

Research Article

Finite-Time Lyapunov Functions and Impulsive Control Design

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In this paper, we introduce finite-time Lyapunov functions for impulsive systems. The relaxed sufficient conditions for asymptotic stability of an equilibrium of an impulsive system are given via finite-time Lyapunov functions. A converse finite-time Lyapunov theorem for controlling the impulsive system is proposed. Three examples are presented to show how to analyze the stability of an equilibrium of the considered impulsive system via finite-time Lyapunov functions. Furthermore, according to the results, we design an impulsive controller for a chaotic system modified from the Lorenz system.

1. Introduction

Impulsive systems have been investigated by researchers, since impulsive systems can describe many practical problems from fields such as engineering, finance, chemistry, and biology (see [1, 2]). In references [1, 3–8], the authors studied the stability of equilibria of impulsive systems. Lyapunov functions are widely used to analyze stability problems of dynamical systems because it is not necessary to compute an analytic solution of the considered system. Many researchers try to relax the conditions imposed on Lyapunov functions. In [1, 2, 9], Lyapunov functions are allowed to be nonincreasing at the impulses. In [10], the conditions imposed on Lyapunov functions are more relaxed. Lyapunov functions could be nonincreasing at some of resetting times. The authors in [6] investigated the stability of hybrid systems by nonmonotonic Lyapunov functions which are not monotonically decreasing along the considered system trajectories. In [11, 12], Lyapunov functions for the considered impulsive system may increase during the continuous part of the trajectory of the state. In [5], Lyapunov functions are monotonically decreasing along the continuous part of the trajectory of the considered system and could increase at the resetting times. Based on the attained results, the authors designed a H_∞ controller for the considered problem. In [13], utilizing results from [2], the author got some sufficient

conditions for impulsive control for a class of systems. In [14, 15], the authors studied chaotic communication systems. Then, they designed impulsive control for the considered systems. In [7], stochastic switched systems with impulses and time delay were investigated. Based on vector Lyapunov functions, input-to-state stability of the considered system was discussed. In [16], the authors investigated switched systems with time delay. Lyapunov functions can be nonincreasing at the switching times. According to the results, adaptive control was designed for the considered system. In [17], the authors presented an overview of the research investigations on impulsive control systems. Using Lyapunov functions with relaxed constraints, the authors in [18] discussed sufficient and necessary conditions for asymptotic stability of an equilibrium of a discrete-time homogeneous dynamical system. The results were extended to discrete-time systems in [19, 20]. A converse Lyapunov theorem was proposed for continuous-time systems via Lyapunov functions with relaxed conditions in [21]. In [22], two ways were designed for the computation of Lyapunov functions with relaxed conditions for continuous-time systems. In [23], the authors discussed input-to-state stability for continuous-time systems via input-to-state stable (ISS) Lyapunov functions with relaxed conditions. In [24], the authors proposed relaxed sufficient conditions for asymptotic stability of an equilibrium of time varying

impulsive systems via indefinite Lyapunov functions and designed an impulsive controller for a chaotic system.

In this paper, we will analyze the stability of equilibria of time invariant impulsive systems by Lyapunov functions with relaxed constraints, later named finite-time Lyapunov functions, and then design impulsive control for a chaotic system adapted from the Lorenz system. The ideas of the paper are inspired by the results discussed above. We obtained some novel results. Finite-time Lyapunov functions can increase during some continuous part of the trajectory of the considered system and have positive jumps at some impulses (see Theorem 1, example). It is worthy to point out that a converse finite-time Lyapunov theorem (see Theorem 2) is proposed. Based on Theorem 2, a finite-time Lyapunov function can be constructed for the considered impulsive system. Moreover, we design an impulse controller to get a chaotic system stabilized based on Theorem 1 and Corollary 1.

This paper is organized as follows. In Section 2, we introduce notations and basic definitions. Finite-time Lyapunov functions for impulsive systems are introduced. The main problem studied in this paper is described. In Section 3, the main results are discussed. We first study how to prove the origin of system (1) is asymptotic stable by finite-time Lyapunov functions. Then, a converse finite-time Lyapunov theorem for impulsive systems is obtained, that is, if the origin of system (1) is asymptotically stable and Condition 1 holds, then there exists a finite-time Lyapunov function for system (1). In Section 4, we show the efficiency of our main results via three examples. Especially, Example 3 shows that finite-time Lyapunov functions can increase along some continuous portion of the trajectory of system (31) and increase at some resetting times. Furthermore, according to our main results, we design impulsive control for a chaotic system in Section 5. We present simulation results of the chaotic system with the designed controller. Some concluding remarks are discussed in Section 6.

2. Notations and Preliminaries

The real numbers, the nonnegative real numbers, and the nonnegative integers are denoted by \mathbb{R} , \mathbb{R}_+ , and \mathbb{Z}_+ , respectively. The Euclidean norm of the real vector $x \in \mathbb{R}^n$ is denoted by $|x|$. For $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we denote 1-norm for x by $|x|_1 = \sum_{i=1, \dots, n} |x_i|$. The open ball of radius r around z in the norm of $|\cdot|$ is defined by $B(z, r) = \{x \in \mathbb{R}^n \mid |x - z| < r\}$. For a set $\Omega \subset \mathbb{R}^n$, the boundary and the interior of Ω are denoted by $\partial\Omega$ and $\text{int } \Omega$, respectively.

It is well known that comparison functions are widely used in stability analysis. Comparison functions are described as follows. If a continuous function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$, then we say it is *positive definite*. A positive definite function is of *class \mathcal{K}* if it is strictly increasing and of *class \mathcal{K}_∞* if it belongs to the class \mathcal{K} and unbounded. We say a continuous function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to *class \mathcal{L}* if $\gamma(r)$ is strictly decreasing to 0 as $r \rightarrow \infty$. A continuous function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is

said to be of *class \mathcal{KL}* if it is of class \mathcal{K}_∞ in the first argument and of class \mathcal{L} in the second argument.

