac{G}{\rho}(\psi - \omega_x)_x - \dfrac{\delta}{\rho}\theta_x \\ 3\Lambda - \Psi \\ \dfrac{D}{I_\rho}(3s - \psi)_{xx} + \dfrac{G}{I_\rho}(\psi - \omega_x) \\ \Lambda \\ \dfrac{D}{I_\rho}s_{xx} - \dfrac{G}{I_\rho}(\psi - \omega_x) - \dfrac{4\gamma}{3I_\rho}s - \dfrac{4\beta}{3I_\rho}\Lambda \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}\Phi_x \\ -\dfrac{\alpha}{\tau}q - \dfrac{1}{\tau}\theta_x \end{pmatrix}. \tag{16}$$

We consider the following spaces:

$$L^2_*(0,1) = \left\{ v \in L^2(0,1): \int_0^1 v(x)\mathrm{d}x = 0 \right\},$$

$$H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1),$$

$$H^2_*(0,1) = \left\{ v \in H^2(0,1): v_x(0) = v_x(1) = 0 \right\}. \tag{17}$$

Let

$$\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1)$$
$$\times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1) \tag{18}$$

be the Hilbert space equipped with the inner product

$$(U, \widetilde{U})_{\mathcal{H}} = \rho \int_0^1 \Phi\widetilde{\Phi}dx + I_\rho \int_0^1 (3\Lambda - \Psi)(3\widetilde{\Lambda} - \widetilde{\Psi})\mathrm{d}x + 3I_\rho \int_0^1 \Lambda\widetilde{\Lambda}dx$$
$$+ \rho_3 \int_0^1 \theta\widetilde{\theta}\mathrm{d}x + \tau \int_0^1 q\widetilde{q}\mathrm{d}x + 4\gamma \int_0^1 s\widetilde{s}\mathrm{d}x + D \int_0^1 (3s - \psi)_x (3\widetilde{s} - \widetilde{\psi})_x \mathrm{d}x \tag{19}$$
$$+ G \int_0^1 (\psi - \omega_x)(\widetilde{\psi} - \widetilde{\omega}_x)\mathrm{d}x + 3D \int_0^1 s_x\widetilde{s}_x\mathrm{d}x.$$

The domain of $\mathcal{A}$ is given by

$$D\mathcal{A} = \left\{ U \in \mathcal{H} \left| \begin{array}{l} \omega \in H^2_*0,1 \cap H^1_*0,1, 3s - \psi, s \in H^20,1 \cap H^1_00,1, \\ \Phi, q \in H^1_*0,1, 3\Lambda - \Psi, \Lambda, \theta \in H^1_00,1 \end{array} \right. \right\}. \tag{20}$$

The well posedness result can be stated in the following theorem.

Theorem 1. *Let $U_0 \in \mathcal{H}$, then problems (1)–(3) admit a unique weak solution $U \in C(\mathbb{R}^+, \mathcal{H})$. In addition, if $U_0 \in D(\mathcal{A})$, then $U \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. It is easy to obtain that, for any $U = (\omega, \Phi, 3s - \psi, 3\Lambda - \Psi, s, \Lambda, \theta, q)^T \in D(\mathcal{A})$,

$$(\mathcal{A}U, U)_{\mathcal{H}} = -4\beta \int_0^1 \Lambda^2 \mathrm{d}x - \alpha \int_0^1 q^2 \mathrm{d}x \le 0, \tag{21}$$

which implies the operator $\mathcal{A}$ is a dissipative operator.

In what follows, we shall show the operator $Id - \mathcal{A}$ is surjective. In other words, given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathcal{H}$, we will seek a solution $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) \in D(\mathcal{A})$ of

$$(Id - \mathcal{A})V = F. \tag{22}$$

We rewrite (21) as

$$\begin{cases} v_1 - v_2 = f_1, \\[2mm] \rho v_2 - G v_{1xx} - G v_{3x} + 3G v_{5x} + \delta v_{7x} = \rho f_2, \\[2mm] v_3 - v_4 = f_3, \\[2mm] I_\rho v_4 - D v_{3xx} - 3G v_5 + G v_3 + G v_{1x} = I_\rho f_4, \\[2mm] v_5 - v_6 = f_5, \\[2mm] \left(I_\rho + \dfrac{4}{3}\beta\right) v_6 - D v_{5xx} - G v_3 - G v_{1x} + \left(3G + \dfrac{4}{3}\gamma\right) v_5 = I_\rho f_6, \\[2mm] \rho_3 v_7 + v_{8x} + \delta v_{2x} = \rho_3 f_7, \\[2mm] (\tau + \alpha) v_8 + v_{7x} = \tau f_8, \end{cases} \tag{23}$$

which implies that

$$v_2 = v_1 - f_1, \tag{24}$$

$$v_4 = v_3 - f_3, \tag{25}$$

$$v_6 = v_5 - f_5, \tag{26}$$

$$v_{7x} = -(\tau + \alpha)v_8 + \tau f_8. \tag{27}$$

We infer from (27) that

$$v_7 = -(\tau + \alpha)\int_0^x v_8(y)\mathrm{d}y + \tau\int_0^x f_8(y)\mathrm{d}y. \tag{28}$$

Replacing (24)–(26) and (28) in (23), we see that

$$
\begin{cases}
\rho v_1 - G v_{1xx} - G v_{3x} + 3G v_{5x} - \delta(\tau + \alpha)v_8 = \rho(f_1 + f_2) - \tau f_8, \\[2mm]
I_\rho v_3 - D v_{3xx} - 3G v_5 + G v_3 + G v_{1x} = I_\rho(f_3 + f_4), \\[2mm]
\left(I_\rho + 3G + \dfrac{4}{3}\beta + \dfrac{4}{3}\gamma\right)v_5 - D v_{5xx} - G v_3 - G v_{1x} = I_\rho(f_5 + f_6) + \dfrac{4}{3}\beta f_5, \\[2mm]
-\rho_3(\tau + \alpha)\displaystyle\int_0^x v_8(y)\mathrm{d}y + v_{8x} + \delta v_{1x} = \rho_3 f_7 - \rho_3\tau\int_0^x f_8(y)\mathrm{d}y + \delta f_{1x}.
\end{cases}
\tag{29}
$$