In this paper, we are going to study the stability property of the following system with impulses described by

$$\begin{cases} \dot{x}(t) = f(x(t)), & t \neq t_k, k \in \{1, 2, 3, \dots\}, \\ x(t_k) = g(x(t_k^-)), & t = t_k, k \in \{1, 2, 3, \dots\}, \end{cases} \quad (1)$$

where $0 < t_1 < t_2 < \dots < t_k < \dots$ are resetting times in $(0, \infty)$ and $\lim_{k \rightarrow \infty} t_k = \infty$. The functions $f, g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are locally Lipschitz continuous and satisfy the conditions $f(0) = 0$, $g(0) = 0$. It is evident that the origin is an equilibrium of system (1). Suppose that a sequence of impulse times $\{t_k\}$ is given; the solution of system (1) corresponding to an initial condition $x_0 = x(0)$ is denoted by $x(t, x_0)$. The limits of $x(t)$ from left and right are denoted by $x(t^-)$ and $x(t^+)$, respectively. It is easy to see that the solution $x(t)$ of system (1) is right continuous, that is, it is continuous in $(0, t_1)$, (t_k, t_{k+1}) , and the following conditions hold.

$$\begin{aligned} x(t_k^-) &= \lim_{\varepsilon \rightarrow 0^+} x(t_k - \varepsilon), \\ x(t_k) &= x(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} x(t_k + \varepsilon). \end{aligned} \quad (2)$$

For a constant $T > 0$ and a sequence of impulse times $\{t_k\}$, we define a positively T -invariant set for system (1).

Definition 1. Given a constant $T > 0$ and a sequence of impulse times $\{t_k\}$, a compact set $\Omega \subset \mathbb{R}^n$ is called a positively T -invariant set for system (1) if for all $x(t, x_0) \in \Omega$, it satisfies $x(t + T, x_0) \in \Omega$ for $t \in \mathbb{R}_+$.

Remark 1. In Definition 1, if $T = 0$ holds, then we call the set Ω a positively invariant set for system (1).

The following definition describes asymptotic stability of system (1) we are interested in.

Definition 2. For system (1), we suppose that a sequence of impulse times $\{t_k\}$ is given. The origin of system (1) is asymptotically stable in a compact set $\Omega \subset \mathbb{R}^n$ if there exists a function $\beta \in \mathcal{KL}$ such that for every initial condition $x_0 \in \Omega$, it holds that

$$|x(t, x_0)| \leq \beta(|x_0|, t), \quad t \geq 0. \quad (3)$$

In the literature, sufficient conditions for asymptotic stability of system (1) were obtained via Lyapunov functions described by the following definition.

Definition 3. Given a sequence of impulse times $\{t_k\}$ for system (1). A continuous function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be a Lyapunov function for system (1) on a compact set $\Omega \subset \mathbb{R}^n$, if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a positive definite function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a continuous function $\theta: \mathbb{R}_+ \rightarrow [0, 1) \subset \mathbb{R}_+$ such that

(i) (C1)

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n. \quad (4)$$

(ii) (C2)

$$\nabla V(x(t, x_0))f(x(t)) \leq -\alpha(V(x(t, x_0))), \quad (5)$$

for $x_0 \in \Omega, t \neq t_k, k \in \{1, 2, 3, \dots\}$.

(ii) (C3)

$$V(g(x(t, x_0))) \leq \theta(V(x(t^-, x_0))), \quad (6)$$

for $x_0 \in \Omega, t = t_k, k \in \{1, 2, 3, \dots\}$.

We are going to relax the conditions imposed on Lyapunov functions in Definition 3 and then demonstrate how to prove asymptotic stability of system (1) via finite-time Lyapunov functions defined by the following definition.

Definition 4. Consider system (1) with a given impulsive time sequence $\{t_k\}$. A continuous function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be a finite-time Lyapunov function for system (1) on a compact set $\Omega \subset \mathbb{R}^n$, if there exist a positive constant T , functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a function $\rho \in \mathcal{K}$ with $\rho < \text{id}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \quad (7)$$

$$V(x(T + t, x_0)) \leq \rho(V(x(t, x_0))), \quad \text{for } t \geq 0, x_0 \in \Omega. \quad (8)$$

Remark 2

- (i) From constraint (8), we have that the function V defined in Definition 4 may increase during some continuous portion of the trajectory and at some impulses. It is clear that the conditions imposed on finite-time Lyapunov functions are more relaxed than those on Lyapunov functions defined by Definition 3.
- (ii) In order to verify condition (8), we have to construct finite-time Lyapunov functions. However, it is not easy to give a formulation of a finite-time Lyapunov function. Under certain conditions, Theorem 2 will explain how to construct a finite-time Lyapunov function. When we check if the condition (8) holds, it is necessary to calculate the solution $x(T + t, x_0)$. For easy computation, the Euler method will be used to calculate the solution $x(T + t, x_0)$ for examples considered in Section 4.

The following impulsive integral inequality of Gronwall type will be used in deducing inequalities in the proofs of our main results.

Lemma 1. Let $t_1, t_2, \dots, t_k, \dots$ be a strictly increasing sequence of impulse times in $(0, \infty)$ and $\lim_{k \rightarrow \infty} t_k = \infty$, the function $m: \mathbb{R}_+ \rightarrow \mathbb{R}$ a continuous function for $t \neq t_k$ and right continuous at $t = t_k$ ($k = 1, 2, \dots$), and the function

$p: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function. Moreover, we assume that

$$m(t) \leq A + \int_0^t p(s)m(s)ds + \sum_{0 < t_k \leq t} \lambda_k m(t_k), \quad (9)$$

$t \geq 0, k > 0, k \in \mathbb{Z}_+$,

where $\lambda_k \geq 0$ and A are constants. Then it holds that

$$m(t) \leq A \prod_{0 < t_k \leq t} (1 + \lambda_k) e^{\int_0^t p(s)ds}, \quad t \geq 0. \quad (10)$$

Proof. The proof is similar to the proof of Theorem 16.1 in [25]. \square

3. Main Results

In this section, we first demonstrate how to prove asymptotic stability of system (1) via finite-time Lyapunov functions defined by Definition 4. Then, we propose a converse finite-time Lyapunov theorem for system (1).

Theorem 1. Consider system (1) with a given impulsive time sequence $\{t_k\}$. Let T be a positive constant, and a compact set $\Omega \subset \mathbb{R}^n$ with $0 \in \text{int}\Omega$ be a positively T -invariant set for system (1). If there exists a finite-time Lyapunov function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ with T for system (1) on Ω , then the origin of system (1) is asymptotically stable in Ω over the given impulse sequence. Furthermore, an estimate of the domain of attraction of the origin of system (1) is given by $\mathcal{D}_a = \{x \in \Omega \mid V(x) \leq \min_{x \in \partial\Omega} V(x)\}$.

Proof. According to the conditions, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a function $\rho \in \mathcal{K}$ with $\rho < \text{id}$ such that inequalities (7) and (8) from Definition 4 hold. For any $t > 0$, $t \in \mathbb{R}_+$, there exists a integer $N > 0$ such that $t = NT + j$, $j \in [0, T)$. Utilizing (8) recursively, we obtain that

$$\begin{aligned} V(x(t, x_0)) &= V(x(NT + j, x_0)) = V(x((N-1)T + T + j, x_0)) \\ &\leq \rho(V(x((N-1)T + j, x_0))) \\ &\dots \\ &\leq \rho^N(V(x(j, x_0))) \\ &\leq \rho^N(\alpha_2(|x(j, x_0)|)), \end{aligned} \quad (11)$$

where ρ^N represents the N -times composition of ρ and α_2 comes from (7). Let $k \in \mathbb{Z}_+$ denote the number of impulsive time $t_i \in [0, j]$ ($i \in \mathbb{Z}_+, i > 0$), and

$$\sigma = \begin{cases} 0, & k = 0, \\ 1, & k \geq 1. \end{cases} \quad (12)$$

The solution $x(t, x_0)$ at time $t = j$ is given by

$$x(j, x_0) = \begin{cases} x_0 + \sum_{d=1}^k g(x(t_d^-)) + \int_0^j f(x(s))ds, & \text{for } k \geq 1, x_0 + \int_0^j f(x(s))ds, & \text{for } k = 0. \end{cases} \quad (13)$$

Using (12), $x(j, x(0))$ is rewritten as

$$x(j, x_0) = x_0 + \sigma \sum_{d=1}^k g(x(t_d^-)) + \int_0^j f(x(s))ds, \quad (14)$$

for any $j \geq 0$.

Then, we have

$$\begin{aligned} |x(j, x_0) - x(0)| &\leq \sigma \sum_{d=1}^k |g(x(t_d^-)) - g(x_0) + g(x_0)| \\ &\quad + \int_0^j |f(x(s)) - f(x_0) + f(x_0)|ds \\ &\leq \sigma \sum_{d=1}^k |g(x(t_d^-)) - g(x_0)| + \sigma \sum_{d=1}^k |g(x_0)| \\ &\quad + \int_0^j |f(x(s)) - f(x_0)|ds + \int_0^j |f(x_0)|ds. \end{aligned} \quad (15)$$

Using the Lipschitz conditions for f, g , and Lemma 1, we get that

$$\begin{aligned} |x(j, x_0) - x_0| &\leq \sigma \sum_{d=1}^k L_g |x(t_d^-) - x_0| + \sigma k |g(x_0)| \\ &\quad + \int_0^j L_f |x(s) - x_0|ds + \int_0^j |f(x_0)|ds \leq C(1 + \sigma L_g)^k e^{L_f j}, \end{aligned} \quad (16)$$

where L_f, L_g are Lipschitz constants for the functions f, g , respectively, and $C = \sigma k \sum_{d=1}^k |g(x_0)| + \int_0^j |f(x_0)|ds$. Therefore, it holds that

$$\begin{aligned} |x(j, x_0)| &\leq C(1 + \sigma L_g)^k e^{L_f j} + |x_0| \leq C(1 + \sigma L_g)^k e^{L_f T} + |x_0| \\ &:= H_T(|x_0|). \end{aligned} \quad (17)$$

Based on the assumptions on the functions f, g , it is obtained that $H_T(|x(0)|)$ is continuous with respect to $|x_0|$. Moreover, it is clear that $H_T(0) = 0$, and $H_T(s) > 0$ for $s > 0$. Thus, $H_T(|x(0)|)$ is a positive definite function with respect to $|x_0|$. It is easy to see that there exists a function $\gamma \in \mathcal{K}_\infty$ such that $H_T(|x_0|) \leq \gamma(|x_0|)$. Then, it is satisfied that $|x(j, x_0)| \leq \gamma(|x(0)|)$ for all $0 \leq j < T$.