We multiply (29) by $\widetilde{v}_1$, $\widetilde{v}_3$, $\widetilde{v}_5$, and $(\tau + \alpha)\int_0^x \widetilde{v}_8(y)\mathrm{d}y$, respectively, and integrate their sum over $(0, 1)$ to get the following variational formulation:

$$\mathcal{B}\big((v_1, v_3, v_5, v_8), (\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)\big) = \mathcal{L}(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8), \tag{30}$$

where the bilinear form $\mathcal{B}: [H_*^1(0, 1) \times H_0^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)]^2 \longrightarrow \mathbb{R}$ is given by

$$
\begin{aligned}
\mathcal{B}&\big((v_1, v_3, v_5, v_8), (\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)\big) \\
&= G\int_0^1 (-v_{1x} - v_3 + 3v_5)(-\widetilde{v}_{1x} - \widetilde{v}_3 + 3\widetilde{v}_5)\mathrm{d}x + \rho\int_0^1 v_1\widetilde{v}_1\mathrm{d}x + I_\rho\int_0^1 v_3\widetilde{v}_3\mathrm{d}x \\
&\quad + (3I_\rho + 4\beta + 4\gamma)\int_0^1 v_5\widetilde{v}_5\mathrm{d}x + (\tau + \alpha)\int_0^1 v_8\widetilde{v}_8\mathrm{d}x + D\int_0^1 v_{3x}\widetilde{v}_{3x}\mathrm{d}x \\
&\quad + 3D\int_0^1 v_{5x}\widetilde{v}_{5x}\mathrm{d}x + \rho_3(\tau + \alpha)^2\int_0^1\left(\int_0^x v_8(y)\mathrm{d}y\right)\left(\int_0^x \widetilde{v}_8(y)\mathrm{d}y\right)\mathrm{d}x,
\end{aligned}
\tag{31}
$$

and the linear form $\mathcal{L}: [H_*^1(0, 1) \times H_0^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)] \longrightarrow \mathbb{R}$ is defined by

$$
\begin{aligned}
\mathcal{L}(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8) &= \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\widetilde{v}_1\mathrm{d}x + I_\rho\int_0^1 (f_3 + f_4)\widetilde{v}_3\mathrm{d}x \\
&\quad + \int_0^1\big[(3I_\rho + 4\beta)f_5 + 3I_\rho f_6\big]\widetilde{v}_5\mathrm{d}x \\
&\quad + \delta(\tau + \alpha)\int_0^1 f_1\widetilde{v}_8\mathrm{d}x \\
&\quad + \int_0^1\left[\rho_3\tau(\tau + \alpha)\int_0^x f_8(y)\mathrm{d}y - \rho_3(\tau + \alpha)f_7\right] \\
&\quad \cdot \left(\int_0^x \widetilde{v}_8(y)\mathrm{d}y\right)\mathrm{d}x.
\end{aligned}
\tag{32}
$$

We denote the Hilbert space V by

$$V = H_*^1(0, 1) \times H_0^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1), \tag{33}$$

equipped with the norm

$$
\begin{aligned}
\big\|(v_1, v_3, v_5, v_8)\big\|_V^2 &= \big\|-v_{1x} - v_3 + 3v_5\big\|_2^2 + \big\|v_1\big\|_2^2 \\
&\quad + \big\|v_8\big\|_2^2 + \big\|v_{3x}\big\|_2^2 + \big\|v_{5x}\big\|_2^2.
\end{aligned}
\tag{34}
$$

It is easy to get that $\mathcal{B}(\cdot, \cdot)$ and $\mathcal{L}(\cdot)$ are bounded. Moreover there exists a positive constant m such that

$$
\begin{aligned}
\mathcal{B}&\big((v_1, v_3, v_5, v_8), (v_1, v_3, v_5, v_8)\big) \\
&= G\int_0^1 (-v_{1x} - v_3 + 3v_5)^2\mathrm{d}x + \rho\int_0^1 v_1^2\mathrm{d}x + I_\rho\int_0^1 v_3^2\mathrm{d}x \\
&\quad + (3I_\rho + 4\beta + 4\gamma)\int_0^1 v_5^2\mathrm{d}x + (\tau + \alpha)\int_0^1 v_8^2\mathrm{d}x + D\int_0^1 v_{3x}^2\mathrm{d}x \\
&\quad + 3D\int_0^1 v_{5x}^2\mathrm{d}x + \rho_3(\tau + \alpha)^2\int_0^1\left(\int_0^x v_8(y)\mathrm{d}y\right)^2\mathrm{d}x \\
&\geq G\int_0^1 (-v_{1x} - v_3 + 3v_5)^2\mathrm{d}x + \rho\int_0^1 v_1^2\mathrm{d}x \\
&\quad + (\tau + \alpha)\int_0^1 v_8^2\mathrm{d}x + D\int_0^1 v_{3x}^2\mathrm{d}x + 3D\int_0^1 v_{5x}^2\mathrm{d}x \\
&\geq m\big\|(v_1, v_3, v_5, v_8)\big\|_V^2.
\end{aligned}
\tag{35}
$$