The idea of the proof of the existence of the function ρ_1 is inspired by Theorem 2.1 in [21] and Lemma 12 in [22]. Because ρ is positive definite, without loss of generality, we can assume that ρ is invertible, that is, ρ is a one-to-one and onto function. According to Theorem 3.16 in [26] and the above discussion, it holds that ρ^{-1} is continuous and $\rho^{-1}(0) = \rho^{-1}(\rho(0)) = 0$. Thus, for $t = NT + j$, we obtain that

$$V(x(t, x_0)) \leq \rho^{(t/T)} \circ \rho^{-1} \circ \alpha_2 \circ \gamma(|x_0|). \quad (18)$$

From the above analysis, the function ρ^{-1} is positive definite. Then, there exists a function $\rho_1 \in \mathcal{K}_\infty$ satisfying $\rho_1(s) \geq \rho^{-1}(s)$ for $s \geq 0$. Thus, it holds that

$$V(x(t, x_0)) \leq \rho^{(t/T)} \circ \rho_1 \circ \alpha_2 \circ \gamma(|x_0|). \quad (19)$$

Since the condition $\alpha_1 \in \mathcal{K}_\infty$ is satisfied, then the function α_1 is a one-to-one and onto function. Therefore, the function α_1^{-1} exists and $\alpha_1^{-1} \in \mathcal{K}_\infty$ holds. Furthermore, utilizing (7), we have that

$$|x(t, x_0)| \leq \alpha_1^{-1} \circ \rho^{(t/T)} \circ \rho_1 \circ \alpha_2 \circ \gamma(|x_0|). \quad (20)$$

Let $\beta(|x_0|, t) = \alpha_1^{-1} \circ \rho^{(t/T)} \circ \rho_1 \circ \alpha_2 \circ \gamma(|x_0|)$. Since $\alpha_1^{-1}, \rho_1, \alpha_2, \gamma$ are \mathcal{K}_∞ functions, for fixed t , the function β increases as the argument $|x_0|$ increases. Because $\rho < \text{id}$ holds, for fixed $|x_0|$, the function β decreases as the argument t increases. Therefore, we obtain that $\beta \in \mathcal{KL}$ is a \mathcal{K}_∞ function in the argument $|x_0|$ and a \mathcal{L} function in the argument t . Then, it holds that

$$|x(t, x_0)| \leq \beta(|x_0|, t). \quad (21)$$

Therefore, the origin of system (1) is asymptotically stable in Ω over the given impulse sequence. Moreover, an estimate of the domain of attraction of the origin of system (1) is obtained by $\mathcal{D}_a = \{x \in \Omega | V(x) \leq \min_{x \in \partial\Omega} V(x)\}$. \square

Remark 3. To make sure $x(T, x_0) \in \Omega$ for all $x_0 \in \Omega$, we have to ensure that the set Ω is a positively T -invariant set for system (1).

In the following, a converse finite-time Lyapunov theorem is investigated. To obtain the desired result, it is necessary to require the following condition.

Condition 1. Consider system (1) with a given impulsive time sequence $\{t_k\}$. There exists a \mathcal{KL} function β which satisfies (2) for system (1) and the inequality

$$\beta(s, T) < s, \quad (22)$$

for some $T > 0$ and $s > 0$.

Theorem 2. Consider system (1) with a given impulsive time sequence $\{t_k\}$. If the origin of system (1) is asymptotically stable in an invariant set $\Omega \subset \mathbb{R}^n$ over the given impulse sequence and Condition 1 holds, then for any function $\eta \in \mathcal{K}_\infty$, the function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ with

$$V(x) = \eta(|x|), \quad \text{for } x \in \mathbb{R}^n \quad (23)$$

satisfies inequalities (7) and (8) with T from Condition 1.

Proof. Because $\eta \in \mathcal{K}_\infty$, it is evident that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that inequality (7) holds. Since Condition 1 is satisfied, then there exists a positive constant T such that inequality (22) holds. Thus, we have that

$$\begin{aligned}
V(x(t+T, x_0)) &= \eta(|x(t+T, x_0)|) = \eta(|x(T, x(t, x_0))|) \\
&\leq \eta(\beta(|x(t, x_0)|, T)) = \eta^\circ \beta(\eta^{-1}(V(x(t, x_0))), T).
\end{aligned} \tag{24}$$

Let $\rho = \eta^\circ \beta(\eta^{-1}(V(x(t, x_0))), T)$. It is obvious that ρ is a positive definite function of the variable $V(x(t, x_0))$. Based on Condition 1, it is obtained that $\rho(s) < \eta \circ \eta^{-1}(s) = s$ holds. Thus, V satisfies inequality (8) with T from Condition 1. \square

Remark 4. Theorem 2 provides a way to construct finite-time Lyapunov functions for systems. However, for the considered system, it is not easy to check if $V(x) = \eta(|x|)$ satisfies inequality (8).

4. Examples

In this section, three examples are presented to illustrate how to analyze stability of impulsive systems with finite-time Lyapunov functions. Based on the definition of finite-time Lyapunov function (see Definition 4), in order to check if a continuous function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a finite-time Lyapunov function for system (1), it is necessary to calculate $V(x(T, x_0))$ in (8) from Definition 4. For a constant $0 < T < +\infty$, we compute the value of $x(T, x_0)$ of system (1) with respect to an initial condition x_0 by the Euler method with the time step denoted by h_t . In order to simplify the computation, for the following examples, 1-norm $|\cdot|_1$ is utilized.

4.1. Example 1. Consider the following one-dimensional system described by

$$\begin{cases} \dot{x}(t) = -x(t), & t \neq t_k, \\ x(t) = 2x(t_k^-), & t = t_k = \frac{3k}{4}, \end{cases} \tag{25}$$

where $x \in \mathbb{R}$, $k > 0$, $k \in \mathbb{Z}_+$.

Let $V_1(x) = |x|$ be a finite-time Lyapunov function candidate for system (25). To show there exists a constant $T > 0$ such that V_1 satisfies (8) from Definition 4, we calculate $V(x(T, x_0))$ with $T = (3/4)$, $h_t = (1/4)$.

$$\begin{cases} x\left(\frac{1}{4}, x_0\right) = \frac{3}{4x_0}, x\left(\frac{1}{2}, x_0\right) = \frac{9}{16x_0}, \\ x(t_1^-, x_0) = x\left(\left(\frac{3}{4}\right)^-, x_0\right) = \frac{27}{64x_0}, x\left(\frac{3}{4}, x_0\right) = \frac{27}{32x_0}, \\ x(1, x_0) \leq \frac{81}{128x_0}, x\left(\frac{5}{4}, x_0\right) \leq \frac{243}{512x_0}, \\ x(t_2^-, x_0) = x\left(\left(\frac{3}{2}\right)^-, x_0\right) = \frac{729}{2048x_0}, x\left(\frac{3}{2}, x_0\right) = \frac{729}{1024x_0}. \end{cases} \tag{26}$$

From the above calculation, the following inequalities hold.

$$\begin{aligned}
V\left(x\left(\frac{3}{4}, x_0\right)\right) &\leq \left(\frac{27}{32}\right)V(x_0), \quad x_0 \in \mathbb{R}, \\
V\left(x\left(\left(\frac{3}{2}\right)^+ + t, x(t, x_0)\right)\right) &\leq \left(\frac{27}{32V}\right)(x(t, x_0)), \quad t \in \mathbb{R}_+, x_0 \in \mathbb{R}.
\end{aligned} \tag{27}$$

Then, it is obvious that V_1 is a finite-time Lyapunov function for system (25) in \mathbb{R} . Based on Theorem 1, we have that the origin of system (25) is asymptotically stable in \mathbb{R} over the given impulse sequence (see Figure 1). Figure 1 clearly shows that $V_1(x) = |x|$ is not a Lyapunov function for system (25).

4.2. Example 2. In this section, we consider the following one-dimensional system described by

$$\begin{cases} \dot{x}(t) = x(t), & t \neq t_k, \\ x(t) = \left(\frac{1}{2}\right)x(t_k^-), & t = t_k = \frac{k}{2}, \end{cases} \tag{28}$$

where $x \in \mathbb{R}$, $k > 0$, $k \in \mathbb{Z}_+$.

Let $V_2(x) = |x|$ be a finite-time Lyapunov function candidate for system (28). To check if V_2 satisfies condition (8) from Definition 4 with $T = (1/2)$, we have to calculate with $h_t = (1/4)$.

$$\begin{cases} x\left(\left(\frac{1}{4}\right)^-, x_0\right) = \left(\frac{5}{4x_0}\right), \\ x(t_1^-, x_0) = x\left(\left(\frac{1}{2}\right)^-, x_0\right) = \left(\frac{25}{16x_0}\right), x\left(\frac{1}{2}, x_0\right) = \left(\frac{25}{32x_0}\right), \\ x\left(\left(\frac{3}{4}\right)^-, x_0\right) = \left(\frac{125}{128x_0}\right), \\ x(t_2^-, x_0) = x\left((1)^-, x_0\right) = \left(\frac{625}{512x_0}\right), x(1, x_0) = \frac{625}{1024x_0}. \end{cases} \tag{29}$$

It is evident that the following inequality is satisfied.

$$V_2\left(x\left(\left(\frac{1}{2}\right)^+ + t, x(t, x_0)\right)\right) \leq V_2(x(t, x_0)), \quad t \in \mathbb{R}_+, x_0 \in \mathbb{R}. \tag{30}$$

According to the above analysis, we conclude that V_2 is a finite-time Lyapunov function for system (28) in \mathbb{R} . Moreover, utilizing Theorem 1, it is attained that the origin of system (28) is asymptotically stable in \mathbb{R} over the given impulse sequence. Figure 2 demonstrates that V_2 is not a Lyapunov function for system (28).

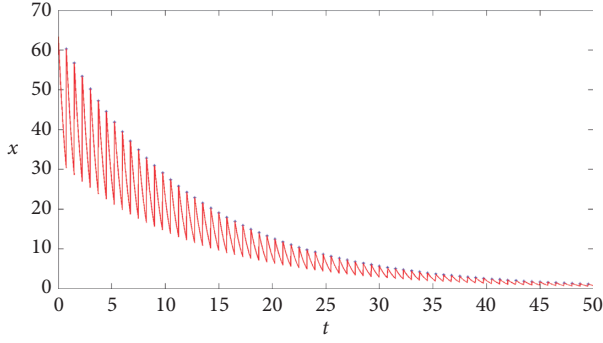


FIGURE 1: The trajectory of the state of system (25) corresponding to the initial condition $x(0) = 64$ and the resetting time $t_k = 0.75k, k = 1, 2, \dots$.

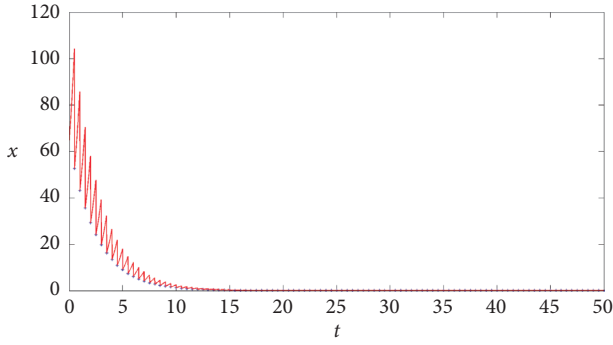


FIGURE 2: The trajectory of the state of system (28) corresponding to the initial condition $x(0) = 64$ and the resetting time $t_k = 0.5k, k = 1, 2, \dots$.

4.3. Example 3. We consider the following two-dimensional system described by

$$\begin{cases} \dot{x}_1(t) = -0.2x_2(t), & t \neq t_k, \\ \dot{x}_2(t) = x_1(t) - x_2(t), & t \neq t_k, \\ x_1(t) = x_1(t_k^-) + \left(\frac{1}{8}\right)\sin(x_1(t_k^-) + x_2(t_k^-))x_2(t_k^-), & t = t_k = 5k, \\ x_2(t) = x_2(t_k^-) + \left(\frac{1}{8}\right)\cos(x_1(t_k^-) + x_2(t_k^-))x_1(t_k^-), & t = t_k = 5k, \end{cases} \quad (31)$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2, k > 0, k \in \mathbb{Z}_+$.