Thus, $\mathscr{B}$ is coercive on $V \times V$. Consequently, using Lax–Milgram theorem, we conclude that (30) has a unique solution:

$$v_1 \in H_*^1(0,1),$$
$$v_3, v_5 \in H_0^1(0,1), \tag{36}$$
$$v_8 \in L_*^2(0,1).$$

Substituting $v_1, v_3, v_5,$ and v_8 into (24)–(26) and (28), respectively, we have

$$v_2 \in H_*^1(0,1),$$
$$v_4, v_6 \in H_0^1(0,1), \tag{37}$$
$$v_7 \in H_0^1(0,1).$$

Let $\widetilde{v}_1 \in H_0^1(0,1)$ and denote

$$\widehat{\widetilde{v}}_1(x) = \widetilde{v}_1(x) - \int_0^1 \widetilde{v}_1(s)\mathrm{d}s, \tag{38}$$

which gives us $\widehat{\widetilde{v}}_1 \in H_*^1(0,1)$. Now we replace $(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)$ by $(\widehat{\widetilde{v}}_1, 0, 0, 0)$ in (30) to obtain

$$G \int_0^1 (-v_{1x} - v_3 + 3v_5)\left(-\widehat{\widetilde{v}}_{1x}\right)\mathrm{d}x + \rho \int_0^1 v_1 \widehat{\widetilde{v}}_1 dx$$
$$= \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\widehat{\widetilde{v}}_1 dx, \tag{39}$$

i.e.,

$$G \int_0^1 v_{1xx}\widehat{\widetilde{v}}_1 dx = \rho \int_0^1 v_1 \widehat{\widetilde{v}}_1 dx - G \int_0^1 v_{3x}\widehat{\widetilde{v}}_1 dx + 3G \int_0^1 v_{5x}\widehat{\widetilde{v}}_1 dx$$
$$- \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\widehat{\widetilde{v}}_1 dx, \quad \forall \widetilde{v}_1 \in H_0^1(0,1), \tag{40}$$

which yields

$$Gv_{1xx} = \rho v_1 - Gv_{3x} + 3Gv_{5x}$$
$$- (\rho f_1 + \rho f_2 - \tau\delta f_8) \in L^2(0,1). \tag{41}$$

Thus,

$$v_1 \in H^2(0,1). \tag{42}$$

Moreover, (39) also holds for any $\phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$Gv_{1x}(1)\phi(1) - Gv_{1x}(0)\phi(0) - G \int_0^1 v_{1xx}\phi \, \mathrm{d}x$$
$$+ \rho \int_0^1 v_1\phi \mathrm{d}x - G \int_0^1 v_{3x}\phi \mathrm{d}x$$
$$+ 3G \int_0^1 v_{5x}\phi \mathrm{d}x - \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\phi \, \mathrm{d}x = 0. \tag{43}$$

Then, we get for any $\phi \in C^1([0,1])$,

$$Gv_{1x}(1)\phi(1) - Gv_{1x}(0)\phi(0) = 0. \tag{44}$$

From (28), we obtain

$$v_7(0) = v_7(1) = 0. \tag{45}$$

Since ϕ is arbitrary, we get that $v_{1x}(0) = v_{1x}(1) = 0$. Hence, $v_1 \in H_*^2(0,1)$. Using similar arguments as above, we can obtain

$$v_3, v_5 \in H^2(0,1) \cap H_0^1(0,1),$$
$$v_7 \in H_0^1(0,1), \tag{46}$$
$$v_8 \in H_*^1(0,1).$$

Thus, $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) \in D(\mathscr{A})$ and $\mathscr{A}$ is maximal. By using Lumer–Philips theorem, see, for example, Liu and Zheng [32] and Pazy [33], we end the proof of the theorem. □

3. Stability

In this section, we study the stability of systems (1)–(3). More precisely, we establish exponential and polynomial decay results depending on χ defined by

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right). \tag{47}$$

The energy functional of systems (1)–(3) is defined by

$$E(t) = E(\omega, \psi, s, \theta, q)$$
$$= \frac{1}{2} \int_0^1 \Big[\rho\omega_t^2 + I_\rho\big[(3s - \psi)_t\big]^2 + 3I_\rho s_t^2 + \rho_3\theta^2$$
$$+ \tau q^2 + 4\gamma s^2 + D\big[(3s - \psi)_x\big]^2 + G(\psi - \omega_x)^2 + 3Ds_x^2\Big]\mathrm{d}x. \tag{48}$$

Now we give our stability results.

Theorem 2 (exponential decay). *Suppose that $\chi = 0$. For any initial data $U_0 \in \mathscr{H}$, there exist two positive constants μ and η such that the energy functional (48) satisfies*

$$E(t) \leq \mu e^{-\eta t}, \quad \forall t \geq 0. \tag{49}$$

Theorem 3 (polynomial decay). *Suppose that $\chi \neq 0$. For any initial data $U_0 \in D(\mathscr{A})$, there exists positive constant μ_0 such that the energy functional (48) satisfies*

$$E(t) \leq \frac{\mu_0}{t}, \quad \forall t > 0. \tag{50}$$

To prove Theorems 1 and 2, we need the following technical lemmas.

3.1. Technical Lemmas

Lemma 1. *It holds that the energy functional $E(t)$ is non-increasing and satisfies*

$$E'(t) = -4\beta \int_0^1 s_t^2 \mathrm{d}x - \alpha \int_0^1 q^2 \mathrm{d}x \leq 0. \tag{51}$$

Proof. Multiplying (1) by ω_t, $(3s - \psi)_t$, s_t, θ, and q, respectively, integrating the results by parts and using boundary condition (1), we easily get (51). □

Lemma 2. *Define the functional $F_1(t)$ by*

$$F_1(t) = I_\rho \int_0^1 (3s - \psi)_t (3s - \psi) \mathrm{d}x \qquad (52)$$
$$- \rho \int_0^1 \omega_t \int_0^x (3s - \psi)(y) \mathrm{d}y \, \mathrm{d}x.$$

Then, we have for any $\varepsilon_1 > 0$,

$$F_1'(t) \le -\frac{D}{2} \int_0^1 \left[(3s - \psi)_x \right]^2 \mathrm{d}x$$
$$+ \varepsilon_1 \int_0^1 \omega_t^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \left[(3s - \psi)_t \right]^2 \mathrm{d}x$$
$$+ \frac{\delta^2 c_*^2}{2D} \int_0^1 \theta^2 \mathrm{d}x, \qquad (53)$$

where $c_* > 0$ is the Poincaré constant.