For system (31), we choose $V_3(x) = |x|_1$ as a finite-time Lyapunov function candidate. Now, we show there exists a constant T such that inequality (8) from Definition 4 is satisfied. The following calculation is done with $T = 5, h_t = 1$. For the calculation, we utilize the following notations: $x_{10} = x_1(0), x_{20} = x_2(0), x_{11} = x_1(h_t, x_0), x_{21} = x_2(h_t, x_0), x_{12} = x_1(2h_t, x_0), x_{22} = x_2(2h_t, x_0), \dots, x_{1k} = x_1(kh_t, x_0), x_{2k} = x_2(kh_t, x_0)$.

$$\begin{cases} x_{11} = x_{10} - 0.2x_{20}, \\ x_{21} = x_{10}, \\ \dots, \\ x_{14} = 0.44x_{10} - 0.12x_{20}, \\ x_{24} = 0.6x_{10} - 0.16x_{20}, \\ x_1(5^-, x_0) = 0.32x_{10} - 0.088x_{20}, \\ x_2(5^-, x_0) = 0.44x_{10} - 0.12x_{20}, \\ |x_{15}| + |x_{25}| \leq \frac{9}{8} \cdot 0.76|x_0|_1 = 0.855|x_0|_1. \end{cases} \quad (32)$$

Thus, we have that

$$\begin{aligned} V_3(x(5, x_0)) &\leq 0.855V_3(x_0), \\ V_3(x(5+t, x_0)) &\leq 0.855V_3(x(t, x_0)), \quad t \in \mathbb{R}_+, x_0 \in \mathbb{R}^2. \end{aligned} \quad (33)$$

From the above calculation, it is obvious that V_3 is a finite-time Lyapunov function for system (31) in \mathbb{R}^2 . Based on Theorem 1, it is obtained that the origin of system (31) is asymptotically stable in \mathbb{R}^2 over the given impulse sequence (see Figures 3 and 4). Figure 5 demonstrates that the function V_3 is not a Lyapunov function for system (31).

5. Impulsive Control of a Chaotic System

In this section, based on Theorem 1, impulsive control is designed for a chaotic system described by

$$\dot{x} = Ax + \gamma(x), \quad (34)$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, and

$$\begin{aligned} A &= \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \\ \gamma(x) &= \begin{pmatrix} x_2x_3 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}, \end{aligned} \quad (35)$$

in which $a, b, c \in \mathbb{R}$ are positive constants. The constants a and b are called the Prandtl number and Rayleigh number, respectively. In [27], the authors studied system (34) adapted from the Lorenz system.

In order to ensure the origin is an asymptotic stable equilibrium, we design an impulsive controller for system (34) described by

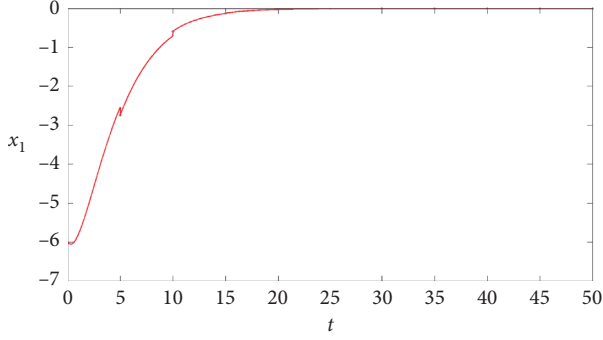


FIGURE 3: The trajectory of the state x_1 of system (31) corresponding to the initial condition $x(0) = (-6, 2)^T$ and the resetting time $t_k = 5k, k = 1, 2, \dots$.

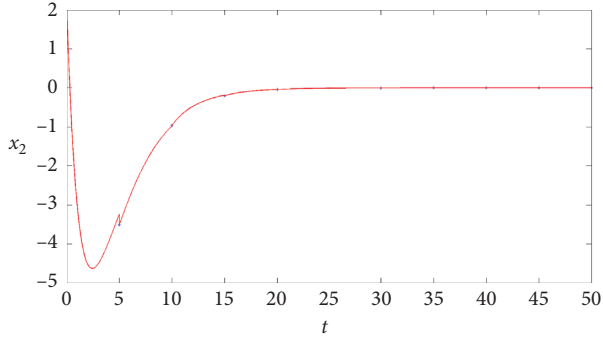


FIGURE 4: The trajectory of the state x_2 of system (31) corresponding to the initial condition $x(0) = (-6, 2)^T$ and the resetting time $t_k = 5k, k = 1, 2, \dots$.

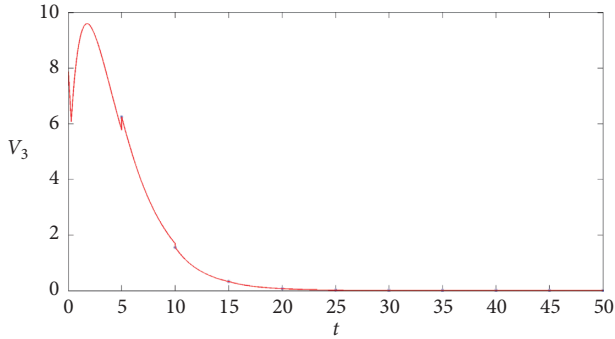


FIGURE 5: The trajectory of the finite-time Lyapunov function V_3 for system (31) with the initial condition $x(0) = (-6, 2)^T$ and the resetting time $t_k = 5k, k = 1, 2, \dots$.

$$\begin{cases} \dot{x}(t) = Ax(t) + \gamma(x(t)), & \text{for } t \neq t_k, \\ x(t) = x(t_k^+) = D(x(t_k^-)), & \text{for } t = t_k, \end{cases} \quad (36)$$

where $t_k (k = 1, 2, \dots)$, $0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$ ($t_k \rightarrow \infty$ as $k \rightarrow \infty$) are impulses, $h = t_1 = t_{k+1} - t_k (k = 1, 2, \dots)$, and

$$D(x) = \begin{pmatrix} d_{11}(x_1) & d_{12}(x_2) & d_{13}(x_3) \\ d_{21}(x_1) & d_{22}(x_2) & d_{23}(x_3) \\ d_{31}(x_1) & d_{32}(x_2) & d_{33}(x_3) \end{pmatrix}, \quad (37)$$

in which $d_{ij}: \mathbb{R} \rightarrow \mathbb{R}$ are continuous. We assume that there exist constants $\bar{d}_{ij} \in \mathbb{R}_+$ ($i, j = 1, 2, 3$) satisfying $|d_{ij}(s)| \leq \bar{d}_{ij}|s|$ for $s \in \mathbb{R}$.

For convenience, the following notations are introduced:

$$\begin{aligned} d_1 &= |\bar{d}_{11}(1 - ah)| + |\bar{d}_{12}ch| + |\bar{d}_{12}hR| + |\bar{d}_{13}hR| \\ &\quad + |\bar{d}_{22}ch| + |\bar{d}_{21}(1 - ah)| + |\bar{d}_{22}hR| + |\bar{d}_{23}hR| \\ &\quad + |\bar{d}_{31}(1 - ah)| + |\bar{d}_{32}ch| + |\bar{d}_{32}hR| + |\bar{d}_{33}hR|, \\ d_2 &= |\bar{d}_{11}ah| + |\bar{d}_{11}hR| + |\bar{d}_{12}(1 - h)| + |\bar{d}_{13}hR| \\ &\quad + |\bar{d}_{22}(1 - h)| + |\bar{d}_{21}hR| + |\bar{d}_{21}ah| + |\bar{d}_{23}hR| \\ &\quad + |\bar{d}_{31}ah| + |\bar{d}_{31}hR| + |\bar{d}_{32}(1 - h)| + |\bar{d}_{33}hR|, \\ d_3 &= |\bar{d}_{11}Rh| + |\bar{d}_{12}hR| + |\bar{d}_{13}(1 - bh)| \\ &\quad + |\bar{d}_{22}Rh| + |\bar{d}_{21}hR| + |\bar{d}_{23}(1 - bh)| \\ &\quad + |\bar{d}_{31}Rh| + |\bar{d}_{32}hR| + |\bar{d}_{33}(1 - bh)|, \end{aligned} \quad (38)$$

where R from Corollary 1 is a positive constant. It is clear that $d_i (i = 1, 2, 3)$ are positive constants.

Corollary 1. Let $R > 0$ be a constant. Consider system (36) on a bounded set $\mathcal{D} \subset B(0, R) \subset \mathbb{R}^3$. Let $M = \max_{i=1,2,3} c_i$. If $M < 1$ is satisfied, then the function $V: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $V(x) = |x|_1$ is a finite-time Lyapunov function for system (36) with $T = h$. Therefore, the origin of system (36) is asymptotically stable in \mathcal{D} over the given impulse sequence.

Proof. Under the conditions, we prove $V(x) = |x|_1$ is a finite-time Lyapunov function for system (36). It is obvious that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|). \quad (39)$$

Now, we have to calculate $V(x(h, x_0))$ for system (36) with $x_0 \in \mathcal{D}$ by the Euler method with the step size h . Let $x_0 = (x_{10}, x_{20}, x_{30})^T$.

$$\begin{cases} x_1(h^-, x_0) = x_{10} + [a(x_{20} - x_{10}) + x_{20}x_{30}]h, \\ x_2(h^-, x_0) = x_{20} + [cx_{10} - x_{20} - x_{10}x_{30}]h, \\ x_3(h^-, x_0) = x_{30} + [x_{10}x_{20} - bx_{30}]h, \\ x(h, x_0) = D(x(h^-)). \end{cases} \quad (40)$$

By calculation, it is obtained that $V(x(h, x_0)) \leq M|x_0|_1 = MV(x_0)$. Thus, the function V satisfies inequality (8) from Definition 4. By Theorem 1, we obtain that the origin of system (36) is asymptotically stable in Ω over the given impulse sequence. \square

Remark 5. In the proof, we calculate $x(h, x_0)$ by the Euler method with the step size h . The reason for letting the step size being h is that it is easy to estimate the value of $|x(h^-, x_0)|_1$.

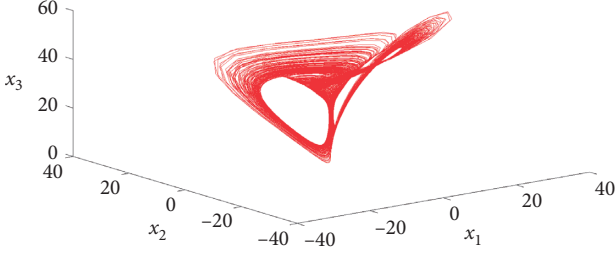


FIGURE 6: The trajectory of the states of system (34) with the coefficients $a = 35$, $b = (8/3)$, and $c = 25$ and the initial condition $x(0) = (3, 4, 5)^T$.

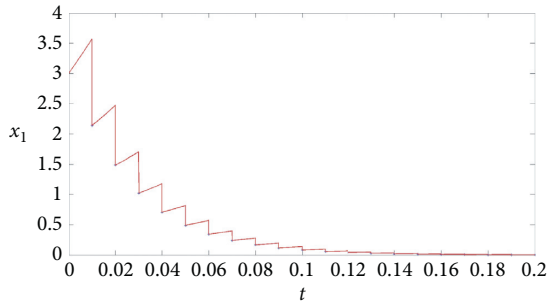


FIGURE 7: The trajectory of the state x_1 of system (36) with the coefficients $a = 35$, $b = (8/3)$, $c = 25$, $t_k = 0.01k$ ($k = 1, 2, \dots$), $d_{ij} = 0$ ($i, j = 1, 2, 3, i \neq j$), and $d_{ii} = 0.6$ ($i = 1, 2, 3$) and the initial condition $x(0) = (3, 4, 5)^T$.