Proof. It follows from (1) that

$$F_1'(t) = D \int_0^1 (3s - \psi)_{xx} (3s - \psi) \mathrm{d}x$$
$$+ G \int_0^1 (\psi - \omega_x)(3s - \psi) \mathrm{d}x + I_\rho \int_0^1 \left[(3s - \psi)_t \right]^2 \mathrm{d}x$$
$$+ G \int_0^1 (\psi - \omega_x) \int_0^x (3s - \psi)(y) \mathrm{d}y \, \mathrm{d}x$$
$$+ \delta \int_0^1 \theta_x \int_0^x (3s - \psi)(y) \mathrm{d}y \, \mathrm{d}x$$
$$- \rho \int_0^1 \omega_t \int_0^x (3s - \psi)_t (y) \mathrm{d}y \, \mathrm{d}x. \qquad (54)$$

Using integration by parts and boundary condition (1), we arrive at

$$F_1'(t) = -D \int_0^1 \left[(3s - \psi)_x \right]^2 \mathrm{d}x + I_\rho \int_0^1 \left[(3s - \psi)_t \right]^2 \mathrm{d}x$$
$$- \delta \int_0^1 \theta (3s - \psi) \mathrm{d}x$$
$$- \rho \int_0^1 \omega_t \int_0^x (3s - \psi)_t (y) \mathrm{d}y \, \mathrm{d}x. \qquad (55)$$

Then, by using Hölder's, Young's, and Poincaré's inequalities, we can get (53) from (57). □

Lemma 3. *The functional $F_2(t)$ defined by*

$$F_2(t) = \rho \int_0^1 (\psi - \omega_x) \int_0^x \omega_t(y) \mathrm{d}y \, \mathrm{d}x, \qquad (56)$$

satisfies for any $\varepsilon_2 > 0$,

$$F_2'(t) \le -\frac{G}{2} \int_0^1 (\psi - \omega_x)^2 \mathrm{d}x + \varepsilon_2 \int_0^1 \psi_t^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 \omega_t^2 \mathrm{d}x$$
$$+ \frac{\delta^2}{2G} \int_0^1 \theta^2 \mathrm{d}x. \qquad (57)$$

Proof. Differentiating $F_2(t)$ with respect to t and using (1), we see that

$$F_2'(t) = \rho \int_0^1 \psi_t \int_0^x \omega_t(y) \mathrm{d}y \, \mathrm{d}x - \rho \int_0^1 \omega_{xt} \int_0^x \omega_t(y) \mathrm{d}y \, \mathrm{d}x$$
$$- G \int_0^1 (\psi - \omega_x) \int_0^x (\psi - \omega_y)_y \mathrm{d}y \, \mathrm{d}x$$
$$- \delta \int_0^1 (\psi - \omega_x) \int_0^x \theta_y \mathrm{d}y \, \mathrm{d}x. \qquad (58)$$

Using integration by parts, we obtain

$$F_2'(t) = \rho \int_0^1 \psi_t \int_0^x \omega_t(y) \mathrm{d}y \, \mathrm{d}x + \rho \int_0^1 \omega_t^2 \mathrm{d}x$$
$$- G \int_0^1 (\psi - \omega_x)^2 \mathrm{d}x - \delta \int_0^1 \theta (\psi - \omega_x) \mathrm{d}x. \qquad (59)$$

Then, by using Young's inequality and Hölder's inequality, we can get (57). □

Lemma 4. *Define the functional $F_3(t)$ by*

$$F_3(t) = \tau \rho_3 \int_0^1 \theta \int_0^x q(y) \mathrm{d}y \, \mathrm{d}x. \qquad (60)$$

Then, we can get for any $\varepsilon_3 > 0$,

$$F_3'(t) \le -\frac{\rho_3}{2} \int_0^1 \theta^2 \mathrm{d}x + \varepsilon_3 \int_0^1 \omega_t^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_3}\right) \int_0^1 q^2 \mathrm{d}x. \qquad (61)$$

Proof. Differentiating F_3 with respect to t and using (1), we obtain

$$F_3'(t) = -\tau \int_0^1 q_x \int_0^x q(y) \mathrm{d}y \, \mathrm{d}x - \tau \delta \int_0^1 \omega_{xt} \int_0^x q(y) \mathrm{d}y \, \mathrm{d}x$$
$$- \rho_3 \alpha \int_0^1 \theta \int_0^x q \, \mathrm{d}y \, \mathrm{d}x - \rho_3 \int_0^1 \theta \int_0^x \theta_y \mathrm{d}y \, \mathrm{d}x. \qquad (62)$$

Integration by parts gives us

$$F_3'(t) = \tau \int_0^1 q^2 \mathrm{d}y + \tau \delta \int_0^1 \omega_t q \mathrm{d}x - \rho_3 \alpha \int_0^1 \theta \int_0^x q \mathrm{d}y \, \mathrm{d}x$$
$$- \rho_3 \int_0^1 \theta^2 \mathrm{d}x. \qquad (63)$$