5.1. Simulation Results. In this section, system (34) is considered as an example with the coefficients $a = 35$, $b = (8/3)$, $c = 25$, and $x_0 = (3, 4, 5)^T$. Figure 6 shows that a chaotic attractor exists for system (34) with the given conditions and is similar to that for the Lorenz system.

An impulsive controller is designed as follows: $t_k = 0.01k$ ($k = 1, 2, \dots$), $d_{ij} = 0$ ($i, j = 1, 2, 3, i \neq j$), $d_{ii} = 0.6$ ($i = 1, 2, 3$). We consider system (36) with the given coefficients on $\mathcal{D} = B(0, 20) \subset \mathbb{R}^3$. The constraints of Corollary 1 are satisfied with $R = 20$, $h = 0.01$, $M = 0.84$. Then, $V(x) = |x|_1$ is a finite-time Lyapunov function for system (36) with the given coefficients. Hence, the origin of system (36) with the given coefficients is asymptotic stable on \mathcal{D} (see Figures 7–9). Figure 10 clearly shows that V is not a Lyapunov function for system (36).

Remark 6. From Figures 7–9, we obtain that the simulation results are similar to that of [24], since the designed impulsive control is similar to each other. However, from examples of Section 4 and this chaotic system, we can conclude that construction of finite-time Lyapunov functions or design of impulsive control based on Theorem 1 is easier than the method proposed in [24]. The reason for this point is that we do not have to calculate the derivative of Lyapunov function along the continuous part of the trajectory of the considered system.

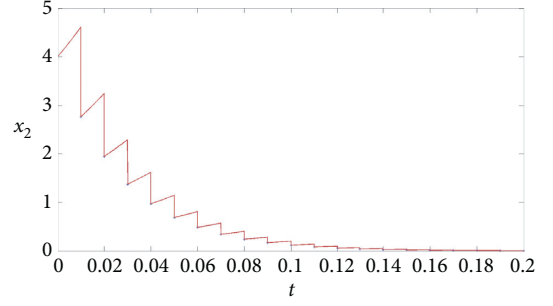


FIGURE 8: The trajectory of the state x_2 of system (36) with the coefficients $a = 35$, $b = (8/3)$, $c = 25$, $t_k = 0.01k$ ($k = 1, 2, \dots$), $d_{ij} = 0$ ($i, j = 1, 2, 3, i \neq j$), and $d_{ii} = 0.6$ ($i = 1, 2, 3$) and the initial condition $x(0) = (3, 4, 5)^T$.

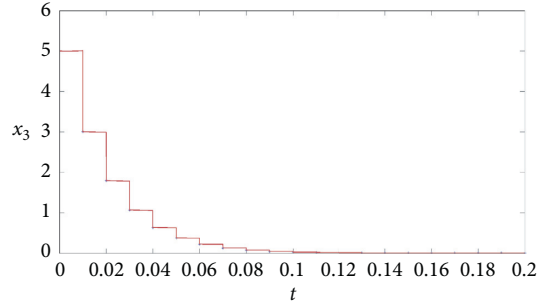


FIGURE 9: The trajectory of the state x_3 of system (36) with the coefficients $a = 35$, $b = (8/3)$, $c = 25$, $t_k = 0.01k$ ($k = 1, 2, \dots$), $d_{ij} = 0$ ($i, j = 1, 2, 3, i \neq j$), and $d_{ii} = 0.6$ ($i = 1, 2, 3$) and the initial condition $x(0) = (3, 4, 5)^T$.

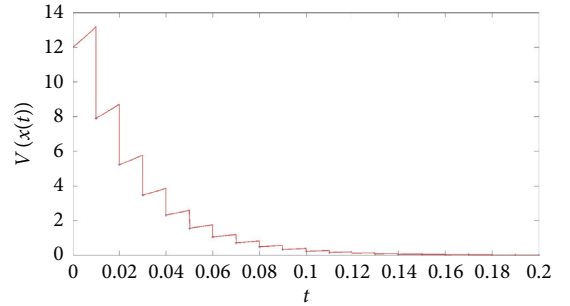


FIGURE 10: The trajectory of the finite-time Lyapunov function V of system (36) with the coefficients $a = 35$, $b = (8/3)$, $c = 25$, $t_k = 0.01k$ ($k = 1, 2, \dots$), $d_{ij} = 0$ ($i, j = 1, 2, 3, i \neq j$), and $d_{ii} = 0.6$ ($i = 1, 2, 3$) and the initial condition $x(0) = (3, 4, 5)^T$.

6. Conclusion

In this paper, we introduced the definition of finite-time Lyapunov function for impulsive systems. It was proved that if there exists a finite-time Lyapunov function for system (1), then the origin of system (1) is asymptotically stable (Theorem 1). It is worthy to point out that the conditions imposed on finite-time Lyapunov functions for system (1) are more relaxed

than those on Lyapunov function which decrease along the continuous part of the trajectory or have negative jumps at all impulses. Finite-time Lyapunov functions can increase during some continuous part of the trajectory of the considered system and have positive jumps at some resetting times. This point was demonstrated by example. A converse finite-time Lyapunov theorem (Theorem 2) was proposed. Three examples were presented to illustrate how to analyze stability of the origin of an impulsive system via finite-time Lyapunov functions. According to our main results, impulsive control was designed to ensure the origin of the considered chaotic system is asymptotically stable. Some simulation results of the chaotic system with impulsive control were presented to show how to design an impulsive controller for the chaotic system by finite-time Lyapunov functions.

Data Availability

The data used to support the findings of this study are included within the article. The data used in computation are stated for each example in the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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