By using Young's inequality and Hölder's inequality, we can get (61). □

Lemma 5. *The functional $F_4(t)$ defined by*

$$F_4(t) = -\rho\rho_3 \int_0^1 \theta \int_0^x \omega_t(y)dy\,dx, \qquad (64)$$

satisfies for any $\varepsilon_4 > 0$,

$$F_4'(t) \leq -\frac{\rho\delta}{2} \int_0^1 \omega_t^2 dx + \varepsilon_4 \int_0^1 (\psi - \omega_x)^2 dx + c\left(1 + \frac{1}{\varepsilon_4}\right) \int_0^1 \theta^2 dx$$

$$+ \frac{\rho}{2\delta} \int_0^1 q^2 dx. \qquad (65)$$

Proof. We take the derivative of F_4 and use (1) and integrate by parts to obtain

$$F_4'(t) = \rho \int_0^1 q_x \int_0^x \omega_t(y)dy\,dx + \rho\delta \int_0^1 \omega_{xt} \int_0^x \omega_t(y)dy\,dx$$

$$+ \rho_3 G \int_0^1 \theta \int_0^x (\psi - \omega_y)_y(y)dy\,dx$$

$$+ \rho_3 \delta \int_0^1 \theta \int_0^x \theta_y(y)$$

$$= -\rho \int_0^1 q\,\omega_t dx - \rho\delta \int_0^1 \omega_t^2 dx$$

$$+ \rho_3 G \int_0^1 \theta(\psi - \omega_x)dx + \rho_3\delta \int_0^1 \theta^2 dx. \qquad (66)$$

Then, using Young's inequality, we can get (65). □

Lemma 6. *Define the functional $F_5(t)$ by*

$$F_5(t) = \tau G\delta I_\rho \int_0^1 (3s - \psi)_t(\psi - \omega_x)dx$$

$$- \tau\delta D\rho \int_0^1 \omega_t(3s - \psi)_x dx$$

$$+ \tau\rho_3(D\rho - GI_\rho) \int_0^1 \theta(3s - \psi)_t dx$$

$$- \tau(D\rho - GI_\rho) \int_0^1 q(3s - \psi)_x dx. \qquad (67)$$

Then, we have for any $\varepsilon_5 > 0$,

$$F_5'(t) \leq -\frac{\tau G\delta I_\rho}{2} \int_0^1 [(3s - \psi)_t]^2 dx + c_1 \int_0^1 s_t^2 dx$$

$$+ c_2 \int_0^1 \theta^2 dx + c_3 \int_0^1 (\psi - \omega_x)^2 dx$$

$$+ \varepsilon_5 \int_0^1 [(3s - \psi)_x]^2 dx + C_{\varepsilon_5} \int_0^1 q^2 dx$$

$$+ \chi \int_0^1 \theta_x(3s - \psi)_x dx, \qquad (68)$$

where c_i $(i = 1, 2, 3)$ are positive constants.

Proof. By differentiating F_5 with respect to t, we have

$$F_5'(t) = \underbrace{\tau G\delta I_\rho \int_0^1 (3s - \psi)_{tt}(\psi - \omega_x)dx}_{:=I_1} + \tau G\delta I_\rho \int_0^1 (3s - \psi)_t(\psi - \omega_x)_t dx$$

$$\underbrace{-\tau\delta D\rho \int_0^1 \omega_{tt}(3s - \psi)_x dx}_{:=I_2} - \tau\delta D\rho \int_0^1 \omega_t(3s - \psi)_{xt} dx$$

$$\underbrace{+\tau\rho_3(D\rho - GI_\rho) \int_0^1 \theta_t(3s - \psi)_t dx}_{:=I_3} + \underbrace{\tau\rho_3(D\rho - GI_\rho) \int_0^1 \theta(3s - \psi)_{tt} dx}_{:=I_4}$$

$$\underbrace{-\tau(D\rho - GI_\rho) \int_0^1 q_t(3s - \psi)_x dx}_{:=I_5} \underbrace{-\tau(D\rho - GI_\rho) \int_0^1 q(3s - \psi)_{xt} dx}_{:=I_6}. \qquad (69)$$

Using equation (1) and integrating by parts, we see that

$$I_1 = -\tau G\delta D \int_0^1 (3s - \psi)_x(\psi - \omega_x)_x dx$$

$$+ \tau G^2 \delta \int_0^1 (\psi - \omega_x)^2 dx, \qquad (70)$$

$$I_2 = \tau\delta DG \int_0^1 (\psi - \omega_x)_x(3s - \psi)_x dx$$

$$+ \tau\delta^2 D \int_0^1 \theta_x(3s - \psi)_x dx, \qquad (71)$$

$$I_3 = -\tau(D\rho - GI_\rho) \int_0^1 q_x(3s - \psi)_t dx$$

$$- \tau\delta(D\rho - GI_\rho) \int_0^1 \omega_{xt}(3s - \psi)_t dx, \qquad (72)$$

$$I_4 = -\frac{\tau\rho_3 D}{I_\rho}(D\rho - GI_\rho) \int_0^1 \theta_x(3s - \psi)_x dx$$

$$+ \frac{\tau\rho_3 G}{I_\rho}(D\rho - GI_\rho) \int_0^1 \theta(\psi - \omega_x)dx, \qquad (73)$$

$$I_5 = \alpha\left(D\rho - GI_\rho\right)\int_0^1 q(3s - \psi)_x dx + \left(D\rho - GI_\rho\right)\int_0^1 \theta_x(3s - \psi)_x dx, \tag{74}$$

$$I_6 = \tau\left(D\rho - GI_\rho\right)\int_0^1 q_x(3s - \psi)_t dx. \tag{75}$$

Inserting (70)–(75) into (69), we can obtain

$$F_5'(t) = \tau G^2\delta\int_0^1 (\psi - \omega_x)^2 dx + \tau G\delta I_\rho\int_0^1 \psi_t(3s - \psi)_t dx$$

$$+ \frac{\tau\rho_3 G}{I_\rho}\left(D\rho - GI_\rho\right)\int_0^1 \theta(\psi - \omega_x)dx$$

$$+ \alpha\left(D\rho - GI_\rho\right)\int_0^1 q(3s - \psi)_x dx$$

$$+ \chi\int_0^1 \theta_x(3s - \psi)_x dx. \tag{76}$$

Recalling $\psi = (\psi - 3s) + 3s$ and using Young's inequality, we conclude that

$$\tau G\delta I_\rho\int_0^1 \psi_t(3s - \psi)_t dx$$

$$= -\tau G\delta I_\rho\int_0^1 \left[(3s - \psi)_t\right]^2 dx + 3\tau G\delta I_\rho\int_0^1 s_t(3s - \psi)_t dx$$

$$\leq -\frac{\tau G\delta I_\rho}{2}\int_0^1 \left[(3s - \psi)_t\right]^2 dx + c_1\int_0^1 s_t^2 dx, \tag{77}$$

$$\frac{\tau\rho_3 G}{I_\rho}\left(D\rho - GI_\rho\right)\int_0^1 \theta(\psi - \omega_x)dx$$

$$\leq c_2\int_0^1 \theta^2 dx + c_3\int_0^1 (\psi - \omega_x)^2 dx, \tag{78}$$

and for any $\varepsilon_5 > 0$,

$$\alpha\left(D\rho - GI_\rho\right)\int_0^1 q(3s - \psi)_x dx \leq \varepsilon_5\int_0^1 \left[(3s - \psi)_x\right]^2 dx$$

$$+ C_{\varepsilon_5}\int_0^1 q^2 dx, \tag{79}$$

which, together with (76)–(78), gives us (68). □

Lemma 7. *The functional $F_6(t)$ defined by*

$$F_6(t) = 3I_\rho\int_0^1 s_t s\, dx + 2\beta\int_0^1 s^2 dx, \tag{80}$$

satisfies

$$F_6'(t) \leq -3\gamma\int_0^1 s^2 dx - 3D\int_0^1 s_x^2 dx$$

$$+ c_4\int_0^1 (\psi - \omega_x)^2 dx + 3I_\rho\int_0^1 s_t^2 dx, \tag{81}$$

where c_4 is a positive constant.

Proof. follows from (1) that

$$F_6'(t) = -3D\int_0^1 s_x^2 dx - 3G\int_0^1 s(\psi - \omega_x)dx$$

$$- 4\gamma\int_0^1 s^2 dx + 3I_\rho\int_0^1 s_t^2 dx. \tag{82}$$

Young's inequality gives us (82). □

3.2. Exponential Stability: Proof of Theorem 1

Proof. We define the functional $\mathscr{L}(t)$ by

$$\mathscr{L}(t) = NE(t) + F_1(t) + N_2 F_2(t) + N_3 F_3(t)$$

$$+ N_4 F_4(t) + N_5 F_5(t) + F_6(t), \tag{83}$$

where N and N_i $(i = 2, 3, 4, 5)$ are positive constants that will be chosen later.

Note that

$$\int_0^1 \psi_t^2 dx = \int_0^1 [(3s - \psi) - 3s]_t^2 dx$$

$$\leq 2\int_0^1 \left[(3s - \psi)_t\right]^2 dx + 18\int_0^1 s_t^2 dx. \tag{84}$$

Replacing (84) in (57) and then combining (51)–(53), (57)–(68), and (82), we obtain

$$\mathscr{L}'(t) \leq -\left(4\beta N - c_1 N_5 - 18\varepsilon_2 N_2 - 3I_\rho\right)\int_0^1 s_t^2 dx$$

$$-\left(\frac{D}{2} - \varepsilon_5 N_5\right)\int_0^1 \left[(3s - \psi)_x\right]^2 dx$$

$$-\left[\alpha N - cN_3\left(1 + \frac{1}{\varepsilon_3}\right) - \frac{\rho}{2\delta}N_4 - C_{\varepsilon_5}N_5\right]\int_0^1 q^2 dx$$

$$-\left[\frac{\rho\delta}{2}N_4 - \varepsilon_1 - \varepsilon_3 N_3 - cN_2\left(1 + \frac{1}{\varepsilon_2}\right)\right]\int_0^1 \omega_t^2 dx$$

$$-\left[\frac{\tau G\delta I_\rho}{2}N_5 - 2\varepsilon_2 N_2 - c\left(1 + \frac{1}{\varepsilon_1}\right)\right]\int_0^1 \left[(3s - \psi)_t\right]^2 dx$$

$$-\left(\frac{G}{2}N_2 - \varepsilon_4 N_4 - c_3 N_5 - c_4\right)\int_0^1 (\psi - \omega_x)^2 dx$$

$$-\left[\frac{\rho_3}{2}N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G}N_2 - cN_4\left(1 + \frac{1}{\varepsilon_4}\right) - c_2 N_5\right]\int_0^1 \theta^2 dx$$

$$- 3D\int_0^1 s_x^2 dx - 3\gamma\int_0^1 s^2 dx. \tag{85}$$

Taking

$$\varepsilon_1 = 1,$$

$$\varepsilon_2 = \frac{\tau G \delta I_\rho}{8 N_2} N_5,$$

$$\varepsilon_3 = \frac{1}{N_3},$$ (86)

$$\varepsilon_4 = \frac{G}{4 N_4} N_2,$$

$$\varepsilon_5 = \frac{D}{4 N_5},$$

we obtain

$$\mathscr{L}'(t) \le -\left(4\beta N - c_1 N_5 - \frac{9}{4}\tau G \delta I_\rho N_5 - 3 I_\rho\right) \int_0^1 s_t^2 dx$$

$$-\frac{D}{4}\int_0^1 \left[(3s - \psi)_x\right]^2 dx$$

$$-\left[\alpha N - c N_3 (1 + N_3) - \frac{\rho}{2\delta} N_4 - C_{\varepsilon_5} N_5\right]\int_0^1 q^2 dx$$

$$-\left[\frac{\rho\delta}{2} N_4 - 2 - c N_2 \left(1 + \frac{8 N_2}{N_5 \tau G \delta I_\rho}\right)\right]\int_0^1 \omega_t^2 dx$$

$$-\left(\frac{\tau G \delta I_\rho}{4} N_5 - 2c\right)\int_0^1 \left[(3s - \psi)_t\right]^2 dx$$

$$-\left(\frac{G}{4} N_2 - c_3 N_5 - c_4\right)\int_0^1 (\psi - \omega_x)^2 dx$$

$$-\left[\frac{\rho_3}{2} N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G} N_2\right.$$

$$\left. - c N_4 \left(1 + \frac{4 N_4}{G N_2}\right) - c_2 N_5\right]\int_0^1 \theta^2 dx$$

$$- 3D\int_0^1 s_x^2 dx - 3\gamma\int_0^1 s^2 dx.$$

(87)

At this point, we first choose $N_5 > 0$ large enough such that

$$\frac{\tau G \delta I_\rho}{4} N_5 - 2c > 0.$$ (88)

For fixed N_5, we take $N_2 > 0$ so large that

$$\frac{G}{4} N_2 - c_3 N_5 - c_4 > 0.$$ (89)

Then, we pick $N_4 > 0$ large so that

$$\frac{\rho\delta}{2} N_4 - 2 - c N_2 \left(1 + \frac{8 N_2}{N_5 \tau G \delta I_\rho}\right) > 0.$$ (90)

And then we choose N_3 so large that

$$\frac{\rho_3}{2} N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G} N_2 - c N_4 \left(1 + \frac{4 N_4}{G N_2}\right) - c_2 N_5 > 0. \quad (91)$$

At last, we take $N > 0$ large enough so that the functional $\mathscr{L}(t)$ is equivalent to the energy functional $E(t)$, i.e., there exist two positive constants:

$$\beta_1 E(t) \le \mathscr{L}(t) \le \beta_2 E(t),$$ (92)

and further so that

$$\alpha N - c N_3 (1 + N_3) - \frac{\rho}{2\delta} N_4 - C_{\varepsilon_5} N_5 > 0.$$

(93)

$$4\beta N - c_1 N_5 - \frac{9}{4}\tau G \delta I_\rho N_5 - 3 I_\rho > 0,$$

Recalling (48), we infer that there exists a positive constant β_3 such that, for any $t > 0$,

$$\mathscr{L}'(t) \le -\beta_3 E(t),$$ (94)

which, along with (92), implies

$$\mathscr{L}'(t) \le -\frac{\beta_3}{\beta_2}\mathscr{L}(t).$$ (95)

Integrating (95) over $(0, t)$, we have, for any $t > 0$,

$$\mathscr{L}(t) \le \mathscr{L}(0) e^{-(\beta_3/\beta_2)=t,}$$ (96)

which, using (95) again, gives us (49). The proof of Theorem 1 is done. □

3.3. Polynomial Stability: Proof of Theorem 2. In this section, we consider the case $\chi \ne 0$ to prove Theorem 2.

Differentiating system (1) with respect to time, we obtain the following system:

$$\begin{cases} \rho\omega_{ttt} + G(\psi - \omega_x)_{xt} + \delta\theta_{xt} = 0, \\\\ I_\rho (3s - \psi)_{ttt} - D(3s - \psi)_{xxt} - G(\psi - \omega_x)_t = 0, \\\\ I_\rho s_{ttt} - D s_{xxt} + G(\psi - \omega_x)_t + \frac{4}{3}\gamma s_t + \frac{4}{3}\beta s_{tt} = 0, \quad (97) \\\\ \rho_3 \theta_{tt} + q_{xt} + \delta\omega_{xtt} = 0, \\\\ \tau q_{tt} + \alpha q_t + \theta_{xt} = 0, \end{cases}$$

which subject to the following boundary conditions:

$$\begin{cases} \omega_{xt}(0, t) = \psi_t(0, t) = s_t(0, t) = \theta_t(0, t) = 0, & t \in (0, \infty), \\ \omega_{xt}(1, t) = \psi_t(1, t) = s_t(1, t) = \theta_t(1, t) = 0, & t \in (0, \infty). \end{cases}$$

(98)

For any initial data $U_0 \in D(\mathscr{A})$, system (97) is well posed. Next, we introduce second-order energy functional $\tilde{E}(t)$ by

$$\tilde{E}(t) = E(\omega_t, \psi_t, s_t, \theta_t, q_t)$$

$$= \frac{1}{2} \int_0^1 \left[\rho \omega_{tt}^2 + I_\rho \left[(3s - \psi)_{tt} \right]^2 + 3I_\rho s_{tt}^2 + \rho_3 \theta_t^2 + \tau q_t^2 + 4\gamma s_t^2 \right.$$

$$\left. + D \left[(3s - \psi)_{xt} \right]^2 + G \left[(\psi - \omega_x)_t \right]^2 + 3D s_{xt}^2 \right] dx. \tag{99}$$

By using the same arguments as in Lemma 3, we can get the second-order energy $\tilde{E}(t)$ defined by (99) is nonincreasing and satisfies

$$\tilde{E}'(t) = -4\beta \int_0^1 s_{tt}^2 dx - \alpha \int_0^1 q_t^2 dx \leq 0. \tag{100}$$

In Lemma 6, we have proved that, for any $\varepsilon_5 > 0$,

$$F_5'(t) \leq -\frac{\tau G \delta I_\rho}{2} \int_0^1 \left[(3s - \psi_t)^2 dx \right] + c_1 \int_0^1 s_t^2 dx$$

$$+ c_2 \int_0^1 \theta^2 dx + c_3 \int_0^1 (\psi - \omega_x)^2 dx \tag{101}$$

$$+ \varepsilon_5 \int_0^1 \left[(3s - \psi)_x \right]^2 dx + C_{\varepsilon_5} \int_0^1 q^2 dx$$

$$+ \chi \int_0^1 \theta_x (3s - \psi)_x dx.$$

Thanks to (1) and Young's inequality, we derive that

$$\int_0^1 \theta_x^2 dx \leq c \int_0^1 q^2 dx + c \int_0^1 q_t^2 dx. \tag{102}$$

Then, for any $\varepsilon_5 > 0$,

$$\chi \int_0^1 \theta_x (3s - \psi)_x dx \leq \varepsilon_5 \int_0^1 \left[(3s - \psi)_x \right]^2 dx + C_{\varepsilon_5} \int_0^1 \theta_x^2 dx$$

$$\leq \varepsilon_5 \int_0^1 \left[(3s - \psi)_x \right]^2 dx + C_{\varepsilon_5} \int_0^1 q^2 dx$$

$$+ C_{\varepsilon 5} \int_0^1 q_t^2 dx. \tag{103}$$

Therefore, the derivative of F_5 satisfies

$$F_5'(t) \leq -\frac{\tau G \delta I_\rho}{2} \int_0^1 \left[(3s - \psi)_t \right]^2 dx + c_1 \int_0^1 s_t^2 dx$$

$$+ c_2 \int_0^1 \theta^2 dx + c_3 \int_0^1 (\psi - \omega_x)^2 dx$$

$$+ 2\varepsilon_5 \int_0^1 \left[(3s - \psi)_x \right]^2 dx + C_{\varepsilon_5} \int_0^1 q^2 dx + C_{\varepsilon_5} \int_0^1 q_t^2 dx. \tag{104}$$

Proof. We define the functional $\tilde{\mathscr{L}}(t)$ by

$$\tilde{\mathscr{L}}(t) = N \left(E(t) + \tilde{E}(t) \right) + F_1(t) + N_2 F_2(t)$$

$$+ N_3 F_3(t) + N_4 F_4(t) + N_5 F_5(t) + F_6(t). \tag{105}$$

It follows from (51)–(53), (57)–(65), and (100)–(104) that

$$\tilde{\mathscr{L}}'(t) \leq -\left(4\beta N - c_1 N_5 - 18\varepsilon_2 N_2 - 3I_\rho \right) \int_0^1 s_t^2 dx - \left(\frac{D}{2} - 2\varepsilon_5 N_5 \right) \int_0^1 \left[(3s - \psi)_x \right]^2 dx$$

$$- \left[\alpha N - c N_3 \left(1 + \frac{1}{\varepsilon_3} \right) - \frac{\rho}{2\delta} N_4 - C_{\varepsilon_5} N_5 \right] \int_0^1 q^2 dx$$

$$- \left[\frac{\rho \delta}{2} N_4 - \varepsilon_1 - \varepsilon_3 N_3 - c N_2 \left(1 + \frac{1}{\varepsilon_2} \right) \right] \int_0^1 \omega_t^2 dx$$

$$- \left[\frac{\tau G \delta I_\rho}{2} N_5 - 2\varepsilon_2 N_2 - c \left(1 + \frac{1}{\varepsilon_1} \right) \right] \int_0^1 \left[(3s - \psi)_t \right]^2 dx \tag{106}$$

$$- \left(\frac{G}{2} N_2 - \varepsilon_4 N_4 - c_3 N_5 - c_4 \right) \int_0^1 (\psi - \omega_x)^2 dx$$

$$- \left[\frac{\rho_3}{2} N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G} N_2 - c N_4 \left(1 + \frac{1}{\varepsilon_4} \right) - c_2 N_5 \right] \int_0^1 \theta^2 dx$$

$$- 3D \int_0^1 s_x^2 dx - 3\gamma \int_0^1 s^2 dx - \left(\alpha N - C_{\varepsilon_5} \right) \int_0^1 q_t^2 dx.$$

With the same choice of constants as in Section 3.2, we further take $N > 0$ so large that

$$\alpha N - C_{\varepsilon_5} > 0. \tag{107}$$

Noting that (48), we know that there exists a positive constant μ_1 such that, for any $t > 0$,

$$\widetilde{\mathscr{L}}'(t) \leq -\mu_1 E(t). \tag{108}$$

Since the energy functional $E(t)$ is positive and non-increasing, we infer (108) that, for any $t > 0$,

$$tE(t) \leq \int_0^t E(s)\mathrm{d}s \leq \frac{1}{\mu_1}\left(\widetilde{\mathscr{L}}(0)t - n\widetilde{\mathscr{L}}q(t)\right) \leq \frac{\widetilde{\mathscr{L}}(0)}{\mu_1}, \tag{109}$$

which gives us

$$E(t) \leq \frac{\mu_0}{t}, \quad \forall t > 0. \tag{110}$$

Here, $\mu_0 = (\widetilde{\mathscr{L}}(0)t/\mu_1) = (E(0) + \widetilde{E}(0)/\mu_1)$. The proof is complete. $\qquad\square$

Remark 1. We point out that the functional $\widetilde{\mathscr{L}}(t)$ is inequivalent to the energy functional $E(t)$. That is to say, (92) does not hold true.

Data Availability

No data were used during this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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